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Neutrosophic $\theta-$Closure Operator

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Abstract. The fundamental intent of this article is to develop the idea of neutrosophic θ-cluster point, neutrosophic θ- closure operator, neutrosophic $\epsilon\theta$q-neibourhood in neutrosophic topological spaces. We characterize some types of functions like neutrosophic θ-continuous, neutrosophic strongly-θ-continuous, neutrosophic weakly continuous functions in terms of $\mathcal{N}\theta$-closure operator are discussed. Further, neutrosophic regular space is also introduced.

Keywords: neutrosophic quasi coincident; $\mathcal{N}_{\epsilon}^{q} - nbd$; $\mathcal{N}_{\epsilon}^{\theta q} - nbd$; $\mathcal{N}\theta$-cluster points; $\mathcal{N}\theta$-closure; $\mathcal{N}\theta$-closed set, $\mathcal{N}$ Strongly θ-continuous; $\mathcal{N}$weakly-continuous.

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1. Introduction

Fuzzy set theory is introduced and studied as a mathematical tool concern with uncertainties where every element had a "degree of membership, truth(t)", by Zadeh [28]. A fuzzy set is one where every element had a "degree of membership" which lies between 0 and 1 . Atanassov [11] developed intuitionistic fuzzy set(IFS) as a generalization of fuzzy sets where besides, the "degree of non-membership" is assigned to each elelment. Both degrees belong to the interval [0,1] with the restriction that their sum is should not exceed 1. In IFS, the "degree of non-membership" depends on the "degree of membership".

Neutrality (i), "the degree of indeterminacy", as an independent notion, was proposed by F. Smarandache [26, 27]. In addition he described neutrosophic set on "three components (t , f, i) = (truth, falsehood, indeterminacy)". In neutrosophic set respectively "degree of membership, indeterminacy and non-membership" assigend to every elememt and it lies between [0,1]*, non-standard unit interval. Unlike in IFS, where the uncertainity depends on both "degree of membership" as well as "non-membership", here the uncertainity is independent of "degree of membership and non-membership". Neutrosophic sets are certainly too general than IFS as there are no restrictions between "degree of membership, degree of indeterminacy and degree of membership".

Neutrosophic notion have many applications in the fields of Information Sytems, Artificial Intelligence , decision making and evaluating airline service quality [1–4].As developments goes on, some researchers [5–9] have extended the idea of neutrosophic set into plithogenic set and applied it in MCDM, MADM and optimization technique supply chain based model. Salama et al [23] developed Neutrosophic topological space in 2012. This gave the way for investigation in terms of neutrosophic topology and its application in decision making problems. The properties of neutrosophic open sets, neutrosophic closed sets, neutrosophic interior operator and neutrosophic closure operator gave the way for applying neutrosophic topology. Researchers established the sets which are close to neutrosophic open sets as well asmneutrosophic closed sets. Like this, Neutrosophic closed sets as well as Neutrosophic continuous mappings were developed in [24]. Arokiarani et al. [10] introduced neutrosophic semi-open (sequentially, pre-open as well as α-open) mappings and discussed their properties. R. Dhavaseelan et al. [12] introduced generalized neutrosophic closed sets. In [14, 15] the concept of neutrosophic generalized α-contra continuous along with neutrosophic Almost α-contra-continuous functions are introduced and studied their properties.Dhavaseelan et al. [16] presented the idea of neutrosophic α^m-continuity. Narmada Devi, et al. [20] presented the idea of Neutrosophic structure ring contra strong precontinuity. The notion of fuzzy θ-closure operator introduced in [19] . Hanafy et al [17] established the notion of intuitionistic fuzzy θ-closure operator and intuionistic fuzzy weakly continuous functions.

The main contribution of the article is

- To establish the notion of neutrosophic θ-closure operator along with its properties in neutrosophic topological spaces.
- Neutrosophic θ-closed set is also defined using the operator defined.
- As application of this new notion, neutrosophic θ-continuous, neutrosophic strongly θ-continuous and neutrosophic weakly continuos functions are charcterized in terms of neutrosophic θ-closure operator.
- At the end we have shown the relation between these neutrosophic continuous functions through implication diagram.

2. Preliminaries

Definition 2.1. [26, 27] For a nonempty fixed set N_X a neutrosophic set [in short, NS] K is an object of the form $K = \{\langle x, \mu_K(x), \sigma_K(x), \gamma_K(x)\rangle : x \in N_X\}$ where $\mu_K(x), \sigma_K(x)$ and

$\gamma_K(x)$ respectively denotes the "degree of membership function ($\mu_K(x)$)", the "degree of indeterminacy ($\sigma_K(x)$)" as well as the "degree of nonmembership ($\gamma_K(x)$)" of each element $x \in N_X$ to the set K.

Remark 2.1. [26,27]

(1) A NS $K = \{\langle j, \mu_K(j), \sigma_K(j), \gamma_K(j)\rangle : j \in K\}$ can be recognized as an ordered triple $\langle \mu_K, \sigma_K, \gamma_K\rangle$ in $]0^-, 1^+[$ on N_X.

(2) For covenience, we write $K = \langle \mu_K, \sigma_K, \gamma_K\rangle$ for the NS set $K = \{\langle j, \mu_K(j), \sigma_K(j), \gamma_K(j)\rangle : j \in N_X\}$.

Definition 2.2. [26,27] Consider a nonempty set N_X along with NSs K as well as H in the form
$K = \{\langle j, \mu_K(j), \sigma_K(j), \gamma_K(j)\rangle : j \in N_X\}$, $H = \{\langle j, \mu_H(j), \sigma_H(j), \gamma_H(j)\rangle : j \in N_X\}$. Then

(a) $K \subseteq H$ iff $\mu_K(j) \le \mu_H(j)$, $\sigma_K(j) \le \sigma_H(j)$ and $\gamma_K(j) \ge \gamma_H(j)$ for every $j \in N_X$;
(b) $K = H$ iff $K \subseteq H$ and $H \subseteq K$;
(c) $\bar{K} = \{\langle j, \gamma_K(j), \sigma_K(j), \mu_K(j)\rangle : j \in N_X\}$; [Complement of K]
(d) $K \cap H = \{\langle j, \mu_K(j) \wedge \mu_H(j), \sigma_K(x) \wedge \sigma_H(j), \gamma_K(j) \vee \gamma_H(j)\rangle : j \in N_X\}$;
(e) $K \cup H = \{\langle j, \mu_K(j) \vee \mu_H(jx), \sigma_K(x) \vee \sigma_H(x), \gamma_K(x) \wedge \gamma_H(j)\rangle : j \in N_X\}$;
(f) $[\,]K = \{\langle j, \mu_K(j), \sigma_K(j), 1 - \mu_K(j)\rangle : j \in N_X\}$;
(g) $\langle\rangle K = \{\langle j, 1 - \gamma_K(j), \sigma_K(j), \gamma_K(j)\rangle : j \in N_X\}$.

Definition 2.3. [26,27] Let $\{K_i : i \in J\}$ be any family of NSs in N_X. Then

(a) $\bigcap K_i = \{\langle x, \wedge\mu_{K_i}(x), \wedge\sigma_{K_i}(x), \vee\gamma_{K_i}(x)\rangle : x \in N_X\}$;
(b) $\bigcup K_i = \{\langle x, \vee\mu_{K_i}(x), \vee\sigma_{K_i}(x), \wedge\gamma_{K_i}(x)\rangle : x \in N_X\}$.

Since our main work is to construct the tools for generating neutrosophic topological spaces, so we present the NSs 0_N and 1_N in N_X as below:

Definition 2.4. [26,27] $0_N = \{\langle x, 0, 0, 1\rangle : x \in N_X\}$ and $1_N = \{\langle x, 1, 1, 0\rangle : x \in N_X\}$.

Definition 2.5. [23] A neutrosophic topology (NT) on a nonempty set N_X is a collection Ω of NSs in N_X satisfy the axioms given below:

(i) $0_N, 1_N \in \Omega$,
(ii) $R_1 \cap R_2 \in \Omega$ for any $R_1, R_2 \in \Omega$,
(iii) $\cup R_i \in \Omega$ for arbitrary collection $\{R_i \mid i \in \Lambda\} \subseteq \Omega$.

Here the ordered pair (N_X, Ω) or only N_X is termed as neutrosophic topological space (NTS) and each NS in Ω is known as neutrosophic open set (NOS). The complement $\overline{R}$ of a NOS R in X is known as neutrosophic closed set (NCS) in N_X.

Definition 2.6. [13] Consider a NS, K in a NTS N_X. Then
$\mathcal{N}int(K) = \bigcup\{R \mid R$ is a NOS in N_X also $R \subseteq K\}$ is referred as neutrosophic interior of K;
$\mathcal{N}cl(K) = \bigcap\{R \mid R$ is a NCS in N_X with $R \supseteq K\}$ is referred as neutrosophic closure of K.

Definition 2.7. [12] Consider a nonempty set as N_X. Whenever t, i, f be "real standard or non standard" subsets of $]0^-, 1^+[$ then the NS $x_{t,i,f}$ is named as neutrosophic point(shortly, NP)in N_X given by

$$x_{t,i,f}(x_p) = \begin{cases} (t, i, f), & \text{whenever } x = x_p \\ (0, 0, 1), & \text{whenever } x \neq x_p \end{cases}$$

for $x_p \in N_X$ is called the support of $x_{t,i,f}$.where t denotes the "degree of membership" ,i the "degree of indeterminacy" and f is the "degree of non-membership" of $x_{t,i,f}$.

3. Neutrosophic $\theta-$Closure Operator

Definition 3.1. (1) A NP $x_{(\alpha,\beta,\lambda)}$ in N_X is termed as quasi-coincident with the NS $\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), \gamma_\Lambda(x)\rangle\}$, represented as $x_{(\alpha,\beta,\lambda)}q\Lambda$ iff $\alpha + \mu_\Lambda > 1$, $\beta + \sigma_\Lambda > 1$ and $\lambda + \gamma_\Lambda < 1$.

(2) Consider $\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), \gamma_\Lambda(x)\rangle\}$ along with $\Gamma = \{\langle x, \mu_\Gamma(x), \sigma_\Gamma(x), \gamma_\Gamma(x)\rangle\}$ as NSs in N_X. Then Λ is said to be quasi-coincident with Γ, indicated as $\Lambda q\Gamma$ iff there exists an element $x \in N_X$ such that $\mu_\Lambda(x) + \mu_\Gamma(x) > 1$,$\sigma_\Lambda(x) + \sigma_\Gamma(x) > 1$ and $\gamma_\Lambda(x) + \gamma_\Gamma(x) < 1$.

The expression "not quasi-coincident" will be summarized as $\widetilde{q}$.

Proposition 3.1. Let Λ and Γ be two NSs along with a NP $x_{(\alpha,\beta,\lambda)}$ in N_X. Then

- i) $\Lambda\widetilde{q}\,\overline{\Gamma}$ iff $\Lambda \subseteq \Gamma$.
- ii) $\Lambda q\,\Gamma$ iff $\Lambda \nsubseteq \Gamma$.
- iii) $x_{(\alpha,\beta,\lambda)} \subseteq \Lambda$ iff $x_{(\alpha,\beta,\lambda)}\widetilde{q}\,\overline{\Lambda}$
- iv) $x_{(\alpha,\beta,\lambda)}q\,\Lambda$ iff $x_{(\alpha,\beta,\lambda)} \nsubseteq \overline{\Lambda}$

Definition 3.2. Let $\mu : N_X \to N_Y$ be a function and $x_{(\alpha,\beta,\lambda)}$ be a NP in N_X. Then the preimage of $x_{(\alpha,\beta,\lambda)}$ under μ, designated as $\mu(x_{(\alpha,\beta,\lambda)})$ is defined by
$\mu(x_{(\alpha,\beta,\lambda)}) = \{\langle y, \mu(x_p)_\alpha, \mu(x_p)_\beta, (1 - \mu(x_p)_{1-\lambda})\rangle : y \in N_Y\}$.

Proposition 3.2. Let $f : N_X \to N_Y$ be a function and $x_{(\alpha,\beta,\lambda)}$ be a NP in N_X.

i) $x_{(\alpha,\beta,\lambda)}qf^{-1}(\Gamma)$ if $f(x_{(\alpha,\beta,\lambda)})q\Gamma$. for any NS Γ in N_Y.
ii) $f(x_{(\alpha,\beta,\lambda)})qf(\Lambda)$ if $x_{(\alpha,\beta,\lambda)}q\Lambda$ for any NS Λ in N_X

Definition 3.3. Let (X, Θ) be a NTS on N_X and $x_{(\alpha,\beta,\lambda)}$ be a NP in N_X. A NS Λ is called $\mathcal{N}\epsilon q - nbd$ of $x_{(\alpha,\beta,\lambda)}$, if there exists a neutrosophic open Γ in N_X such that $x_{(\alpha,\beta,\lambda)}q\Gamma$ and $\Gamma \subseteq \Lambda$. The family of all $\mathcal{N}\epsilon q - nbd$ of $x_{(\alpha,\beta,\lambda)}$ is indicated as $\mathcal{N}N^q_\epsilon(x_{(\alpha,\beta,\lambda)})$.

Definition 3.4. A NP $x_{(\alpha,\beta,\lambda)}$ is known as neutrosophic θ-cluster point($\mathcal{N}\theta$-cluster point,for short) of a NS Λ iff for each Γ in $\mathcal{N}\epsilon q-nbd$ of $x_{(\alpha,\beta,\lambda)}$ and $\mathcal{N}cl(\Gamma)q\Lambda$. The set of all $\mathcal{N}\theta$-cluster points of Λ is named as neutrosophic θ closure of Λ and denoted by $\mathcal{N}cl_\theta$.

A NS Λ will be $\mathcal{N}\theta-$closed set($\mathcal{N}\theta$CS for short) iff $\Lambda = \mathcal{N}cl_\theta(\Lambda)$. The complement of a $\mathcal{N}\theta-$closed set is $\mathcal{N}\theta-$open set (in short $\mathcal{N}\theta$OS).

Proposition 3.3. Let (N_X, Θ) be a NTS and let Λ and Γ be two NSs in N_X. Then

i) $\Lambda \subseteq \Gamma \Rightarrow \mathcal{N}cl_\theta(\Lambda) \subseteq \mathcal{N}cl_\theta(\Gamma)$
ii) $\Lambda \cup \Gamma \Rightarrow \mathcal{N}cl_\theta(\Lambda) \cup \mathcal{N}cl_\theta(\Gamma)$
iii) $\mathcal{N}int_\theta(\Lambda) = \overline{\mathcal{N}cl_\theta(\overline{\Lambda})}$

Definition 3.5. A NS Λ of a NTS N_X is named as $\mathcal{N}\epsilon\theta q-nbd$ of a NP $x_{(\alpha,\beta,\lambda)}$ if there arises a $\mathcal{N}\epsilon q-nbd$ Γ of $x_{(\alpha,\beta,\lambda)}$ such that $\mathcal{N}cl(\Gamma)\widetilde{q}\overline{\Lambda}$. The family of all $\mathcal{N}\epsilon\theta q-nbd$ of $x_{(\alpha,\beta,\lambda)}$ is represented as $\mathcal{N}N_\epsilon^{\theta q}(x_{(\alpha,\beta,\lambda)})$.

Remark 3.1. For any NS Λ in a NTS N_X, $\mathcal{N}cl(\Lambda) \subseteq \mathcal{N}cl_\theta(\Lambda)$.

Proposition 3.4. If Λ is a NOS in a NTS N_X, then $\mathcal{N}cl(\Lambda) = \mathcal{N}cl_\theta(\Lambda)$.

Proof. It is enough to prove $\mathcal{N}cl(\Lambda) \supseteq \mathcal{N}cl_\theta(\Lambda)$. Consider $x_{(\alpha,\beta,\lambda)}$ be a NP in N_X so as t $x_{(\alpha,\beta,\lambda)} \notin \mathcal{N}cl(\Lambda)$, then there exists $\Gamma \in \mathcal{N}N_\epsilon^q(x_{(\alpha,\beta,\lambda)})$ such that $\Gamma\widetilde{q}\Lambda$ and hence $\Gamma \subseteq \overline{\Lambda}$. Then $\mathcal{N}cl(\Gamma) \subseteq \overline{\mathcal{N}int(\Lambda)} \subseteq \overline{\Lambda}$, as Λ is a NOS in N_X. Thus $\mathcal{N}cl(\Gamma)\widetilde{q}\Lambda$ which implies $x_{(\alpha,\beta,\lambda)} \notin \mathcal{N}cl_\theta(\Lambda)$. Then $\mathcal{N}cl_\theta(\Lambda) \subseteq \mathcal{N}cl(\Lambda)$.Thus $\mathcal{N}cl(\Lambda) = \mathcal{N}cl_\theta(\Lambda)$. □

Proposition 3.5. Let (N_X, Θ) be a NTS, the conditions are satisfied

i) Finite union and arbitrary intersection of neutrosophic θ-closed sets in N_X is a $\mathcal{N}\theta$CS.
ii) For two neutrosophic sets Λ and Γ in N_X, if $\Lambda \subseteq \Gamma$, then $\mathcal{N}cl_\theta(\Lambda) \subseteq \mathcal{N}cl_\theta(\Gamma)$.
iii) 0_N and 1_N are neutrosophic θ-closed sets.

Corollary 3.1. Let Λ be a NS in NTS N_X. $\mathcal{N}cl_\theta(\Lambda)$ is evidently NCS.The converse of the Corollary doesn't hold .

Example 3.1. For $N_X = \{k_1, k_2, k_3\}$ NSs Λ, Γ and K in N_X are defined as :
$\Lambda = \langle x, (\frac{k_1}{0.6}, \frac{k_2}{0.6}, \frac{k_3}{0.2}), (\frac{k_1}{0.6}, \frac{k_2}{0.6}, \frac{k_3}{0.2}), (\frac{k_1}{0.3}, \frac{k_2}{0.4}, \frac{k_3}{0.1})\rangle$,
$\Gamma = \langle x, (\frac{k_1}{0.4}, \frac{k_2}{0.3}, \frac{k_3}{0.3}), (\frac{k_1}{0.4}, \frac{k_2}{0.3}, \frac{k_3}{0.3}), (\frac{k_1}{0.5}, \frac{k_2}{0.6}, \frac{k_3}{0.3})\rangle$ and
$K = \langle x, (\frac{k_1}{0.3}, \frac{k_2}{0.3}, \frac{k_3}{0.1}), (\frac{k_1}{0.3}, \frac{k_2}{0.3}, \frac{k_3}{0.1}), (\frac{k_1}{0.6}, \frac{k_2}{0.7}, \frac{k_3}{0.1})\rangle$. Then the family $\Theta = \{0_N, 1_N, \Lambda, \Gamma\}$ is NT on N_X. So, (N_X, Θ) is NTSs. Let $x_{(0.6,0.6,0.3)}(k_1)$ and $x_{(0.8,0.8,0.1)}(k_1)$ are neutrosophic points in N_X. Here
$x_{(0.6,0.6,0.3)}(k_1) \in \mathcal{N}cl_\theta(K)$, that is $x_{(0.6,0.6,0.3)}(k_1)q\Lambda \subseteq \Lambda$ and $\mathcal{N}cl(\Lambda) = 1_N\ q\ K$. Now $x_{(0.8,0.8,0.1)}(k_1) \notin \mathcal{N}cl_\theta(K)$, that is $x_{(0.8,0.8,0.1)}(k_1)q\Gamma$, $\mathcal{N}cl(\Gamma) = \overline{\Gamma}\widetilde{q}K$. But $x_{(0.8,0.8,0.1)}(k_1) \in \mathcal{N}cl_\theta(x_{(0.6,0.6,0.3)}(k_1)) \subseteq \mathcal{N}cl_\theta(\mathcal{N}cl_\theta(K))$. Hence $\mathcal{N}cl_\theta(K)$ is not $\mathcal{N}\theta$CS.

Proposition 3.6. A NS Λ is $\mathcal{N}\theta$OS in NTS N_X iff for each NP $x_{(\alpha,\beta,\lambda)}$ in N_X with $x_{(\alpha,\beta,\lambda)}q\Lambda$, Λ is a $\mathcal{N}\epsilon\theta q - nbd$ of $x_{(\alpha,\beta,\lambda)}$.

Proposition 3.7. For any NS Λ in a NTS (N_X, Θ), $\mathcal{N}cl_\theta(\Lambda) = \cap\{\mathcal{N}cl_\theta(\Gamma) : \Gamma \in \Theta$ and $\Lambda \subseteq \Gamma\}$.

Proof. Obviously $\mathcal{N}cl_\theta(\Lambda) \subseteq \cap\{\mathcal{N}cl_\theta(\Gamma) : \Gamma \in \Theta$ and $\Lambda \subseteq \Gamma\}$.
Now, let $x_{(\alpha,\beta,\lambda)} \in \cap\{\mathcal{N}cl_\theta(\Gamma) : \Gamma \in \Theta$ and $\Lambda \subseteq \Gamma\}$, but $x_{(\alpha,\beta,\lambda)} \notin \mathcal{N}cl_\theta(\Lambda)$. Consequently there arises a $\mathcal{N}\epsilon q - nbd\eta$ of $x_{(\alpha,\beta,\lambda)}$ so that $\mathcal{N}cl(\eta)\tilde{q}\Lambda$ and hence by Proposition3.1, $\Lambda \subseteq \overline{\mathcal{N}cl(\eta)}$. Then $x_{(\alpha,\beta,\lambda)} \in \mathcal{N}cl_\theta(\overline{\mathcal{N}cl(\eta)})$ and consequently, $\mathcal{N}cl(\eta)q\overline{\mathcal{N}cl(\eta)}$.Which is a contradiction. □

Definition 3.6. A NTS N_X is named as neutrosophic regular($\mathcal{NRS}$ in short) iff for each $x_{(\alpha,\beta,\lambda)}$ in N_X and each $\mathcal{N}\epsilon q - nbd\ \eta$ of $x_{(\alpha,\beta,\lambda)}$, there arises $\mathcal{N}\epsilon q - nbd\ \Gamma$ of $x_{(\alpha,\beta,\lambda)}$ such that $\mathcal{N}cl(\Gamma) \subseteq \eta$.

Proposition 3.8. A NTS N_X is $\mathcal{NRS}$ iff for each NS Λ in N_X, $\mathcal{N}cl(\Lambda) = \mathcal{N}cl_\theta(\Lambda)$.

Proof. Let N_X be a $\mathcal{NRS}$. It is true that $\mathcal{N}cl(\Lambda) \subseteq \mathcal{N}cl_\theta(\Lambda)$ for any NS Λ. Now, consider $x_{(\alpha,\beta,\lambda)}$ be NP in N_X with $x_{(\alpha,\beta,\lambda)} \in \mathcal{N}cl_\theta(\Lambda)$ and let Γ be a $\mathcal{N}\epsilon q - nbd$ of $x_{(\alpha,\beta,\lambda)}$. Then by $\mathcal{NRS}$ X, there exists $\mathcal{N}\epsilon q - nbd\ \eta$ of $x_{(\alpha,\beta,\lambda)}$ such that $\mathcal{N}cl(\eta) \subseteq \Gamma$. Now, $x_{(\alpha,\beta,\lambda)} \in \mathcal{N}cl_\theta(\Lambda)$ implies $\mathcal{N}cl(\eta)q\Lambda$ implies $\Gamma q\Lambda$ implies $x_{(\alpha,\beta,\lambda)} \in \mathcal{N}cl(\Lambda)$. Hence $\mathcal{N}cl_\theta(\Lambda) \subseteq \mathcal{N}cl(\Lambda)$. Thus $\mathcal{N}cl(\Lambda) = \mathcal{N}cl_\theta(\Lambda)$.

Contrarily, let $x_{(\alpha,\beta,\lambda)}$ be a NP in N_X and Λ be a $\mathcal{N}\epsilon q - nbd$ of $x_{(\alpha,\beta,\lambda)}$. Thereupon $x_{(\alpha,\beta,\lambda)} \notin \overline{\Lambda} = \mathcal{N}cl(\overline{\Lambda}) = \mathcal{N}cl_\theta(\overline{\Lambda})$. Thus there exists a $\mathcal{N}\epsilon q - nbd\ \eta$ of $x_{(\alpha,\beta,\lambda)}$ such that $\mathcal{N}cl(\eta)\tilde{q}\overline{\Lambda}$ and then $\mathcal{N}cl(\eta) \subseteq \Lambda$. Hence N_X is $\mathcal{NRS}$. □

4. **Applications**

Here we characterize some types of functions in terms of $\mathcal{N}\theta$-closure operator as application. Using this operator, we characterize neutrosophic strongly-θ-continuous, neutrosophic weakly continuous functions.

Definition 4.1. A function $f : (N_X, \Theta) \to (N_Y, \Xi)$ is termed as neutrosophic strongly $\theta-$continuous ($\mathcal{N}Str\theta$-continuous, for short), if for each NP $x_{(\alpha,\beta,\lambda)}$ in N_X and $\Gamma \in \mathcal{N}N^q_\epsilon(f(x_{(\alpha,\beta,\lambda)}))$, there exists $\Lambda \in \mathcal{N}N^q_\epsilon(x_{(\alpha,\beta,\lambda)})$ such that $f(\mathcal{N}cl(\Lambda)) \subseteq \Gamma$.

Proposition 4.1. For a function $\mu : (N_X, \Theta) \to (N_Y, \Xi)$ the conditions are equivalent :

i) μ is $\mathcal{N}Str\theta$-continuous.
ii) $\mu(\mathcal{N}cl_\theta(\Lambda)) \subseteq \mathcal{N}cl(\mu(\Lambda))$ for each NS $\Lambda \in N_Y$.
iii) $\mathcal{N}cl_\theta(\mu^{-1}(\Gamma)) \subseteq \mu^{-1}(\mathcal{N}cl(\Gamma))$ for each NS $\Gamma \in N_Y$.

iv) $\mu^{-1}(\Gamma)$ is a $\mathcal{N}\theta CS$ in N_X for each $\mathcal{N}CS$ $\Gamma \in N_Y$.

v) $\mu^{-1}(\Gamma)$ is a $\mathcal{N}\theta OS$ in N_X for every $\mathcal{N}OS$ $\Gamma \in N_Y$.

Proof. $i) \Rightarrow ii)$ Let $x_{(\alpha,\beta,\lambda)} \in \mathcal{N}cl_\theta(\Lambda)$ and $\Omega \in \mathcal{N}N^q_\epsilon(\beta(x_{(\alpha,\beta,\lambda)}))$. By (i) , there exists $\eta \in \mathcal{N}N^q_\epsilon(x_{(\alpha,\beta,\lambda)})$ such that $\beta(\mathcal{N}cl(\eta)) \subseteq \Omega$. Now, using Definition 3.4 and Proposition 3.1, we have $x_{(\alpha,\beta,\lambda)} \in \mathcal{N}cl_\theta(\Lambda) \Rightarrow \mathcal{N}cl(\eta)q\Lambda \Rightarrow \beta(\mathcal{N}cl(\eta)q\mu(\Lambda) \Rightarrow \Omega q\mu(\Lambda) \Rightarrow \mu(x_{(\alpha,\beta,\lambda)}) \in \mathcal{N}cl(\mu(\Lambda)) \Rightarrow x_{(\alpha,\beta,\lambda)} \in \mu^{-1}(\mathcal{N}cl(\mu(\Lambda)))$. Hence $\mathcal{N}cl_\theta(\Lambda \subseteq \mu^{-1}(\mathcal{N}cl(\mu(\Lambda)))$ and so $\mu(\mathcal{N}cl_\theta(\Lambda)) \subseteq \mathcal{N}cl(\beta(\Lambda))$

$ii) \Rightarrow iii)$Is obvious by substituting $\Lambda = \mu^{-1}(\Lambda)$.

$iii) \Rightarrow iv)$ Take Γ be a $\mathcal{N}$CS in N_Y. By (iii), we have $\mathcal{N}cl_\theta(\mu^{-1}(\Gamma)) \subseteq \beta^{-1}(\mathcal{N}cl(\Gamma)) = \mu^{-1}(\Gamma)$ which implies that $\mu^{-1}(\Gamma) = \mathcal{N}cl_\theta(\Gamma)$. Hence $\mu^{-1}(\Gamma)$ is a $\mathcal{N}\theta$CS in N_X.

$iv) \Rightarrow v)$Let $\overline{\Gamma}$ as a $\mathcal{N}$OS in N_Y. By (iii), we have $\overline{\mathcal{N}cl_\theta(\mu^{-1}(\overline{\Gamma}))} \supseteq \overline{\mu^{-1}(\mathcal{N}cl(\overline{\Gamma}))} = \overline{\mu^{-1}(\overline{\Gamma})}$ which implies that $\overline{\mu^{-1}(\overline{\Gamma})} = \overline{\mathcal{N}cl_\theta(\overline{\Gamma})}$. Hence $\overline{\mu^{-1}(\overline{\Gamma})}$ is a $\mathcal{N}\theta$OS in N_X.

$v) \Rightarrow i)$ Consider $x_{(\alpha,\beta,\lambda)}$ be a NP and $\Omega \in \mathcal{N}^q_\epsilon(\beta(x_{(\alpha,\beta,\lambda)}))$. By (v), $\mu^{-1}(\Omega)$ is a $\mathcal{N}\theta$OS in N_X. Now, using Proposition 3.1, we have $\mu(x_{(\alpha,\beta,\lambda)})q\Omega \Rightarrow x_{(\alpha,\beta,\lambda)}q\mu^{-1}(\Omega) \Rightarrow x_{(\alpha,\beta,\lambda)} \notin \overline{\mu^{-1}(\Omega)}$. Hence $\overline{\mu^{-1}(\Omega)}$ is a $\mathcal{N}\theta$CS, such that $x_{(\alpha,\beta,\lambda)} \notin \overline{\mu^{-1}(\Omega)}$. Then there exists $\eta \in \mathcal{N}^q_\epsilon(\beta(x_{(\alpha,\beta,\lambda)}))$ such that $\mathcal{N}cl(\eta)\tilde{q}\overline{\mu^{-1}(\Omega)}$ which implies that $\mu(\mathcal{N}cl(\eta)) \subseteq \Omega$. Hence μ is a $\mathcal{N}Str\theta$-continuous. □

Definition 4.2. A function $\beta : (N_X, \Theta) \to (Y, \Xi)$ is termed as neutrosophic weakly continuous[$\mathcal{N}w$-continuous for short], iff for each $\mathcal{N}$OS Λ in Y, $\beta^{-1}(\Lambda) \subseteq \mathcal{N}int(\beta^{-1}(\mathcal{N}cl(\Lambda)))$.

Proposition 4.2. Let $\beta : (N_X, \Theta) \to (N_Y, \Xi)$ be a function. Then for a NS Γ in N_Y.$\beta(\overline{\beta^{-1}(\Gamma)}) \subseteq \overline{\Gamma}$, wherein equality holds if β is surjective.

Proposition 4.3. Let $\mathfrak{D}$ be a NS and $x_{(\alpha,\beta,\lambda)}$ be NP in a NTS (N_X, Θ). Then the function $f : (N_X, \Theta) \to (N_Y, \Xi)$ if $x_{(\alpha,\beta,\lambda)} \in \mathfrak{D}$ then $f(x_{(\alpha,\beta,\lambda)}) \in f(\mathfrak{D})$.

Proposition 4.4. The successive results are equivalent for a function $\beta : (N_X, \Theta) \to (N_Y, \Xi)$:

a) β is a $\mathcal{N}w$-continuous.

b) $\beta(\mathcal{N}cl(\mathfrak{D})) \subseteq \mathcal{N}cl_\theta(\beta(\mathfrak{D}))$ for each NS $\mathfrak{D}$ in N_X.

c) $\mathcal{N}cl(\beta^{-1}(\mathfrak{G})) \subseteq \beta^{-1}(\mathcal{N}cl_\theta(\mathfrak{G}))$ for each NS $\mathfrak{G}$ in N_Y.

d) $\mathcal{N}cl(\beta^{-1}(\mathfrak{G})) \subseteq \beta^{-1}(\mathcal{N}cl(\mathfrak{G}))$ for each NOS $\mathfrak{G}$ in N_Y.

Proposition 4.5. Let $f : (N_X, \Theta) \to (N_Y, \Xi)$ be a $\mathcal{N}w$-continuous function, then

i) $f^{-1}(\Gamma)$ is a $\mathcal{N}CS$ in N_X, for every $\mathcal{N}\theta CS$ Γ in N_Y.

ii) $f^{-1}(\Gamma)$ is a $\mathcal{N}OS$ in N_X, for each $\mathcal{N}\theta OS$ Γ in N_Y.

Definition 4.3. A function $\mu : (N_X, \Theta) \rightarrow (N_Y, \Xi)$ is known as neutrosophic θ continuus($\mathcal{N}\theta-$continuous, for short), iff for each NP $x_{(\alpha,\beta,\lambda)}$ in N_X and each $\Gamma \in \mathcal{N}^q_\epsilon(\mu(x_{(\alpha,\beta,\lambda)}))$, there arises $\Lambda \in \mathcal{N}^q_\epsilon(x_{(\alpha,\beta,\lambda)})$ so as $\mu(\mathcal{N}cl(\Lambda)) \subseteq \mathcal{N}cl(\Gamma)$.

Proposition 4.6. For $\mu : (N_X, \Theta) \rightarrow (N_Y, \Xi)$, the successive results are identical:

a) μ is a $\mathcal{N}\theta$-continuous.
b) $\mu(\mathcal{N}cl_\theta(\mathfrak{D})) \subseteq \mathcal{N}cl_\theta(\mu(\mathfrak{D}))$ for each NS $\mathfrak{D}$ inN_X.
c) $\mathcal{N}cl_\theta(\mu^{-1}(\mathfrak{G})) \subseteq \mu^{-1}(\mathcal{N}cl_\theta(\mathfrak{G}))$ for everry NS $\mathfrak{G}$ in N_Y.
d) $\mathcal{N}cl_\theta(\mu^{-1}(\mathfrak{G})) \subseteq \mu^{-1}(\mathcal{N}cl(\mathfrak{G}))$ for each NOS $\mathfrak{G}$ in N_Y.

Remark 4.1. Based on the above results we have implication diagram as shown below.

$$\begin{array}{ccccc} \mathcal{N}Str-\text{continuous} & \Longrightarrow & \mathcal{N}\text{-continuous} & \Longrightarrow & \mathcal{N}w\text{-continuous} \\ & & \Downarrow & & \\ & & \mathcal{N}\theta\text{-continuous} & & \end{array}$$

5. Conclusion

This research article presents and establishes the idea of neutrosophic θ-closure operator in neutrosophic topogical spaces. Using this operator neutrosophic θ-closed set is defined. Some results are discussed and further more, as applications of these concepts, certain functions like neutrosophic θ-continuous, neutrosophic strongly θ-continuous together with neutrosophic weakly continuous are characterized interms of neutrosophic θ-closure operator. Neutrosophic regular space is also introduced and characterized interms of neutrosophic θ-closure operator. In future, using this operator, one can define the neutrosophic θ-generalized closed set and do the further interesting research.

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