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Dual Artificial Variable-Free Simplex Algorithm for Solving Neutrosophic Linear Programming Problems

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Abstract. This paper presents a simplified form of dual simplex algorithm for solving linear programming problems with fuzzy and neutrosophic numbers which supplies some great benefits over phase 1 of traditional dual simplex algorithm. For instance, it could start with any infeasible basis of linear programming problems; it doesn't need any kind of artificial variables or artificial constraints, so the number of variables of the proposed method is less than the number of variables in the traditional dual simplex algorithm, therefore; the run time for the proposed algorithm is also faster than the phase 1 of traditional dual simplex algorithm, and the proposed method overcomes the traditional dual simplex algorithm for both the fuzzy approach and the neutrosophic approach according to the iterations number. We also use numerical examples to compare between the fuzzy and the neutrosophic approaches, the results show that the neutrosophic approach is more accurate than the fuzzy approach. Furthermore, the proposed algorithm overcomes the phase 1 of traditional dual simplex algorithm for both the fuzzy and neutrosophic approach.

Keywords: Fuzzy Number; Neutrosophic Number; Rank Function; Dual Artificial Variable Free version of Simplex Method.

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1. Introduction

Linear programming is the most frequently applied operations research technique. A linear programming model represents real world situations with some sets of parameters determined by experts and decision makers while in real world applications certainty, reliability and precision are often illusory concepts, therefore experts and decision makers cannot determine the exact value of parameters, or they may not be in a position to specify the objective functions or constraints precisely. By implying fuzzy and neutrosophic set theory to linear programming, which leads to fuzzy and neutrosophic linear programming, the so-called problems are being overcome. All of this causes us to resort to fuzzy and neutrosophic numbers that deal with uncertain information. Neutrosophic Set (NS) [26] is a generalization of the fuzzy set [27] and intuitionistic fuzzy set [3]; each element of set had a truth, indeterminacy and falsity membership functions. So, neutrosophic set can assimilate inaccurate, vague and maladjusted information efficiently and effectively.

After the pioneering work on fuzzy linear programming by Tanaka et al. [17,18] and Zimmermann [19], several kinds of fuzzy linear programming problems have appeared in the literatures and different methods have been proposed to solve such problems [16,23,29]. One important class of these methods that has been high-lighted by many researches is based on comparing of fuzzy numbers using ranking functions. Based on this idea, Maleki et al. [23] proposed a simple method for solving fuzzy number linear programming (FNLP) problems. After that, many various approaches appeared that deal with the vague and imprecise information such as intuitionistic fuzzy set and neutrosophic set.

Arsham [5] introduced the simplex method without using artificial variables. First, the basic feasible variable set (BVS) is determined to be the empty set. Then, the non-basic variable is chosen to be the basic variable one by one until the BVS is full. After the problem has the complete BVS, the simplex method is performed. However, this method has the mistake as shown by Enge and Huhn [21] in 1998.

Pan [24] presented the simplex algorithm by avoiding artificial variables. The algorithm starts when the initial basis gives primal and dual infeasible solutions by adjusting negative reduced costs to a single positive value. Then, the dual solution is feasible and the dual simplex method is performed. After the optimal solution is found in this step, the original reduced costs are restored and the simplex method is performed.

Abdel-Basset et.al [1] proposed the neutrosophic simplex algorithm that solves the neutrosophic linear programming (NLP). They introduced a comparison between fuzzy approach and neutrosophic approach by using numerical examples. On the other hand, their manuscript has some incorrect assumptions.

Akanksha Singh et.al [25] spotted some incorrect assumptions in Abdel-Basset's manuscript [1]. They suggested the required modifications in Abdel-Basset's method. On the other hand, [21] used different rank functions to compare between fuzzy approach and neutrosophic approach, which makes this comparison not fair. Therefore, in this essay, the authors emphasis use the same rank function.

Elsayed Badr et.al [2] proposed a novel method that deal with initial non basic solution. This method is called neutrosophic two-phase method and it solves the linear programming problems with neutrosophic numbers. They used the same rank function when they compared between fuzzy approach and neutrosophic approach, which makes the comparison is fair.

For more details about the linear programming, the reader can refer to [6,7,11-13,15]. On the other hand, for more details about the fuzzy linear programming, the reader is referred to [2,9,10,20]. Finally, for more details about the neutrosophic linear programming, the reader may refer to [8].

In this paper, we apply dual artificial variable-free simplex algorithm for solving linear programming problems with fuzzy and neutrosophic numbers, which has several advantages, for instance, it could start with any infeasible basis of linear programming problem. This algorithm follows the same pivoting sequence as of dual simplex phase 1 without showing any explicit description of artificial variables which also makes it space efficient. The proposed algorithm reduces the size of the problem and reduces the execution time to solve the problem. Then the CPU time for the proposed method is also faster than the phase 1 of traditional dual simplex method. So, the proposed method can reduce the computational time. We also compare between the neutrosophic approach and the fuzzy approach using numerical examples.

The remaining parts of this work are organized as follows: In sec. 2, the fundament concepts of fuzzy and neutrosophic sets have been presented, and a new technique which converts the fuzzy representation to the neutrosophic representation has been proposed. Akanksha Singh *et al.*'s modifications [25] and a new neutrosophic dual artificial variable-free simplex algorithm (NDAVFSA) are proposed in Sec. 3. In Sec. 4, a numerical example that shows the importance of the proposed modification for primal neutrosophic simplex method has been introduced, and the superiority of the proposed algorithm (NDAVFSA) on the primal neutrosophic simplex algorithm has been shown. Finally, we introduce conclusions and the future work in Sec. 5.

2. Preliminaries

In this section, three subsections have been introduced. First one is representation of the fuzzy numbers. Second, the representation of the neutrosophic numbers. Finally, we show that how do to convert the fuzzy numbers and neutrosophic numbers to crisp number.

2.1. Fuzzy Representation

We review the fundamental notions of fuzzy set theory, initiated by Bellman and Zadeh [22].

2.1.1. Definition

A convex fuzzy set $\tilde{A}$ on $\mathbb{R}$ is a fuzzy number if the following conditions hold:

(a) Its membership function is piecewise continuous.

(b) There exist three intervals [a, b], [b, c], [c, d] such that $\mu_{\tilde{a}}$ is increasing on [a, b], equal to 1 on $[b, c]$, decreasing on $[c, d]$ and equal to 0 elsewhere.

2.1.2. Definition

Let $\tilde{a} = (a^L, a^U, \alpha, \beta)$ denote the trapezoidal fuzzy number, where

$(a^L - \alpha, a^U + \beta)$ is the support of $\tilde{a}$and $[a^L, a^U]$ its core.

Remark 1: We denote the set of all trapezoidal fuzzy numbers by F ($\mathbb{R}$) as shown as in figure 1.

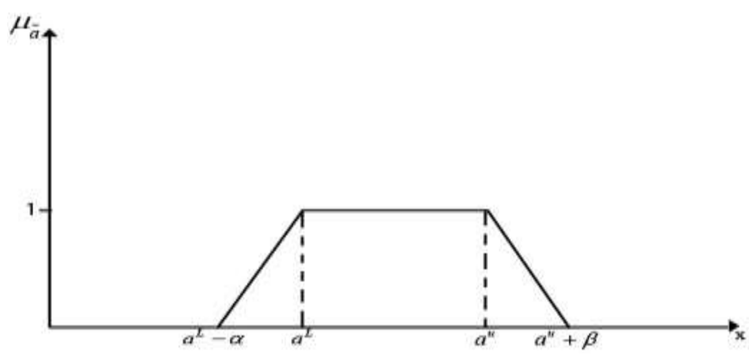

Figure 1: Truth membership function of trapezoidal fuzzy number ã

2.1.3. Definition

Let $\tilde{a} = (a^L, a^U, \alpha, \beta)$ and $\tilde{b} = (b^L, b^U, \gamma, \theta)$ be two trapezoidal fuzzy numbers, the arithmetic operation on the trapezoidal fuzzy number are defined as:

$x\tilde{a} = (xa^L, xa^U, x\alpha, x\beta);\ x > 0, \quad x \in \mathbb{R}.$

$x\tilde{a} = (xa^U, xa^L, -x\beta, -x\alpha);\ x < 0, \quad x \in \mathbb{R}.$

$\tilde{a}+\tilde{b}=(a^L, a^U, \alpha, \beta)+ (b^L, b^U, \gamma, \theta)= [a^L + b^L, a^U + b^U, \alpha + \gamma, \beta + \theta]$

2.2. Neutrosophic Representation

In this subsection, some basic definitions in the neutrosophic set theory are introduced:

2.2.1. Definition [1]

the trapezoidal neutrosophic number $\tilde{A}$ is a neutrosophic set in $\mathbb{R}$ with the following truth (T), indeterminacy (I) and falsity (F) membership functions as shown in figure 2:

$$T_{\tilde{A}}(x) = \begin{cases} \frac{\alpha_{\tilde{A}}(x-a_1)}{a_2-a_1} : a_1 \le x \le a_2 \\ \alpha_{\tilde{A}} \quad : a_2 \le x \le a_3 \\ \alpha_{\tilde{A}}(\frac{x-a_3}{a_4-a_3}) : a_3 \le x \le a_4 \\ 0 \quad \text{otherwise} \end{cases} \qquad I_{\tilde{A}}(x) = \begin{cases} \frac{(a_2-x+\theta_{\tilde{A}}(x-a_1'))}{a_2-a_1'} : a_1' \le x \le a_2 \\ \theta_{\tilde{A}} \quad : a_2 \le x \le a_3 \\ \frac{(x-a_3+\theta_{\tilde{A}}(a_4'-x))}{a_4'-a_3} : a_3 \le x \le a_4' \\ 1 \quad \text{otherwise} \end{cases}$$

$$F_{\tilde{A}}(x) = \begin{cases} \frac{(a_2 - x + \beta_{\tilde{A}}(x - a_1''))}{a_2 - a_1''} : a_1'' \le x \le a_2 \\ \beta_{\tilde{A}} \quad : a_2 \le x \le a_3 \\ \frac{(x - a_3 + \beta_{\tilde{A}}(a_4'' - x))}{a_4'' - a_3} : a_3 \le x \le a_4'' \\ 1 \quad \text{otherwise} \end{cases}$$

where $\alpha_{\tilde{A}}$, $\theta_{\tilde{A}}$ and $\beta_{\tilde{A}}$ represent the maximum degree of truthiness, minimum degree of indeterminacy and minimum degree of falsity, respectively, $\alpha_{\tilde{A}}$, $\theta_{\tilde{A}}$ and$\beta_{\tilde{A}} \in [0,1]$

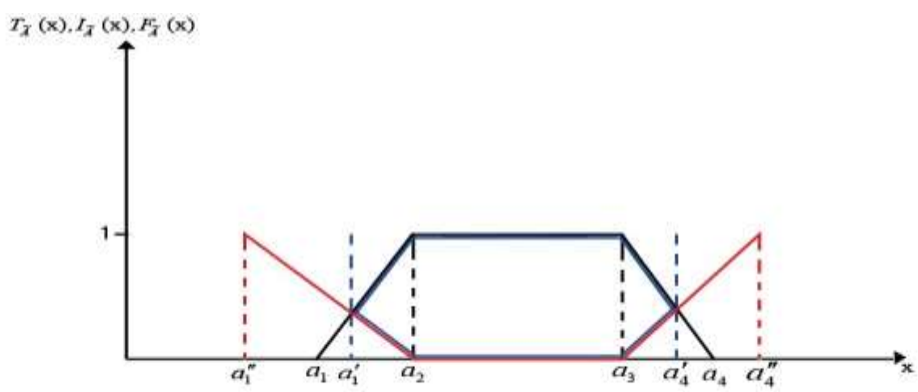

Figure 2: Truth, indeterminacy and falsity membership functions of trapezoidal neutrosophic number $\tilde{A}$

2.2.2. Definition [1]

the mathematical operations on two trapezoidal neutrosophic numbers. $\widetilde{A} =< a_1, a_2, a_3, a_4; \alpha_{\widetilde{A}}, \theta_{\widetilde{A}}, \beta_{\widetilde{A}} >$ and $\tilde{B} = < b_1, b_2, b_3, b_4; \alpha_{\widetilde{B}}, \theta_{\widetilde{B}}, \beta_{\widetilde{B}} >$ are as follows:

$$\widetilde{A} + \widetilde{B} = < (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4); \alpha_{\widetilde{A}} \wedge \alpha_{\widetilde{A}}, \theta_{\widetilde{A}} \vee \theta_{\widetilde{A}}, \beta_{\widetilde{A}} \vee \beta_{\widetilde{A}} >$$

$$\widetilde{A} - \widetilde{B} = < (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1); \alpha_{\widetilde{A}} \wedge \alpha_{\widetilde{A}}, \theta_{\widetilde{A}} \vee \theta_{\widetilde{A}}, \beta_{\widetilde{A}} \vee \beta_{\widetilde{A}} >$$

$$\widetilde{A}^{-1} = < (\frac{1}{a_4}, \frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_1}); \alpha_{\widetilde{A}}, \theta_{\widetilde{A}}, \beta_{\widetilde{A}} > \text{where } \widetilde{A} \neq 0)$$

$$\lambda\widetilde{A} = \begin{cases} < \lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4; \alpha_{\widetilde{A}}, \theta_{\widetilde{A}}, \beta_{\widetilde{A}} > : \lambda > 0 \\ < \lambda a_4, \lambda a_3, \lambda a_2, \lambda a_1; \alpha_{\widetilde{A}}, \theta_{\widetilde{A}}, \beta_{\widetilde{A}} > : \lambda < 0 \end{cases}$$

$$\widetilde{A}\widetilde{B} = \begin{cases} < (a_1b_1, a_2b_2, a_3b_3, a_4b_4); \alpha_{\widetilde{A}} \wedge \alpha_{\widetilde{A}}, \theta_{\widetilde{A}} \vee \theta_{\widetilde{A}}, \beta_{\widetilde{A}} \vee \beta_{\widetilde{A}} > \\ \quad \text{if } (a_4 > 0, b_4 > 0) \\ < (a_1b_4, a_2b_3, a_3b_2, a_4b_1); \alpha_{\widetilde{A}} \wedge \alpha_{\widetilde{A}}, \theta_{\widetilde{A}} \vee \theta_{\widetilde{A}}, \beta_{\widetilde{A}} \vee \beta_{\widetilde{A}} > \\ \quad \text{if } (a_4 < 0, b_4 > 0) \\ < (a_4b_4, a_3b_3, a_2b_2, a_1b_1); \alpha_{\widetilde{A}} \wedge \alpha_{\widetilde{A}}, \theta_{\widetilde{A}} \vee \theta_{\widetilde{A}}, \beta_{\widetilde{A}} \vee \beta_{\widetilde{A}} > \\ \quad \text{if } (a_4 < 0, b_4 < 0) \end{cases}$$

$$\frac{\widetilde{A}}{\widetilde{B}} \begin{cases} < \left(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}\right); \alpha_{\widetilde{A}} \wedge \alpha_{\widetilde{A}}, \theta_{\widetilde{A}} \vee \theta_{\widetilde{A}}, \beta_{\widetilde{A}} \vee \beta_{\widetilde{A}} > \\ \quad \text{if } (a_4 > 0, b_4 > 0) \\ < \left(\frac{a_4}{b_4}, \frac{a_3}{b_3}, \frac{a_2}{b_2}, \frac{a_1}{b_1}\right); \alpha_{\widetilde{A}} \wedge \alpha_{\widetilde{A}}, \theta_{\widetilde{A}} \vee \theta_{\widetilde{A}}, \beta_{\widetilde{A}} \vee \beta_{\widetilde{A}} > \\ \quad \text{if } (a_4 < 0, b_4 > 0) \\ < \left(\frac{a_4}{b_1}, \frac{a_3}{b_2}, \frac{a_2}{b_3}, \frac{a_1}{b_4}\right); \alpha_{\widetilde{A}} \wedge \alpha_{\widetilde{A}}, \theta_{\widetilde{A}} \vee \theta_{\widetilde{A}}, \beta_{\widetilde{A}} \vee \beta_{\widetilde{A}} > \\ \quad \text{if } (a_4 < 0, b_4 < 0) \end{cases}$$

2.3. Transfer from Fuzzy Representation to Neutrosophic Representation [25]

The goal of this section is to explain how to convert fuzzy numbers representation into neutrosophic numbers representation. This transformation is used for simplicity and to make the comparison fair. There are many types of techniques to transfer from fuzzy to neutrosophic representation such as, ranking functions and ∝-cut technique.

2.3.1. Definition.

Ranking function is a viable approach for ordering fuzzy numbers and neutrosophic numbers. It is known that there are many ranking functions for ordering the fuzzy numbers and neutrosophic numbers.

In this subsection, we explain how to apply technique to convert from fuzzy number to neutrosophic number:

From Figure 1 and Figure 2 we can illustrate the following relations between the two representations:

$$a_1 = a_2 - \alpha, a_2 = a^L, a_3 = a^U \text{ and } a_4 = a_3 + \beta \quad (1)$$

Assuming that the ranking function is used for ordering the fuzzy numbers as follows:

$$R(\tilde{a}) = \frac{a^L + a^U}{2} + \frac{\beta - \alpha}{4} \qquad (2)$$

$\beta - \alpha = a_4 - a_3 - (a_2 - a_1) = a_4 - a_3 - a_2 + a_1$

$R(\tilde{a}) = \frac{a_2+a_3}{2} + \frac{a_4-a_3-a_2+a_1}{4} = \frac{a_2+a_3+a_4+a_1}{4}$

From the relations (1) & (2) we can express the rank function is used for ordering the neutrosophic numbers as follows:

$$R(\tilde{a}) = \frac{1}{4}\sum_{i=1}^{4} \tilde{a}_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}}) \qquad (3)$$

From (1), we can convert fuzzy numbers representation into neutrosophic numbers representation. On the other hand from (2) and (3), we can use the same function for both fuzzy numbers and neutrosophic numbers to obtain a fair comparison between them.

(i) Assuming that $T_{\tilde{A}} = 1$, $I_{\tilde{A}} = 0$, $\tilde{F}_A = 0$, then the TrNN $\tilde{a} =< a_1, a_2, a_3, a_4; T_{\tilde{A}}, I_{\tilde{A}}, F_{\tilde{A}} >$ equal to number $\tilde{a} =< a_1, a_2, a_3, a_4; 1,0,0 >$ and hence, in this case,

The expression $R(\tilde{a}) = \frac{1}{4}\sum_{i=1}^{4} a_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$ is equivalent to the expression

$$R(\tilde{a}) = \frac{1}{4}\sum_{i=1}^{4} \tilde{a}_i + 1$$

(ii) Furthermore, it is well known the fact that if $a_1 = a_2 = a_3 = a_4$ then the trapezoidal neutrosophic number $\tilde{A} =< a_1, a_2, a_3, a_4; 1,0,0 >$ will be transformed into a real number A = (a, a, a, a; 1, 0, 0) and hence, in this case, the expression $R(\tilde{a}) = \frac{1}{4}\sum_{i=1}^{4} a_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$ is equivalent to the expression $R(A) = a + 1 \neq a$

The following table represents the fuzzy ranking function, and the corresponding neutrosophic ranking function and the corresponding real ranking function.

Table 1: fuzzy ranking function into it's corresponding neutrosophic ranking function

Fuzzy Rank Function	**Corresponding Neutrosophic Rank Function**	**Corresponding Real Rank function of constraints**
$R(\tilde{a}) = (\frac{a^l + a^u}{2} + \frac{\beta - \alpha}{4})$	$R(\tilde{a}) = \frac{1}{4}\sum_{j=1}^{4} a_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$	$R(a) = a + 1$

3. Algorithms

In this section; firstly, we present Akanksha Singh *et al.*'s modifications [25] and the proposed modification about the mathematical incorrect assumptions, considered by Abdel-Basset *et al.* [1] in their proposed method to convert from neutrosophic numbers into real numbers. Secondly, we propose a new fuzzy dual artificial variable free simplex algorithm. Finally, we develop this algorithm in order to solve linear programming with neutrosophic numbers (neutrosophic dual artificial variable free simplex algorithm).

3.1. Akanksha Singh *et al.*'s modifications [25]

The following table presents Akanksha Singh et al.'s modifications to convert from neutrosophic number to crisp number.

Table 2: Akanksha Singh et al.'s modifications.

No	NLPP- (Type)	NLPP- (Form)	Exact Crisp LPP
1	The coefficients of the objective function are represented by trapezoidal neutrosophic numbers	$Max\backslash \text{Min}\ \left[\sum_{j=1}^{n} = \tilde{c}_j x_j\right]$ s.t $\sum_{j=1}^{n} a_{ij} x_j\ \le, \ge,\ b_j$, $i = 1,2,\ldots\ldots,m; x_j \ge 0$, $j = 1,2,\ldots\ldots,n.$	$Max\ /Min\ [\sum_{j=1}^{n} R(\tilde{c}_j x_j) - \sum_{j=1}^{n} T_{\tilde{c}_j} x_j + \sum_{j=1}^{n} I_{\tilde{c}_j} x_j + \sum_{j=1}^{n} F_{\tilde{c}_j} x_j + \min_{1\le j\le n}\left\{T_{\tilde{c}_j} x_j\right\} - \max_{1\le j\le n}\left\{I_{\tilde{c}_j} x_j\right\} - \max_{1\le j\le n}\left\{F_{\tilde{c}_j} x_j\right\}]$ $s.t.$ $\sum_{j=1}^{n} a_{ij} x_j\ \le, \ge, = b_j$, $i = 1,2,\ldots\ldots,m;$ $x_j \ge 0$, $j = 1,2,\ldots\ldots,n.$
2	The coefficients of constraints variables and right hand side are represented by trapezoidal neutrosophic numbers	$Max\backslash \text{Min}\ \left[\sum_{j=1}^{n} = c_j x_j\right]$ s.t. $\sum_{j=1}^{n} \tilde{a}_{ij} x_j\ \le, \ge, = \tilde{b}_j$, $i = 1,2,\ldots\ldots,m;\ \ x_j \ge 0$, $j = 1,2,\ldots\ldots,n.$	Max / Min $\sum_{j=1}^{n} c_j x_j$ s.t. $\left[\sum_{j=1}^{n} R(\tilde{a}_{ij} x_j) - \sum_{j=1}^{n} T_{\tilde{a}_{ij}} x_j + \sum_{j=1}^{n} I_{\tilde{a}_{ij}} x_j + \sum_{j=1}^{n} F_{\tilde{a}_{ij}} x_j + \min_{1\le j\le n}\left\{T_{\tilde{a}_{ij}} x_j\right\} - \max_{1\le j\le n}\left\{I_{\tilde{a}_{ij}} x_j\right\} - \max_{1\le j\le n}\left\{F_{\tilde{a}_{ij}} x_j\right\}\right] \le, \ge, = R(\tilde{b}_i)$ $x_j \ge 0$, $j = 1,2,\ldots\ldots,n.$
3	All parameters are represented by trapezoidal neutrosophic numbers, except variables are exemplified only by real values	$Max\backslash \text{Min}\ \left[\sum_{j=1}^{n} = \tilde{c}_j x_j\right]$ s.t. $\sum_{j=1}^{n} \tilde{a}_{ij} x_j\ \le, \ge, = \tilde{b}_j$, $i = 1,2,\ldots\ldots,m; x_j \ge 0$, $j = 1,2,\ldots\ldots,n.$	$Max\ /Min\ [\sum_{j=1}^{n} R(\tilde{c}_j x_j) - \sum_{j=1}^{n} T_{\tilde{c}_j} x_j + \sum_{j=1}^{n} I_{\tilde{c}_j} x_j + \sum_{j=1}^{n} F_{\tilde{c}_j} x_j + \min_{1\le j\le n}\left\{T_{\tilde{c}_j} x_j\right\} - \max_{1\le j\le n}\left\{I_{\tilde{c}_j} x_j\right\} - \max_{1\le j\le n}\left\{F_{\tilde{c}_j} x_j\right\}]$ s.t. $(\sum_{j=1}^{n} R(\tilde{a}_{ij}) x_j) + 1\ \le, \ge, = R(\tilde{b}_j)$, $i = 1,2,\ldots\ldots,m;$ $x_j \ge 0$, $j = 1,2,\ldots\ldots,n.$
4	The coefficients of objective function and constraints variables are represented by real numbers and right hand side are represented by trapezoidal neutrosophic numbers	$Max\backslash \text{Min}\ \left[\sum_{j=1}^{n} = c_j x_j\right]$ s.t. $\sum_{j=1}^{n} a_{ij} x_j\ \le, \ge, = \tilde{b}_j$, $i = 1,2,\ldots\ldots,m; x_j \ge 0$, $j = 1,2,\ldots\ldots,n.$	Max / Min $\sum_{j=1}^{n} c_j x_j$ s.t. $R\left[\sum_{j=1}^{n}(a_{ij} x_j)\right] \le, \ge, = R(\tilde{b}_i)$ $x_j \ge 0$, $j = 1,2,\ldots\ldots,n.$

Remark 2:

- If $R(a) = a + 1$ and the coefficients of the objective function & constraints variables are real, then the fuzzy linear programming problem is equivalent to the neutrosophic linear programming problem.
- ~ : represents the presence of neutrosophic numbers within the matrices or vectors.
- NLPP: neutrosophic linear programming problem.

3.2. A novel Neutrosophic Dual Artificial Variable Free Simplex Algorithm.

Nayatullah et al [22] proposed a streamlined artificial variable free version of simplex algorithm (AVFSA) for solving the linear programming problems with real numbers. In this section, we propose a new algorithm which solves linear programming with neutrosophic numbers (Neutrosophic Dual Artificial Variable-Free Simplex Algorithm NDAVFSA). The proposed algorithm overcame traditional neutrosophic dual simplex algorithms.

Algorithm 1: Neutrosophic Dual Artificial variable -Free Simplex Algorithm (NDAVFSA)

Step 0: (Initialization)

- Converting fuzzy numbers into neutrosophic numbers according to Section 2.3.1 [25]
- Apply Akanksha Singh et al.'s modifications according to Section 3.1

Step 1: Let $\tilde{K}$ be a maximal subset of $\tilde{B}$ such that $\tilde{B} = \{ j : d_{0j} < 0 , j \epsilon \tilde{N}\}$. If $\tilde{K} = \varphi$ then $D(\tilde{B})$ is dual feasible. Exit.

Step 2: Denote the basic variables y_k by $-y_k^-$ and compute dual infeasibility objective vector $W'(\tilde{B}) \epsilon R^{\tilde{B}}$ such that $w'_i = \sum_{j \in k} d_{ij},\ i \in \tilde{B}$. Place w' to the right of the dictionary $D(\tilde{B})$.

Step 3: Let $\tilde{L} \subseteq \tilde{B}$ such that $\tilde{L} = \{ i : w'_i < 0,\ i \in \tilde{B} \}$. If $\tilde{L} = \varphi$ then $D(\tilde{B})$ is dual inconsistent. Exit.

Step 4: (Choice of entering variable)

Choose r ∈ $\tilde{L}$ such that $w'_r \leq w'_i\ \forall i \in \tilde{L}$ (Ties are broken arbitrarily)

Step 5: (Choice of leaving variable)

Choose $k_1 \in \tilde{K}$ and $k_2 \in \tilde{N} \setminus \tilde{K}$ such that:

k_1= arg max $\{\{\frac{d_{0j}}{d_{rj}} | (d_{0j} \leq 0, d_{rj} > 0)\}, j \in \tilde{K} \}$

k_2= arg max $\{\{\frac{d_{0j}}{d_{rj}} | (d_{0j} \geq 0, d_{rj} < 0)\}, j \in \tilde{N} \setminus \tilde{K} \}$ Set $\tilde{K}$: = arg max $\{\frac{d_{0k_1}}{\tilde{d}_{rk_1}}, \frac{d_{0k_2}}{d_{rk_2}}\}$

Step 6: Make a pivot on (r, k) ($\Rightarrow$ Set $\tilde{B} := (\tilde{B} \cup \{k\})\setminus\{r\}, \tilde{N} := (\tilde{N} \cup \{r\})\setminus\{k\}$ and update $D(\tilde{B})$ along with the appended $w'(\tilde{B})$).

Step 7: If $k \in \tilde{K}$, set $\tilde{K} := \tilde{K} \setminus \{k\}$ and $w'_k := w'_k + 1$, replace notation of $-y_k^-$ by y_k

Step 8: Pivot operation

For $r \in \tilde{B}, k \in \tilde{N}$ and (r, k) being the position of the pivot element d_{rk} ($\neq 0$) of D, then one can obtain an updated equivalent short table $D(\tilde{B}')$ with a new basis $\tilde{B}' := (\tilde{B} \cup \{k\})\setminus\{r\}$ and the new non-basis $\tilde{N}' := (\tilde{N} \cup \{r\})\setminus\{k\}$ by performing the following operations on $D(\tilde{B})$.

$$d'_{rk} := \frac{1}{d_{rk}}$$

$$d'_{rj} := \frac{d_{rj}}{d_{rk}}, j \in \tilde{N}\setminus\{k\}$$

$$d'_{ik} := -\frac{d_{ik}}{d_{rk}}, i \in \tilde{B}\setminus\{r\}$$

$$d'_{ij} := d_{ij} - d_{rj} \times \frac{d_{ik}}{d_{rk}}, i \in \tilde{B}\setminus\{r\}, j \in \tilde{N}\setminus\{k\}$$

The above replacement is known as a pivot operation on (r, k).

Step 9: If $\widetilde{K} = \varphi$, then $D(\tilde{B})$ is dual feasible. Exit.
Otherwise, go to step 3.

Step 10: If phase 1 ends with an objective value equal to zero, it implies that all artificial variables have attained a value zero (means all infeasibilities have been removed) and our current basis is feasible to the original problem, then we return to the original objective and proceed with simplex phase 2. **Otherwise**, we conclude that the problem has no solution.

4. Numerical Examples and Results Analysis

In this study, we solve well-known fuzzy and neutrosophic linear programming problem that presented in [28] with the traditional and proposed method.

$$\begin{aligned} &Max\ \tilde{z} = (1,5,2,4)x_1 + (10,12,2,6)x_2 \\ &s.t. \\ &3x_1 + 10x_2 \le 10 \\ &x_1 - x_2 \ge 2 \\ &x_1, x_2 \ge 0 \end{aligned} \qquad P_1$$

In the upcoming two subsections, problem P_1 will be solved using fuzzy & neutrosophic dual artificial variable-free simplex method respectively, uses the same rank function and compare between the results.

4.1. Solving (P_1) using Fuzzy Dual Artificial Variable-Free Simplex Method

Putting P_1 in the standard dual form, we have:

$$\begin{aligned} &Min\ \tilde{z} = (-5,-1,4,2)x_1 + (-12,-10,6,2)x_2 \\ &s.t. \\ &3x_1 + 10x_2 + x_3 = 10 \\ &-x_1 + x_2 \quad + x_4 = -2 \\ &x_1, x_2, x_3, x_4 \ge 0 \end{aligned} \qquad D1$$

By adding non-negative slack variablesx_3, x_4, the associated short table of D1 can be constructed as shown below. The dual variables y_3, y_4 have been demonstrated explicitly as it is required to observe dual variables too.

Here y is the dual objective variable. Objective coefficients (z) of primal non basic variables are the values of dual basic variables, and values of primal basic variables are coefficients of dual non-basic variables in dual objective.

b $\mathbf{x_1}$ $\mathbf{x_2}$

$$\begin{array}{c} \mathbf{z} \\ \mathbf{x_3} \\ \mathbf{x_4} \\ \end{array}\begin{bmatrix} 0 & (-5,-1,4,2) & (-12,-10,6,2) \\ 10 & 3 & 10 \\ -2 & -1 & 1 \\ & -y_1' & -y_2' \end{bmatrix}\begin{array}{c} \\ \mathbf{y_3} \\ \mathbf{y_4} \\ \\ \end{array}$$

Here k = {1, 2}, replace $-y_k{}^{-} \rightarrow y_k$

$$\begin{array}{c} \\ \mathbf{z} \\ \mathbf{x_3} \\ \mathbf{x_4} \\ \end{array}\begin{array}{c} \begin{array}{cccc} \mathbf{b} & \mathbf{x_1} & \mathbf{x_2} & \mathbf{w'} \end{array} \\ \begin{bmatrix} 0 & (-5,-1,4,2) & (-12,-10,6,2) & (11,17,4,10) \\ 10 & 3* & 10 & -13 \\ -2 & -1 & 1 & 0 \\ & -y_1' & -y_2' & \end{bmatrix} \end{array}\begin{array}{c} \\ \\ \mathbf{y_3} \\ \mathbf{y_4} \\ \\ \end{array}$$

Initial table:

Here B = {3, 4} and N = {1, 2}, according to most negative dual coefficient rule k = 1, so leaving dual basic variable is y_1 and according to artificial variable free dual minimum ratio test r = 3, the entering dual basic variable is ‘y_3‘. Perform the pivot operation on (3, 1). Replace $-y_1{}^{-} \rightarrow y_1$, k = {1, 2}\ {1} = {2}, $w'_1 := w'_1 + 1$.

$$\begin{array}{c} \\ \mathbf{z} \\ \mathbf{x_1} \\ \mathbf{x_4} \\ \end{array}\begin{array}{c} \begin{array}{cccc} \mathbf{b} & \mathbf{x_3} & \mathbf{x_2} & \mathbf{w'} \end{array} \\ \begin{bmatrix} (5,4,6,104/3) & (-3,-5,6,80/3) & (-1,-3,4,32/3) & (-1,-3,4,32/3) \\ 10/3 & 1/3 & 10/3 & -10/3 \\ 4/3 & 1/3 & 13/3* & -13/3 \\ & y_3 & -y_2' & \end{bmatrix} \end{array}\begin{array}{c} \\ \\ \mathbf{y_1} \\ \mathbf{y_4} \\ \\ \end{array}$$

Iteration 2:

Here, k = 2 and r = 4perform pivot operation on (4, 2). Since k ϵ k, replace$-y_2{}^{-} \rightarrow y_2$; k = {2}\ {2} = {}, $w'_2 := w'_2 + 1$.

$$\begin{array}{c} \\ \mathbf{z} \\ \mathbf{x_1} \\ \mathbf{x_2} \\ \end{array}\begin{array}{c} \begin{array}{cccc} \mathbf{b} & \mathbf{x_3} & \mathbf{x_4} & \mathbf{w'} \end{array} \\ \begin{bmatrix} (4,16,-3,53/13) & (7,9,20,-94/13) & (1/5,1/3,-1/3,-71/65) & 0 \\ 30/13 & 1/13 & -10/13 & 0 \\ 4/13 & 1/13 & 3/13 & 0 \\ & y_3 & y_4 & - \end{bmatrix} \end{array}\begin{array}{c} \\ \\ \mathbf{y_1} \\ \mathbf{y_2} \\ \\ \end{array}$$

Dual feasibility is achieved; the complementary dual feasible solution is (x_1, x_2) = (30/13, 4/13).

Resolve (P_1) using neutrosophic dual artificial variable-free simplex method uses the same rank function and we will compare between them.

4.2. Solving $(\boldsymbol{P_1})$ using Neutrosophic Dual Artificial Variable-Free Simplex method

First: We will convert the fuzzy numbers into neutrosophic numbers. Then, using the following rank function:

$$R(\breve{a}) = \frac{1}{4}\sum_{i=1}^{4} \tilde{a}_i + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})$$

$$\begin{aligned} &Min\ \tilde{z} = R[(-1,1,5,9)]x_1 + R[8,10,12,18]x_2 \\ &s.t. \\ &\quad 3x_1 + 10x_2 \le 10 \\ &\quad x_1 - x_2 \ge 2 \\ &\quad\ x_1, x_2 \ge 0 \end{aligned} \qquad \text{D1}$$

$$\begin{aligned} &Min\ z = 9/2x_1 + 13x_2 - 1 \\ &s.t. \\ &\quad 3x_1 + 10x_2 \le 10 \\ &\quad x_1 - x_2 \ge 2 \\ &\quad\ x_1, x_2 \ge 0 \end{aligned} \qquad \text{D2}$$

Putting (D2) in the standard form:

$$\begin{aligned} &Min\ z = 9/2x_1 + 13x_2 - 1 \\ &s.t. \\ &\quad 3x_1 + 10x_2 + x_3 = 10 \\ &\quad -x_1 + x_2 \qquad +x_4 = -2 \\ &\qquad\quad x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{array}{c} \\ \mathbf{z} \\ \mathbf{x_3} \\ \mathbf{x_4} \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{b} & \mathbf{x_1} & \mathbf{x_2} \end{array} \\ \left[\begin{array}{ccc} 0 & -9/2 & -13 \\ 10 & 3 & 10 \\ -2 & -1 & 1 \\ & -y_1' & -y_2' \end{array}\right] \end{array} \begin{array}{c} \\ \\ \mathbf{y_3} \\ \mathbf{y_4} \\ \\ \end{array}$$

Here k = {1, 2}, replace $-y_k{}^{-} \to y_k$

$$\begin{array}{c} \\ \mathbf{z} \\ \mathbf{x_3} \\ \mathbf{x_4} \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{b} & \mathbf{x_1} & \mathbf{x_2} & \mathbf{w'} \end{array} \\ \left[\begin{array}{cccc} 0 & -9/2 & -13 & 35/2 \\ 10 & 3 & 10\ * & -13 \\ -2 & -1 & 1 & 0 \\ & -y_1' & -y_2' & \end{array}\right] \end{array} \begin{array}{c} \\ \\ \mathbf{y_3} \\ \mathbf{y_4} \\ \\ \end{array}$$

Initial table:

Here B = {3, 4} and N = {1, 2}, according to most negative dual coefficient rule k = 2, so leaving dual basic variable is y_2 and according to artificial variable-free dual minimum ratio test r = 3, the entering dual basic variable is 'y_3'. Perform the pivot operation on (3, 1).
Replace $-y_2{}^{-} \to \boldsymbol{y_2}$, k = {1, 2}\ {2} = {1}, $w'_2 := w'_2 + 1$.

$$\begin{array}{c} \\ \mathbf{z} \\ \mathbf{x_2} \\ \mathbf{x_4} \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{b} & \mathbf{x_1} & \mathbf{x_3} & \mathbf{w'} \end{array} \\ \left[\begin{array}{cccc} 13 & -3/5 & 13/10 & 3/5 \\ 1 & 3/10\ * & 1/10 & -3/10 \\ -3 & -13/10 & -1/10 & 13/10 \\ & -y_1' & y_3 & \end{array}\right] \end{array} \begin{array}{c} \\ \\ \mathbf{y_2} \\ \mathbf{y_4} \\ \\ \end{array}$$

Iteration 2:

Here, k = 1 and r = 2 perform pivot operation on (2, 1). Since k ϵ k, replace$-x_1^- \rightarrow x_1$; k = {1}\ {1} = {}$w'_1 := w'_1 + 1$.

$$\begin{array}{c} \\ \mathbf{z} \\ \mathbf{x_1} \\ \mathbf{x_4} \\ \\ \end{array} \begin{bmatrix} \mathbf{b} & \mathbf{x_1} & \mathbf{x_3} & \mathbf{w'} \\ 15 & 2 & 3/2 & 0 \\ 10/3 & 10/3 & 1/3 & 0 \\ 4/3 & 13/3 & 1/3 & 0 \\ & y_1 & y_3 & \end{bmatrix} \begin{array}{c} \\ \\ \mathbf{y_1} \\ \mathbf{y_4} \\ \\ \end{array}$$

Dual feasibility is achieved; the complementary dual feasible solution is (x_1,x_2)= (10/3, 0).

$Max\ z = 9/2\ x_1 + 13\ x_2 - 1 = 15 - 1 = 14$

Table 3: *A comparison between two-phase algorithm, Fuzzy and Neutrosophic D*AVFSA

	Two-Phase Simplex Algorithm	**Fuzzy Dual Art Simplex Algorithm**	**Neutrosophic Dual Art Simplex Algorithm**
no (iteration)	5	3	3
Z	11.8	11.8	14
$\mathbf{x_1}$	30/13	30/13	10/3
$\mathbf{x_2}$	4/13	4/13	0

In table 3, a good comparisons have been made between two-phase simplex algorithm, fuzzy dual artificial variable-free simplex algorithm and neutrosophic dual artificial variable-free simplex algorithm; we noticed that the neutrosophic approach is more accurate than the fuzzy approach. On the other hand, the proposed algorithm overcomes the traditional two phase simplex algorithm for both the fuzzy approach and the neutrosophic approach according to the iterations number.

Conclusion

In this work, a new algorithm (Dual Artificial Variable-Free Simplex Algorithm) has been proposed, which solves linear programming problems with fuzzy and neutrosophic numbers. In this algorithm, the artificial variables are virtually present but their presence is not revealed to the user in the form of extra columns in the simplex table. It also follows the same pivoting sequence as of simplex phase 1 without showing any explicit description of artificial variables or artificial constraints but it could be easily seen that numbers of computations are noticeably reduced and the proposed algorithm overcame the traditional simplex algorithm for both the neutrosophic approach and the fuzzy approach according to the iterations number. We also compared between the neutrosophic approach and the fuzzy approach using numerical examples. Finally, the numerical examples show that the neutrosophic approach is more accurate than the fuzzy approach. In future work, we propose new hybrid methods such as using the cosine simplex method for phase 2 or using a traditional simplex algorithm for phase 2 while phase 1 uses the proposed method was proposed in this paper. We expect that these hybrid methods may overcome the traditional method.

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Conflicts of Interest
The authors declare that there are no conflicts of interest regarding the publication of this paper.

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