Neutrosophic Sets and Systems, Vol. 47, 2021
University of New Mexico

Generalized Pythagorean Neutrosophic Sets In the Study of Group Theory

Somen Debnath[1,*]
[1]Department of Mathematics, Umakanta Academy, Agartala-799001; Tripura, INDIA
*Correspondence: somen008@rediffmail.com; Mob. : (+91) 8787301661

Abstract: In 2019, Jansi et al. present the notion of the Pythagorean neutrosophic set (PNS) as an extension of a neutrosophic set with dependent neutrosophic components whenever $0 \le \mu_A(x)^2 + \varsigma_A(x)^2 + \gamma_A(x)^2 \le 2$. But due to the more complexity involved in a decision-making problem, there is a serious need to generalize the PNS for dealing with indeterminate, incomplete, and inconsistent data present in the belief system. The main objective of this paper is to elicit the notion of (α, β, δ)-Pythagorean neutrosophic set as a generalization of PNS. The (α, β, δ)-PNS provides a more powerful tool to model the various types of uncertainty with high precision and accuracy. Concerning to the idea of (α, β, δ)-PNS, we propose a new (α, β, δ)-Pythagorean neutrosophic subgroup (PNSG) and thus investigate some properties based on the proposed subgroup. Moreover, we discuss the impact of (α, β, δ)-Pythagorean neutrosophic subgroups in solving real-world problems with an aid of a suitable example.

Keywords: Pythagorean neutrosophic set; (α, β, δ)-Pythagorean neutrosophic set; Pythagorean neutrosophic subgroup, (α, β, δ)-Pythagorean neutrosophic subgroup, (α, β, δ)-Pythagorean neutrosophic normal subgroup.

1. Introduction

Before the invention of the fuzzy set (FS), suggested by Zadeh[1], we deal with uncertainty in an unorganized manner. The FS provides a general framework to handle uncertainty systematically. It makes a huge impact on real decision-making problems. Later, Zimmermann [2] elaborately studied the FS theory and discussed its practical applications. Every FS is a set of couples in which for every member of the set of discourse there exists a membership value, belongs to [0, 1]. So, we need to assign a membership function to design the uncertainty. Thus, the FS is useful in modeling vagueness with a proper methodology. It grows rapidly over the decades and it has huge applications in different disciplines such as medicine, economics, social science, computer science, engineering, etc. Relying on the need and the importance of FS theory in the advancement of modern technologies and researches, by embedding this idea, many new theories such as fuzzy logic [3], fuzzy graph

[4], fuzzy topology [5], fuzzy optimization [6], fuzzy image processing [7], fuzzy neural network [8], fuzzy sets in artificial intelligence [9], fuzzy Boolean algebra [10], linguistic fuzzy logic game theory [11], etc. have been developed and they have huge applications in decision-making. The fuzzy set theory has been extended to the intuitionistic fuzzy set(IFS) proposed by Atanassov [12], which incorporates the concept of non-membership as well as membership. Some other extensions of fuzzy sets are given in [13-16].

Nowadays the concept of two-valued logic does not justify the concept of imprecision in various situations. Smarandache[17] established the neutrosophic set(NS) as a result of this. There are three independent components in the neutrosophic set. It is a recent topic to handle uncertainty, incompleteness, and indeterminacy. Wang et al. [18] defined the single-valued neutrosophic set for research purposes. Also, [19-24] discusses some more aspects of the neutrosophic set.

Group theory is the study of groups where several sets are equipped with an operation. It is the building block of abstract algebra and it has been used in nearly all branches of mathematics such as cryptography, number theory, harmonic analysis, algebraic geometry, crystallography, etc. In the study of combinatory, we use a symmetric group. We also use the concept of group theory in physics, Chemistry, Molecular biology, Material science, etc. Rosenfeld proposed fuzzy groups [25] in 1971, introducing the notion of degree in the fuzzy set. Anthony et al. [26] redefined fuzzy groups. Das [27], in 1981, gave the idea of fuzzy level subgroup, Ajmal et al. [28] introduced fuzzy coset and fuzzy normal subgroup, Zhan and Tan [29] studied intuitionistic fuzzy subgroup, complex intuitionistic fuzzy groups were proposed by Husban et al. [30]. In a different way, Agboola et al. [31] defined neutrosophic groups and subgroups.

As an extension of fuzzy set, Atanassov developed the notion of intuitionistic fuzzy set in [12], where $\mu(x)+\gamma(x)\leq 1$ and $\mu(x)$, $\gamma(x)\in[0, 1]$. But, there are some situations under which $\mu(x)+\gamma(x)>1$. To put it another way, neither the fuzzy set nor the intuitionistic fuzzy set can resolve such uncertainty. Yager[32] proposed the Pythagorean fuzzy set(PFS) as a result of this. It is the latest tool developed to deal with imprecision with a wider scope of applications. It is a generalization of the fuzzy set and intuitionistic fuzzy set. The Pythagorean fuzzy set has a close connection with the intuitionistic fuzzy set. Ejegwa [33] gave an application of PFS in career placement, Liang et al. [34] investigated the use of PFS to extend TOPSIS to multi-criteria decision making. Lin et al.[35] studied the Pythagorean TOPSIS method and its application, which is based on unique correlation measurements. Garg[36] presented the generalized Pythagorean fuzzy geometric aggregation operator using Einstein t-norm and t-co-norm for multi-criteria decision-making. In 2018, Garg [37] defined the linguistic Pythagorean fuzzy sets and their applications, Naz et al.[38] used a novel approach to decision-making with Pythagorean fuzzy information. The complex Pythagorean fuzzy environment for decision-making was proposed by Akram and Naz[39]. The generalized interval-valued Pythagorean fuzzy aggregation operators were discovered by Rahman et al. [40]. The Pythagorean neutrosophic fuzzy graph was proposed by Ajay et al.[41]. To deal with indeterminacy along with incompleteness, in 2019, Jansi et al.[42] introduced the PNS withT and F as dependent components where the truth-membership, falsity -membership, and indeterminate-membership satisfy the criteria $0\leq(\mu(x))^2+(\gamma(x))^2+(\varsigma(x))^2\leq 2$, with $\mu(x)$, $\gamma(x)$ and $\varsigma(x)\in[0, 1]$.

The primary objective for writing this study is to present a novel strategy for generalizing the PNS concept described in [42]. When the PNS condition fails, i.e., $(\mu(x))^2+(\gamma(x))^2+(\varsigma(x))^2>2$, we have to figure out what to do. For example, information about an object in a neutrosophic environment is delivered to the decision-maker in the form $(0.9, 0.8, 0.8)$, which PNS cannot address because $0.9^2 + 0.8^2 + 0.8^2 > 2$. This led to the foundation of introducing $(\alpha, \beta, \delta)-$ PNSs and formation of their associated groups and subgroups. Then, we study some properties of $(\alpha, \beta, \delta)-$ Pythagorean neutrosophic groups and justify them with examples. We also discuss the concept of $(\alpha, \beta, \delta)-$ Pythagorean neutrosophic coset, $(\alpha, \beta, \delta)-$ Pythagorean neutrosophic normal subgroup. Finally, we give a real-world example that justifies the newly proposed theory and its contribution to the field of group theory appropriately.

The remainder of the paper is organized as follows: The proposed study requires certain basic definitions, which are included in Section 2. Section 3 examines some of the features and assertions of different sorts of (α, β, δ) -Pythagorean neutrosophic subgroups. A real-life implementation of the suggested study has been carried out in section 4. Section 5 presents the proposed study's conclusion and future scope.

2. Preliminaries

In this section, we provide the foundation of knowledge that helps us to unveil the newly proposed theory.

2.1 Definition [1, 2]

Let X be a crisp set and A be a subset of X. Then the fuzzy set defined on X is defined as a set of ordered pairs of the form $\{(x, \mu_A(x)) : x \in X\}$, where $\mu_A : X \to [0,1]$.

2.2 Definition [12]

Let X be a crisp set and A be a subset of X. Then the intuitionistic fuzzy set defined on X is defined as a set of ordered triples of the type $\{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$, where $\mu_A : X \to [0,1], \gamma_A : X \to [0,1]$ and satisfies the condition $0 \le \mu_A(x) + \gamma_A(x) \le 1$.

2.3 Definition [17]

Let x be a generic element in X, a universe of discourse. A neutrosophic set A in X is characterized by a truth-membership function T_A, an indeterminacy-membership function I_A, and a falsity-membership function F_A. Here, T_A, I_A and F_A are real standard and non-standard subsets of [0,1] and its set-theoretic representation is given by

$$A = \{\langle x, (T_A(x), I_A(x), F_A(x))\rangle : x \in X, T_A(x), I_A(x), F_A(x) \in [0,1]\}$$

As there are no restrictions on the sum of $T_A(x), I_A(x)$ and $F_A(x)$, as a result, we take $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

2.4 Definition [42]

Let X be a universal set. A neutrosophic set A on X is of the type $A = \{\langle x, (T_A(x), I_A(x), F_A(x))\rangle : x \in X, T_A(x), I_A(x), F_A(x) \in [0,1]\}$.

If $T_A(x)$ and $F_A(x)$ are both dependent components, and $I_A(x)$ is an independent component, then $0 \le T_A(x) + I_A(x) + F_A(x) \le 2, \forall x \in X$.

2.5 Definition [32]

Let X be a crisp set. A PFS χ on X is an object of the type $\chi = \{(x, \mu(x), \gamma(x)) : x \in X\}$, where $\mu(x), \gamma(x) \in [0,1]$ which satisfies the condition $0 \le \mu^2(x) + \gamma^2(x) \le 1$.

2.6 Definition [42]

Let X be a universal set. A PNS A, with $\mu_A(x)$ and $\gamma_A(x)$ are dependent neutrosophic components and $\varsigma_A(x)$ is an independent component, is an object on X is of the form $A = \{\langle x, (\mu_A(x), \varsigma_A(x), \gamma_A(x))\rangle : x \in X\}$, where $\mu_A(x), \varsigma_A(x)$ and $\gamma_A(x) \in [0,1]$ and $0 \le (\mu_A(x))^2 + (\varsigma_A(x))^2 + (\gamma_A(x))^2 \le 2$

2.7 Definition [25]

Let $(G, \circ)$ be a group and μ be a fuzzy subset of G. Then μ is regarded as a fuzzy subgroup of $(G, \circ)$ if $\mu(x \circ y) \ge \mu(x) \wedge \mu(y)$ and $\mu(x^{-1}) \ge \mu(x)$, $\forall x, y \in G$.

2.8 Definition [29]

Let $(G, \circ)$ be a group and $A = \{(x, \mu(x), \gamma(x)) : x \in G\}$ be an intuitionistic fuzzy set of G. Then A is described as an intuitionistic fuzzy subgroup of $(G, \circ)$ if

$\mu(x \circ y) \ge \mu(x) \wedge \mu(y)$ and $\gamma(x \circ y) \le \mu(x) \vee \mu(y)$

$\mu(x^{-1}) \ge \mu(x)$ and $\gamma(x^{-1}) \le \gamma(x)$, $\forall x, y \in G$.

2.9 Definition [32]

Let $A=\left\{\left(x,\mu\left(x\right),\gamma\left(x\right)\right):x\in \breve{G}\right\}$ be a PFS of a group $(\breve{G},\circ)$, then A is said to be a Pythagorean fuzzy subgroup (PFSG) of $\breve{G}$ if it fulfills the conditions

$$\mu^2\left(x\circ y\right)\geq \mu^2\left(x\right)\wedge \mu^2\left(y\right) \text{ and } \gamma^2\left(x\circ y\right)\leq \gamma^2\left(x\right)\vee \gamma^2\left(y\right)$$

$$\mu^2\left(x^{-1}\right)\geq \mu^2\left(x\right) \text{ and } \gamma^2\left(x^{-1}\right)\leq \gamma^2\left(x\right), \quad \forall x,y\in \breve{G}$$

2.10 Definition

Let $(\breve{G},\circ)$ be any group and $A=\left\{\left(x,\mu\left(x\right),\varsigma\left(x\right),\gamma\left(x\right)\right):x\in \breve{G}\right\}$ be a NS of $\breve{G}$. Then A is called a neutrosophic subgroup of $(\breve{G},\circ)$ if

$$\mu\left(x\circ y\right)\leq \mu\left(x\right)\vee \mu\left(y\right), \varsigma\left(x\circ y\right)\leq \varsigma\left(x\right)\vee \varsigma\left(y\right) \text{ and } \gamma\left(x\circ y\right)\geq \gamma\left(x\right)\wedge \gamma\left(y\right)$$

$$\mu\left(x^{-1}\right)\leq \mu\left(x\right), \varsigma\left(x^{-1}\right)\leq \varsigma\left(x\right) \text{ and } \gamma\left(x^{-1}\right)\geq \gamma\left(x\right), \quad \forall x,y\in \breve{G}$$

2.11 Definition

Let $(\breve{G},\circ)$ be a group and $A=\left\{\left\langle x,\left(\mu_A\left(x\right),\varsigma_A\left(x\right),\gamma_A\left(x\right)\right)\right\rangle:x\in \breve{G}\right\}$ be a PNS of $\breve{G}$. Then A is called a Pythagorean neutrosophic subgroup (PNSG) of $\breve{G}$ if it satisfies the following

$$\mu^2\left(x\circ y\right)\geq \mu^2\left(x\right)\wedge \mu^2\left(y\right), \varsigma^2\left(x\circ y\right)\geq \varsigma^2\left(x\right)\wedge \varsigma^2\left(y\right) \text{ and } \gamma^2\left(x\circ y\right)\leq \gamma^2\left(x\right)\vee \gamma^2\left(y\right)$$

$$\mu^2\left(x^{-1}\right)\geq \mu^2\left(x\right), \varsigma^2\left(x^{-1}\right)\geq \varsigma^2\left(x\right) \text{ and } \gamma^2\left(x^{-1}\right)\leq \gamma^2\left(x\right), \quad \forall x,y\in \breve{G}.$$

3. Different Types of (α,β,δ)-Pythagorean neutrosophic Subgroups (PNSGs) and their properties

This section contains the (α,β,δ)-PNSs, (α,β,δ)-PNSGs and investigate some of their properties.

3.1 Definition

Let X be a crisp set and α,β and $\delta\in[0,1]$ such that $0\leq \alpha^2+\beta^2+\delta^2\leq 2$. A (α,β,δ)-PNS χ in X is an item of the form

$\chi = \left\{\left(x,\mu^{\alpha}\left(x\right),\varsigma^{\beta}\left(x\right),\gamma^{\delta}\left(x\right)\right):x\in X\right\}$, where $\mu^{\alpha}\left(x\right) = \mu\left(x\right)\vee\alpha$, $\varsigma^{\beta}\left(x\right) = \varsigma\left(x\right)\vee\beta$ and $\gamma^{\delta}\left(x\right)=\gamma\left(x\right)\wedge\delta$, where $0\leq\left(\mu^{\alpha}\left(x\right)\right)^2+\left(\varsigma^{\beta}\left(x\right)\right)^2+\left(\gamma^{\delta}\left(x\right)\right)^2\leq 2$

We are now going to describe some operations on (α,β,δ)-PNSs.

Let χ_1 and χ_2 be two $(\alpha,\beta,\delta)-$PNSs such that $\chi_1=\left\{\left(x,\mu_1^{\alpha}(x),\varsigma_1^{\beta}(x),\gamma_1^{\delta}(x)\right):x\in X\right\}$ and $\chi_2=\left\{\left(x,\mu_2^{\alpha}(x),\varsigma_2^{\beta}(x),\gamma_2^{\delta}(x)\right):x\in X\right\}$ then we have the following operations defined on two $(\alpha,\beta,\delta)-$PNSs

(a) $\chi_1\cup\chi_2=\left\{\left(x,\left\langle\mu_1^{\alpha}(x)\vee\mu_2^{\alpha}(x),\varsigma_1^{\beta}(x)\vee\varsigma_2^{\beta}(x),\gamma_1^{\delta}(x)\wedge\gamma_2^{\delta}(x)\right\rangle\right):x\in X\right\}$

(b) $\chi_1\cap\chi_2=\left\{\left(x,\left\langle\mu_1^{\alpha}(x)\wedge\mu_2^{\alpha}(x),\varsigma_1^{\beta}(x)\wedge\varsigma_2^{\beta}(x),\gamma_1^{\delta}(x)\vee\gamma_2^{\delta}(x)\right\rangle\right):x\in X\right\}$

(c) $(\chi_1\cup\chi_2)^c=\chi_1^c\cap\chi_2^c$ and $(\chi_1\cap\chi_2)^c=\chi_1^c\cup\chi_2^c$ (Demorgan's laws)

(d) $\chi_1\subseteq\chi_2$ if $\mu_1^{\alpha}(x)\le\mu_2^{\alpha}(x),\varsigma_1^{\beta}(x)\le\varsigma_2^{\beta}(x)$ and $\gamma_1^{\delta}(x)\ge\gamma_2^{\delta}(x)$

(e) $\chi_1=\chi_2$ if $\mu_1^{\alpha}(x)=\mu_2^{\alpha}(x),\varsigma_1^{\beta}(x)=\varsigma_2^{\beta}(x)$ and $\gamma_1^{\delta}(x)=\gamma_2^{\delta}(x)$

Proof: Proofs of (a), (b), (d), and (e) are obvious. We only show the proof of (c) given in the following:

We define the complement of χ_1 and χ_2 as follows

$\chi_1^c=\left\{\left(x,\gamma_1^{\delta}(x),\varsigma_1^{\beta}(x),\mu_1^{\alpha}(x)\right):x\in X\right\}$ and $\chi_2^c=\left\{\left(x,\gamma_2^{\delta}(x),\varsigma_2^{\beta}(x),\mu_2^{\alpha}(x)\right):x\in X\right\}$

Then, $(\chi_1\cup\chi_2)^c=\left\{\left(x,\left\langle\mu_1^{\alpha}(x)\vee\mu_2^{\alpha}(x),\varsigma_1^{\beta}(x)\vee\varsigma_2^{\beta}(x),\gamma_1^{\delta}(x)\wedge\gamma_2^{\delta}(x)\right\rangle\right):x\in X\right\}^c$

$$=\left\{\left(x,\left\langle\gamma_1^{\delta}(x)\vee\gamma_2^{\delta}(x),\varsigma_1^{\beta}(x)\wedge\varsigma_2^{\beta}(x),\mu_1^{\alpha}(x)\wedge\mu_2^{\alpha}(x)\right\rangle\right):x\in X\right\}$$

$$\chi_1^c\cap\chi_2^c=\left\{\left(x,\left\langle\gamma_1^{\delta}(x)\vee\gamma_2^{\delta}(x),\varsigma_1^{\beta}(x)\wedge\varsigma_2^{\beta}(x),\mu_1^{\alpha}(x)\wedge\mu_2^{\alpha}(x)\right\rangle\right):x\in X\right\}$$

Thus, $(\chi_1\cup\chi_2)^c=\chi_1^c\cap\chi_2^c$

We can also demonstrate that $(\chi_1\cap\chi_2)^c=\chi_1^c\cup\chi_2^c$

3.1.1 Example

Let $X=\{a,b,c\}$ be a crisp set. The membership-function (μ), indeterminacy-function (ς) and the non-membership function (γ) on X are all defined as follows:

$\mu(a)=0.8, \varsigma(a)=0.5, \gamma(a)=0.8$

$\mu(b)=0.7, \varsigma(a)=0.6, \gamma(a)=0.8$

$\mu(c)=0.8, \varsigma(c)=0.9, \gamma(c)=0.8$

Clearly, $\mu(x)+\varsigma(x)+\gamma(x)>2, \forall x \in X$.

By definition **2.4**, $N=\left\{\left(x,\left(\mu(x),\varsigma(x),\gamma(x)\right)\right): x \in X\right\}$ is not a NS with the dependent neutrosophic component.

Again, $\mu^2(c)+\varsigma^2(c)+\gamma^2(c)=2.09>2$. So, by definition **2.6**, N is not a PNS.

Taking, $\alpha=0.7, \beta=0.6$ and $\delta=0.7$ such that $\alpha^2+\beta^2+\delta^2=1.34<2$.

Now, $\mu^{\alpha}(a)=\mu(a)\vee\alpha=0.8$, $\varsigma^{\beta}(a)=\varsigma(a)\vee\beta=0.6$ and $\gamma^{\delta}(a)=\gamma(a)\wedge\delta=0.7$

We can easily verify that $0\le\mu^{\alpha}(x)+\varsigma^{\beta}(x)+\gamma^{\delta}(x)\le 2, \forall x\in X$.

Thus, the set $N=\left\{\left(x,\left(\mu(x),\varsigma(x),\gamma(x)\right)\right): x \in X\right\}$ is $(\alpha,\beta,\delta)-$PNS.

3.2 Definition

Let $(\breve{G},\circ)$ be a group and $\breve{N}=\left\{\left\langle x,\left(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right)\right\rangle : x\in G\right\}$ be a $(\alpha,\beta,\delta)-$PNS.

Then, $\breve{N}$ is said to be the (α,β,δ) – PNSG of $\breve{G}$ if it satisfies the following conditions:

$\mu^{\alpha}(x\circ y)\le\mu^{\alpha}(x)\vee\mu^{\alpha}(y);\varsigma^{\beta}(x\circ y)\le\varsigma^{\beta}(x)\vee\varsigma^{\beta}(y)$ and $\gamma^{\delta}(x\circ y)\ge\gamma^{\delta}(x)\wedge\gamma^{\delta}(y)$

$\mu^{\alpha}\left(x^{-1}\right)\le\mu^{\alpha}(x);\varsigma^{\beta}\left(x^{-1}\right)\le\varsigma^{\beta}(x)$ and $\gamma^{\delta}\left(x^{-1}\right)\ge\gamma^{\delta}(x)$, $\forall x,y\in\breve{G}$.

3.2.1 Example

Let $(\breve{G},*)$ be a group where $\breve{G}=\left\{1,\omega,\omega^2\right\}$ and $'*'$ denotes the usual multiplication. In this group 1 is the identity element and ω represents the cube root of unity such that $\omega^3=1$. Let us define the membership functions as follows

$\mu(1)=0.8, \varsigma(1)=0.7, \gamma(1)=0.6$

$\mu(\omega)=0.7, \varsigma(\omega)=0.6, \gamma(\omega)=0.5$

$\mu(\omega^2)=0.6, \varsigma(\omega^2)=0.8, \gamma(\omega^2)=0.7$

We take, $\alpha=0.65, \beta=0.55$ and $\delta=0.67$

Now,

$\mu^{\alpha}(1*\omega)=\mu^{\alpha}(\omega)=\mu(\omega)\vee\alpha=0.7$;

$\varsigma^{\beta}(1*\omega)=\varsigma^{\beta}(\omega)=\varsigma(\omega)\vee\beta=0.6$; $\gamma^{\delta}(1*\omega)=\gamma^{\delta}(\omega)=\gamma(\omega)\wedge\delta=0.5$

Again, $\mu^{\alpha}(1) \vee \mu^{\alpha}(\omega)$=0.8; $\varsigma^{\beta}(1) \vee \varsigma^{\beta}(\omega)$=0.7 and $\gamma^{\delta}(1) \wedge \gamma^{\delta}(\omega)$=0.5

$\mu^{\alpha}(\omega^{-1}) = \mu^{\alpha}(\omega^{2}) = 0.65 \leq \mu^{\alpha}(\omega) = 0.7$; $\varsigma^{\beta}(\omega^{-1}) = \varsigma^{\beta}(\omega^{2}) = 0.8 \leq \varsigma^{\beta}(\omega) = 0.55$ and

$\gamma^{\delta}(\omega^{-1}) = \gamma^{\delta}(\omega^{2}) = 0.67 \geq \gamma^{\delta}(\omega) = 0.5$

So, for all $x, y \in \breve{G}$, we can easily verify that the set $N = \{\langle x, (\mu^{\alpha}, \varsigma^{\beta}, \gamma^{\delta}) \rangle : x \in G\}$ is a (α, β, δ) –PNSG of $\breve{G}$.

It can easily verify that every PNSG is a (α, β, δ) –PNSG, but the converse is not true.

3.2.2 Example

Consider the Klein's four-group $\breve{G} = \{e, a, b, \mathrm{c}\}$ with four elements, in which each element is self-inverse and the composition $'*'$ on any two elements results in the third element. Let us check whether it satisfies the criteria of being (α, β, δ) –PNSG or not.

Let us consider the membership values of each element of $\breve{G}$ as follows:

$\mu(e) = 0.6, \varsigma(e) = 0.7, \gamma(e) = 0.5$

$\mu(a) = 0.65, \varsigma(a) = 0.55, \gamma(a) = 0.45$

$\mu(b) = 0.35, \varsigma(b) = 0.4, \gamma(b) = 0.75$

$\mu(c) = 0.6, \varsigma(c) = 0.7, \gamma(c) = 0.55$

We consider $\alpha = 0.5, \beta = 0.6$ and $\delta = 0.65$

$\mu^{\alpha}(\mathrm{a*b}) = \mu^{\alpha}(c) = \mu(c) \vee \alpha = 0.6; \varsigma^{\beta}(\mathrm{a*b}) = \varsigma^{\beta}(c) = \varsigma(c) \vee \beta = 0.7$

$\gamma^{\delta}(\mathrm{a*b}) = \gamma^{\delta}(c) = \gamma(c) \wedge \delta = 0.55$

$\mu^{\alpha}(a) \vee \mu^{\alpha}(b)$=0.65; $\varsigma^{\beta}(a) \vee \varsigma^{\beta}(b)$=0.6 and $\gamma^{\delta}(a) \wedge \gamma^{\delta}(b)$=0.45

Since, the condition $\varsigma^{\beta}(\mathrm{a*b}) \leq \varsigma^{\beta}(a) \vee \varsigma^{\beta}(b)$ does not hold, therefore the group $\breve{G} = \{e, a, b, \mathrm{c}\}$ is not a (α, β, δ) –PNSG.

3.2.3Example

Let us consider the group = U_{10} $\{[a] \in Z_{10} : 0 < a < 10 \,\&\, \gcd(a,10) = 1\}$. Then, $U_{10} = \{[1],[3],[7],[9]\}$.

We can easily show that it is (α, β, δ) –PNSG.

The calculation part is left as an exercise for the readers.

3.3 Theorem

If $A=\left\{\left(x,\mu(x),\varsigma(x),\gamma(x)\right):x\in G\right\}$ is a neutrosophic subgroup of the group $(G,\circ)$ then

$N=\left\{\left\langle x,\left(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right)\right\rangle:x\in G\right\}$ is a (α,β,δ) –PNSG of $(G,\circ)$.

Proof. Since $\left\{\left(x,\mu(x),\varsigma(x),\gamma(x)\right):x\in G\right\}$ is a neutrosophic subgroup of the group $(G,\circ)$,

Then,

$\mu^2(x\circ y)\le\mu^2(x)\vee\mu^2(y)$, $\varsigma^2(x\circ y)\le\varsigma^2(x)\vee\varsigma^2(y)$ and $\gamma^2(x\circ y)\ge\gamma^2(x)\wedge\gamma^2(y)$

$\mu^2(x^{-1})\le\mu^2(x)$, $\varsigma^2(x^{-1})\le\varsigma^2(x)$ and $\gamma^2(x^{-1})\ge\gamma^2(x)$, $\forall x,y\in G$

Now,

$$\mu^{\alpha}(x\circ y)=\mu(x\circ y)\vee\alpha\le\left[\mu(x)\vee\alpha\right]\vee\left[\mu(y)\vee\alpha\right]=\mu^{\alpha}(x)\vee\mu^{\alpha}(y)$$

Similarly,

$\varsigma^{\beta}(x\circ y)\le\varsigma^{\beta}(x)\vee\varsigma^{\beta}(y)$ and $\gamma^{\delta}(x\circ y)\ge\gamma^{\delta}(x)\wedge\gamma^{\delta}(y)$

Again,

$$\mu^{\alpha}(x^{-1})=\mu(x^{-1})\vee\alpha\le\mu(x)\vee\alpha=\mu^{\alpha}(x)$$

Similarly, $\varsigma^{\beta}(x^{-1})\le\varsigma^{\beta}(x)$ and $\gamma^{\delta}(x^{-1})\ge\gamma^{\delta}(x)$, $\forall x,y\in G$.

This proves the theorem.

3.4 Proposition A (α,β,δ) -PNS $N=\left\{\left\langle x,\left(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right)\right\rangle:x\in G\right\}$ of $(G,\circ)$ is a (α,β,δ) – PNSG of $(G,\circ)$ iff

$$\mu^{\alpha}(x\circ y^{-1})\le\mu^{\alpha}(x)\vee\mu^{\alpha}(y^{-1});\varsigma^{\beta}(x\circ y^{-1})\le\varsigma^{\beta}(x)\vee\varsigma^{\beta}(y^{-1})\text{ and }\gamma^{\delta}(x\circ y^{-1})\ge\gamma^{\delta}(x)\wedge\gamma^{\delta}(y^{-1})$$

Proof. It is obvious.

3.5 Theorem The intersection of two (α,β,δ) –PNSGs of $(G,\circ)$ is a (α,β,δ) –PNSG of $(G,\circ)$.

Proof.

Let $N_1=\left\{\mu_1^{\alpha},\varsigma_1^{\beta},\gamma_1^{\delta}\right\}$ and $N_2=\left\{\mu_2^{\alpha},\varsigma_2^{\beta},\gamma_2^{\delta}\right\}$ be two (α,β,δ) –PNSGs of $(G,\circ)$.

Then, their intersection defined as

$N_1\cap N_2=\left\{\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right\}$, where $\mu^{\alpha}=\mu_1^{\alpha}\vee\mu_2^{\alpha}$, $\varsigma^{\beta}=\varsigma_1^{\beta}\vee\varsigma_2^{\beta}$ and $\gamma^{\delta}=\gamma_1^{\delta}\wedge\gamma_2^{\delta}$.

Now, for all $x,y\in G$,

$$\begin{aligned}\mu^{\alpha}\left(x\circ y^{-1}\right)&=\mu_1^{\alpha}\left(x\circ y^{-1}\right)\vee\mu_2^{\alpha}\left(x\circ y^{-1}\right)\leq\left(\mu_1^{\alpha}\left(x\right)\vee\mu_1^{\alpha}\left(y\right)\right)\vee\left(\mu_2^{\alpha}\left(x\right)\vee\mu_2^{\alpha}\left(y\right)\right)\\&=\left(\mu_1^{\alpha}\left(x\right)\vee\mu_2^{\alpha}\left(x\right)\right)\vee\left(\mu_1^{\alpha}\left(y\right)\vee\mu_2^{\alpha}\left(y\right)\right)\\&=\mu^{\alpha}\left(x\right)\vee\mu^{\alpha}\left(y\right)\end{aligned}$$

Similarly, $\varsigma^{\beta}\left(x\circ y^{-1}\right)\leq\varsigma^{\beta}\left(x\right)\vee\varsigma^{\beta}\left(y\right)$ and $\gamma^{\delta}\left(x\circ y^{-1}\right)\geq\gamma^{\delta}\left(x\right)\wedge\gamma^{\delta}\left(y\right)$

This proves the theorem.

Note: The union of two (α,β,δ) –PNSGs of $(\overset{\square}{G},\circ)$ may not be a (α,β,δ) –PNSG of $(\overset{\square}{G},\circ)$.

3.6 Definition

Let $N^{\square}=\left(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right)$ be a $\left(\alpha,\beta,\delta\right)$-PNSG of a group $\left(\overset{\square}{G},\circ\right)$. Then for all $x,y\in\overset{\square}{G}$, $N^{\square}$ is said to be a normalized $\left(\alpha,\beta,\delta\right)$-PNSG of a group $\left(\overset{\square}{G},\circ\right)$, if the following conditions hold:

(a) $\mu^{\alpha}\left(x\circ y\right)\leq\mu^{\alpha}\left(x\right)\vee\mu^{\alpha}\left(y\right),\varsigma^{\beta}\left(x\circ y\right)\leq\varsigma^{\beta}\left(x\right)\vee\varsigma^{\beta}\left(y\right)$ and $\gamma^{\delta}\left(x\circ y\right)\geq\gamma^{\delta}\left(x\right)\wedge\gamma^{\delta}\left(y\right)$

(b) $\mu^{\alpha}\left(x^{-1}\right)=\mu^{\alpha}\left(x\right),\varsigma^{\beta}\left(x^{-1}\right)=\varsigma^{\beta}\left(x\right)$ and $\gamma^{\delta}\left(x^{-1}\right)=\gamma^{\delta}\left(x\right)$

(c) $\mu^{\alpha}\left(e^{\square}\right)=1,\varsigma^{\beta}\left(e^{\square}\right)=1$ and $\gamma^{\delta}\left(e^{\square}\right)=0$, where $e^{\square}$ is the identity element.

3.7 Theorem

Let $N^{\square}=\left(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right)$ be a $\left(\alpha,\beta,\delta\right)$ -PNSG of a group $\left(\overset{\square}{G},\circ\right)$. Then the set $M^{\square}=\left\{x\in\overset{\square}{G}:\mu^{\alpha}\left(x\right)=\mu^{\alpha}\left(e^{\square}\right),\varsigma^{\beta}\left(x\right)=\varsigma^{\beta}\left(e^{\square}\right),\gamma^{\delta}\left(x\right)=\gamma^{\delta}\left(e^{\square}\right)\right\}$ forms a $\left(\alpha,\beta,\delta\right)$-PNSG of a group $\left(\overset{\square}{G},\circ\right)$.

Proof: Clearly, the set $M^{\square}$ is the non-empty set, as $e^{\square}\in M^{\square}$.

Since, $M^{\square}$ is the $\left(\alpha,\beta,\delta\right)$-PNSG of a group $\left(\overset{\square}{G},\circ\right)$, then for all $x,y\in\overset{\square}{G}$

$$\begin{aligned}\mu^{\alpha}\left(x\circ y^{-1}\right)&\leq\mu^{\alpha}\left(x\right)\vee\mu^{\alpha}\left(y^{-1}\right)\\&=\mu^{\alpha}\left(x\right)\vee\mu^{\alpha}\left(y\right)\\&=\mu^{\alpha}\left(e^{\square}\right)\end{aligned}$$

Similarly, $\varsigma^{\beta}\left(x\circ y^{-1}\right)\le\varsigma^{\beta}\left(e^{\square}\right)$ and $\gamma^{\delta}\left(x\circ y^{-1}\right)\ge\gamma^{\delta}\left(e^{\square}\right)$

Thus, $x\circ y^{-1}\in M^{\square}$.

Therefore, $\left(M^{\square},\circ\right)$ is the subgroup of $\left(G^{\square},\circ\right)$

3.8 Definition

Let $N^{\square}=\left(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right)$ be a (α,β,δ) -PNSG of a group $\left(\overset{\square}{G},\circ\right)$. Then for all $x\in\overset{\square}{G}$, (α,β,δ)-Pythagorean neutrosophic left coset(PNLC) of $N^{\square}$ is the (α,β,δ)-PNS such that $xN^{\square}$ = $\left(x\mu^{\alpha},x\varsigma^{\beta},x\gamma^{\delta}\right)$ and it is defined by $\left(x\mu^{\alpha}\right)(k)=\mu^{\alpha}\left(x^{-1}\circ k\right),\left(x\varsigma^{\beta}\right)(k)=\varsigma^{\beta}\left(x^{-1}\circ k\right)$ and $\left(x\gamma^{\delta}\right)(k)=\gamma^{\delta}\left(x^{-1}\circ k\right)$ for all $k\in\overset{\square}{G}$.

Similarly, the (α,β,δ)-Pythagorean neutrosophic right coset(PNRC) of $N^{\square}$ is defined by

$(k)\left(x\mu^{\alpha}\right)=\mu^{\alpha}\left(k\circ x^{-1}\right),(k)\left(x\varsigma^{\beta}\right)=\varsigma^{\beta}\left(k\circ x^{-1}\right)$ and $(k)\left(x\gamma^{\delta}\right)=\gamma^{\delta}\left(k\circ x^{-1}\right)$ for all $k\in\overset{\square}{G}$.

3.9 Definition

Let $N^{\square}=\left(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right)$ be a (α,β,δ)-PNSG of a group $\left(\overset{\square}{G},\circ\right)$. Then $N^{\square}$ is a (α,β,δ)-Pythagorean neutrosophic normal subgroup(PNNSG) of the group $\left(\overset{\square}{G},\circ\right)$, if every (α,β,δ)- PNLC of $N^{\square}$ is also a (α,β,δ)- PNRC of $N^{\square}$.

3.9.1 Example

Let us consider the group $\overset{\square}{G}=\left(Z_4,+_4\right)$, denotes the integers modulo 4 under addition. Firstly we construct the composition table as follows:

$+_4$	*0*	*1*	*2*	*3*
0	*0*	*1*	*2*	*3*
1	*1*	*2*	*3*	*0*
2	*2*	*3*	*0*	*1*
3	*3*	*0*	*1*	*2*

For $3 \in \breve{G}$ and for all $k \in \breve{G}$, we define the (α,β,δ)-PNLC and the (α,β,δ)-PNRC as follows:

$(3\mu^{\alpha})(k)=\mu^{\alpha}\left(3^{-1}+_4 k\right),(3\varsigma^{\beta})(k)=\varsigma^{\beta}\left(3^{-1}+_4 k\right)$ and $(3\gamma^{\delta})(k)=\gamma^{\delta}\left(3^{-1}+_4 k\right)$

Again, $(k)(3\mu^{\alpha})=\mu^{\alpha}\left(k+_4 3^{-1}\right),(k)(3\varsigma^{\beta})=\varsigma^{\beta}\left(k+_4 3^{-1}\right)$ and $(k)(3\gamma^{\delta})=\gamma^{\delta}\left(k+_4 3^{-1}\right)$

For k=0,

$$(3\mu^{\alpha})(0)=\mu^{\alpha}\left(3^{-1}+_4 0\right)=\mu^{\alpha}(1+_4 0)=\mu^{\alpha}(1)$$

$$(0)(3\mu^{\alpha})=\mu^{\alpha}\left(0+_4 3^{-1}\right)=\mu^{\alpha}(0+_4 1)=\mu^{\alpha}(1)$$

Therefore, $(3\mu^{\alpha})(0)=(0)(3\mu^{\alpha})$

Similarly, we can show that $(3\varsigma^{\beta})(0)=(0)(3\varsigma^{\beta})$ and $(3\gamma^{\delta})(0)=(0)(3\gamma^{\delta})$

We also verify the above result for, k=1, 2, 3.

Therefore, the group $\breve{G}=(Z_4,+_4)$ is a (α,β,δ)-PNNSG.

3.10 Proposition

Let $N^{\breve{}}=(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta})$ be a (α,β,δ)-PNSG of $(\breve{G},\circ)$. Then for every $x,y\in\breve{G}$, $N^{\breve{}}$ is a (α,β,δ)-PNNSG of a group $(\breve{G},\circ)$ iff $\mu^{\alpha}(x\circ y)=\mu^{\alpha}(y\circ x),\varsigma^{\beta}(x\circ y)=\varsigma^{\beta}(y\circ x)$ and $\gamma^{\delta}(x\circ y)=\gamma^{\delta}(y\circ x)$

Proof:

Let $N^{\breve{}}=(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta})$ be a (α,β,δ)-PNSG of a group $(\breve{G},\circ)$. Then for every $x\in\breve{G}$, $xN^{\breve{}}=N^{\breve{}}x$. Now, for every $x,k\in\breve{G}$, $(x\mu^{\alpha})(k)=(\mu^{\alpha}x)(k)$, $(x\varsigma^{\beta})(k)=(\varsigma^{\beta}x)(k)$, and $(x\gamma^{\delta})(k)=(\gamma^{\delta}x)(k)$.

Also, for every $x,k\in\breve{G}$, $\mu^{\alpha}\left(x^{-1}\circ k\right)=\mu^{\alpha}\left(k\circ x^{-1}\right)$, $\varsigma^{\beta}\left(x^{-1}\circ k\right)=\varsigma^{\beta}\left(k\circ x^{-1}\right)$, and $\gamma^{\delta}\left(x^{-1}\circ k\right)=\gamma^{\delta}\left(k\circ x^{-1}\right)$.

Therefore, $\mu^{\alpha}(x\circ y)=\mu^{\alpha}\left(x\circ\left(y^{-1}\right)^{-1}\right)=\mu^{\alpha}\left(\left(y^{-1}\right)^{-1}\circ x\right)=\mu^{\alpha}(y\circ x)$

Similarly, $\varsigma^{\beta}\left(x \circ y\right)=\varsigma^{\beta}\left(y \circ x\right)$, and $\gamma^{\delta}\left(x \circ y\right)=\gamma^{\delta}\left(y \circ x\right)$.

Conversely, for every $x, \mathrm{y} \in \overset{\square}{G}$, $\mu^{\alpha}\left(x \circ y\right)=\mu^{\alpha}\left(y \circ x\right)$, $\varsigma^{\beta}\left(x \circ y\right)=\varsigma^{\beta}\left(y \circ x\right)$, and $\gamma^{\delta}\left(x \circ y\right)=\gamma^{\delta}\left(y \circ x\right)$.

This gives, $\mu^{\alpha}\left(x \circ\left(y^{-1}\right)^{-1}\right)=\mu^{\alpha}\left(\left(y^{-1}\right)^{-1} \circ x\right)$, $\varsigma^{\beta}\left(x \circ\left(y^{-1}\right)^{-1}\right)=\varsigma^{\beta}\left(\left(y^{-1}\right)^{-1} \circ x\right)$, and $\gamma^{\delta}\left(x \circ\left(y^{-1}\right)^{-1}\right)=\gamma^{\delta}\left(\left(y^{-1}\right)^{-1} \circ x\right)$.

Put $y^{-1}=z$. Then for every $x, \mathrm{z} \in \overset{\square}{G}$, $\mu^{\alpha}\left(x \circ z^{-1}\right)=\mu^{\alpha}\left(z^{-1} \circ x\right)$, $\varsigma^{\beta}\left(x \circ z^{-1}\right)=\varsigma^{\beta}\left(z^{-1} \circ x\right)$, and $\gamma^{\delta}\left(x \circ z^{-1}\right)=\gamma^{\delta}\left(z^{-1} \circ x\right)$.

$\Rightarrow\left(\mu^{\alpha} z\right)(x)=\left(z \mu^{\alpha}\right)(x),\left(\varsigma^{\beta} z\right)(x)=\left(z \varsigma^{\beta}\right)(x),\ \left(\gamma^{\delta} z\right)(x)=\left(z \gamma^{\delta}\right)(x)$

$\Rightarrow \mu^{\alpha} z=z \mu^{\alpha}, \varsigma^{\beta} z=z \varsigma^{\beta}$, and $\gamma^{\delta} z=z \gamma^{\delta}$

$\Rightarrow N^{\square} z=z N^{\square}$ for every $\mathrm{z} \in \overset{\square}{G}$.

As a result, $N^{\square}$ is a (α, β, δ)-PNNSG of a group $\left(\overset{\square}{G}, \circ\right)$.

3.11 Proposition

Let $N^{\square}=\left(\mu^{\alpha}, \varsigma^{\beta}, \gamma^{\delta}\right)$ be a (α, β, δ)-PNSG of a group $\left(\overset{\square}{G}, \circ\right)$. Then for every $x, y \in \overset{\square}{G}$, $N^{\square}$ is a (α, β, δ)-PNNSG of a group $\left(\overset{\square}{G}, \circ\right)$ iff

$$\mu^{\alpha}\left(y \circ x \circ y^{-1}\right)=\mu^{\alpha}(x), \varsigma^{\beta}\left(y \circ x \circ y^{-1}\right)=\varsigma^{\beta}(x) \text{ and } \gamma^{\delta}\left(y \circ x \circ y^{-1}\right)=\gamma^{\delta}(x)$$

Proof:

Let $N^{\square}=\left(\mu^{\alpha}, \varsigma^{\beta}, \gamma^{\delta}\right)$ be a (α, β, δ)-PNSG of a group $\left(\overset{\square}{G}, \circ\right)$.

Then for every $x, y \in \overset{\square}{G},$ $\mu^{\alpha}\left(x \circ y\right)=\mu^{\alpha}\left(y \circ x\right)$, $\varsigma^{\beta}\left(x \circ y\right)=\varsigma^{\beta}\left(y \circ x\right)$, and $\gamma^{\delta}\left(x \circ y\right)=\gamma^{\delta}\left(y \circ x\right)$.

Therefore,

$$\mu^{\alpha}\left(y\circ x\circ y^{-1}\right)=\mu^{\alpha}\left(\left(y\circ x\right)\circ y^{-1}\right)$$
$$=\mu^{\alpha}\left(y^{-1}\circ\left(y\circ x\right)\right)$$
$$=\mu^{\alpha}\left(y^{-1}\circ y\circ x\right)$$
$$=\mu^{\alpha}\left(e\circ x\right)$$
$$=\mu^{\alpha}\left(x\right)$$

Similarly, we can write $\varsigma^{\beta}\left(y\circ x\circ y^{-1}\right)=\varsigma^{\beta}\left(x\right)$, and $\gamma^{\delta}\left(y\circ x\circ y^{-1}\right)=\gamma^{\delta}\left(x\right)$.

Conversely,

for all $x,y\in\overset{\square}{G}$, $\mu^{\alpha}\left(y\circ x\circ y^{-1}\right)=\mu^{\alpha}\left(x\right),\varsigma^{\beta}\left(y\circ x\circ y^{-1}\right)=\varsigma^{\beta}\left(x\right)$ and $\gamma^{\delta}\left(y\circ x\circ y^{-1}\right)=\gamma^{\delta}\left(x\right)$

Therefore,

$$\mu^{\alpha}\left(x\circ y\right)=\mu^{\alpha}\left(y^{-1}\circ y\circ x\circ y\right)$$
$$=\mu^{\alpha}\left(\left(y^{-1}\right)\circ\left(y\circ x\right)\circ\left(y^{-1}\right)^{-1}\right)$$
$$=\mu^{\alpha}\left(y\circ x\right)$$

Similarly, $\varsigma^{\beta}\left(x\circ y\right)=\varsigma^{\beta}\left(y\circ x\right)$, and $\gamma^{\delta}\left(x\circ y\right)=\gamma^{\delta}\left(y\circ x\right)$.

Hence by using proposition **3.10,** $N^{\square}$ is a $\left(\alpha,\beta,\delta\right)$-PNNSG of a group $\left(\overset{\square}{G},\circ\right)$.

3.12 Theorem

Let $N^{\square}=\left(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right)$ be a $\left(\alpha,\beta,\delta\right)$ -PNSG of a group $\left(\overset{\square}{G},\circ\right)$. Then

$S=\left\{x\in\overset{\square}{G}:\mu^{\alpha}\left(x\right)=\mu^{\alpha}\left(e^{\square}\right),\varsigma^{\beta}\left(x\right)=\varsigma^{\beta}\left(e^{\square}\right)\text{ and }\gamma^{\delta}\left(x\right)=\gamma^{\delta}\left(e^{\square}\right)\right\}$ is a normal subgroup of the group $\left(\overset{\square}{G},\circ\right)$.

Proof.

As $e^{\square}\in S$, S is a non-empty set. Clearly, S is a subgroup of $\left(\overset{\square}{G},\circ\right)$.

Let $x\in\overset{\square}{G}$ and $s\in S$, then $\mu^{\alpha}\left(s\right)=\mu^{\alpha}\left(e^{\square}\right),\varsigma^{\beta}\left(s\right)=\varsigma^{\beta}\left(e^{\square}\right)$ and $\gamma^{\delta}\left(s\right)=\gamma^{\delta}\left(e^{\square}\right)$

By proposition **3.11**,

$$\mu^{\alpha}\left(x\circ s\circ x^{-1}\right)=\mu^{\alpha}\left(s\right)=\mu^{\alpha}\left(e\right),\varsigma^{\beta}\left(x\circ s\circ x^{-1}\right)=\varsigma^{\beta}\left(s\right)=\varsigma^{\beta}\left(e\right)\text{and }\gamma^{\delta}\left(x\circ s\circ x^{-1}\right)=\gamma^{\delta}\left(s\right)=\gamma^{\delta}\left(e\right)$$

Therefore, $x\circ s\circ x^{-1}\in S$.

Thus, $(S,\circ)$ is a normal subgroup of $\left(G,\circ\right)$.

4. An Application

When we are dealing with an object that appears symmetric, group theory can help us for its study and analysis. This concept applies to geometric figures, which remain invariant under some transformations (reflection or rotation or both). In mathematics, a dihedral group is the group of symmetry of a regular polygon which includes rotation and reflection. Decorative motifs on floor tiles, buildings, and artwork are frequently based on dihedral groups. Chemists and mineralogists utilize dihedral groups to categorize the structure of molecules and crystals, respectively. Many advertising agencies are employed symmetric groups to design the logo for the companies.

In this section, we study the dihedral group for an asymmetric molecule having a tetrahedron structure. We express the dihedral group in terms of (α,β,δ) -Pythagorean neutrosophic subgroup. For this, we take the dihedral group D_4 (symmetries of squares).

D_4 is a symmetric group of order 8 and it is defined by $D_4=\left\{R_0,R_{90},R_{180},R_{270},H,V,D,D^{/}\right\}$, where R_0 is the rotation of 0°, R_{90} is the rotation of 90°, R_{180} is the rotation of 180°, R_{270} is the rotation of 270°, H is a reflection about x-axis, V is a reflection about y-axis, D is a reflection about main diagonal, and $D^{/}$ is the reflection about other diagonal. Now we assign the membership, indeterminacy, and non-membership values to each element of D_4, which is displayed in the form of a table given by,

	R_0	R_{90}	R_{180}	R_{270}	H	V	D	$D^{/}$
μ	*0.8*	*0.65*	*0.75*	*0.9*	*0.4*	*0.7*	*0.85*	*0.5*
ς	*0.7*	*0.55*	*0.67*	*0.5*	*0.8*	*0.6*	*0.55*	*0.9*
γ	*0.6*	*0.9*	*0.7*	*0.8*	*0.9*	*0.8*	*0.75*	*0.7*

Table1. Membership, indeterminacy, and non-membership values to each element of D_4

We take α =0.8, β =0.6 and δ =0.7. For the elements H and V,

$$\mu^{\alpha}\left(H\circ V^{-1}\right)=\mu^{\alpha}\left(R_{180}\right)=\mu\left(R_{180}\right)\vee\alpha=0.8\leq\mu^{\alpha}\left(H\right)\vee\mu^{\alpha}\left(V^{-1}\right)=0.8$$

$$\varsigma^{\beta}\left(H\circ V^{-1}\right)=\varsigma^{\beta}\left(R_{180}\right)=\varsigma\left(R_{180}\right)\vee\beta=0.6\leq\varsigma^{\beta}\left(H\right)\vee\varsigma^{\beta}\left(V^{-1}\right)=0.8$$

$$\gamma^{\delta}\left(H\circ V^{-1}\right)=\gamma^{\delta}\left(R_{180}\right)=\gamma\left(R_{180}\right)\wedge\delta=0.7\geq\gamma^{\delta}\left(H\right)\wedge\gamma^{\delta}\left(V^{-1}\right)=0.7$$

For every pair of elements in D_4, the above conditions hold.

Therefore, $N^{\square}=\left(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right)$ is a-PNSG.

5. Conclusions

PFS, PIFS, PNS, etc. are the latest mathematical tools that are developed to design the uncertainty with high precision. But, there exist some events where all these tools are unable to make a decision. This led to the motivation of the introduction of (α,β,δ)-PNS-as an extension of a PNS. Concerning to this notion we derive (α,β,δ) -PNSGs and study their various results and properties. We also discuss (α,β,δ)-PNNSGs, (α,β,δ)-PNCs. Finally, we give an application where we use the dihedral group and by using it we model the (α,β,δ)-PNSG. In the future, there are a lot of scopes to use this concept in the study of cryptography, crystallography, graph theory, molecular chemistry, natural science, artificial intelligence, data mining, etc.

Funding: There is no external funding for this research.

Conflicts of Interest: The author declares no conflict of interest.

References

[1] Zadeh, L. A. (1965). Fuzzy set. Information and Control, 8, 338-353.

[2] Zimmermann, H. J. (2001). Fuzzy set theory and applications, 4th Rev. ed. Boston. Kluwer Academic Publishers.

[3] Singh, H., Gupta, M.M., Meitzler, T., Hou, Z.G., Garg, K.K., Solo, A.M.G., & Zadeh, L.A. (2013). Real-life applications of fuzzy logic. Advances in Fuzzy Systems, 2013, 1-3.

[4] Sunitha, M.S., & Mathew, S. (2013). Fuzzy graph theory: a survey. Annals of Pure and Applied Mathematics, 4, 92-110.

[5] Sostak, A. P. (1996). Basic structures of fuzzy topology. Journal of Mathematical Sciences, 78, 662-701.

[6] Verma, M., & Shukla, K. K. (2015). Application of fuzzy optimization to the orienteering problem. Advances in Fuzzy Systems, 2015, 1-12.

[7] Jeon, G., Anisetti, M., Damiani, E., & Monga, O. (2018). Real-time image processing systems using fuzzy and rough sets techniques. Soft Computing, 22, 1381-1384.

[8] Xu, L.H., Yu, Y. L., & Wu, Q. D. (1999). General fuzzy neural network: basic structure, algorithms and its applications. IFAC Proceedings, 32, 5255-5260.

[9] Kandel, A., & Schneider, M. (1989). Fuzzy sets and their applications to artificial intelligence. Advances in Computers, 28, 69-105.

[10] Coulon, J., & Coulon, J. L. (1984). Fuzzy Boolean algebras. Journal of Mathematical Analysis and Applications, 99, 248-256.

[11] Arfi, B. (2006). Linguistic fuzzy-logic game theory. Journal of Conflict Resolution, 50, 28-57.

[12] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20, 87-96.

[13] Sebastian, S., & Ramakrishnan, T. V. (2011). Multi-fuzzy sets: an extension of fuzzy sets. Fuzzy Information and Engineering, 3, 35-43.

[14] Deschrijver, G., & Kerre, E. E. (2003). On the relationship between some extensions of fuzzy set theory. Fuzzy Sets and Systems, 133, 227-235.

[15] Gorzalczany, M. B. (1987). A method of inference in approximate reasoning based on interval-valued fuzzy sets. Fuzzy Sets and Systems, 21, 1-17.

[16] Atanassov, K. T., & Gargov, G. (1989). Interval-valued intuitionistic fuzzy sets. Fuzzy Sets and Systems, 31, 343-349.

[17] Smarandache, F. (2006). Neutrosophic set-a generalization of the intuitionistic fuzzy set. IEEE International Conference on Granular Computing, Atlanta, Georgia, USA, 1-7.

[18] Wang, H., Smarandache, F., Zhang, Y. Q., & Sunderraman, R. (2010). Single valued neutrosophic sets. Review of the Air Force Academy, 1, 10-14.

[19] Deli, I., Broumi, S., & Smarandache, F. (2015). On neutrosophic refined sets and their applications in medical diagnosis. Journal of New Theory, 6, 88-98.

[20] Zhang, H. Y., Wang, J. Q., & Chen, X. H. (2014). Interval neutrosophic sets and their applications in multicriteria decision-making problems. The Scientific World Journal, 2014, 1-15.

[21] Deli, I., Ali, M., & Smarandache, F. (2015). Bipolar neutrosophic sets and their application based on multi-criteria decision-making problems. Proceedings of the International Conference on Advances Mechatronic Systems, Beijing, China, August, 22-24.

[22] Ali, M., & Smarandache, F. (2017). Complex neutrosophic set. Neural Computing and Applications, 28, 1817-1834.

[23] Mondal, K., Pramanik, S., & Giri, B. C. (2018). Interval neutrosophic similarity measure based MADM strategy and its application to MADM problems. Neutrosophic Sets and Systems, 19, 47-56.

[24] Biswas, P., Pramanik, S., & Giri, B. C. (2019). Non-linear programming approach for single-valued neutrosophic TOPSIS method. New Mathematics and Natural Computation, 15, 307-326.

[25] Rosenfeld, A. (1971). Fuzzy groups. Journal of Mathematical Analysis and Applications, 35, 512-517.

[26] Anthony, J. M., & Sherwood, H. (1982). Fuzzy groups redefined. Journal of Mathematical Analysis and Applications, 69, 124-130.

[27] Das, P. S. (1981). Fuzzy groups and level subgroups. Journal of Mathematical Analysis and Applications, 84, 264-269.

[28] Ajmal, N., & Prajapati, A. S. (1992). Fuzzy cosets and fuzzy normal subgroups. Information Sciences, 64, 17-25.

[29] Zhan, J., & Tan, Z. (2004). Intuitionistic M-fuzzy groups. Soochow Journal of Mathematics, 30, 85-90.

[30] Husban, R. A., Salleh, A. R., & Ahmed, A. G. B. (2016). Complex intuitionistic fuzzy group. Global Journal of Pure and Applied Mathematics, 12, 4929-4949.

[31] Agboola, A. A. A., Akwu, A. D., & Oyebo, Y. T. (2012). Neutrosophic groups and subgroups. International Journal of Mathematical Combinatorics, 3, 1-9.

[32] Yager, R. R. (2013). Pythagorean fuzzy subsets. 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), 57-61. doi:10.1109/ifsa-nafips.2013.6608375.

[33] Ejegwa, P. A. (2019). Pythagorean fuzzy set and its application in career placements based on academic performance using max-min-max composition. Complex and Intelligent Systems, 5, 165-175.

[34] Liang, D., & Xu, Z. (2017). The new extension of TOPSIS method for multiple criteria decision-making with hesitant Pythagorean fuzzy sets. Applied Soft Computing, 60, 167-179.

[35] Lin, Y. L., Ho, L. H., Yeh, S. L., & Chen, T. Y. (2019). A pythagorean fuzzy TOPSIS method based on novel correlation measures and its application to multiple criteria decision analysis of inpatient stroke rehabilitation. International Journal of Computational Intelligence Systems, 12, 410-425.

[36] Garg, H. (2017). Generalized Pythagorean fuzzy geometric aggregation operators using Einstein t-norm and t-conorm for multicriteria decision making process. International Journal of Intelligent Systems, 32, 597-630.

[37] Garg, H. (2018). Linguistic pythagorean fuzzy sets and its application in multiattribute decision making process. International Journal of Intelligent Systems, 33, 1234-1263.

[38] Naz, S., Ashraf, S., & Akram, M. (2018). A novel approach to decision-making with pythagorean fuzzy information. Mathematics, 6, 1-28.

[39] Akram, M., & Naz, S. (2019). A novel decision-making approach under complex pythagorean fuzzy environment. Mathematical and Computational Applications, 24, 1-33.

[40] Rahman, K., & Abdullah, S. (2018). Generalized interval-valued pythagorean fuzzy aggregation operators and their application to group decision-making. Granular Computing, 4, 15-25.

[41] Ajay, D., & Chellamani, P. (2020). Pythagorean Neutrosophic Fuzzy Graphs. *International Journal of Neutrosophic Science (IJNS)*, *11*(2).

[42] Jansi, R., Mohana, K., & Smarandache, F. (2019). Correlation measure for pythagorean neutrosophic sets with T and F as dependent neutrosophic components. Neutrosophic Sets and Systems, 30, 202-212.

Received: Aug 17, 2021. Accepted: Dec 1, 2021