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Norms and Delta-Equalities of Complex Neutrosophic Sets

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Abstract: The purpose of this paper is to put forward the basics results of complex fuzzy sets (CFSs) such as union, intersection, complement, product into complex neutrosophic sets because as the CFSs and complex intuitionistics sets does give the erroneous and inconvenient information about uncertainty and periodicity and also there are results related to different norms. Moreover we give some results about the distance measures of complex neutrosophic sets and define some notions.

Keywords: CFSs, complex neutrosophic sets, distance measures, delta-equalities.

1. Introduction

Lotfi A. Zadeh [19] introduced a fuzzy set (FS) in 1965. FS was designed to manipulate ambiguity, fuzziness, vagueness, crispness, and uncertainty in different aspects of life. It has great significance in the field of genetic algorithm in chemical industry.

Krassimir Atanassov [3] generalized the concept of L. A. Zadeh and introduced an intuitionistic fuzzy set (IFS) in which instead of the truth function of each element there is also the falsehood function. It indicates that statement can be true or false, yes or no, right or wrong, feasible or not.

De et al., [6] in 2001, use the idea of a fuzzy set for modeling in real life problems, like marketing, psychological investigations, and determination of diagnosis [16] etc. IFS has great significance in career determination. In IFS the concept of distance measure also introduced but there was a problem to deal when both the informations contain uncertainties of yes and no at a time and at a time neither yes nor no. Thus F. Smarandache [15] gave the solution of this problem by introducing new FS called a neutrosophic fuzzy set (NFS) which is a framework for unification of a FS and an IFS or it is a bridge between FS and IFS. Neutrosophy is the philosophys branch, in which we deal with the scope, nature, and origin of neutral along with ideational spectra. Neutrosophy has a great engineering application like in medicine, military, airspace, cybernetics etc. A neutrosophic set is that which contain truth function T , indeterministic function I and falsehood function F .

A neutrosophic fuzzy set yields three type of chances like win, lose, draw or accept, reject, pending or positive, negative, zero etc. NS is the extension of some FSs like interval valued fuzzy sets (IVFSs) [16], conventional FSs [19], paradoxist sets [15] and IFS [3]. Wang et al., [17] gave more information about NS by presenting the single valued NS which has a lot of application in engineering and social problems and have additional benefit to interpret vagueness, crispness, and uncertainty. For more details about neutrosophic sets one can refer [1], [7], [9], [10], [11] and [14].
After that Ramot et al., [12] gave the idea of a complex fuzzy set (CFS) for handling problems having amplitude term where the complex mapping is used a instead of real valued mapping and is defined as

$$\mu_s(x) = r_s(x)e^{i\omega_s(x)}, \quad i = \sqrt{-1}$$

where amplitude term $r_s(x)$ and phase term $\omega_s(x)$ are the real valued function having the range $[0,1]$, and the range of $\mu_s(x)$ is expanded to a circle of radius 1. In a CFS amplitude term conserve the crispness idea together with the phase term which declare the periodicity in a CFS. The phase term makes it different from conventional fuzzy set [19], IFSs [3], and cubic set because it gives constructive and destructive interference which concludes that a complex fuzzy set has wavelike character. G. Zhang et al., [12] defined several important properties in complex fuzzy sets like union, intersection, complement, product, some norms like quasi-triangular norm, s-norm, t-norm etc.
After this Alkouri and Saleh [2] extended a CFS into a complex intuitionistic set and it contains complex valued truth function together with the complex valued falsehood function. They differ the idea of a FS in a way such that an IFS have two phase terms instead of one. F. Smarandache introduced a complex neutrosophic set (CNS) which contains truth function T, indeterministic function I and falsehood function F having the range is extended to unit circle. CNSs contain amplitude terms together with the three phase terms and can work with information containing uncertainties, crispness and vagueness in periodicity.
Pappis [pappis] for the first time worked on the concept of proximity measure and approximately equal fuzzy set whose work was generalized by Hong and Hwang [hong]. Later on Cai [cai],[4] felt that both were using the same concept so he changed that approach and expressed as special measure is used for defining $\delta - equalities$. Two FSs A and B are called $\delta - equal$ if they are $1 - \delta$ part away. Zhang et al. [18] used this concept of $\delta - equality$ for applications in signal processing which certify $\delta - equality$ of CFSs practically.
We are extending the work of G. Zhang et al., [18] from CFSs into complex neutrosophic sets and investigate some useful results.

Definition *CNS S is defined on a X, distinguished by degree of truth , indeterminate function and falsehood function respectively. The truth function, indeterminate function and falsehood function are defined as*

$$T_S(x) = p_S(x)e^{i\mu_S(x)}, I_S(x) = q_S(x)e^{i\nu_S(x)}, F_S(x) = r_S(x)e^{i\omega_S(x)},$$

where $p_S(x)$ *represents a FS and* $\mu_S(x)$ *is any real function. Similarly for indeterminacy and falsity* $q_S(x)$ *&* $\nu_S(x)$ and $r_S(x)$ $\omega_S(x)$, *such that*

$$0^- \le p_S(x) + q_S(x) + r_S(x) \le 3^+.$$

CNS S is defined to be

$S = \{(x, T_S(x) = a_T, I_S(x) = a_I, F_S(x) = a_I)? x \in X\}$, where

$$T_S: X \to \{a_T: a_T \in C, |a_T| \leq 1\},$$
$$I_S: X \to \{a_I: a_I \in C, |a_I| \leq 1\},$$
$$F_S: X \to \{a_F: a_F \in C, |a_F| \leq 1\},$$
$$\text{and } T_S(x) + I_S(x) + F_S(x) \leq 3^+.$$

Definition (1) *A function* $(0,1] \times (0,1] \to [0,1]$ *is a quasi-triangular norm* T *if following holds:*

(i) $T(1,1) = 0$

(ii) $T(a,b) = T(b,a)$

(iii) $T(a,b) \leq T(c,d)$, *whenever,* $a \leq c, b \leq d$

(iv) $T(T(a,b),c) = T\big(a,(b,c)\big)$

(2) *A function* $(0,1] \times (0,1] \to [0,1]$ *is a triangular norm* T *if it satisfies previous* $(i) - (iv)$ *conditions together with*

(v) $T(0,0) = 0$

(3) *A function* $(0,1] \times (0,1] \to [0,1]$ *is s-norm if it satisfies triangular norm's conditions together with*

(vi) $T(a,0) = a$

(4) *A function* $(0,1] \times (0,1] \to [0,1]$ *is t-norm if it satisfies triangular norm's conditions together with*

(vii) $T(a,1) = a.$

Definition *The union for CNSs is defined as: Assume*

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$

be the complex neutrosophic sets on X *such that*

$$T_A(x) = p_A(x)e^{i\mu_A(x)}, I_A(x) = q_A(x)e^{i\nu_A(x)}, F_A(x) = r_A(x)e^{i\omega_A(x)},$$
$$T_B(x) = p_B(x)e^{i\mu_B(x)}, I_B(x) = q_B(x)e^{i\nu_B(x)}, F_B(x) = r_B(x)e^{i\omega_B(x)},$$

be complex valued truth, indeterminate and falsehood functions respectively, then union of A *and* B *be represented as*

$$A \cup B = \{x, T_{A\cup B}(x), I_{A\cup B}(x), F_{A\cup B}(x), x \in X\}$$

where $T_{A\cup B(x)}$, $I_{A\cup B(x)}$, $F_{A\cup B(x)}$ *are defined as*

$$T_{A\cup B}(x) = [p_A(x) \vee p_B(x)]e^{i\mu_{T_{A\cup B}}(x)},$$
$$I_{A\cup B}(x) = [q_A(x) \wedge q_B(x)]e^{i\nu_{I_{A\cup B}}(x)},$$
$$F_{A\cup B}(x) = [r_A(x) \wedge r_B(x)]e^{i\omega_{F_{A\cup B}}(x)},$$

where $\vee$ *represent the max operator and* $\wedge$ *represent min operator.*

Proposition. *The complex neutrosophic union is s-norm.*

Proof Here we prove only $(iii)\&(iv)$ properties because others are quite easy

(ii) Let

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$
$$C = \{x, T_C(x), I_C(x), F_C(x), x \in X\},$$

be the complex neutrosophic sets on X such that

$$T_A(x) = p_A(x)e^{i\mu_A(x)},\ T_B(x) = p_B(x)e^{i\mu_B(x)},\ T_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$I_A(x) = p_A(x)e^{i\mu_A(x)},\ I_B(x) = p_B(x)e^{i\mu_B(x)},\ I_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$F_A(x) = p_A(x)e^{i\mu_A(x)},\ F_B(x) = p_B(x)e^{i\mu_B(x)},\ F_C(x) = p_C(x)e^{i\mu_C(x)},$$

we suppose that

$$|p_A(x)| \leq |p_B(x)|, |r_A(x)| \leq |r_B(x)|, |q_A(x)| \leq |q_B(x)|,$$
$$\mu_A(x) \leq \mu_B(x), \nu_A(x) \leq \nu_B(x), \omega_A(x) \leq \omega_B(x), \text{ for all } x \in X.$$

Thus

$$|T_{A\cup C}(x)| = max(p_A(x), p_C(x)) \leq max(p_B(x), p_C(x)) = |T_{B\cup C}(x)|, \text{ for all } x \in X.$$

Similarly

$$|I_{A\cup C}(x)| = max(q_A(x), q_C(x)) \leq max\left(q_B(x), p_{qC}(x)\right) = |I_{B\cup C}(x)|, \text{ for all } x \in X, \text{ and}$$

$$|F_{A\cup C}(x)| = max(r_A(x), r_C(x)) \leq max(r_B(x), r_C(x)) = |F_{B\cup C}(x)|\ , \text{ for all } x \in X.$$

Also

$$|\mu_{A\cup C}(x)| = max(\mu_A(x), \mu_C(x)) \leq max(\mu_B(x), \mu_C(x)) = |\mu_{B\cup C}(x)|, \text{ for all } x \in X,$$

$$|\nu_{A\cup C}(x)| = max\big(\nu_A(x), \nu_C(x)\big) \leq max\big(\nu_C(x), \nu_C(x)\big) = |\nu_{B\cup C}(x)|, \text{ for all } x \in X,$$

$$|\omega_{A\cup C}(x)| = max\big(\omega_A(x), \omega_C(x)\big) \leq max\big(\omega_B(x), \omega_C(x)\big) = |\omega_{B\cup C}(x)|, \text{ for all } x \in X.$$

(iv)

Let

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$
$$C = \{x, T_C(x), I_C(x), F_C(x), x \in X\},$$

be the complex neutrosophic sets on X, such that

$$T_A(x) = p_A(x)e^{i\mu_A(x)},\ T_B(x) = p_B(x)e^{i\mu_B(x)},\ T_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$I_A(x) = p_A(x)e^{i\mu_A(x)},\ I_B(x) = p_B(x)e^{i\mu_B(x)},\ I_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$F_A(x) = p_A(x)e^{i\mu_A(x)},\ F_B(x) = p_B(x)e^{i\mu_B(x)},\ F_C(x) = p_C(x)e^{i\mu_C(x)}.$$

Therefore

$$T_{(A\cup B)\cup C}(x) = p_{(A\cup B)\cup C}(x)e^{i\mu_{(A\cup B)\cup C}(x)}$$

$$= max\,[p_{A\cup B}(x), p_C(x)]e^{i\,max\,[\mu_{A\cup B}(x), \mu_C(x)]}$$

$$= max\left[max\big(p_A(x), p_B(x)\big), p_C(x)\right]e^{i\,max\,[max\,(\mu_A(x), \mu_B(x)), \mu_C(x)]}$$

$$= max\left[\big(p_A(x)\big), max\big(p_B(x), p_C(x)\big)\right]e^{i\,max[(\mu_A(x)), max\,(\mu_B(x), \mu_C(x))]}$$

$$= max\,[p_A(x), p_{B\cup C}(x)]e^{i\,max\,[\mu_A(x), \mu_{B\cup C}(x)]}$$

$$= p_{A\cup(B\cup C)}(x)e^{i\mu_{A\cup(B\cup C)}(x)} = T_{A\cup(B\cup C)}(x).$$

Following the same procedure we can prove for indeterminacy and falsehood functions.

Corollary *Let* $C_\alpha \in X, \alpha \in I$ *and*

$$T_{C_\alpha}(x) = p_{C_\alpha}(x)e^{i\mu_{C_\alpha}(x)}, I_{C_\alpha}(x) = q_{C_\alpha}e^{i\nu_{C_\alpha}(x)}, F_{C_\alpha}(x) = r_{C_\alpha}e^{i\omega_{C_\alpha}(x)}$$

Then $\bigcup_{\alpha\in I} C_\alpha \in X.$ *Thus*

$$T \bigcup_{\alpha \in I} C_\alpha(x) = \sup_{\alpha \in I} p_{C_\alpha}(x) e^{i \sup_{\alpha \in I} \mu_{C_\alpha}(x)},$$

$$I \bigcup_{\alpha \in I} C_\alpha(x) = \inf_{\alpha \in I} q_{C_\alpha}(x) e^{i \sup_{\alpha \in I} \nu_{C_\alpha}(x)},$$

$$F \bigcup_{\alpha \in I} C_\alpha(x) = \inf_{\alpha \in I} r_{C_\alpha}(x) e^{i \sup_{\alpha \in I} \omega_{C_\alpha}(x)}.$$

Proof It is trivial.

Definition *The intersection of CNSs is defined as*

Let

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$

be CNSs on X *such that*

$$T_A(x) = p_A(x) e^{i\mu_A(x)}, I_A(x) = q_A e^{i\nu_A(x)}, F_A(x) = r_A e^{i\omega_A(x)},$$
$$T_B(x) = p_B(x) e^{i\mu_B(x)}, I_B(x) = q_B e^{i\nu_B(x)}, F_B(x) = r_B e^{i\omega_B(x)},$$

is represented as

$$A \cap B = \{x, T_{A\cap B}(x), I_{A\cap B}(x), F_{A\cap B}(x), x \in X\},$$

where $T_{A\cap B(x)}$, $I_{A\cap B(x)}$, $F_{A\cap B(x)}$ *are defined as*

$$T_{A\cap B}(x) = [p_A(x) \wedge p_B(x)] e^{i\mu_{T_{A\cap B}}(x)},$$
$$I_{A\cap B}(x) = [q_A(x) \vee q_B(x)] e^{i\nu_{I_{A\cap B}}(x)},$$
$$F_{A\cap B}(x) = [r_A(x) \vee r_B(x)] e^{i\omega_{F_{A\cap B}}(x)},$$

where $\vee$ *is a maximum operator &* $\wedge$ *is a minimum operator.*

Proposition *If* A *and* B *are CNSs on* X*. Then* $\overline{A\cap B} = \overline{A} \cup \overline{B}$.

Proof For membership function

$$T_{\overline{A\cap B}}(x) = p_{\overline{A\cap B}}(x) e^{i\mu_{\overline{A\cap B}}(x)} = \left(1 - p_{A\cap B}(x)\right) e^{i\left(2\pi - \mu_{A\cap B}(x)\right)}$$

$$= \left(1 - \min\left(p_A(x), p_B(x)\right)\right) e^{i\left(2\pi - \min\left(\mu_A(x), \mu_B(x)\right)\right)}$$

$$= \max\left(1 - p_A(x), 1 - p_B(x)\right) e^{i\max\left(2\pi - \left(\mu_A(x), 2\pi - \mu_B(x)\right)\right)}$$

$$= max\left(p_{\overline{A}}(x), p_{\overline{B}}(x)e^{i\,max\left(2\pi-\left(\mu_A(x),2\pi-\mu_B(x)\right)\right)}\right) = T_{\overline{A}\cup\overline{B}}(x).$$

For indeterminacy function

$$I_{\overline{A\cap B}}(x) = q_{\overline{A\cap B}}(x)e^{i\nu_{\overline{A\cap B}}(x)} = \left(1 - q_{A\cap B}(x)\right)e^{i\left(2\pi-\nu_{A\cap B}(x)\right)}$$

$$= \left(1 - max\left(p_A(x), p_B(x)\right)\right)e^{i\left(2\pi-min\left(\nu_A(x),\nu_B(x)\right)\right)}$$

$$= min\left(1 - q_A(x), 1 - q_B(x)\right)e^{i\,max\left(2\pi-\left(\nu_A(x),2\pi-\nu_B(x)\right)\right)}$$

$$= min\left(q_{\overline{A}}(x), q_{\overline{B}}(x)e^{i\,max\left(2\pi-\left(\nu_A(x),2\pi-\nu_B(x)\right)\right)}\right) = I_{\overline{A}\cup\overline{B}}(x).$$

Similarly we can prove for falsehood function.

Proposition *The complex neutrosophic intersection on* X *is t-norm.*

Proof Here we prove only (i) & (iv) properties because others are quite easy

(i) Let

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$
$$C = \{x, T_C(x), I_C(x), F_C(x), x \in X\},$$

be the complex neutrosophic sets on X such that

$$T_A(x) = p_A(x)e^{i\mu_A(x)},\ T_B(x) = p_B(x)e^{i\mu_B(x)},\ T_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$I_A(x) = p_A(x)e^{i\mu_A(x)},\ I_B(x) = p_B(x)e^{i\mu_B(x)},\ I_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$F_A(x) = p_A(x)e^{i\mu_A(x)},\ F_B(x) = p_B(x)e^{i\mu_B(x)},\ F_C(x) = p_C(x)e^{i\mu_C(x)},$$

Now we suppose

$$|p_A(x)| \leq |p_B(x)|, |q_A(x)| \leq |q_B(x)|, |r_A(x)| \leq |r_B(x)|,$$
$$\mu_A(x) \leq \mu_B(x),\ \nu_A(x) \leq \nu_B(x),\ \omega_A(x) \leq \omega_B(x), \text{ for all } x \in X.$$

Thus

$$|T_{A\cap C}(x)| = min\left(p_A(x), p_C(x)\right) \leq min\left(p_B(x), p_C(x)\right) = |T_{B\cap C}(x)|, \text{ for all } x \in X.$$

Similarly

$$|I_{A\cap C}(x)| = max\left(q_A(x), q_C(x)\right) \leq max\left(q_B(x), p_C(x)\right) = |I_{B\cap C}(x)|, \text{ for all } x \in X,$$

$$|F_{A\cap C}(x)| = max(r_A(x), r_C(x)) \le max(r_B(x), r_C(x)) = |F_{B\cap C}(x)|, \text{ for all } x \in X.$$

Likewise

$$|\mu_{A\cup C}(x)| = min(\mu_A(x), \mu_C(x)) \le min(\mu_B(x), \mu_C(x)) = |\mu_{B\cap C}(x)|, \text{ for all } x \in X,$$

$$|\nu_{A\cap C}(x)| = min(\nu_A(x), \nu_C(x)) \le min(\nu_C(x), \nu_C(x)) = |\nu_{B\cap C}(x)| \text{ for all } x \in X,$$

$$|\omega_{A\cap C}(x)| = min(\omega_A(x), \omega_C(x)) \le min(\omega_B(x), \omega_C(x)) = |\omega_{B\cap C}(x)| \text{ for all } x \in X.$$

(iv) Let

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$
$$C = \{x, T_C(x), I_C(x), F_C(x), x \in X\},$$

be complex neutrosophic sets on X such that

$$T_A(x) = p_A(x)e^{i\mu_A(x)},\ T_B(x) = p_B(x)e^{i\mu_B(x)},\ T_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$I_A(x) = p_A(x)e^{i\mu_A(x)},\ I_B(x) = p_B(x)e^{i\mu_B(x)},\ I_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$F_A(x) = p_A(x)e^{i\mu_A(x)},\ F_B(x) = p_B(x)e^{i\mu_B(x)},\ F_C(x) = p_C(x)e^{i\mu_C(x)}.$$

Thus

$$T_{(A\cap B)\cup C}(x) = p_{(A\cap B)\cap C}(x)e^{i\mu_{(A\cap B)\cap C}(x)}$$

$$= min[p_{A\cap B}(x), p_C(x)]e^{i\, min[\mu_{A\cap B}(x), \mu_C(x)]}$$

$$= min\left[min(p_A(x), p_B(x)), p_C(x)\right]e^{i\, min[min(\mu_A(x), \mu_B(x)), \mu_C(x)]}$$

$$= min\left[(p_A(x)), min(p_B(x), p_C(x))\right]e^{i\, min[(\mu_A(x)), min(\mu_B(x), \mu_C(x))]}$$

$$= min[p_A(x), p_{B\cap C}(x)]e^{i\, min[\mu_A(x), \mu_{B\cap C}(x)]}$$

$$= p_{A\cap(B\cap C)}(x)e^{i\mu_{A\cap(B\cap C)}(x)} = T_{A\cap(B\cap C)}(x).$$

Following the same procedure we can prove for indeterminacy and falsehood functions.

Corollary *Let* $C_\alpha \in X, \alpha \in I$ *and*

$$T_{C_\alpha}(x) = p_{C_\alpha}(x)e^{i\mu_{C_\alpha}(x)}, I_{C_\alpha}(x) = q_{C_\alpha}e^{i\nu_{C_\alpha}(x)}, F_{C_\alpha}(x) = r_{C_\alpha}e^{i\omega_{C_\alpha}(x)},$$

Then $\bigcup_{\alpha \in I} C_\alpha \in X$ *thus*

$$T_{\bigcup_{\alpha\in I} C_\alpha}(x) = \inf_{\alpha\in I} p_{C_\alpha}(x)e^{i\sup_{\alpha\in I}\mu_{C_\alpha}(x)},$$

$$I_{\bigcup_{\alpha\in I} C_\alpha}(x) = \sup_{\alpha\in I} q_{C_\alpha}(x)e^{i\sup_{\alpha\in I}\nu_{C_\alpha}(x)},$$

$$F_{\bigcup_{\alpha\in I} C_\alpha}(x) = \sup_{\alpha\in I} r_{C_\alpha}(x)e^{i\sup_{\alpha\in I}\omega_{C_\alpha}(x)}.$$

Proof It is trivial.

Corollary *Let* $C_{\alpha\beta} \in X, \alpha \in I_1, \beta \in I_2$ *and*

$$T_{C_{\alpha\beta}}(x) = p_{C_{\alpha\beta}}(x)e^{i\mu_{C_{\alpha\beta}}(x)}, I_{C_{\alpha\beta}}(x) = q_{C_{\alpha\beta}}e^{i\nu_{C_{\alpha\beta}}(x)}, F_{C_{\alpha\beta}}(x) = r_{C_{\alpha\beta}}e^{i\omega_{C_{\alpha\beta}}(x)},$$

where I_1 *and* I_2 *are arbitrary index sets. Then* $\bigcup_{\alpha\in I_1}\bigcap_{\alpha\in I_2} C_{\alpha\beta} \in X$, $\bigcap_{\alpha\in I_1}\bigcup_{\alpha\in I_2} C_{\alpha\beta} \in X$ *. Then*

$$T_{\bigcup_{\alpha\in I_1}\bigcap_{\alpha\in I_2} C_\alpha}(x) = \sup_{\alpha\in I_1}\inf_{\alpha\in I_2} p_{C_\alpha}(x)e^{i\sup_{\alpha\in I_1}\inf_{\alpha\in I_2}\mu_{C_\alpha}(x)},$$

$$I_{\bigcup_{\alpha\in I_1}\bigcap_{\alpha\in I_2} C_\alpha}(x) = \inf_{\alpha\in I_1}\sup_{\alpha\in I_2} q_{C_\alpha}(x)e^{i\sup_{\alpha\in I_1}\inf_{\alpha\in I_2}\nu_{C_\alpha}(x)},$$

$$F_{\bigcup_{\alpha\in I_1}\bigcap_{\alpha\in I_2} C_\alpha}(x) = \inf_{\alpha\in I_1}\sup_{\alpha\in I_2} r_{C_\alpha}(x)e^{i\sup_{\alpha\in I_1}\inf_{\alpha\in I_2}\omega_{C_\alpha}(x)}.$$

Or

$$T_{\bigcap_{\alpha\in I_1}\bigcup_{\alpha\in I_2} C_\alpha}(x) = \inf_{\alpha\in I_1}\sup_{\alpha\in I_2} p_{C_\alpha}(x).e^{i\inf_{\alpha\in I_1}\sup_{\alpha\in I_2}\mu_{C_\alpha}(x)},$$

$$I_{\bigcap_{\alpha\in I_1}\bigcup_{\alpha\in I_2} C_\alpha}(x) = \sup_{\alpha\in I_1}\inf_{\alpha\in I_2} q_{C_\alpha}(x).e^{i\inf_{\alpha\in I_1}\sup_{\alpha\in I_2}\nu_{C_\alpha}(x)},$$

$$F_{\bigcap_{\alpha\in I_1}\bigcup_{\alpha\in I_2} C_\alpha}(x) = \sup_{\alpha\in I_1}\inf_{\alpha\in I_2} r_{C_\alpha}(x).e^{i\inf_{\alpha\in I_1}\sup_{\alpha\in I_2}\omega_{C_\alpha}(x)}.$$

Proof It is trivial.

Definition *The product of CNSs is defined as*

Let

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$

be complex valued NSs such that

$$T_A(x) = p_A(x)e^{i\mu_A(x)}, I_A(x) = q_A(x)e^{i\nu_A(x)}, F_A(x) = r_A(x)e^{i\omega_A(x)},$$
$$T_B(x) = p_B(x)e^{i\mu_B(x)}, I_B(x) = q_B(x)e^{i\nu_B(x)}, F_B(x) = r_B(x)e^{i\omega_B(x)},$$

is denoted as

$$A \circ B = \{x, T_{A\circ B}(x), I_{A\circ B}(x), F_{A\circ B}(x), x \in X\},$$

where $T_{A\circ B(x)}$, $I_{A\circ B(x)}$, $F_{A\circ B(x)}$ *are defined as*

$$T_{A\circ B}(x) = p_{A\circ B}(x)e^{i\mu_{A\circ B}(x)} = [p_A(x).p_B(x)]e^{i2\pi\left(\frac{\mu_A(x)}{2\pi}.\frac{\mu_B(x)}{2\pi}\right)},$$

$$I_{A\circ B}(x) = q_{A\circ B}(x)e^{i\nu_{A\circ B}(x)} = [q_A(x).q_B(x)]e^{i2\pi\left(\frac{\nu_A(x)}{2\pi}.\frac{\nu_B(x)}{2\pi}\right)},$$

$$F_{A\circ B}(x) = r_{A\circ B}(x)e^{i\omega_{A\circ B}(x)} = [r_A(x).r_B(x)]e^{i2\pi\left(\frac{\omega_A(x)}{2\pi}.\frac{\omega_B(x)}{2\pi}\right)}.$$

Proposition *The complex neutrosophic product on* X *is t-norm.*

Proof Here we prove only $(iii)\&(iv)$ properties because others are quite easy

(iii) Let

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$
$$C = \{x, T_C(x), I_C(x), F_C(x), x \in X\},$$

be the CNSs on X such that

$$T_A(x) = p_A(x)e^{i\mu_A(x)},\ T_B(x) = p_B(x)e^{i\mu_B(x)},\ T_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$I_A(x) = p_A(x)e^{i\mu_A(x)},\ I_B(x) = p_B(x)e^{i\mu_B(x)},\ I_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$F_A(x) = p_A(x)e^{i\mu_A(x)},\ F_B(x) = p_B(x)e^{i\mu_B(x)},\ F_C(x) = p_C(x)e^{i\mu_C(x)},$$

Now, we suppose that

$$|p_A(x)| \le |p_B(x)|, |q_A(x)| \le |q_B(x)|, |r_A(x)| \le |r_B(x)|,$$
$$\mu_A(x) \le \mu_B(x),\ \nu_A(x) \le \nu_B(x),\ \omega_A(x) \le \omega_B(x), \text{ for all } x \in X.$$

Thus

$$|T_{A\circ C}(x)| = |p_A(x)|.|p_C(x)| \le |p_B(x)|.|p_C(x)| = |T_{B\circ C}(x)|, \text{ for all } x \in X.$$

Similarly

$$|I_{A\circ C}(x)| = |q_A(x)|.|q_C(x)| \le |q_B(x)|.|q_C(x)| = |I_{B\circ C}(x)|, \text{ for all } x \in X,$$
$$|F_{A\cap C}(x)| = |r_A(x)|.|r_C(x)| \le |r_B(x)|.|r_C(x)| = |F_{B\circ C}(x)|, \text{ for all } x \in X.$$

Likewise

$$|\mu_{A\circ C}(x)| = 2\pi\left(\frac{\mu_{A(x)}}{2\pi}.\frac{\mu_{C(x)}}{2\pi}\right) \le 2\pi\left(\frac{\mu_{B(x)}}{2\pi}.\frac{\mu_{C(x)}}{2\pi}\right) = |\mu_{B\circ C}(x)|, \text{ for all } x \in X,$$

$$|\nu_{A\circ C}(x)| = 2\pi\left(\frac{\nu_{A(x)}}{2\pi}.\frac{\nu_{C(x)}}{2\pi}\right) \le 2\pi\left(\frac{\nu_{B(x)}}{2\pi}.\frac{\nu_{C(x)}}{2\pi}\right) = |\nu_{B\circ C}(x)|, \text{ for all } x \in X,$$

$$|\omega_{A\circ C}(x)| = 2\pi\left(\frac{\omega_{A(x)}}{2\pi}.\frac{\omega_{C(x)}}{2\pi}\right) \le 2\pi\left(\frac{\omega_{B(x)}}{2\pi}.\frac{\omega_{C(x)}}{2\pi}\right) = |\omega_{B\circ C}(x)|, \text{ for all } x \in X.$$

(iv) Let

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$
$$C = \{x, T_C(x), I_C(x), F_C(x), x \in X\},$$

be complex neutrosophic sets on X such that

$$T_A(x) = p_A(x)e^{i\mu_A(x)},\ T_B(x) = p_B(x)e^{i\mu_B(x)},\ T_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$I_A(x) = p_A(x)e^{i\mu_A(x)},\ I_B(x) = p_B(x)e^{i\mu_B(x)},\ I_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$F_A(x) = p_A(x)e^{i\mu_A(x)},\ F_B(x) = p_B(x)e^{i\mu_B(x)},\ F_C(x) = p_C(x)e^{i\mu_C(x)}.$$

We have

$$T_{A\circ(B\circ C)}(x) = p_{A\circ(B\circ C)}(x).e^{i.\mu_{A\circ(B\circ C)}(x)}$$
$$= [p_A(x).p_{B\circ C}(x)].e^{i2\pi\left(\frac{\mu_{A(x)}}{2\pi}.\frac{\mu_{B\circ C(x)}}{2\pi}\right)}$$
$$= [p_A(x).(p_B(x).p_C(x))].e^{i2\pi\left(\frac{\mu_{A(x)}}{2\pi}.2\pi\frac{\left(\frac{\mu_{B(x)}}{2\pi}.\frac{\mu_{C(x)}}{2\pi}\right)}{2\pi}\right)}$$
$$= [(p_A(x).p_B(x)).p_C(x)].e^{i2\pi\left(2\pi\frac{\left(\frac{\mu_{A(x)}}{2\pi}.\frac{\mu_{B(x)}}{2\pi}\right)}{2\pi}.\frac{\mu_{C(x)}}{2\pi}\right)}$$
$$= [p_{A\circ B}(x).p_C(x)].e^{i2\pi\left(\frac{\mu_{A\circ B}(x)}{2\pi}.\frac{\mu_{C(x)}}{2\pi}\right)}$$
$$= p_{(A\circ B)\circ C}(x).e^{\mu_{(A\circ B)\circ C}} = T_{(A\circ B)\circ C}(x).$$

Following the same procedure we can prove for indeterminacy and falsehood functions.

Corollary *Let* $C_\alpha \in X, \alpha \in I$ *and*

$$T_{C_\alpha}(x) = p_{C_\alpha}(x)e^{i\mu_{C_\alpha}(x)}, I_{C_\alpha}(x) = q_{C_\alpha}e^{i\nu_{C_\alpha}(x)}, F_{C_\alpha}(x) = r_{C_\alpha}e^{i\omega_{C_\alpha}(x)},$$

Then $\underset{\alpha\in I}{\Pi} C_\alpha = C_1(x) \circ C_2(x) \circ \ldots \circ C_\alpha(x) \in X$. *Thus*

$$T \underset{\alpha\in I}{\Pi} C_\alpha(x) = p_{C_1(x)} \cdot p_{C_2(x)} \cdots p_{C_\alpha(x)} e^{i2\pi\left(\frac{\mu_{C1}(x)}{2\pi} \cdot \frac{\mu_{C2}(x)}{2\pi} \cdots \frac{\mu_{C\alpha}(x)}{2\pi}\right)},$$

$$I \underset{\alpha\in I}{\Pi} C_\alpha(x) = q_{C_1(x)} \cdot q_{C_2(x)} \cdots q_{C_\alpha(x)} e^{i2\pi\left(\frac{\nu_{C1}(x)}{2\pi} \cdot \frac{\nu_{C2}(x)}{2\pi} \cdots \frac{\nu_{C\alpha}(x)}{2\pi}\right)},$$

$$F \underset{\alpha\in I}{\Pi} C_\alpha(x) = r_{C_1(x)} \cdot r_{C_2(x)} \cdots r_{C_\alpha(x)} e^{i2\pi\left(\frac{\omega_{C1}(x)}{2\pi} \cdot \frac{\omega_{C2}(x)}{2\pi} \cdots \frac{\omega_{C\alpha}(x)}{2\pi}\right)}.$$

Proof It is trivial.

Definition *Let* An *be* N *CNSs on* X $(n = 1,2,\ldots,N)$ *and*

$$T_{A_n}(x) = p_A(x)e^{i\mu_{A_n}(x)}, I_{A_n}(x) = q_{A_n}(x)e^{i\nu_{A_n}(x)}, F_{A_n}(x) = r_{A_n}(x)e^{i\mu_{\omega A_n}(x)},$$

The Cartesian product of An *, denoted as* $A_1 \times A_2 \times \ldots \times A_N$, *defined as*

$$T_{A_1\times A_2\times\ldots\times A_N}(x) = p_{A_1\times A_2\times\ldots\times A_N}(x)e^{i\mu_{A_1\times A_2\times\ldots\times A_N}(x)}$$

$$= min\left(p_{A_1}(x_1), p_{A_2}(x_2), \ldots, p_{A_N}(x_N)\right)e^{i\, min\left(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_N}(x_N)\right)}.$$

Similarly

$$I_{A_1\times A_2\times\ldots\times A_N}(x) = q_{A_1\times A_2\times\ldots\times A_N}(x)e^{i\nu_{A_1\times A_2\times\ldots\times A_N}(x)}$$

$$= max\left(q_{A_1}(x_1), q_{A_2}(x_2), \ldots, q_{A_N}(x_N)\right)e^{i\, max\left(\nu_{A_1}(x_1), \nu_{A_2}(x_2), \ldots, \nu_{A_N}(x_N)\right)},$$

and

$$F_{A_1\times A_2\times\ldots\times A_N}(x) = r_{A_1\times A_2\times\ldots\times A_N}(x)e^{i\omega_{A_1\times A_2\times\ldots\times A_N}(x)}$$

$$= max\left(r_{A_1}(x_1), r_{A_2}(x_2), \ldots, r_{A_N}(x_N)\right)e^{i\, max\left(\omega_{A_1}(x_1), \omega_{A_2}(x_2), \ldots, \omega_{A_N}(x_N)\right)},$$

where $x = (x_1, x_2, \ldots, x_N) \in \underbrace{X \times X \times \ldots \times X}_{N}.$

Delta-equalities of Complex Neutrosophic Sets

Definition *The distance of CNS is a function* $d = CN \times CN \to [0,1]$ *such that for any* $A, B, C \in CN$

$(i)\, d(A,B) \geq 0$ if and only if $A = B$,

$(ii)\, d(A,B) = d(B,A)$,

$(iii)\, d(A,B) \leq d(A,C) + d(C,B)$,

where $d(A,B)$ *is defined as*

$$d(A,B) = max \left(\begin{array}{c} max\left(\sup_{x\in X}|p_A(x) - p_B(x)|, \sup_{x\in X}|q_A(x) - q_B(x)|, \sup_{x\in X}|r_A(x) - r_B(x)|\right), \\ max\left(\begin{array}{c} \frac{1}{2\pi}\sup_{x\in X}|\mu_A(x) - \mu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x) - \nu_B(x)|, \\ \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x) - \omega_B(x)| \end{array} \right). \end{array} \right)$$

Definition *Let*

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$

be the complex neutrosophic sets on X *such that*

$$T_A(x) = p_A(x)e^{i\mu_A(x)}, I_A(x) = q_A(x)e^{i\nu_A(x)}, F_A(x) = r_A(x)e^{i\mu_{\omega_A}(x)},$$
$$T_B(x) = p_B(x)e^{i\mu_B(x)}, I_B(x) = q_B(x)e^{i\nu_B(x)}, F_B(x) = r_B(x)e^{i\omega_B(x)},$$

be complex valued truth, indeterminate and falsehood functions respectively. Then A *and* B *are said to be* $\delta - equal$ *if and only if* $d(A,B) \leq 1$ *where* $0 \leq \delta \leq 1$, *which is denoted by* $A = (\delta)B$.

Lemma *Let*

$$\delta_1 * \delta_2 = max(0, \delta_1 + \delta_2 - 1); 0 \leq \delta_1, \delta_2 \leq 1,$$

then the following results hold,

(1) $0 * \delta_1 = 0$; *for all* $\delta_1 \in [0,1]$,

(2) $1 * \delta_1 = \delta_1$; *for all* $\delta_1 \in [0,1]$,

(3) $0 \leq \delta_1 * \delta_2 \leq 1$; *for all* $\delta_1, \delta_2 \in [0,1]$,

(4) $\delta_1 \leq \delta_1' \Rightarrow \delta_1 * \delta_2 \leq \delta_1' * \delta_2$; *for all* $\delta_1, \delta_1', \delta_2 \in [0,1]$,

(5) $\delta_1 * \delta_2 = \delta_2 * \delta_1$; *for all* $\delta_1, \delta_2 \in [0,1]$,

(6) $(\delta_1 * \delta_2) * \delta_3 = \delta_2 * (\delta_1 * \delta_3)$; *for all* $\delta_1, \delta_2, \delta_3 \in [0,1]$.

Proof It is *trivial.*

Lemma *For the complex valued bounded function* f, g *on a set* X. *We have*

$$\left|\sup_{x\in U} f(x) - \sup_{x\in U} g(x)\right| \leq \sup_{x\in U}|f(x) - g(x)|,$$

$$\left|\inf_{x\in U} f(x) - \inf_{x\in U} g(x)\right| \leq \inf_{x\in U}|f(x) - g(x)|.$$

Theorem *If* $A=(\delta_1)A^{'}$ *and* $B=(\delta_2)B^{'}$ *, then* $A\cup B=(min(\delta_1,\delta_2))A^{'}\cup B^{'}$.

Proof

$$d(A,A^{'})=max\left(\begin{array}{c}max\left(\sup_{x\in X}|p_A(x)-p_{A^{'}}(x)|,\sup_{x\in X}|q_A(x)-q_{A^{'}}(x)|,\sup_{x\in X}|r_A(x)-r_{A^{'}}(x)|\right),\\ max\left(\begin{array}{c}\frac{1}{2\pi}\sup_{x\in X}|\mu_A(x)-\mu_{A^{'}}(x)|,\frac{1}{2\pi}\sup_{x\in X}|\nu_A(x)-\nu_{A^{'}}(x)|,\\ \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x)-\omega_{A^{'}}(x)|\end{array}\right)\end{array}\right)$$

$$\leq 1-\delta_1$$

$$d(B,B^{'})=max\left(\begin{array}{c}max\left(\sup_{x\in X}|p_B(x)-p_{B^{'}}(x)|,\sup_{x\in X}|q_B(x)-q_{B^{'}}(x)|,\sup_{x\in X}|r_B(x)-r_{B^{'}}(x)|\right),\\ max\left(\begin{array}{c}\frac{1}{2\pi}\sup_{x\in X}|\mu_B(x)-\mu_{B^{'}}(x)|,\frac{1}{2\pi}\sup_{x\in X}|\nu_B(x)-\nu_{B^{'}}(x)|,\\ \frac{1}{2\pi}\sup_{x\in X}|\omega_B(x)-\omega_{B^{'}}(x)|\end{array}\right)\end{array}\right)$$

$$\leq 1-\delta_2.$$

Therefore

$$\sup_{x\in X}|p_A(x)-p_{A^{'}}(x)|\leq 1-\delta_1,\frac{1}{2\pi}\sup_{x\in X}|\mu_A(x)-\mu_{A^{'}}(x)|\leq 1-\delta_1,$$

$$\sup_{x\in X}|q_A(x)-q_{A^{'}}(x)|\leq 1-\delta_1,\frac{1}{2\pi}\sup_{x\in X}|\nu_A(x)-\nu_{A^{'}}(x)|\leq 1-\delta_1,$$

$$\sup_{x\in X}|r_B(x)-r_{B^{'}}(x)|\leq 1-\delta_1,\frac{1}{2\pi}\sup_{x\in X}|\omega_A(x)-\omega_{A^{'}}(x)|\leq 1-\delta_1,$$

$$\sup_{x\in X}|p_B(x)-p_{B^{'}}(x)|\leq 1-\delta_2,\frac{1}{2\pi}\sup_{x\in X}|\mu_B(x)-\mu_{B^{'}}(x)|\leq 1-\delta_2,$$

$$\sup_{x\in X}|q_B(x)-q_{B^{'}}(x)|\leq 1-\delta_2,\frac{1}{2\pi}\sup_{x\in X}|\nu_B(x)-\nu_{B^{'}}(x)|\leq 1-\delta_2,$$

$$\sup_{x\in X}|r_B(x)-r_{B^{'}}(x)|\leq 1-\delta_2,\frac{1}{2\pi}\sup_{x\in X}|\omega_B(x)-\omega_{B^{'}}(x)|\leq 1-\delta_2.$$

For membership function

$$\sup_{x\in X}|p_{A\cup B}(x)-p_{A^{'}\cup B^{'}}(x)|=\sup_{x\in X}\left|max(p_A(x),p_B(x))-max\left(p_{A^{'}}(x),p_{B^{'}}(x)\right)\right|$$

$$= \begin{cases} \sup_{x\in X}|p_A(x) - p_{A'}(x)|, \text{if } p_A(x) \geq p_B(x) \text{ and } p_{A'}(x) \geq p_{B'}(x) \\ \sup_{x\in X}|p_A(x) - p_{B'}(x)|, \text{if } p_A(x) \geq p_B(x) \text{ and } p_{B'}(x) \geq p_{A'}(x) \\ \sup_{x\in X}|p_B(x) - p_{A'}(x)|, \text{if } p_B(x) > p_A(x) \text{ and } p_{A'}(x) \geq p_{B'}x, \\ \sup_{x\in X}|p_B(x) - p_{B'}(x)|, \text{if } p_B(x) > p_A(x) \text{ and } p_{B'}(x) \geq p_{A'}(x) \end{cases}$$

$$\leq \begin{cases} 1-\delta_1, \text{if } p_A(x) \geq p_B(x) \text{ and } p_{A'}(x) \geq p_{B'}(x) \\ \sup_{x\in X}|p_A(x) - p_{B'}(x)|, \text{if } p_A(x) \geq p_B(x) \text{ and } p_{B'}(x) \geq p_{A'}(x) \\ \sup_{x\in X}|p_B(x) - p_{A'}(x)|, \text{if } p_B(x) > p_A(x) \text{ and } p_{A'}(x) \geq p_{B'}(x) \\ 1-\delta_2, \text{if } p_B(x) > p_A(x) \text{ and } p_{B'}(x) \geq p_{A'}(x) \end{cases}$$

(iii) Consider the case $p_A(x) \geq p_B(x)$ and $p_{A'}(x) > p_{B'}(x)$

(1) $p_A(x) \geq p_{B'}(x) \geq 0$, then $p_A(x) \geq p_{A'}(x) \geq p_A(x) \geq p_{B'}(x) \geq 0$ from $p_{B'}(x) \geq p_{A'}(x)$, therefore

$$\sup_{x\in X}|p_A(x) - p_{B'}(x)| = \sup_{x\in X}\left(p_A(x) - p_{B'}(x)\right) \leq \sup_{x\in X}\left(p_A(x) - p_{A'}(x)\right) \leq$$

$$\sup_{x\in X}|p_A(x) - p_{B'}(x)| \leq 1-\delta_1.$$

(2) $p_A(x) \geq p_{B'}(x) \leq 0$ then $p_{B'}(x) \geq p_B(x) \geq p_{B'}(x) \geq p_A(x) \geq 0$ from $p_B(x) \leq p_A(x)$, therefore

$$\sup_{x\in X}|p_A(x) - p_{B'}(x)| = \sup_{x\in X}\left(p_{B'}(x) - p_A(x)\right) \leq \sup_{x\in X}\left(p_{B'}(x) - p_B(x)\right) \leq$$

$$\sup_{x\in X}|p_{B'}(x) - p_B(x)| \leq 1-\delta_2.$$

Thus if
$p_A(x) \geq p_B(x)$ *and* $p_{B'}(x) > p_{A'}(x)$.

We have

$$\sup_{x\in X}|p_A(x) - p_{B'}(x)| \leq \max(1-\delta_1, 1-\delta_2) = 1 - \min(\delta_1, \delta_2).$$

(iv) Similarly for the case

$$\sup_{x\in X}|p_B(x) - p_{A'}(x)| \leq \max(1-\delta_1, 1-\delta_2) = 1 - \min(\delta_1, \delta_2).$$

Now if $p_B(x) > p_A(x)$ and $p_{A'}(x) \geq p_{B'}(x)$, thus

$$\sup_{x\in X}\left|p_{A\cup B}(x) - p_{A'\cup B'}(x)\right| \leq max(1-\delta_1, 1-\delta_2) = 1 - min(\delta_1, \delta_2).$$

On same steps we can prove for indeterminacy function and falsehood function, likewise

$$\frac{1}{2\pi}\sup_{x\in X}\left|\mu_{A\cup B}(x) - \mu_{A'\cup B'}(x)\right| = \frac{1}{2\pi}\sup_{x\in X}\left|max\left(\mu_A(x), \mu_B(x)\right) - max\left(\mu_{A'}(x), \mu_{B'}(x)\right)\right|$$

$$= \begin{cases} \frac{1}{2\pi}\sup_{x\in X}\left|\mu_A(x) - \mu_{A'}(x)\right|, \text{ if } \mu_A(x) \geq \mu_B(x) \text{ and } \mu_{A'}(x) \geq \mu_{B'}(x) \\ \frac{1}{2\pi}\sup_{x\in X}\left|\mu_A(x) - \mu_{B'}(x)\right|, \text{ if } \mu_A(x) \geq \mu_B(x) \text{ and } \mu_{B'}(x) \geq \mu_{A'}(x) \\ \frac{1}{2\pi}\sup_{x\in X}\left|\mu_B(x) - \mu_{A'}(x)\right|, \text{ if } \mu_B(x) > \mu_A(x) \text{ and } \mu_{A'}(x) \geq \mu_{B'}(x) \\ \frac{1}{2\pi}\sup_{x\in X}\left|\mu_B(x) - \mu_{B'}(x)\right|, \text{ if } \mu_B(x) > \mu_A(x) \text{ and } \mu_{B'}(x) \geq \mu_{A'}(x) \end{cases}$$

$$\leq \begin{cases} 1-\delta_1, \text{ if } \mu_A(x) \geq \mu_B(x) \text{ and } \mu_{A'}(x) \geq \mu_{B'}(x) \\ \frac{1}{2\pi}\sup_{x\in X}\left|\mu_A(x) - \mu_{B'}(x)\right|, \text{ if } \mu_A(x) \geq \mu_B(x) \text{ and } \mu_{B'}(x) \geq \mu_{A'}(x) \\ \frac{1}{2\pi}\sup_{x\in X}\left|\mu_B(x) - \mu_{A'}(x)\right|, \text{ if } \mu_B(x) > \mu_A(x) \text{ and } \mu_{A'}(x) \geq \mu_{B'}(x) \\ 1-\delta_2, \text{ if } \mu_B(x) > \mu_A(x) \text{ and } \mu_{B'}(x) \geq \mu_{A'}(x) \end{cases}$$

(i) Consider the case $\mu_A(x) \geq \mu_B(x)$ and $\mu_{A'}(x) > \mu_{B'}(x)$

(1) $\mu_A(x) \geq \mu_{B'}(x) \geq 0$,then $\mu_A(x) \geq \mu_{A'}(x) \geq \mu_A(x) \geq \mu_{B'}(x) \geq 0$ from $\mu_{B'}(x) > \mu_{A'}(x)$, therefore

$$\frac{1}{2\pi}\sup_{x\in X}\left|\mu_A(x) - \mu_{B'}(x)\right| = \frac{1}{2\pi}\sup_{x\in X}\left(\mu_A(x) - \mu_{B'}(x)\right) \leq \frac{1}{2\pi}\sup_{x\in X}\left(\mu_A(x) - \mu_{A'}(x)\right)$$
$$\leq \frac{1}{2\pi}\sup_{x\in X}\left|\mu_A(x) - \mu_{B'}(x)\right| \leq 1 - \delta_1.$$

(2) $\mu_A(x) \geq \mu_{B'}(x) \leq 0$ then $\mu_{B'}(x) \geq \mu_B(x) \geq \mu_{B'}(x) \geq \mu_A(x) \geq 0$ from $\mu_B(x) \leq \mu_A(x)$, therefore

$$\frac{1}{2\pi}\sup_{x\in X}\left|\mu_A(x) - \mu_{B'}(x)\right| = \frac{1}{2\pi}\sup_{x\in X}\left(\mu_{B'}(x) - \mu_A(x)\right) \leq \frac{1}{2\pi}\sup_{x\in X}\left(\mu_{B'}(x) - \mu_B(x)\right)$$

$$\leq \frac{1}{2\pi} \sup_{x \in X} \left| \mu_{B'}(x) - \mu_B(x) \right| \leq 1 - \delta_2.$$

Thus if

$\mu_A(x) \geq \mu_B(x)$ and $\mu_{B'}(x) > \mu_{A'}(x)$.

We have

$$\frac{1}{2\pi} \sup_{x \in X} \left| \mu_A(x) - \mu_{B'}(x) \right| \leq max(1 - \delta_1, 1 - \delta_2) = 1 - min(\delta_1, \delta_2).$$

(ii) Similarly for the case

$$\frac{1}{2\pi} \sup_{x \in X} \left| \mu_B(x) - \mu_{A'}(x) \right| \leq max(1 - \delta_1, 1 - \delta_2) = 1 - min(\delta_1, \delta_2).$$

Now if $\mu_B(x) > \mu_A(x)$ and $\mu_{A'}(x) \geq \mu_{B'}(x)$ thus

$$\frac{1}{2\pi} \sup_{x \in X} \left| \mu_{A \cup B}(x) - \mu_{A' \cup B'}(x) \right| \leq max(1 - \delta_1, 1 - \delta_2) = 1 - min(\delta_1, \delta_2).$$

On same steps we can prove for indeterminacy function and falsehood function, likewise

$$d(A \cup B, A' \cup B')$$

$$= max\left(\begin{array}{c} max\left(\begin{array}{c} \sup_{x \in X} \left| p_{A \cup B}(x) - p_{A' \cup B'} \right|, \sup_{x \in X} \left| q_{A \cup B}(x) - q_{A' \cup B'}(x) \right|, \\ \sup_{x \in X} \left| r_{A \cup B}(x) - r_{A' \cup B'}(x) \right| \end{array} \right), \\ max\left(\begin{array}{c} \frac{1}{2\pi} \sup_{x \in X} \left| \mu_{A \cup B}(x) - \mu_{A' \cup B'}(x) \right|, \frac{1}{2\pi} \sup_{x \in X} \left| \nu_{A \cup B}(x) - \nu_{A' \cup B'}(x) \right|, \\ \frac{1}{2\pi} \sup_{x \in X} \left| \omega_{A \cup B}(x) - \omega_{A' \cup B'}(x) \right| \end{array} \right) \end{array} \right)$$

$$\leq max(1 - \delta_1, 1 - \delta_2) = 1 - min(\delta_1, \delta_2).$$

Thus $A \cup B = (min(\delta_1, \delta_2)) A' \cup B'$.

Corollary

If $A_\alpha = (\delta_\alpha) B_\alpha, \alpha \in I$, *then* $\bigcup_{\alpha \in I} A_\alpha = \left(\inf_{\alpha \in I} (\delta_\alpha) \right) \bigcup_{\alpha \in I} B_\alpha$.

Proof Using lemma mod sup less or equal inf mod, we get

$$d\left(\bigcup_{\alpha \in I} A_\alpha, \bigcup_{\alpha \in I} B_\alpha \right)$$

$$= max\begin{pmatrix} max\begin{pmatrix} \sup_{x\in X}\left|p \underset{\alpha\in I}{\cup} A_\alpha(x) - p \underset{\alpha\in I}{\cup} B_\alpha\right|, \sup_{x\in X}\left|q \underset{\alpha\in I}{\cup} A_\alpha(x) - q \underset{\alpha\in I}{\cup} B_\alpha(x)\right|, \\ \sup_{x\in X}\left|r \underset{\alpha\in I}{\cup} A_\alpha(x) - r \underset{\alpha\in I}{\cup} B_\alpha(x)\right| \end{pmatrix}, \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}\left|\mu \underset{\alpha\in I}{\cup} A_\alpha(x) - \mu \underset{\alpha\in I}{\cup} B_\alpha\right|, \frac{1}{2\pi}\sup_{x\in X}\left|\nu \underset{\alpha\in I}{\cup} A_\alpha(x) - \nu \underset{\alpha\in I}{\cup} B_\alpha(x)\right|, \\ \frac{1}{2\pi}\sup_{x\in X}\left|\omega \underset{\alpha\in I}{\cup} A_\alpha(x) - \omega \underset{\alpha\in I}{\cup} B_\alpha(x)\right| \end{pmatrix} \end{pmatrix}$$

$$= max\begin{pmatrix} \begin{pmatrix} max & \sup_{x\in X}\left|p\sup_{\alpha\in I} A_\alpha(x) - p\sup_{\alpha\in I} B_\alpha\right|, \sup_{x\in X}\left|q\sup_{\alpha\in I} A_\alpha(x) - q\sup_{\alpha\in I} B_\alpha(x)\right|, \\ & \sup_{x\in X}\left|r\sup_{\alpha\in I} A_\alpha(x) - r\sup_{\alpha\in I} B_\alpha(x)\right| \end{pmatrix}, \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}\left|\mu\sup_{\alpha\in I} A_\alpha(x) - \mu\sup_{\alpha\in I} B_\alpha\right|, \frac{1}{2\pi}\sup_{x\in X}\left|\nu\sup_{\alpha\in I} A_\alpha(x) - \nu\sup_{\alpha\in I} B_\alpha(x)\right|, \\ \frac{1}{2\pi}\sup_{x\in X}\left|\omega\sup_{\alpha\in I} A_\alpha(x) - \omega\sup_{\alpha\in I} B_\alpha(x)\right| \end{pmatrix} \end{pmatrix}$$

$$\leq max\begin{pmatrix} \begin{pmatrix} max & \sup_{x\in X}\sup_{\alpha\in I}|pA_\alpha(x) - pB_\alpha(x)|, \sup_{x\in X}\inf_{\alpha\in I}|qA_\alpha(x) - qB_\alpha(x)|, \\ & \sup_{x\in X}\inf_{\alpha\in I}|rA_\alpha(x) - rB_\alpha(x)| \end{pmatrix}, \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}\sup_{\alpha\in I}|\mu A_\alpha(x) - \mu B_\alpha|, \frac{1}{2\pi}\sup_{x\in X}\sup_{\alpha\in I}|\nu A_\alpha(x) - \nu B_\alpha(x)|, \\ \frac{1}{2\pi}\sup_{x\in X}\sup_{\alpha\in I}|\omega A_\alpha(x) - \omega B_\alpha(x)| \end{pmatrix} \end{pmatrix}$$

$$= max\begin{pmatrix} \begin{pmatrix} max & \sup_{\alpha\in I}\sup_{x\in X}|pA_\alpha(x) - pB_\alpha(x)|, \inf_{\alpha\in I}\sup_{x\in X}|qA_\alpha(x) - qB_\alpha(x)|, \\ & \inf_{\alpha\in I}\sup_{x\in X}|rA_\alpha(x) - rB_\alpha(x)| \end{pmatrix}, \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{\alpha\in I}\sup_{x\in X}|\mu A_\alpha(x) - \mu B_\alpha|, \frac{1}{2\pi}\sup_{\alpha\in I}\sup_{x\in X}|\nu A_\alpha(x) - \nu B_\alpha(x)|, \\ \frac{1}{2\pi}\sup_{\alpha\in I}\sup_{x\in X}|\omega A_\alpha(x) - \omega B_\alpha(x)| \end{pmatrix} \end{pmatrix}$$

$$= max\begin{pmatrix} max\left(\sup_{\alpha\in I}(1-\delta_\alpha), \inf_{\alpha\in I}(1-\delta_\alpha), \inf(1-\delta_\alpha)\right), \\ max\left(\sup_{\alpha\in I}(1-\delta_\alpha), \sup_{\alpha\in I}(1-\delta_\alpha), \sup(1-\delta_\alpha)\right) \end{pmatrix}$$

$$= max\left(\sup_{\alpha\in I}(1-\delta_\alpha), \sup_{\alpha\in I}(1-\delta_\alpha)\right)$$

$$= \sup_{\alpha\in I}(1-\delta_\alpha) = 1 - \inf_{\alpha\in I}\delta_\alpha.$$

Theorem *If* $A = (\delta)B$, *then* $\overline{A} = (\delta)\overline{B}$.

Proof As

$$d(\overline{A},\overline{B})$$

$$= max\begin{pmatrix} max\left(\sup_{x\in X}|p_{\overline{A}}(x)-p_{\overline{B}}(x)|, \sup_{x\in X}|q_{\overline{A}}(x)-q_{\overline{B}}(x)|, \sup_{x\in X}|r_{\overline{A}}(x)-r_{\overline{B}}(x)|\right), \\ max\left(\frac{1}{2\pi}\sup_{x\in X}|\mu_{\overline{A}}(x)-\mu_{\overline{B}}(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_{\overline{A}}(x)-\nu_{\overline{B}}(x)|, \frac{1}{2\pi}\sup_{x\in X}|\omega_{\overline{A}}(x)-\omega_{\overline{B}}(x)|\right)\end{pmatrix}$$

$$= max\begin{pmatrix} max\begin{pmatrix}\sup_{x\in X}|(1-p_A(x))-(1-p_B(x))|, \sup_{x\in X}|(1-q_A(x))-(1-q_B(x))|, \\ \sup_{x\in X}|(1-r_A(x))-(1-r_B(x))|\end{pmatrix}, \\ max\begin{pmatrix}\frac{1}{2\pi}\sup_{x\in X}|(2\pi-\mu_A(x))-(2\pi-\mu_B(x))|, \frac{1}{2\pi}\sup_{x\in X}|(2\pi-\nu_A(x))-(2\pi-\nu_B(x))|, \\ \frac{1}{2\pi}\sup_{x\in X}|(2\pi-\omega_A(x))-(2\pi-\omega_B(x))|\end{pmatrix}\end{pmatrix}$$

$$= max\begin{pmatrix} max\left(\sup_{x\in X}|p_A(x)-p_B(x)|, \sup_{x\in X}|q_A(x)-q_B(x)|, \sup_{x\in X}|r_A(x)-r_B(x)|\right), \\ max\left(\frac{1}{2\pi}\sup_{x\in X}|\mu_A(x)-\mu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x)-\nu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x)-\omega_B(x)|\right)\end{pmatrix}$$

$$= d(A,B) \leq 1-\delta.$$

Theorem *If* $A = (\delta_1)A'$ *and* $B = (\delta_2)B'$ *, then* $A \cap B = (min(\delta_1,\delta_2))A' \cap B'$.

Proof By use of previous theorem a complement equals del b complement, we have

$\overline{A} = (\delta_1)\overline{A}'$, $\bar{B} = (\delta_1)\bar{B}'$ and

$\overline{A} \cup \overline{B} = min(\delta_1,\delta_2)\overline{A}' \cup \bar{B}'$.

Thus

$$A \cap B = \overline{\overline{A} \cup \bar{B}}$$

$$= \left(min(\delta_1,\delta_2)\right)\overline{\overline{A}' \cup \bar{B}'}$$

$$= \left(min(\delta_1,\delta_2)\right)A' \cap B'.$$

Corollary *If* $A_\alpha = (\delta_\alpha) B_\alpha, \alpha \in I,$ *where I is an index set, then* $\underset{\alpha\in I}{\cap} A_\alpha = \left(\inf_{\alpha\in I}(\delta_\alpha)\right) \underset{\alpha\in I}{\cap} B_\alpha.$

Proof From above corollary union alpha equals inf union beta, we have

$$d\left(\underset{\alpha\in I}{\cup} A_\alpha, \underset{\alpha\in I}{\cup} B_\alpha\right) = 1 - \inf_{\alpha\in I} \delta_\alpha, \text{ and}$$

$$\overline{A_\alpha} = (\delta_\alpha)\overline{B_\alpha}, \text{ for all } \alpha \in I, \text{ and}$$

$$\underset{\alpha\in I}{\cup} \overline{A_\alpha} = \left(\inf_{\alpha\in I}(\delta_\alpha)\right) \underset{\alpha\in I}{\cup} \overline{B_\alpha}.$$

Thus

$$\underset{\alpha\in I}{\cap} A_\alpha = \overline{\underset{\alpha\in I}{\cup} \overline{A_\alpha}} = \left(\inf_{\alpha\in I}(\delta_\alpha)\right) \overline{\underset{\alpha\in I}{\cup} \bar{B}_\alpha}$$

$$= \left(\inf_{\alpha\in I}(\delta_\alpha)\right) \underset{\alpha\in I}{\cap} B.$$

Corollary *If* $A_{\alpha\beta} = (\delta_{\alpha\beta}) B_{\alpha\beta}, \alpha \in I_1, \beta \in I_2,$ *where* I_1 *and* I_2 *are index sets, then*

$$\underset{\alpha\in I_1}{\cup} \underset{\alpha\in I_2}{\cap} A_{\alpha\beta} = \left(\inf_{\alpha\in I_1} \inf_{\alpha\in I_2}(\delta_{\alpha\beta})\right) \underset{\alpha\in I_1}{\cup} \underset{\alpha\in I_2}{\cap} B_{\alpha\beta},$$

$$\underset{\alpha\in I_1}{\cap} \underset{\alpha\in I_2}{\cup} A_{\alpha\beta} = \left(\inf_{\alpha\in I_1} \inf_{\alpha\in I_2}(\delta_{\alpha\beta})\right) \underset{\alpha\in I_1}{\cap} \underset{\alpha\in I_2}{\cup} B_{\alpha\beta}.$$

Proof By using corollary union alpha equals inf union beta and intersection alpha equals inf intersection beta we can easily prove it.

Theorem *If* $A = (\delta_1)A'$ *and* $B = (\delta_2)B'$ *, then* $A \circ B = (\delta_1 * \delta_2) A' \circ B'.$

Proof As $A = (\delta_1)A'$ and $B = (\delta_2)B'$, so we have

$$d(A, A') = \max\begin{pmatrix} \max\left(\sup_{x\in X}|p_A(x) - p_{A'}(x)|, \sup_{x\in X}|q_A(x) - q_{A'}(x)|, \sup_{x\in X}|r_A(x) - r_{A'}(x)|\right), \\ \max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}|\mu_A(x) - \mu_{A'}(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x) - \nu_{A'}(x)|, \\ \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x) - \omega_{A'}(x)| \end{pmatrix} \end{pmatrix}$$

$$\leq 1-\delta_1$$

$$d(B,B') = max\begin{pmatrix} max\left(\sup_{x\in X}|p_B(x)-p_{B'}(x)|, \sup_{x\in X}|q_B(x)-q_{B'}(x)|, \sup_{x\in X}|r_B(x)-r_{B'}(x)|\right), \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}|\mu_B(x)-\mu_{B'}(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_B(x)-\nu_{B'}(x)|, \\ \frac{1}{2\pi}\sup_{x\in X}|\omega_B(x)-\omega_{B'}(x)| \end{pmatrix} \end{pmatrix}$$

$$\leq 1-\delta_2.$$

Therefore

$$\sup_{x\in X}|p_A(x)-p_{A'}(x)| \leq 1-\delta_1, \frac{1}{2\pi}\sup_{x\in X}|\mu_A(x)-\mu_{A'}(x)| \leq 1-\delta_1,$$

$$\sup_{x\in X}|q_A(x)-q_{A'}(x)| \leq 1-\delta_1, \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x)-\nu_{A'}(x)| \leq 1-\delta_1,$$

$$\sup_{x\in X}|r_B(x)-r_{B'}(x)| \leq 1-\delta_1, \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x)-\omega_{A'}(x)| \leq 1-\delta_1,$$

$$\sup_{x\in X}|p_B(x)-p_{B'}(x)| \leq 1-\delta_2, \frac{1}{2\pi}\sup_{x\in X}|\mu_B(x)-\mu_{B'}(x)| \leq 1-\delta_2,$$

$$\sup_{x\in X}|q_B(x)-q_{B'}(x)| \leq 1-\delta_2, \frac{1}{2\pi}\sup_{x\in X}|\nu_B(x)-\nu_{B'}(x)| \leq 1-\delta_2,$$

$$\sup_{x\in X}|r_B(x)-r_{B'}(x)| \leq 1-\delta_2, \frac{1}{2\pi}\sup_{x\in X}|\omega_B(x)-\omega_{B'}(x)| \leq 1-\delta_2.$$

We have,

$$d(A\circ B, A'\circ B')$$

$$= max\begin{pmatrix} max\left(\sup_{x\in X}|p_{A\circ B}(x)-p_{A'\circ B'}(x)|, \sup_{x\in X}|q_{A\circ B}(x)-q_{A'\circ B'}(x)|, \sup_{x\in X}|r_{A\circ B}(x)-r_{A'\circ B'}(x)|\right), \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}|\mu_{A\circ B}(x)-\mu_{A'\circ B'}(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_{A\circ B}(x)-\nu_{A'\circ B'}(x)|, \\ \frac{1}{2\pi}\sup_{x\in X}|\omega_{A\circ B}(x)-\omega_{A'\circ B'}(x)| \end{pmatrix} \end{pmatrix}$$

$$= max\begin{pmatrix} max\begin{pmatrix} \sup_{x\in X}\left|p_A . p_B(x) - p_{A'} . p_{B'}(x)\right|, \sup_{x\in X}\left|q_A . q_B(x) - q_{A'} . q_{B'}(x)\right|, \\ \sup_{x\in X}\left|r_A . r_B(x) - r_{A'} . r_{B'}(x)\right| \end{pmatrix}, \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}\left|2\pi\left(\frac{\mu_A}{2\pi}.\frac{\mu_B}{2\pi}\right) - 2\pi\left(\frac{\mu_{A'}}{2\pi}.\frac{\mu_{B'}}{2\pi}\right)\right|, \\ \frac{1}{2\pi}\sup_{x\in X}\left|2\pi\left(\frac{\nu_A}{2\pi}.\frac{\nu_B}{2\pi}\right) - 2\pi\left(\frac{\nu_{A'}}{2\pi}.\frac{\nu_{B'}}{2\pi}\right)\right|, \\ \frac{1}{2\pi}\sup_{x\in X}\left|2\pi\left(\frac{\omega_A}{2\pi}.\frac{\omega_B}{2\pi}\right) - 2\pi\left(\frac{\omega_{A'}}{2\pi}.\frac{\omega_{B'}}{2\pi}\right)\right| \end{pmatrix} \end{pmatrix}$$

$$= max\begin{pmatrix} max\begin{pmatrix} \sup_{x\in X}\left|p_A . p_B(x) - p_A . p_{B'}(x) + p_A . p_{B'}(x) - p_{A'} . p_{B'}(x)\right|, \\ \sup_{x\in X}\left|q_A . q_B(x) - q_A . q_{B'}(x) + q_A . q_{B'}(x) - q_{A'} . q_{B'}(x)\right|, \\ \sup_{x\in X}\left|r_A . r_B(x) - r_A . r_B(x) + r_A . r_B(x) - r_{A'} . r_{B'}(x)\right| \end{pmatrix}, \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}\left|\frac{\mu_A(x) . \mu_B(x)}{2\pi} - \frac{\mu_A(x) . \mu_{B'}(x)}{2\pi} + \frac{\mu_A(x) . \mu_{B'}(x)}{2\pi} - \frac{\mu_{A'}(x) . \mu_{B'}(x)}{2\pi}\right|, \\ \frac{1}{2\pi}\sup_{x\in X}\left|\frac{\nu_A(x) . \nu_B(x)}{2\pi} - \frac{\nu_A(x) . \nu_{B'}(x)}{2\pi} + \frac{\nu_A(x) . \nu_{B'}(x)}{2\pi} - \frac{\nu_{A'}(x) . \nu_{B'}(x)}{2\pi}\right|, \\ \frac{1}{2\pi}\sup_{x\in X}\left|\frac{\omega_A(x) . \omega_B(x)}{2\pi} - \frac{\omega_A(x) . \omega_{B'}(x)}{2\pi} + \frac{\omega_A(x) . \omega_{B'}(x)}{2\pi} - \frac{\omega_{A'}(x) . \omega_{B'}(x)}{2\pi}\right| \end{pmatrix} \end{pmatrix}$$

$$= max\begin{pmatrix} max\begin{pmatrix} \sup_{x\in X}\left|p_A(x)\left(p_B(x) - p_{B'}(x)\right) + p_{B'}(x)\left(p_A(x) - p_{A'}(x)\right)\right|, \\ \sup_{x\in X}\left|q_A(x)\left(q_B(x) - q_{B'}(x)\right) + q_{B'}(x)\left(q_A(x) - q_{A'}(x)\right)\right|, \\ \sup_{x\in X}\left|r_A(x)\left(r_B(x) - r_B(x)\right) + r_B(x)\left(r_A(x) - r_{A'}(x)\right)\right| \end{pmatrix}, \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}\left|\frac{\mu_A(x)}{2\pi}\left(\mu_B(x) - \mu_{B'}(x)\right) + \frac{\mu_{B'}(x)}{2\pi}\left(\mu_A(x) - \mu_{A'}(x)\right)\right|, \\ \frac{1}{2\pi}\sup_{x\in X}\left|\frac{\nu_A(x)}{2\pi}\left(\nu_B(x) - \nu_{B'}(x)\right) + \frac{\nu_{B'}(x)}{2\pi}\left(\nu_A(x) - \nu_{A'}(x)\right)\right|, \\ \frac{1}{2\pi}\sup_{x\in X}\left|\frac{\omega_A(x)}{2\pi}\left(\omega_B(x) - \omega_{B'}(x)\right) + \frac{\omega_{B'}(x)}{2\pi}\left(\omega_A(x) - \omega_{A'}(x)\right)\right| \end{pmatrix} \end{pmatrix}$$

$$\leq max\left(\begin{array}{c} max\left(\begin{array}{c}\left|\sup_{x\in X}\left(p_B(x)-p_{B'}(x)\right)+\sup_{x\in X}\left(p_A(x)-p_{A'}(x)\right)\right|,\\ \left|\sup_{x\in X}\left(q_B(x)-q_{B'}(x)\right)+\sup_{x\in X}\left(q_A(x)-q_{A'}(x)\right)\right|,\\ \left|\sup_{x\in X}\left(r_B(x)-r_B(x)\right)+\sup_{x\in X}\left(r_A(x)-r_{A'}(x)\right)\right|\end{array}\right),\\ max\left(\begin{array}{c}\frac{1}{2\pi}\left|\sup_{x\in X}\left(\mu_B(x)-\mu_{B'}(x)\right)+\sup_{x\in X}\left(\mu_A(x)-\mu_{A'}(x)\right)\right|,\\ \frac{1}{2\pi}\left|\sup_{x\in X}\left(\nu_B(x)-\nu_{B'}(x)\right)+\sup_{x\in X}\left(\nu_A(x)-\nu_{A'}(x)\right)\right|,\\ \frac{1}{2\pi}\left|\sup_{x\in X}\left(\omega_B(x)-\omega_{B'}(x)\right)+\sup_{x\in X}\left(\omega_A(x)-\omega_{A'}(x)\right)\right|\end{array}\right)\end{array}\right)$$

$$\leq max\left(\begin{array}{c} max\left((1-\delta_2)+(1-\delta_1),(1-\delta_2)+(1-\delta_1),(1-\delta_2)+(1-\delta_1)\right),\\ max\left((1-\delta_2)+(1-\delta_1),(1-\delta_2)+(1-\delta_1),(1-\delta_2)+(1-\delta_1)\right)\end{array}\right)$$

$$= max\left((1-\delta_2)+(1-\delta_1),(1-\delta_2)+(1-\delta_1)\right)$$

$$= 1-(\delta_1+\delta_2-1).$$

As $d\left(A\circ B, A'\circ B'\right)\leq 1$, so $d\left(A\circ B, A'\circ B'\right)\leq 1-\delta_1 * \delta_2$.

Corollary $A_\alpha = (\delta_\alpha)B_\alpha$, $\forall$ $\alpha\in I$ *, where I is an index set, then*

$$A_1\circ A_2\circ\ldots\circ A_\alpha = (\delta_1 * \delta_2 *\ldots\delta_\alpha)B_1\circ B_2\circ\ldots\circ B_\alpha\ .$$

Proof It follows from theorem AB equal delta AB.

Theorem *If* $A_n = (\delta_n)A_n'$, $n = 1,2,\ldots,N$ *then* $A_1\times A_2\times\ldots\times A_N = \left(\inf_{1\leq n\leq N}\delta_n\right)A_1'\times A_2'\times\ldots\times A_N'$.

Proof As $A_n = (\delta_n)A_n', n = 1,2,\ldots,N$. Therefore

$$d\left(A_n, A_n'\right) = max\left(\begin{array}{c} max\left(\begin{array}{c}\sup_{x\in X}\left|p_{A_n}(x)-p_{A_n'}(x)\right|,\sup_{x\in X}\left|q_{A_n}(x)-q_{A_n'}(x)\right|,\\ \sup_{x\in X}\left|r_{A_n}(x)-r_{A_n'}(x)\right|\end{array}\right),\\ max\left(\begin{array}{c}\frac{1}{2\pi}\sup_{x\in X}\left|\mu_{A_n}(x)-\mu_{A_n'}(x)\right|,\frac{1}{2\pi}\sup_{x\in X}\left|\nu_{A_n}(x)-\nu_{A_n'}(x)\right|,\\ \frac{1}{2\pi}\sup_{x\in X}\left|\omega_{A_n}(x)-\omega_{A_n'}(x)\right|\end{array}\right)\end{array}\right)$$

$$\leq 1-\delta_n, \text{ for any } n = 1,2,\ldots,N.$$

Therefore

$$\sup_{x\in X}\left|p_{A_n}(x)-p_{A_n'}(x)\right|\leq 1-\delta_n, \frac{1}{2\pi}\sup_{x\in X}\left|\mu_{A_n}(x)-\mu_{A_n'}(x)\right|\leq 1-\delta_n,$$

$$\sup_{x\in X}\left|q_{A_n}(x)-q_{A_n'}(x)\right|\le 1-\delta_n, \frac{1}{2\pi}\sup_{x\in X}\left|\nu_{A_n}(x)-\nu_{A_n'}(x)\right|\le 1-\delta_n,$$

$$\sup_{x\in X}\left|r_{A_n}(x)-r_{A_n'}(x)\right|\le 1-\delta_n, \frac{1}{2\pi}\sup_{x\in X}\left|\omega_{A_n}(x)-\omega_{A_n'}(x)\right|\le 1-\delta_n.$$

Then by lemma mod sup less or equal inf mod

$$d\left(A_1\times A_2\times\ldots\times A_N, A_1'\times A_2'\times\ldots\times A_N'\right)$$

$$= \max\begin{pmatrix} \max\begin{pmatrix} \sup_{x\in X\times X\ldots X}\left|p_{A_1\times A_2\times\ldots\times A_N}(x)-p_{A_1'\times A_2'\times\ldots\times A_N'}(x)\right|, \\ \sup_{x\in X\times X\ldots X}\left|q_{A_1\times A_2\times\ldots\times A_N}(x)-q_{A_1'\times A_2'\times\ldots\times A_N'}(x)\right|, \\ \sup_{x\in X\times X\ldots X}\left|r_{A_1\times A_2\times\ldots\times A_N}(x)-r_{A_1'\times A_2'\times\ldots\times A_N'}(x)\right| \end{pmatrix}, \\ \max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X\times X\ldots X}\left|\mu_{A_1\times A_2\times\ldots\times A_N}(x)-\mu_{A_1'\times A_2'\times\ldots\times A_N'}(x)\right|, \\ \frac{1}{2\pi}\sup_{x\in X\times X\ldots X}\left|\nu_{A_1\times A_2\times\ldots\times A_N}(x)-\nu_{A_1'\times A_2'\times\ldots\times A_N'}(x)\right|, \\ \frac{1}{2\pi}\sup_{x\in X\times X\ldots X}\left|\omega_{A_1\times A_2\times\ldots\times A_N}(x)-\omega_{A_1'\times A_2'\times\ldots\times A_N'}(x)\right| \end{pmatrix} \end{pmatrix}$$

$$= \max\begin{pmatrix} \max\begin{pmatrix} \sup_{x\in X\times X\ldots X}\left|\min_{1\le n\le N}p_{A_n}(x)-\min_{1\le n\le N}p_{A_n'}(x)\right|, \\ \sup_{x\in X\times X\ldots X}\left|\max_{1\le n\le N}q_{A_n}(x)-\max_{1\le n\le N}q_{A_n'}(x)\right|, \\ \frac{1}{2\pi}\sup_{x\in X\times X\ldots X}\left|\max_{1\le n\le N}r_{A_n}(x)-\max_{1\le n\le N}r_{A_n'}(x)\right| \end{pmatrix}, \\ \max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X\times X\ldots X}\left|\min_{1\le n\le N}\mu_{A_n}(x)-\min_{1\le n\le N}\mu_{A_n'}(x)\right|, \\ \frac{1}{2\pi}\sup_{x\in X\times X\ldots X}\left|\max_{1\le n\le N}\nu_{A_n}(x)-\max_{1\le n\le N}\nu_{A_n'}(x)\right|, \\ \sup_{x\in X\times X\ldots X}\left|\max_{1\le n\le N}\omega_{A_n}(x)-\max_{1\le n\le N}\omega_{A_n'}(x)\right| \end{pmatrix} \end{pmatrix}$$

$$\le \max\begin{pmatrix} \max\begin{pmatrix} \sup_{1\le n\le N}\sup_{x_n\in X_n}\left|\min_{1\le n\le N}p_{A_n}(x)-\min_{1\le n\le N}p_{A_n'}(x)\right|, \\ \sup_{1\le n\le N}\sup_{x_n\in X_n}\left|\max_{1\le n\le N}q_{A_n}(x)-\max_{1\le n\le N}q_{A_n'}(x)\right|, \\ \sup_{1\le n\le N}\sup_{x_n\in X_n}\left|\max_{1\le n\le N}r_{A_n}(x)-\max_{1\le n\le N}r_{A_n'}(x)\right| \end{pmatrix}, \\ \max\begin{pmatrix} \frac{1}{2\pi}\sup_{1\le n\le N}\sup_{x_n\in X_n}\left|\min_{1\le n\le N}\mu_{A_n}(x)-\min_{1\le n\le N}\mu_{A_n'}(x)\right|, \\ \frac{1}{2\pi}\sup_{1\le n\le N}\sup_{x_n\in X_n}\left|\max_{1\le n\le N}\nu_{A_n}(x)-\max_{1\le n\le N}\nu_{A_n'}(x)\right|, \\ \frac{1}{2\pi}\sup_{1\le n\le N}\sup_{x_n\in X_n}\left|\max_{1\le n\le N}\omega_{A_n}(x)-\max_{1\le n\le N}\omega_{A_n'}(x)\right| \end{pmatrix} \end{pmatrix}$$

$$\leq max\left(\sup_{1\leq n\leq N}(1-\delta_n), \sup_{1\leq n\leq N}(1-\delta_n)\right) = 1 - \inf_{1\leq n\leq N}\delta_n.$$

Conclusion *We worked on basic operations of CNSs. First we discussed some properties like union, intersection, complement, Cartesian product and investigated some results related to norms. Moreover we worked on the distance measures which are used to defined* $\delta -$ *equalities of CNSs. Some results such as union, intersection, complement, product on* $\delta -$ *equality also presented. We hope that the theory developed in this paper can be used in computing, data analysis, socio economic problems, medical diagnosis and other problems related to Decision Analysis.*

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