NSS

Neutrosophic Sets and Systems,Vol.38,2020

University of New Mexico

On NeutroNilpotentGroups

Elahe Mohammadzadeh[1] and Akbar Rezaei[2,]

[1,2]Department of Mathematics, Payame Noor University, P.O. Box 19395-3697, Tehran, Iran
e-mails: [1]mohamadzadeh36@gmail.com,[2]rezaei@pnu.ac.ir

*Correspondence: rezaei@pnu.ac.ir; Tel.: (+989131405381)

Abstract. In this paper, we introduce the notion of commutator of two elements in a specific NeutroGroup. Then we define the notion of a NeutroNilpotentGroup and we study some of their properties. Moreover, we show that the intersection of two NeutroNilpotentGroups is a NeutroNilpotentGroup. Also, we show that the quotient of a NeutroNilpotentGroup is a NeutroNilpotentGroup. Specially, using NeutroHomomorphism we prove the NeutroNilpotentcy is closed with respect to homomorphic image.

Keywords: NeutroGroup; NeutroSubgroup; NeutroNilpotentGroup; NeutroQuotientGroup; NeutroGroup Homomorphism.

---

1. Introduction

One of the most important concepts in the study of groups is the notion of nilpotency [6]. Nilpotent groups arose in Galois theory, as well as in the classification of groups. By Galois theory, certain problems in field theory reduced to group theory. In [10, 11], Smarandache introduced the notions of NeutroDefined, AntiDefined laws, NeutroAxiom and AntiAxiom. Then in [9], he studied NeutroAlgebras and AntiAlgebras. Rezaei et al. in [5], proved that there are $(2^n - 1)$ NeutroAlgebras and $(3^n - 2^n)$ AntiAlgebras in a classical algebra S with n operations and axioms all together, where $n \geq 1$. Agboola et al. in [1], studied NeutroGroups $(NG, *)$ where the law of composition and axioms defined on NG may either be only partially defined (partially true), or partially undefined (partially false), or totally undefined (totally false) with respect to $*$. Moreover, they considered three NeutroAxioms (NeutroAssociativity, existence of NeutroNeutral element and existence of NeutroInverse element) to show the difference between groups and NeutroGroups. Also, in [3], Agboola studied NeutroRings by considering three NeutroAxioms (NeutroAbelianGroup (additive), NeutroSemigroup (multiplicative) and

NeutroDistributivity (multiplication over addition)). Scholars applied the notion of NeutroAxioms and NeutroLaw on Rings, Subgrings, Ideals, QuotientRings and Ring Homomorphism to present some new notions and several results are obtained (see [3], [7]). In this paper, we consider a class of NeutroGroups was introduced in [1], and define the notion of NeutroNilpotentGroups. Moreover, we investigate elementary properties of NeutroNilpotentGroups. Specially, we show that the intersection of two NeutroNilpotentGroups is a NeutroNilpotentGroup. Also, we prove the NeutroNilpotency is closed with respect to homomorphic image.

2. Preliminaries

We recall some basic definitions and results which are proposed by the pioneers of this subject.

Definition 2.1 ([8]). (i) A classical operation is well defined for all the set$'s$ elements.

(ii) A NeutroOperation is an operation partially well defined, partially indeterminate, and partially outer defined on the given set.

(iii) A classical law/axiom defined on a nonempty set is totally true (i.e. true for all set$'s$ elements).

(iv) A NeutroLaw/NeutroAxiom (or NeutrosophicLaw/NeutrosophicAxiom) defined on a nonempty set is a law/axiom that is true for some set$'s$ elements (degree of truth (T)), indeterminate for other set$'s$ elements (degree of indeterminacy (I)), or false for the other set$'s$ elements (degree of falsehood (F)), where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the classical axiom.

(v) A NeutroAlgebra is an algebra that has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements and false for other elements).

Definition 2.2 ([4]). For a nonempty set G and a binary operation $*$ on G the couple $(G, *)$ is called a classical group if the following conditions hold:

(G1) $x * y \in G$ for all $x, y \in G$.

(G2) $x * (y * z) = (x * y) * z$ for all $x, y, z \in G$.

(G3) There exists $e \in G$ such that $x * e = e * x = x$ for all $x \in G$.

(G4) There exists $y \in G$ such that $x * y = y * x = e$ for all $x \in G$, where e is the neutral element of G.

If for all $x, y \in G$, (G5) $x * y = y * x$, then $(G, *)$ is called an abelian group.
Note that $x * y$ will be written as xy for all $x, y \in G$.

Definition 2.3 ([6]). A group $(G, *)$ is called nilpotent if it has a central series, that is, a normal series $e = G_0 \leq G_1 \leq \cdots \leq G_n = G$ such that G_{i+1}/G_i is contained in the center of G/G_i for all i. The length of a shortest centeral series of G is the nilpotent class of n.

Definition 2.4 ([6]). Let $(G, *)$ be a group and $x_1, \ldots, x_n$ be elements of G. Commutator of x_1 and x_2 is $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$. A commutator of weight $n \geq 2$ is defined by $[x_1, \ldots, x_n] = [x_1, \ldots, x_{n-1}], x_n]$, where by convention $[x_1] = x_1$.

A NeutroGroup is an alternative of a group that has either one NeutroOperation (partially well-defined, partially indeterminate and partially outer-defined), or at least one NeutroAxiom (NeutroAssociativity, NeutroNeutralElement or NeutroInverseElement) with no AntiOperation (is an operation outer-defined for all the set$'s$ elements (totally falsehood)) or AntiAxion (is an axiom that is false for all set$'s$ elements). It is possible to define NeutroGroup in another way by considering only one NeutroAxiom or by considering two NeutroAxioms or etc.

Definition 2.5. Let NG be a nonempty set and $*$ be a binary operation on NG. The couple $(NG, *)$ is called a NeutroGroup if the following conditions are satisfied:

(NG1) There exists some triplet $(x, y, z) \in NG$ such that $x*(y*z) = (x*y)*z$ and $u*(v*w) \neq (u * v) * w$ for some $(u, v, w) \in NG$ or there exists some $(r, s, t) \in NG$ such that $r * (s * t)$ =indeterminate or $(r * s) * t$ =indeterminate (NeutroAssociativity).

(NG2) There exists at least an element $a \in NG$ that has a single neutral element i.e., we have $e \in NG$ such that $a * e = e * a = a$ and for $b \in NG$ there does not exist $e \in NG$ such that $b * e = e * b \neq b$ or there exists $e_1, e_2 \in NG$ such that $b * e_1 = e_1 * b = b$ or $b * e_2 = e_2 * b = b$ with $e_1 \neq e_2$ or there exists at least an element $c \in NG$ that there is $d \in NG$ such that $c * d = d * c$ =indeterminate (NeutroNeutralElement).

(NG3) There exists an element $a \in NG$ that has an inverse $b \in NG$ w.r.t. a unit element $e \in NG$ i.e., $a * b = b * a = e$, or there exists at least one element $b \in NG$ that has two or more inverses $c, d \in NG$ w.r.t. some unit element $u \in NG$ i.e., $b * c = c * b = u$, $b * d = d * b = u$ or there exists at least one element $r \in NG$ that has one element $s \in NG$ such that $r * s = s * r$=indeterminate (NeutroInverseElement).

(NG4) There exists some duplet $(a, b) \in NG$ such that $a * b = b * a$ and there exists some duplet $(c, d) \in NG$ such that $c * d \neq d * c$, or there exists some $(r, s) \in NG$, $r * s$ =indeterminate or $s*r$ =indeterminate, then $(NG, *)$ is called a NeutroAbelianGroup (NeutroAbelianGroup).

Example 2.6. Let $U = \{a, b, c, d, e, f\}$ be a universe of discourse and $NG = \{a, b, c, d\}$ be a subset of U. Define the operation $*_1$ on NG in table 1. Then $*_1$ is a NeutroLow since $c *_1 d =$ indeterminate. Also,

$$a *_1 (b *_1 c) = (a *_1 b) *_1 c \text{ and } c *_1 (a *_1 c) = c *_1 d = \text{indeterminate}$$

TABLE 1. The table of NeutoGroup $(NG, *_1)$

$*_1$	**a**	**b**	**c**	**d**
a	b	c	d	a
b	c	d	a	c
c	d	a	b	?
d	a	b	?	a

Thus, $(NG, *_1)$ is a NeutroGroup.

Note that $x * y$ will be written as xy for all $x, y \in NG$.

Theorem 2.7 ([1]). *Let $(NH, *)$ be a NeutroSubgroup of the NeutroGroup $(NG, *)$. The sets $(NG/NH)_l = \{xNH : x \in NG\}$ and $(NG/NH)_r = \{NHx : x \in NG\}$ are two NeutroGroups with operations $\circ_l$, $\circ_r$ where for any xNH, $yNH \in (NG/NH)_l$, NHx, $NHy \in (NG/NH)_r$, $x, y \in NG$ we have*

$$xNH \circ_l yNH = xyNH, \quad NHx \circ_r NHy = NHxy.$$

Definition 2.8 ([1]). Let $(NG, *)$ and $(NK, \circ)$ be two NeutroGroups. The mapping $\varphi : NG \to NK$ is called a NeutroGroup Homomorphism if for every duplet $(x, y) \in G$, we have $\varphi(x * y) = \varphi(x) \circ \varphi(y)$.

In addition, if φ is a NeutroBijection, then φ is called a NeutroGroup Isomorphism. NeutroGroup Epimorphism, NeutroGroup Monomorphism, NeutroGroup Endomorphism are defined similarly.

Theorem 2.9 ([1]). *Let $(NG, *)$ and $(NK, \circ)$ be NeutroGroups and let e_{NG} and e_{NH} be NeutroNeutralElements in NG and NK respectively. Suppose that $\varphi : NG \to NK$ is a NeutroGroup Homomorphism. Then $\varphi(e_{NG}) = e_{NK}$.*

From now on, NG is a NeutroGroup with tree NeutroAxioms (NeutroAssociativity, NeutroNeutralElement and NeutroInverseElement). Also, for all $x \in NG$, N_x and I_x represent the NeutroNeutralElement and the NeutroInverseElement respectively.

3. Some Results On NeutroNilpotentGroups

In this section, we introduce the notion of commutator of two elements in a NeutroGroup and study a new concept as *NeutroNilpotentGroups* and their properties are given.

Let x, y be elements of a NeutroGroup NG. The commutator of x, y, denoted by $[x, y]$, is the element $I_x I_y xy$, i.e., $[x, y] = I_x I_y xy$. If I_x or I_y does not exist, then put $I_x = x$ and $I_y = y$.

Also, for any $x, y_1, \ldots, y_n \in NG$, define the commutator $[x, y_1, \ldots, y_n]$ by $[x, y_1, \ldots, y_n] = [[x, y_1, \ldots, y_{n-1}], y_n]$.

TABLE 2. The table of NeutoNilpotentGroup $(NG, *_2)$

$*_2$	**a**	**b**	**c**	**d**
a	b	c	d	a
b	c	d	a	c
c	d	a	b	d
d	a	b	c	a

TABLE 3. The table of NeutoAbelianGroup $(NG, *_3)$

$*_3$	**a**	**b**	**c**	**e**
a	b	a	b	a
b	c	f	c	b
c	d	c	e	c
e	a	b	c	e

Definition 3.1. A NeutroGroup $(NG, *)$ is called *NeutroNilpotentGroup* if $Z_n(NG) = NG$ for some $n \in \mathbb{N}$, where

$$Z_n(NG) = \{x \in NG : [x, g_1, g_2, \ldots, g_n] = N_z \text{ for at least one } g_1, \ldots, g_n, z \in NG\}.$$

The smallest such n is called the NeutroNilpotency of NG.

Note that, if NG is a NeutroNilpotentGroup, then for any $x \in NG$ there exists at least one $g_1, \ldots, g_n, z \in NG$ such that $[x, g_1, g_2, \ldots, g_n] = N_z$.

Example 3.2. Let $U = \{a, b, c, d, e, f\}$ be a universe of discourse and $NG = \{a, b, c, d\}$ be a subset of U. Define the operation $*_2$ on NG in table 2. Since $[a, b] = d$, $[d, b] = d$, $[c, c] = d$ and $[b, b] = a$, we have $[c, d, b] = [b, b, b] = [a, b] = d = N_a$, $[d, b, b] = [d, b] = N_a$ and $[a, b, b] = [d, b] = N_a$. Therefore, NG is a NeutroNilpotentGroup of class 2.

Example 3.3. Let $U = \{a, b, c, d, e, f\}$ be a universe of discourse and let $NG = \{e, a, b, c\}$ be a subset of U. Define the operation $*_3$ on NG in table 3. Since $[a, b, a] = e = N_c$, $[b, a, e] = [e, e] = e$, $[c, e, c] = [c, c] = e$, $[e, a, a] = [b, a] = e$, we have NG is NeutroAbelianGroup and a NeutroNilpotentGroup of class 2.

In what follows we have a non Abelian NeutroNilpotentGroup.

Example 3.4. Let $U = \{a, b, c, d\}$ be a universe and $NG = \{a, b, c\}$ be a NeutroGroup by the Cayley table 4. Then $H = \{a, b\}$, by the operation $*_4$, is a NeutroSubgroup of NG (see [1]). Since $N_a = a$, $I_a = a$, N_b, I_b does not exist, we have $[a, a] = aaaa = a = N_a$ and $[b, b] = a = N_a$. Therefore, H is a NeutroNilpotentSubgroup that is not an AbelianNeutroGroup.

Table 4. The table of non Abelian NeutoAbelianGroup $(NG, *_4)$

$*_4$	**a**	**b**	**c**
a	a	c	b
b	c	a	c
c	a	c	d

Table 5. The table of NeutoSubgroup $(H, *_4)$

$*_4$	**a**	**b**
a	a	c
b	c	a

Table 6. The table of NeutoSubgroup $(NH, *_5)$ of $(NG, *_5)$

$*_5$	**a**	**c**	**d**
a	b	d	a
c	d	b	d
d	a	c	a

Theorem 3.5. *Let NG and NK be two NeutroGroups. Then $Z_n(NG \times NK) = Z_n(NG) \times Z_n(NK)$. Moreover, $NG \times NK$ is a NeutroNilpotentGroup of class n if and only if NG and NK are NeutroNilpotentGroups of class n.*

Proof. Assume $(x, y) \in Z_n(NG \times NK)$, $z \in NG$ and $t \in NK$. Then for some $(x_1, y_1), \ldots, (x_n, y_n) \in NG \times NK$, we have

$$\begin{aligned} (N_z, N_t) &= [(x, y), (x_1, y_1), \ldots, (x_n, y_n)] = ([x, x_1, \ldots, x_n], [y, y_1, \ldots, y_n]) \\ &\Leftrightarrow [x, x_1, \ldots, x_n] = N_z, [y, y_1, \ldots, y_n] = N_t \\ &\Leftrightarrow x \in Z_n(NG), \, y \in Z_n(NK) \\ &\Leftrightarrow (x, y) \in Z_n(NG) \times Z_n(NK). \end{aligned}$$

Therefore, $Z_n(NG \times NK) = Z_n(NG) \times Z_n(NK)$.
Moreover, $NG \times NK$ is NeutroNilpotentGroup if and only if $Z_n(NG \times NK) = NG \times NK = Z_n(NG) \times Z_n(NK)$ if and only if NG and NK are NeutroNilpotentGroups. □

In what follows we have a NeutroSubgroup that is not NeutroNilpotentGroup.

Example 3.6. Consider the NeutroGroup NG from Example 3.2. Define the operation $*_5$ on NG in table 6. Then $NH = \{a, c, d\}$ is a NeutroSubgroup of NG (see [1]). Since $[a, d] = a$, $[a, a]$ and $[a, c]$ does not exist, we get $[a, g_1, \ldots, g_n]$ does not exist for any $g_1, \ldots, g_n \in NH$, and so $a \notin Z_n(NH)$ i.e., NH is not NeutroNilpotentGroup.

Theorem 3.7. *Let NH be a NeutroSubgroup of the NeutroNilpotentGroup NG. Then NeutroQuotientGroups $(NG/NH)_l$ and $(NG/NH)_r$ are NeutroNilpotentGroups.*

Proof. Assume NH be a NeutroSubgroup of NG and $gH \in (NG/NH)_l$. Since NG is a NeutroNilpotentGroup, we have $[g, g_1, \ldots, g_n] = N_z$ for some $g_1, \ldots, g_n, z \in NG$, and so $[gNH, g_1NH, \ldots, g_nNH] = [g, g_1, \ldots, g_n]NH = N_zNH$. Since $(zNH) \circ_l (N_zNH) = (z * N_z)NH = zNH = (N_z)NH \circ_l zNH$, we get $(N_z)NH$ is a NeutroNaturalElement of $(NG/NH)_l$. Therefore, $(NG/NH)_l$ is a NeutroNilpotentGroup. Similarly, $(NG/NH)_r$ is a NeutroNilpotentGroup. □

We recall that the intersection of two NeutroGroups is a NeutroGroup (see [1]). Now we have the following:

Theorem 3.8. *Let NG and NK be two NeutroNilpotentGroups. Then $NG \cap NK$ is a NeutroNilpotentGroup.*

Proof. Straightforward. □

Theorem 3.9. *Let NH be a NeutroNilpotentSubgroup of a NeutroGroup NG and for all $x, t \in NG$ we have*

$$xNH = NH \Rightarrow x \in NH, \quad (N_t)NH = NH.$$

If $(NG/NH)_l$ is a NeutroNilpotentQuotientGroup, then NG is a NeutroNilpotentGroup.

Proof. Assume $(NG/NH)_l$ is NeutroNilpotentGroup of class n and NH is NeutroNilpotentGroup of class m. Then for any $xNH \in (NG/NH)_l$, there exist $g_1NH, \ldots, g_nNH \in (NG/NH)_l$ such that $[xNH, g_1NH, \ldots, g_nNH] = (N_z)NH$, where $z \in NG$. Then $[x, g_1, \ldots, g_n]NH = (N_z)NH = NH$, and so $[x, g_1, \ldots, g_n] \in NH$. Since NH is NeutroNilpotentGroup, we get there exist $k_1, \ldots, k_m \in NH$ such that $[[x, g_1, \ldots, g_n], k_1, \ldots, k_m] = N_t$, for some $t \in NG$. Consequently, NG is NeutroNilpotentGroup of class $n + m$. □

Theorem 3.10. *Let NH be a NeutroNilpotentSubgroup of a NeutroGroup NG and for all $x, t \in NG$ we have*

$$NHx = NH \Rightarrow x \in NH, \quad NH(N_t) = NH.$$

If $(NG/NH)_r$ is a NeutroNilpotentQuotientGroup, then NG is a NeutroNilpotentGroup.

Proof. Similar to the proof of Theorem 3.9. □

Theorem 3.11. *Every homomorphic image of a NeytroNilpotentGroup is NeutroNilpotentGroup.*

Proof. Assume NH be a NeutroSubgroup of a NeutroNilpotentGroup NG and e_1, e_2 be NeutroNeutralElements in NG and NH, respectively. Suppose that $\psi : NG \to NH$ is a NeutroGroup Epimorphism. Then for any $h \in NH$, there exists $x \in NG$ such that $h = \psi(x)$. Since NG is NeutroNilpotentGroup, for $x \in NG$, there exist $g_1, \ldots,$ $g_n \in NG$ such that $[x, g_1, \ldots, g_n] = e_1$. Take $k_1 = \psi(g_1), \ldots, k_n = \psi(g_n)$. Therefore, $[h, k_1, \ldots, k_n] = \psi([x, g_1, \ldots, g_n]) = \psi(e_1) = e_2$, and so NH is a NeutroNilpotenGroup. □

4. Conclusion

In this paper, we defined a class of NeutroGroups, named NeutroNilpotentGroups, and their elementary properties were presented. The intersection of two NeutroSubgroups is not necessarily a NeutroSubgroup while their union is a NeutroSubgroup. We hope to study NeutroSolvabelGroups, NeutroEngelGroups in our future works.

Acknowledgments: The authors wish to thank Editors-in-Chief, and are grateful to Prof. A.A.A. Agboola, and the referees for their valuable suggestions and discussions on this work.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Agboola, A.A.A. Introduction to NeutroGroups, International Journal of Neutrosophic Science, vol. 6, no. 1, pp. 41–47, 2020. DOI: 10.5281/zenodo.3840761
2. Agboola, A.A.A. NeutroGroups (revisited), International Journal of Neutrosophic Science, Preprint.
3. Agboola, A.A.A. Introduction to NeutroRings, International Journal of Neutrosophic Science, vol. 7, no. 2, pp. 62–73, 2020. DOI:10.5281/zenodo.3877121
4. Hungerford, T.W. Algebra, vol, 73, 1974, Graduate Texts in Mathematics. New York: Springer Verlag. DOI:10.1007/978-1-4612-6101-8. (Originally published by Holt, Rinehart and Winston, Inc.)
5. Rezaei, A.; Smarandache, F. On Neutro-BE-algebras and Anti-BE-algebras (revisited), International Journal of Neutrosophic Science, vol. 4, no. 1, pp. 8–15, 2020. DOI: 10.5281/zenodo.3751862
6. Robinson, D.J.S. A Course in the theory of groups, Springer-Verlag, 1980.
7. Smarandache, F. Generalizations and Alternatives of Classical Algebraic Structures to NeutroAlgebraic Structures and AntiAlgebraic Structures, Journal of Fuzzy Extension & Applications, vol. 1, no. 2, pp. 85–87, 2020.
8. Smarandache, F. NeutroAlgebra is a Generalization of Partial Algebra, International Journal of Neutrosophic Science, vol. 2, no. 1, pp. 08–17, 2020.
9. Smarandache, F. Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures (revisited), Neutrosophic Sets and Systems, vol. 31, pp. 1–16, 2020. DOI: 10.5281/zenodo.3638232
10. Smarandache, F. Introduction to NeutroAlgebraic Structures, in Advances of Standard and Nonstandard Neutrosophic Theories, Pons Publishing House Brussels, Belgium, Ch. 6, pp. 240–265, 2019.
11. Smarandache, F. A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability, (3rd edition), American Research Press, Rehoboth, 2003. http://fs.unm.edu/eBook-Neutrosophic6.pdf

Received: June 29, 2020. Accepted: Nov 20, 2020