



# Sumudu Transform for Solving Second Order Ordinary Differential Equation under Neutrosophic Initial Conditions

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**Abstract:** The ordinary differential equation of second order is being used in many engineering disciplines and sciences to model many real-life problems. These problems are mostly uncertain, vague and incomplete and thus they require some more advanced tool for modelling. Neutrosophic logic becomes the solution of all these kind of uncertain problems as it describe the conditions of uncertainty which occurs during the process of modelling on the basis of grade of membership of truth values, indeterminacy values and falsity values, that means it consider all the uncertain parameters on the basis of these degrees. In this research paper, we have considered the ordinary differential equation of second order with neutrosophic numbers as initial conditions of spring mass system is solved using Sumudu transform method which has advantage of unit preserving property over the well established Laplace Transform method. The solution obtained at various computational points by this method is shown in the form of table. Furthermore, the results obtained at different  $(\alpha, \beta, \gamma)$ -cut and time values are also depicted graphically and are verified analytically by de-fuzzifying the data.

**Keywords:** Fuzzy numbers; Neutrosophic numbers; Neutrosophic triangular numbers; Strongly-generalized differentiability; Sumudu transform.

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## 1. Introduction

In our daily lives we encounter many situations that are mostly vague, uncertain, ambiguous, incomplete, and inconsistent. With this limited and incomplete information, it becomes problematic to model and find the solution of the problem in a precise manner. To deal with these kinds of situations, L.A. Zadeh [1], in 1965 discovered the fuzzy set theory as the extension of classical set theory. This theory is more powerful than classical set theory in the sense that it considers uncertain environmental conditions as membership values, whereas classical set theory only studies true or false values and do not analyze any values between them. In real life situations, we often get information in the form of ambiguous words like good, very good, bad, and very bad, etc., but all these facts may differ from one person to another, because it is related to human thinking and hence depends on the human point of view. In fuzzy set theory these terms are known as linguistic terms and to these linguistic terms some membership grades are assigned according to their significance. All of these linguistic terms together with their membership degrees are written in the form of ordered pairs and finally fuzzy sets are formed using these ordered pairs. Sometimes, we have to deal with fuzzified numerical data also. For example, when we ask students how many hours do you self-study in a day? Then they use statements such as about 50 minutes a day, about 40 minutes or 50 minutes a day, or more than 50 minutes a day, etc. and these types of numbers are known as

fuzzy numbers. The set of real numbers act as a superset of these fuzzy numbers. These real values actually represent the grade of membership of a fuzzy set  $A$ , well defined over the universal set  $X$  and grade of membership  $\mu_A(x) \in [0,1]$ .

In some practical problems, considering only the membership value is not enough, it is also necessary to consider the non-membership value. The fuzzy sets are defined by considering the elements which considers only grade of membership of any information and grade of non-membership is not considered. Atanassov [2] studied in this direction and introduced another type of fuzzy set known as intuitionistic fuzzy set (IFS), which is the natural extension of fuzzy set and is more applicable in real life situation. Intuitionistic fuzzy sets considers as an extension of fuzzy sets, because it not only provide the information which belongs to the set but also which does not belong to the set. For example, suppose we want to collect the information of liking of any particular subject among students of class  $A$  and a questionnaire has been prepared for this purpose, which is distributed to the students so that they can fill it and then submit. The student can either fill the plus sign response to show the liking, minus sign to show dislikes or there is also one option to show nothing. In this way, for every student  $X$ , two responses are recorded, viz;  $A(x)$  = number of acceptances/likes,  $N(x)$  = number of non-acceptances/ dislikes. Another concept is also available in the world of uncertainty, which is known as Neutrosophic set theory, which studies the cause, description, and possibility of neutral thoughts. Neutrosophic sets deal with belongingness of truth values, indeterminate values and false values and it was first introduced by Florentin Smarandache [3]. In Neutrosophic logic, grade of membership of Truth values (T), Indeterminate values (I) and False values (F) has been defined within the non-standard interval  $]^{-}0, 1^{+}[$ . Neutrosophic set theory with non-standard interval works well in philosophical point of view. But practically when we deal with science and engineering problems, it is not possible to define data within this non-standard interval. To overcome this problem single valued Neutrosophic sets was defined by the researcher Wang et al. [4] by considering unit interval  $[0,1]$  in its standard form. The values within this interval are called Neutrosophic numbers. Aal SIA et al. [5], Deli and Subas [6], Ye [7] and Chakraborty et al. [8], etc., defined different kind of Neutrosophic numbers. Abdel-Basset et al. [9-13] defined advanced Neutrosophic numbers and presented results on recent pandemic COVID-19, decision making problems, supply chain model, industrial and management problems. In this way, lots of work has been done for the development of the Neutrosophic set theory with applications in real life problems (see for instance [14-18]).

Neutrosophic logic becomes one of the important and valuable tools in almost all area of science and engineering to model various real life phenomenon using differential equations with uncertain and imprecise parameters. Fuzzy differential equations (FDE) were introduced by Dubois et al. [19-21], by considering only membership values. For defining FDE, fuzzy numbers and corresponding fuzzy functions were discovered by Chang et al. [22]. Further the concept of intuitionistic fuzzy differential equations [23-25] came into the existence containing both membership and non-membership values as its parameters. To study the solutions of these fuzzy differential equations, the necessity arises to understand the concept of derivatives in fuzzy environment. In this direction lots of work has been reported in the literature, such as differentials for fuzzy functions were discussed by Puri and Ralescu [26] and Fuzzy Calculus is studied by Goetschel and Voxman [27], etc. Fuzzy derivative concept is used in the solution of ordinary differential equations of first order with initial conditions by Seikkala and Kaleva [28-29]. Buckley and Feuring [30-31] have solved ordinary differential equation of  $n$ th order containing fuzzy initial conditions. In 2005, Bede and Gal [32] worked on fuzzy-valued function and defined generalized differentiability for that. Further he provided the solution of fuzzy differential equations with this generalized differentiability using the lower-upper representation of a fuzzy numbers. Using generalized Hukuhara derivative in 2009, Stefanini et al. [33] represent generalization of fuzzy interval valued function. In this way the advancement of the theory of differential equations in a fuzzy environment has taken place.

It is clear that fuzzy differential equations deal with uncertainty by considering only membership values and intuitionistic fuzzy differential equations which considers only membership and non-membership values but none of them considers indeterminacy. Thus, it was needed to develop neutrosophic differential equations theory to model all three values, i.e., membership, indeterminacy and non-membership. Smarandache introduced Neutrosophic Calculus [34] containing all the basic concepts such as limit, continuity, differentiability, important functions such as exponential and logarithmic, concept of differentials and integrals in the neutrosophic environment. This theory is the growing field and researchers started working in this area. One can find that the theory of neutrosophic differential equations is studied by Sumathi and Priya [35] in the year 2018 and also one recent paper of Sumathi and Sweety [36], which uses trapezoidal neutrosophic numbers for solving differential equations of first order having one independent variable.

One can find a rich literature based on the theory of fuzzy differential equations, which contains various tools for the solution of fuzzy differential equations. To find the appropriate solutions of fuzzy differential equations various analytical, semi-analytical and numerical schemes are suggested and used in the recent literature. Salahshour et al. [37] defined Laplace transform for a fuzzy differential equation and Mondal et al. [38-39] solved fuzzy differential equations using Laplace transform. In 2016, Tapaswini et al. [40] solved fuzzy differential equation using analytical method. Ahmadian et al. in 2018, [41] uses Runge-Kutta method of 4th order for the solution of fuzzified differential equation. Similarly, Sahni et al. [42] solved fuzzy differential equation of second order using trapezoidal Intuitionistic fuzzy numbers.

In 1993, Watugala [43] introduced a new transform, known as Sumudu transform, which is now being used as a tool for solving fuzzy differential equations. This transform have two important properties, viz; scale property and unit-preserving property, so that it cannot restore the new frequency domain and solves the differential equations. After that many fuzzified differential equations have been solved using this transform (see for instance [44]-[48]). It is needed to contribute more and more work for the development of the theory of Neutrosophic differential equations as it covers more real life situations. In this paper, we have attempted to solve ordinary differential equation of second order based on the spring mass system in a neutrosophic environment using Sumudu transform method. The solution is calculated at various levels of cut – points and time values. The results are shown graphically and further compared with the solution obtained by considering the crisp set values.

## 2. Preliminaries

**Definition 2.1(Fuzzy set)[19].** Let  $X$  be any set which is non-empty. A fuzzy set  $M$  over the elements of the set  $X$  is defined as  $= \{(x, \mu_M(x)) | x \in M, \mu_M(x) \in [0,1]\}$ , where the symbol  $\mu_M(x)$  denotes the grade of membership of the element  $x \in M$ .

**Definition 2.2(Support)[19].** Let  $X$  is any Universal set. The crisp set formed from the set of all points in  $X$  having grade of membership which is not zero is called as the support of the fuzzy.

**Definition 2.3(Core)[19].** The core related to fuzzy set  $M$  is defined as the set of all points of the Universal set, whose grade of membership is 1.

**Definition: 2.4 (Convex set)[19].** If  $X \in R$ , a fuzzy set  $M$  is convex, if for  $\mu_M(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_M(x_1), \mu_M(x_2))$ , where  $\lambda \in [0,1]$ .

**Definition 2.5(Fuzzy number)[19].** A fuzzy number is a generalization of the crisp number in which there is collection of possible values and not any single value and are connected to each other, where each of the possible value has its own weight between 0 and 1.

**Definition 2.6(Alpha-cut)[19].** The  $\alpha$ -level cut of the fuzzy set  $\widetilde{M}$  of  $X$  is a crisp set  $M_\alpha$  that contains all the elements of  $X$  that have membership values in  $M$  greater than or equal to  $\alpha$ , i.e.,  $M_\alpha = \{x: \mu_{\widetilde{M}}(x) \geq \alpha, x \in X, \alpha \in [0,1]\}$ .

**Definition 2.7(Triangular fuzzy number)[42].** A triangular fuzzy number  $A$  is a subset of fuzzy number in  $R$  with the following function defined as:

$$\mu_A(x) = \begin{cases} 0 & \text{for } x \leq p \\ \frac{x-p}{q-p} & \text{for } p \leq x \leq q \\ \frac{r-x}{r-q} & \text{for } q \leq x \leq r \\ 0 & \text{for } r \leq x \end{cases}$$

where  $p \leq q \leq r$  and a triangular fuzzy number is denoted by  $A_{Tr}((p, q, r))$ .

**Definition 2.8(Intuitionistic fuzzy number-IFS)[2].** Let  $U$  be a non empty Universal set. An intuitionistic fuzzy set is represented by  $M = \{(x, \mu_M(x), \omega_M(x)) | x \in U\}$ , where value  $\mu_M(x)$  denotes the membership value of  $x$  in  $M$ , and value  $\omega_M(x)$  denotes the non-membership value of  $x$  in  $M$ .

**Definition 2.9( $\alpha, \beta$ )-cut[2].** The  $\alpha, \beta$ -level set of the fuzzy set  $\widetilde{M}$  of  $X$  is a crisp set and  $M_{\alpha, \beta}$  contains all the elements of  $X$  that have membership values in  $M$  greater than or equal to  $\alpha$  and non-membership values in  $M$  greater than or equal to  $\beta$ , i.e.,  $M_{\alpha, \beta} = \{x: \mu_{\widetilde{M}}(x) \geq \alpha, x \in X, \alpha \in [0,1], x: \omega_{\widetilde{M}}(x) \geq \beta, x \in X, \beta \in [0,1]\}$

**Definition 2.10 (Neutrosophic Set)[36].** Let  $U$  be a universal set. A neutrosophic set  $M$  on  $U$  is defined as  $M = \{T_M(x), I_M(x), F_M(x): x \in U\}$ , where  $T_M(x), I_M(x), F_M(x): U \rightarrow ]-0, 1+[$  represents the grade of membership values, grade of indeterminacy value, and grade of non-membership value respectively of the element  $x \in U$ , such that  $-0 \leq T_M(x) + I_M(x) + F_M(x) \leq 3^+$ .

**Definition 2.11 (Single-Valued Neutrosophic Set (SVNS)) [36].** Let  $U$  be a universal set. Let  $M$  be any SVNS defined on the elements of  $U$ , then  $M = \{T_M(x), I_M(x), F_M(x): x \in U\}$ , where  $T_M(x), I_M(x), F_M(x) : U \rightarrow [0,1]$  represents the grade of membership, indeterminacy, and non-membership, respectively of the element  $x \in U$ .

**Definition 2.12 ( $\alpha, \beta, \gamma$ )-cut[36].** The  $(\alpha, \beta, \gamma)$ -cut of neutrosophic set is denoted by  $F(\alpha, \beta, \gamma)$ , where  $\alpha, \beta, \gamma \in [0,1]$  and are fixed numbers, such that  $\alpha + \beta + \gamma \leq 3$  and is defined as  $F(\alpha, \beta, \gamma) = \{ T_M(x), I_M(x), F_M(x): x \in U, T_M(x) \geq \alpha, I_M(x) \leq \beta, F_M(x) \leq \gamma \}$ .

**Definition 2.13 (Neutrosophic Number)[36].** A neutrosophic set  $M$  defined over the universal set of real numbers  $R$  is said to be neutrosophic number if it has the following properties:

- 1)  $M$  is normal: if there exist  $x_0 \in R$ , such that  $T_M(x_0)=1$  ( $I_M(x_0) = F_M(x_0)=0$ )
- 2)  $M$  is convex set for the truth function  $T_M(x)$ , i.e.,  $T_M(\mu x_1+(1-\mu)x_2) \geq \min(T_M(x_1), T_M(x_2))$ ,  $\forall x_1, x_2 \in R$ ,  $\mu \in [0, 1]$ .
- 3)  $M$  is concave set for the indeterminacy function and false function  $I_M(x)$  and  $F_M(x)$ , i.e.,  
 $I_M(\mu x_1+(1-\mu)x_2) \geq \max(I_M(x_1), I_M(x_2))$ ,  $\forall x_1, x_2 \in R$ ,  $\mu \in [0, 1]$ ,  
 $F_M(\mu x_1+(1-\mu)x_2) \geq \max(F_M(x_1), F_M(x_2))$ ,  $\forall x_1, x_2 \in R$ ,  $\mu \in [0, 1]$ .

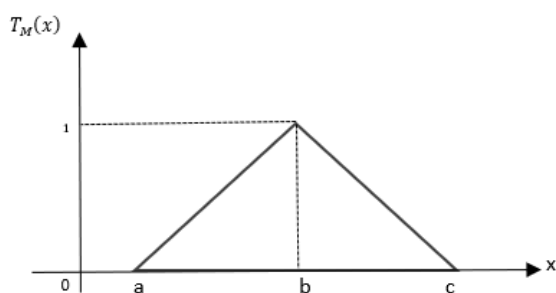
**Definition 2.14 (Triangular Neutrosophic Number)[36].** A neutrosophic number in  $R$  is a superset of the triangular neutrosophic number  $M$ , having truth  $T_M(x)$ , indeterminacy  $I_M(x)$  and false  $F_M(x)$  membership function defined as

$$T_M(x) = \begin{cases} \left(\frac{x-a}{b-a}\right) u_m, & \text{for } a \leq x \leq b \\ u_m, & \text{for } x = b \\ \left(\frac{c-x}{c-b}\right) u_m, & \text{for } b \leq x \leq c \\ 0 & \text{otherwise} \end{cases}$$

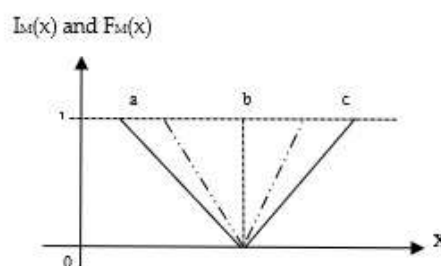
$$I_M(x) = \begin{cases} \left(\frac{b-x}{b-a}\right) v_m, & \text{for } a \leq x \leq b \\ v_m, & \text{for } x = b \\ \left(\frac{x-c}{c-b}\right) v_m, & \text{for } b \leq x \leq c \\ 1 & \text{otherwise} \end{cases}$$

$$F_M(x) = \begin{cases} \left(\frac{b-x}{b-a}\right) w_m, & \text{for } a \leq x \leq b \\ w_m, & \text{for } x = b \\ \left(\frac{x-c}{c-b}\right) w_m, & \text{for } b \leq x \leq c \\ 1 & \text{otherwise} \end{cases}$$

where  $a \leq b \leq c$ . A neutrosophic number in a triangular form is denoted by  $M_{TN}((a, b, c); u_m, v_m, w_m)$ . Here the truth membership function, i.e.  $T_M(x)$  increases in a linear way for  $x \in [a, b]$  and decreases in a linear form for  $x \in [b, c]$ . The inverse behaviour is seen for  $I_M(x)$  and  $F_M(x)$  for  $x \in [a, b]$  and for  $x \in [b, c]$ , where  $\alpha \in [0, u_m]$ ,  $0 < u_m < 1$ ,  $\beta \in [0, v_m]$ ,  $0 < v_m < 1$ ,  $\gamma \in [0, w_m]$ ,  $0 < w_m < 1$ .



(a)  $T_M(x)$  as triangular form



(b)  $I_M(x)$  and  $F_M(x)$  as triangular form

**Definition 2.15**( $\alpha, \beta, \gamma$ )-cut of a Triangular Neutrosophic Number [36].

The  $(\alpha, \beta, \gamma) -$  cut of a triangular neutrosophic number  $M_{TN}(a, b, c); u_m, v_m, w_m$  is defined as follows:

$M_{(\alpha, \beta, \gamma)} = [M_1(\alpha), M_2(\alpha)]; [M_3(\beta), M_4(\beta)]; [M_5(\gamma), M_6(\gamma)]$ ,  $0 \leq \alpha + \beta + \gamma \leq 3$ , where,

$$M_{(\alpha, \beta, \gamma)} = [(a + \alpha(b - a))u_M, (c - \alpha(c - b))u_M],$$

$$[(b - \beta(b - a))v_M, (b + \beta(c - b))v_M],$$

$$[(b - \gamma(b - a))w_M, (b + \gamma(c - b))w_M].$$

**Definition 2.16**(Differentiability) [36]. For a fuzzy valued function  $f : (a, b) \rightarrow R$  at the point  $x_0$ ,

the differentiability is defined as follows:  $g'(x_0) = \lim_{h \rightarrow 0} \frac{g(x_0+h) - g(x_0)}{h}$  and  $g'(x_0)$  is D<sub>1</sub>-differentiable at  $x_0$ . If  $[g(x_0)]_\alpha = [g_1(x_0, \alpha), g_2(x_0, \alpha)]$  and  $g'(x_0)$  is D<sub>2</sub>-differentiable at  $x_0$  and if  $[g(x_0)]_\alpha = [g_2(x_0, \alpha), g_1(x_0, \alpha)]$  for all  $\alpha \in [0, 1]$ .

**Definition 2.17** (Generalized differentiability)[36]. The second-order derivative of a fuzzy value

function  $g : (a, b) \rightarrow R$  at  $x_0$  is defined as follows:  $g''(x_0) = \lim_{h \rightarrow 0} \frac{g'(x_0+h) - g'(x_0)}{h}$  and  $g'(x_0)$  is

D<sub>1</sub>-differentiable at  $x_0$  if

$$g''(x_0, \alpha) = \begin{cases} (g'_1(x_0, \alpha), g'_2(x_0, \alpha)) & \text{if } g \text{ is } D_1 - \text{differentiable on } (a, b) \\ (g'_2(x_0, \alpha), g'_1(x_0, \alpha)) & \text{if } g \text{ is } D_2 - \text{differentiable on } (a, b) \end{cases}$$

for all  $\alpha \in [0, 1]$  and  $g'(x_0)$  is D<sub>2</sub>-differentiable at  $x_0$  if

$$g''(x_0, \alpha) = \begin{cases} (g'_2(x_0, \alpha), g'_1(x_0, \alpha)) & \text{if } g \text{ is } D_1 - \text{differentiable on } (a, b) \\ (g'_1(x_0, \alpha), g'_2(x_0, \alpha)) & \text{if } g \text{ is } D_2 - \text{differentiable on } (a, b) \end{cases}$$

for all  $\alpha \in [0, 1]$ .

**3. Neutrosophic Sumudu Transform**[NST]

Let  $f(u)$  is a neutrosophic valued function which is continuous. Suppose that  $g(u)e^{-t}$  be improper neutrosophic Riemann integrable on  $[0, \infty)$  then  $\int_0^\infty g(u)e^{-t} dt$  is called neutrosophic Sumudu transform and it is defined as,  $G(u) = S[g(t)] = \int_0^\infty g(u)e^{-t} dt$ , ( $u \in [-\tau, \tau]$ )

where variable  $u$  is used to factor the variable  $t$  in the argument of the neutrosophic valued function.

We have,

$g(t, r) = \{g_T(t, r), g_I(t, r), g_F(t, r)\}$ , which are denoted in neutrosophic triangular form as

$$g_T(t, r) = \{\underline{g}_T(t, r), \overline{g}_T(t, r)\} g_I(t, r) = \{\underline{g}_I(t, r), \overline{g}_I(t, r)\} g_F(t, r) = \{\underline{g}_F(t, r), \overline{g}_F(t, r)\}$$

$$\int_0^\infty g_T(u)e^{-t} dt = \left( \int_0^\infty \underline{g}_T(u)e^{-t} dt, \int_0^\infty \overline{g}_T(u)e^{-t} dt, \right)$$

$$\int_0^\infty g_I(u)e^{-t} dt = \left( \int_0^\infty \underline{g}_I(u)e^{-t} dt, \int_0^\infty \overline{g}_I(u)e^{-t} dt, \right)$$

$$\int_0^\infty g_F(ut)e^{-t} dt = \left( \int_0^\infty \underline{g}_F(ut)e^{-t} dt, \int_0^\infty \overline{g}_F(ut)e^{-t} dt \right)$$

also using definition of classical Sumudu transform.

$$s[\underline{g}_T(t, r)] = \int_0^\infty \underline{g}_T(ut)e^{-t} dt$$

$$s[\overline{g}_T(t, r)] = \int_0^\infty \overline{g}_T(ut)e^{-t} dt,$$

$$s[\underline{g}_I(t, r)] = \int_0^\infty \underline{g}_I(ut)e^{-t} dt$$

$$s[\overline{g}_I(t, r)] = \int_0^\infty \overline{g}_I(ut)e^{-t} dt,$$

$$s[\underline{g}_F(t, r)] = \int_0^\infty \underline{g}_F(ut)e^{-t} dt$$

$$s[\overline{g}_F(t, r)] = \int_0^\infty \overline{g}_F(ut)e^{-t} dt,$$

then it follows  $S[g(t)] = (s[\underline{g}(t, r), s[\overline{g}(t, r)])]$ .

### 3.1 Some basic results on fuzzy differential equation using Sumudu transform

The following theorems are useful in our results:-

**Theorem 3.1.1[44].** Let  $g'(t)$  be a continuous neutrosophic valued function and  $g(t)$  is the primitive of  $g'(t)$  on  $[0, \infty)$  then,

$$S[g'(t)] = \frac{S[g(t)]}{u} -_h \frac{g(0)}{u}, \text{ where } g \text{ is (a) differentiable or,}$$

$$S[g'(t)] = \frac{(-g(0))}{u} -_h \frac{(-S[g(t)])}{u}, \text{ where } g \text{ is (b) differentiable.}$$

where “ $-_h$ ” is notation of gh-differentiability.

**Theorem 3.1.2[44].** Let  $g(t), g'(t)$  be an continuous neutrosophic valued function on  $[0, \infty)$  and that  $g''(t)$  be piece wise continuous neutrosophic valued function on  $[0, \infty)$  then,

$$S[g''(t)] = \frac{S[g(t)]}{u^2} -_h \frac{g(0)}{u} -_h \frac{g'(0)}{u}, \text{ where } g \text{ is (a) differentiable and } g' \text{ is (a) differentiable or}$$

$$S[g''(t)] = -\frac{g(0)}{u^2} -_h \left[ -\frac{S[g(t)]}{u^2} \right] -_h \frac{g'(0)}{u}, \text{ where } g \text{ is (a) differentiable and } g' \text{ is (b) differentiable or}$$

$$S[g''(t)] = -\frac{g(0)}{u^2} -_h \left[ -\frac{S[g(t)]}{u^2} \right] -_h \frac{g'(0)}{u}, \text{ where } g \text{ is (b) differentiable and } g' \text{ is (a) differentiable or}$$

$$S[g''(t)] = \frac{S[g(t)]}{u^2} -_h \frac{g(0)}{u^2} -_h \frac{g'(0)}{u}, \text{ where } g \text{ is (b) differentiable and } g' \text{ is (b) differentiable.}$$

where “ $-_h$ ” is notation of gh-differentiability.

**Theorem 3.1.3.** Let  $g : R \rightarrow G(R)$  be a continuous neutrosophic valued function and denote

$$g_T(x) = [\underline{g}_{T\alpha}(x), \overline{g}_{T\alpha}(x)] \text{ for each } \alpha \in [0, 1],$$

$$g_I(x) = [\underline{g}_{I\beta}(x), \bar{g}_{I\beta}(x)] \text{ for each } \beta \in [0,1],$$

$$g_F(x) = [\underline{g}_{F\gamma}(x), \bar{g}_{F\gamma}(x)] \text{ for each } \gamma \in [0,1]. \text{ Then:}$$

(1) If  $g_T$  is (1)-differentiable, then  $\underline{g}_{T\alpha}$  and  $\bar{g}_{T\alpha}$  are differentiable function and  $f'(x) = [\underline{f}_{T\alpha}(x), \bar{f}_{T\alpha}(x)]$

(2) If  $g_T$  is (2)-differentiable, then  $\underline{g}_{T\alpha}$  and  $\bar{g}_{T\alpha}$  are differentiable function and  $f'(x) = [\underline{f}_{T\alpha}(x), \bar{f}_{T\alpha}(x)]$

(3) If  $g_I$  is (1)-differentiable, then  $\underline{g}_{I\beta}$  and  $\bar{g}_{I\beta}$  are differentiable function and  $f'(x) = [\underline{f}_{I\beta}(x), \bar{f}_{I\beta}(x)]$

(4) If  $g_I$  is (2)-differentiable, then  $\underline{g}_{I\beta}$  and  $\bar{g}_{I\beta}$  are differentiable function and  $f'(x) = [\underline{f}_{I\beta}(x), \bar{f}_{I\beta}(x)]$

(5) If  $g_F$  is (1)-differentiable, then  $\underline{g}_{F\gamma}$  and  $\bar{g}_{F\gamma}$  are differentiable function and  $f'(x) = [\underline{f}_{F\gamma}(x), \bar{f}_{F\gamma}(x)]$

(6) If  $g_F$  is (2)-differentiable, then  $\underline{g}_{F\gamma}$  and  $\bar{g}_{F\gamma}$  are differentiable function and  $f'(x) = [\underline{f}_{F\gamma}(x), \bar{f}_{F\gamma}(x)]$

### 3.2 Solution of General Second Order Ordinary Differential Equation in a Neutrosophic Environment using Sumudu Transform

Let us consider a general ordinary differential equation of second order given as follows:

$$y''(t) = f(t, y(t), y'(t)) \tag{1}$$

with the initial conditions  $y(t_0) = y_0, y'(t_0) = z_0$ , where  $f : [t_0, P] \times R \rightarrow R$ .

Suppose that initial values  $y_0$  and  $z_0$  are uncertain and are defined in terms of lower and upper bound of truth, indeterminacy and falsity, i.e. neutrosophic number.

Thus from equation no. 1, we have the following fuzzy initial value differential equation:

$$y''(t) = g(t, y(t), y'(t)), 0 \leq t \leq P$$

$$y_T(t_0) = y_0 = [\underline{y}_{T\alpha}(0), \bar{y}_{T\alpha}(0)], 0 < \alpha \leq 1, y'_T(t_0) = z_0 = [\underline{z}_{T\alpha}(0), \bar{z}_{T\alpha}(0)], 0 < \alpha \leq 1, \tag{2}$$

$$y_I(t_0) = y_0 = [\underline{y}_{I\beta}(0), \bar{y}_{I\beta}(0)], 0 < \beta \leq 1, y'_I(t_0) = z_0 = [\underline{z}_{I\beta}(0), \bar{z}_{I\beta}(0)], 0 < \beta \leq 1, \tag{3}$$



$$y_F(t_0) = y_0 = [\underline{y}_{F\gamma}(0), \bar{y}_{F\gamma}(0)], 0 < \gamma \leq 1, y'_F(t_0) = z_0 = [\underline{z}_{F\gamma}(0), \bar{z}_{F\gamma}(0)], 0 < \gamma \leq 1, \tag{4}$$

where  $g: [t_0, P] \times F(R) \rightarrow F(R)$  is a function of continuous manner.

By applying Neutrosophic Sumudu Transform on given second order differential equation, we have

$$S[y''(t)] = s[g(t, y(t), y'(t))]$$

**Case 1:** Let  $y''(t)$  is (1)-differentiable, and using the above theorem we have  $y''(t) = [\underline{y}'_{\alpha}(t), \bar{y}'_{\alpha}(t)]$ .

The differential equation is then reduced to the following:

$$\begin{aligned} \underline{y}'_{T\alpha}(t) &= \underline{g}_{T\alpha}(t, y(t), y'(t)), \underline{y}_{T\alpha}(t_0) = \underline{y}_{T\alpha}(0) \\ \bar{y}'_{T\alpha}(t) &= \bar{g}_{T\alpha}(t, y(t), y'(t)), \bar{y}_{T\alpha}(t_0) = \bar{y}_{T\alpha}(0) \\ \underline{y}'_{I\beta}(t) &= \underline{g}_{I\beta}(t, y(t), y'(t)), \underline{y}_{I\beta}(t_0) = \underline{y}_{I\beta}(0) \\ \bar{y}'_{I\beta}(t) &= \bar{g}_{I\beta}(t, y(t), y'(t)), \bar{y}_{I\beta}(t_0) = \bar{y}_{I\beta}(0) \\ \underline{y}'_{F\gamma}(t) &= \underline{g}_{F\gamma}(t, y(t), y'(t)), \underline{y}_{F\gamma}(t_0) = \underline{y}_{F\gamma}(0) \\ \bar{y}'_{F\gamma}(t) &= \bar{g}_{F\gamma}(t, y(t), y'(t)), \bar{y}_{F\gamma}(t_0) = \bar{y}_{F\gamma}(0) \end{aligned}$$

Using the Sumudu transform for solving, we get

$$s[y''(t)] = \frac{s[y(t)] - {}_h y(t_0) - {}_h u y'(t_0)}{u^2}.$$

The following six first order ordinary differential equations are developed, two for both Truth,

Indeterminacy and falsity and is defined as

$$\begin{aligned} S[\underline{g}_{T\alpha}(t, y(t), y'(t))] &= \frac{s[\underline{y}_{T\alpha}(t)] - {}_h \underline{y}_{T\alpha}(0) - {}_h u \underline{y}'_{T\alpha}(0)}{u^2} \\ S[\bar{g}_{T\alpha}(t, y(t), y'(t))] &= \frac{s[\bar{y}_{T\alpha}(t)] - {}_h \bar{y}_{T\alpha}(0) - {}_h u \bar{y}'_{T\alpha}(0)}{u^2} \\ S[\underline{g}_{I\beta}(t, y(t), y'(t))] &= \frac{s[\underline{y}_{I\beta}(t)] - {}_h \underline{y}_{I\beta}(0) - {}_h u \underline{y}'_{I\beta}(0)}{u^2} \\ S[\bar{g}_{I\beta}(t, y(t), y'(t))] &= \frac{s[\bar{y}_{I\beta}(t)] - {}_h \bar{y}_{I\beta}(0) - {}_h u \bar{y}'_{I\beta}(0)}{u^2} \\ S[\underline{g}_{F\gamma}(t, y(t), y'(t))] &= \frac{s[\underline{y}_{F\gamma}(t)] - {}_h \underline{y}_{F\gamma}(0) - {}_h u \underline{y}'_{F\gamma}(0)}{u^2} \\ S[\bar{g}_{F\gamma}(t, y(t), y'(t))] &= \frac{s[\bar{y}_{F\gamma}(t)] - {}_h \bar{y}_{F\gamma}(0) - {}_h u \bar{y}'_{F\gamma}(0)}{u^2} \end{aligned}$$

To solve this, we assume that

$$S[\underline{y}_{T\alpha}(t)] = L_{T\alpha}^1(u), S[\overline{y}_{T\alpha}(t)] = U_{T\alpha}^1(u),$$

$$S[\underline{y}_{I\beta}(t)] = L_{I\beta}^1(u), S[\overline{y}_{I\beta}(t)] = U_{I\beta}^1(u),$$

$$S[\underline{y}_{F\gamma}(t)] = L_{F\gamma}^1(u), S[\overline{y}_{F\gamma}(t)] = U_{F\gamma}^1(u)$$

where,  $L_{T\alpha}^1(u), U_{T\alpha}^1(u), L_{I\beta}^1(u), U_{I\beta}^1(u), L_{F\gamma}^1(u), U_{F\gamma}^1(u)$  are solution of differential equations.

By using inverse neutrosophic Sumudu transform, we have

$$\underline{y}_{T\alpha}(t), \overline{y}_{T\alpha}(t), \underline{y}_{I\beta}(t), \overline{y}_{I\beta}(t), \underline{y}_{F\gamma}(t), \overline{y}_{F\gamma}(t)$$

and it follows that,

$$\underline{y}_{T\alpha}(t) = S^{-1}[L_{T\alpha}^1(u)], \overline{y}_{T\alpha}(t) = S^{-1}[U_{T\alpha}^1(u)]$$

$$\underline{y}_{I\beta}(t) = S^{-1}[L_{I\beta}^1(u)], \overline{y}_{I\beta}(t) = S^{-1}[U_{I\beta}^1(u)]$$

$$\underline{y}_{F\gamma}(t) = S^{-1}[L_{F\gamma}^1(u)], \overline{y}_{F\gamma}(t) = S^{-1}[U_{F\gamma}^1(u)]$$

**Case 2:** Let  $y''(t)$  be (2)-differentiable, then from above theorem we have  $y''(t) = [\underline{y}'_{-\alpha}(t), \overline{y}'_{\alpha}(t)]$ .

Reducing the second order ordinary differential equations into first order ordinary differential equation, we have the following differential equations to be solved,

$$\underline{y}'_{-T\alpha}(t) = \underline{g}_{T\alpha}(t, y(t), y'(t)), \underline{y}_{-T\alpha}(t_0) = \underline{y}_{-T\alpha}(0)$$

$$\overline{y}'_{T\alpha}(t) = \overline{g}_{T\alpha}(t, y(t), y'(t)), \overline{y}_{T\alpha}(t_0) = \overline{y}_{T\alpha}(0)$$

$$\underline{y}'_{-I\beta}(t) = \underline{g}_{I\beta}(t, y(t), y'(t)), \underline{y}_{-I\beta}(t_0) = \underline{y}_{-I\beta}(0)$$

$$\overline{y}'_{I\beta}(t) = \overline{g}_{I\beta}(t, y(t), y'(t)), \overline{y}_{I\beta}(t_0) = \overline{y}_{I\beta}(0)$$

$$\underline{y}'_{-F\gamma}(t) = \underline{g}_{F\gamma}(t, y(t), y'(t)), \underline{y}_{-F\gamma}(t_0) = \underline{y}_{-F\gamma}(0)$$

$$\overline{y}'_{F\gamma}(t) = \overline{g}_{F\gamma}(t, y(t), y'(t)), \overline{y}_{F\gamma}(t_0) = \overline{y}_{F\gamma}(0)$$

Using  $[y''(t)] = \frac{(-y(t_0))_{-h}(-s[y(t)]_{-h}uy'(t_0))}{u^2}$ ,

we get the following six first order ordinary differential equations as,

$$S[\underline{g}_{T\alpha}(t, y(t), y'(t))] = \frac{(-\underline{y}_{-T\alpha}(0))_{-h}(-s[\underline{y}_{-T\alpha}(t)]_{-h}u\underline{y}'_{-T\alpha}(0))}{u^2}$$

$$S[\overline{g}_{T\alpha}(t, y(t), y'(t))] = \frac{(-\overline{y}_{T\alpha}(0))_{-h}(-s[\overline{y}_{T\alpha}(t)]_{-h}u\overline{y}'_{T\alpha}(0))}{u^2}$$

$$S[\underline{g}_{I\beta}(t, y(t), y'(t))] = \frac{(-\underline{y}_{-I\beta}(0))_{-h}(-s[\underline{y}_{-I\beta}(t)]_{-h}u\underline{y}'_{-I\beta}(0))}{u^2}$$

$$\begin{aligned}
 S \left[ \underline{g}_{I\beta}(t, y(t), y'(t)) \right] &= \frac{\left( -\underline{y}_{I\beta}(0) \right)_{-h} \left( -s \left[ \underline{y}_{I\beta}(t) \right] \right)_{-h} u \underline{y}'_{I\beta}(0)}{u^2} \\
 S \left[ \underline{g}_{F\gamma}(t, y(t), y'(t)) \right] &= \frac{\left( -\underline{y}_{F\gamma}(0) \right)_{-h} \left( -s \left[ \underline{y}_{F\gamma}(t) \right] \right)_{-h} u \underline{y}'_{F\gamma}(0)}{u^2} \\
 S \left[ \overline{g}_{F\gamma}(t, y(t), y'(t)) \right] &= \frac{\left( -\overline{y}_{F\gamma}(0) \right)_{-h} \left( -s \left[ \overline{y}_{F\gamma}(t) \right] \right)_{-h} u \overline{y}'_{F\gamma}(0)}{u^2}
 \end{aligned}$$

To solve this, we assume that

$$\begin{aligned}
 S \left[ \underline{y}_{T\alpha}(t) \right] &= L_{T\alpha}^1(u), & S \left[ \overline{y}_{T\alpha}(t) \right] &= U_{T\alpha}^1(u), \\
 S \left[ \underline{y}_{I\beta}(t) \right] &= L_{I\beta}^1(u), & S \left[ \overline{y}_{I\beta}(t) \right] &= U_{I\beta}^1(u), \\
 S \left[ \underline{y}_{F\gamma}(t) \right] &= L_{F\gamma}^1(u), & S \left[ \overline{y}_{F\gamma}(t) \right] &= U_{F\gamma}^1(u)
 \end{aligned}$$

where,  $L_{T\alpha}^1(u), U_{T\alpha}^1(u), L_{I\beta}^1(u), U_{I\beta}^1(u), L_{F\gamma}^1(u), U_{F\gamma}^1(u)$  are the Sumudu transform for Truth, Indeterminacy and falsity solution of equations.

By using inverse neutrosophic Sumudu transform, we have  $\underline{y}_{T\alpha}(t), \overline{y}_{T\alpha}(t), \underline{y}_{I\beta}(t), \overline{y}_{I\beta}(t), \underline{y}_{F\gamma}(t), \overline{y}_{F\gamma}(t)$  and it follows:

$$\begin{aligned}
 \underline{y}_{T\alpha}(t) &= S^{-1} \left[ L_{T\alpha}^1(u) \right], \overline{y}_{T\alpha}(t) = S^{-1} \left[ U_{T\alpha}^1(u) \right] \\
 \underline{y}_{I\beta}(t) &= S^{-1} \left[ L_{I\beta}^1(u) \right], \overline{y}_{I\beta}(t) = S^{-1} \left[ U_{I\beta}^1(u) \right] \\
 \underline{y}_{F\gamma}(t) &= S^{-1} \left[ L_{F\gamma}^1(u) \right], \overline{y}_{F\gamma}(t) = S^{-1} \left[ U_{F\gamma}^1(u) \right]
 \end{aligned}$$

### 3.3 Crisp and Fuzzy Solution for Ordinary differential equations of spring mass system

Consider an elastic string vertically tied to a rigid support. The object attached on the other side of an elastic string pulls the string downwards to a distance  $s$  from its usual length  $b$ . The place at which the string extended from its usual position is called as equilibrium position. Now, as the mass of the body pulls the spring downwards, the restoring force of the spring acts as a restraint and opposes the stretching of the spring. Mathematically, it can be written as  $F = ks$  where  $k > 0$  is the proportionality constant or spring constant,  $F$  is the restoring force and the  $s$  is the extension in the spring from its equilibrium position. Let ' $m$ ' be the mass of the object and  $y(t)$  represents the displacement, then according to Newton second law of motion, we have  $= m \frac{d^2y}{dt^2}$ . Suppose no forces are acting on the system, except the force of gravity, then the differential modeling such type of system can be written as

$$\begin{aligned}
 m \frac{d^2y}{dt^2} &= \sum (\text{force acting on the system}) \\
 &= -ks - ky + mg
 \end{aligned}$$

At equilibrium  $ks = mg$ , so after calculation we get differential equation as,

$$m \frac{d^2y}{dt^2} + ky = 0$$

To solve such second order ordinary differential equations, we require two initial conditions  $y(0) = p$  and  $y'(0) = q$ . Thus, the problem of finding the displacement  $y(t)$  is reduced to solving the differential equations of the form,

$$m \frac{d^2y}{dt^2} + ky = 0, \quad y(0) = p, \quad y'(0) = q.$$

If we consider the damping force  $\left(n \frac{dy}{dt}\right)$  which is a function of velocity of the motion that helps to reduce the vibrations, then the above differential equation reduces to

$$m \frac{d^2y}{dt^2} + n \frac{dy}{dt} + ky = 0, \quad y(0) = p, \quad y'(0) = q.$$

Solving the above initial value problem provides the displacement  $y(t)$  in terms of constants  $m, k$  and  $n$ . The above differential equation can be solved by the method of substitution, which provides us the characteristic equation as

$$mr^2 + nr + k = 0,$$

with solutions of the auxiliary equation as  $r_{1,2} = \frac{1}{2m}(-n \pm \sqrt{n^2 - 4mk})$ .

The quantity inside the square root, i.e.  $n^2 - 4mk$  classifies the solution into three cases:

**Case 1.** If  $n^2 - 4mk > 0$ , it is an over damped situation, as the proportionality constant  $k$  is very small as compared to damping coefficient  $n$ .

**Case 2.** If  $n^2 - 4mk = 0$ , the situation is critically damped and the resulting motion is oscillatory and the damping coefficient  $n$  slightly decreases.

**Case 3.** If  $n^2 - 4mk < 0$ , the behaviour of the motion is under damped and the value of spring constant is very large as compared to damping coefficient  $n$ .

#### 4. Application

In this section a problem of spring mass system is considered. The differential equation formed for this system is solved in Neutrosophic environment and then compared it with crisp solution. It is shown here that, Neutrosophic environment includes differential equations with initial conditions containing parameters of belongingness, non-belongingness and indeterminacy, so that it provide more precise solution than crisp environment.

**Problem Statement:** A body of mass  $8lb$  is tied to a spring of length  $4ft$ . At equilibrium position, the length of the spring has  $6ft$ . Let the damping force is defined as  $F_r = 2dy/dt$  and the body is released from the equilibrium position with a down ward initial velocity of  $1ft/s$ , find the displacement  $y(t)$  for any time  $t$  analytically and using fuzzy Sumudu transform.

**Solution.**

The differential equation of this spring-mass system is  $my'' + ny' + ky = 0$ , where the mass of the body is  $m = 8/32 = 1/4 \text{ slug}$ , the spring constant with  $8 = k \times 2$ , so  $k = 4lb/ft$ . The resistive force  $F_R = n \, dy/dt = 2dy/dt$  with the initial conditions  $y(0) = 0, y'(0) = 1$ .

After putting these values, we get  $y''(t) + 8y'(t) + 16y(t) = 0$

**Crisp solution**

The analytical solutions is given by  $y(t) = C_1e^{-4t} + tC_2e^{-4t}$ . Using initial value  $y(0) = 0, y'(0) = 1$ , in the solution, we get  $y(t) = te^{-4t}$ . At  $t = 0.1$ , we get  $y(0.1) = 0.067$ .

**Neutrosophic solution**

Consider the Neutrosophic initial value problem,  $y''(t) + 8y'(t) + 16y(t) = 0$

$$\begin{aligned} y_T(0) &= [\alpha - 1, 1 - \alpha], & y'_T(0) &= [\alpha, 1 - \alpha] \\ y_I(0) &= [-0.5\beta, 0.5\beta], & y'_I(0) &= [1 - 0.5\beta, 1 + 0.5\beta] \\ y_F(0) &= [-0.2\gamma, 0.2\gamma], & y'_F(0) &= [1 - 0.2\gamma, 1 + 0.2\gamma] \end{aligned}$$

Using Neutrosophic Sumudu Transform, we get the expression in Truth, Indeterminacy and Falsity as,

$$\begin{aligned} \underline{y}_T(u, \alpha) &= \left( \frac{(\alpha - 1)(1 + 8u)}{(1 + 4u)^2} + \frac{\alpha u}{(1 + 4u)^2} \right) \\ \bar{y}_T(u, \alpha) &= \left( \frac{(1 - \alpha)(1 + 8u)}{(1 + 4u)^2} + \frac{(2 - \alpha)u}{(1 + 4u)^2} \right) \\ \underline{y}_I(u, \beta) &= \left( \frac{-0.5\beta}{(1 + 4u)^2} - \frac{4\beta u}{(1 + 4u)^2} + \frac{(1 - 0.5\beta)u}{(1 + 4u)^2} \right) \\ \bar{y}_I(u, \beta) &= \left( \frac{0.5\beta}{(1 + 4u)^2} + \frac{4\beta u}{(1 + 4u)^2} + \frac{(1 + 0.5\beta)u}{(1 + 4u)^2} \right) \\ \underline{y}_F(u, \gamma) &= \left( \frac{-0.2\gamma}{(1 + 4u)^2} - \frac{1.6\gamma u}{(1 + 4u)^2} + \frac{(1 - 0.2\gamma)u}{(1 + 4u)^2} \right) \\ \bar{y}_F(u, \gamma) &= \left( \frac{0.2\gamma}{(1 + 4u)^2} + \frac{1.6\gamma u}{(1 + 4u)^2} + \frac{(1 + 0.2\gamma)u}{(1 + 4u)^2} \right) \end{aligned}$$

Now the solution as per lower and upper bound for truth value, indeterminacy value and false value respectively, are:

$$\begin{aligned} \underline{y}_T(t, \alpha) &= ((\alpha - 1)e^{-4t} + (5\alpha - 4)te^{-4t}) \\ \bar{y}_T(t, \alpha) &= ((1 - \alpha)e^{-4t} + (6 - 5\alpha)te^{-4t}) \\ \underline{y}_I(t, \beta) &= ((-0.5\beta)e^{-4t} + (1 - 2.5\beta)te^{-4t}) \\ \bar{y}_I(t, \beta) &= ((0.5\beta)e^{-4t} + (1 + 2.5\beta)te^{-4t}) \\ \underline{y}_F(t, \gamma) &= ((-0.2\gamma)e^{-4t} + (1 - \gamma)te^{-4t}) \\ \bar{y}_F(t, \gamma) &= ((0.2\gamma)e^{-4t} + (1 + \gamma)te^{-4t}) \end{aligned}$$

*4.1. Numerical Observation and Graphical Representation*

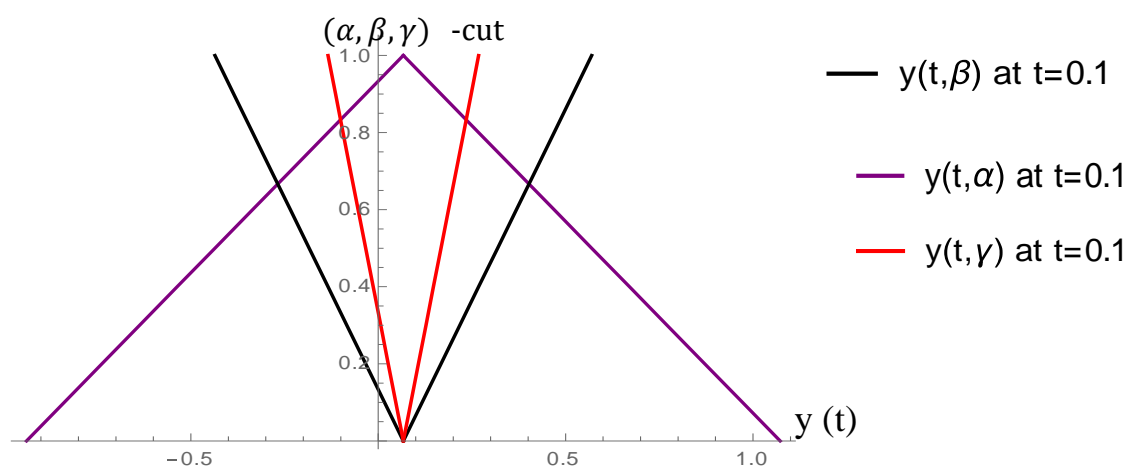
The fuzzy differential equation is solved for different  $(\alpha, \beta, \gamma)$ -cut values. For the solution the step size of 0.1 is considered. It is shown in the table 1 for  $t = 0.1$ . From table 1, it is observed that the for truth membership  $y_T(t, \alpha)$ , lower and upper bound both show an inverse behaviour, i.e one is

increasing and the other is decreasing. The crisp solution matches with the fuzzy solution at  $\alpha$  – cut, with value 1.0. It is observed that for indeterminacy membership  $y_I(t,\beta)$  and for False membership  $y_F(t,\gamma)$ , lower bound is decreasing and the upper bound is increasing at  $\beta, \gamma$  – cut, with value 0, the crisp solution matches with the fuzzy solution.

**Table 1. The solutions for lower and upper bound at  $t = 0.1$  and its comparison with the crisp solution.**

$(\alpha, \beta, \gamma)$ -cut	$\underline{y}_T(t, \alpha)$	$\overline{y}_T(t, \alpha)$	$\underline{y}_I(t, \beta)$	$\overline{y}_I(t, \beta)$	$\underline{y}_F(t, \delta)$	$\overline{y}_F(t, \gamma)$	Exact solution
0	-0.938448	1.072510	0.067032	0.067032	0.0670300	0.0670320	0.067032
0.1	-0.83790	0.971964	0.016758	0.117306	0.0469224	0.0871416	...
0.2	-0.737352	0.871416	0.033516	0.167580	0.0268128	0.1072510	...
0.3	-0.636804	0.770868	-0.083790	0.217850	0.0067032	0.1273610	...
0.4	-0.536256	0.670303	-0.134064	0.268128	-0.0134060	0.1474700	...
0.5	-0.435708	0.569772	-0.184338	0.318402	-0.0335160	0.1675800	...
0.6	-0.335160	0.469224	-0.234612	0.368676	-0.0536256	0.1876902	...
0.7	-0.234612	0.368676	-0.284886	0.418958	-0.0737352	0.2077991	...
0.8	-0.134064	0.268128	-0.335160	0.469224	-0.0938448	0.2279091	...
0.9	-0.033516	0.167580	-0.385434	0.519498	-0.1139540	0.2480180	...
1	0.067032	0.067032	-0.435708	0.569772	-0.1340640	0.2681281	0.067032

In figure 1, the graph is plotted for different  $(\alpha, \beta, \gamma)$  – cut values at  $t = 0.1$ . It is observed that as the  $\alpha$ -cut values are increasing, the solution approaches to the exact solution and as the  $(\beta, \gamma)$ -cut values are decreasing the solution approaches to the exact solution.



**Figure 1. Graph for different values of  $(\alpha, \beta, \gamma)$  -cuts and at time  $t = 0.1$**

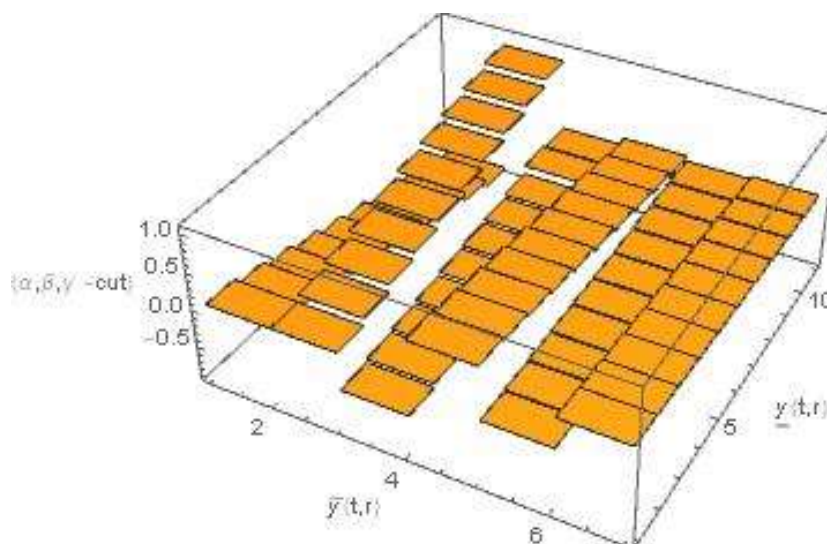


Figure 2. The lower and upper bounds for different values of  $(\alpha, \beta, \gamma)$  - cut for time 0 to 0.5 .

From figure 2, we can observed that, the motion of the spring decreases with the increase in time for truth membership and motion of spring increases with the decreases in time for false and indermancy. We can further study the behaviour of motion of the spring under external force for different  $(\alpha, \beta, \gamma)$  -cut values with varying time under neutrosophic initial values.

### 5. Conclusion and Future works

In this paper, the ordinary differential equation of mechanical spring mass system with neutrosophic initial conditions is solved using Sumudu transform method. The solution of neutrosophic environment obtained is compared with the crisp solution and is more generalized. The results are represented for different  $(\alpha, \beta, \gamma)$  -cut values in table 1. The behavior is also depicted in the form of graphs, for different  $(\alpha, \beta, \gamma)$  -cut values with varying time. This study helps in solving various other ordianry differential equations such as simultaneous differential equation, differential equation with variable coefficients under neutosophic environment. The solution of a differential equation helps in understanding the behaviour of physical systems under an uncertain environment. For non-linear differential equations, our intuition is that it cannot be applied.

### Conflict of interest

The Authors have no conflict of interest.

### Nomenclature and Symbols

$\mu_M(x)$	Fuzzy membership function of set M
$M_\alpha$	The $\alpha$ - level set of the fuzzy set M
U	Universal set
$\mu_M(x)$	Fuzzy membership function of set M
$\omega_M(x)$	Fuzzy Non-membership function of set M

$M_{\alpha,\beta}$	The $\alpha, \beta$ - level set of the fuzzy set M
X	Crisp set
$T_M(x)$	Truth membership function of neutrosophic fuzzy set M
$I_M(x)$	Indeterminacy membership function of neutrosophic fuzzy set M
$F_M(x)$	False membership function of neutrosophic fuzzy set M
$F(\alpha,\beta,\gamma)$ ,	$(\alpha,\beta,\gamma)$ - cut of neutrosophic set
$M(\alpha,\beta,\gamma)$	$(\alpha, \beta, \gamma)$ cut of a triangular neutrosophic number $M_{TN}$
$g(ut)e^{-t}$	improper neutrosophic Riemann integrable
$\underline{g}(t, r)$	Lower bound of fuzzy membership
$\bar{g}(t, r)$	Upper bound of fuzzy membership
$\neg_h$	Generalized-differentiability
$F_R$	Resistive force
m	Mass of a body
k	Spring constant
$\in$	Belongs to
$A_{Tr}\langle(p, q, r)\rangle$	Triangular fuzzy number
$M_{TN}\langle(a, b, c); u_m, v_m, w_m\rangle$ .	Triangular neutrosophic number
$S[g(t)]$	Sumudu transform of function "g"
$L_{T\alpha}(u), U_{T\alpha}(u)$	Lower and upper bound solution of Sumudu transform with respect to $\alpha$ cut for Truth membership function of neutrosophic fuzzy set
$L_{T\beta}(u), U_{T\beta}(u)$	Lower and upper bound solution of Sumudu transform with respect to $\beta$ -cut for Truth membership function of neutrosophic fuzzy set
$L_{T\gamma}(u), U_{T\gamma}(u)$	Lower and upper bound solution of Sumudu transform with respect to $\gamma$ -cut for Truth membership function of neutrosophic fuzzy set

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