NSS

University of New Mexico

Interval neutrosophic sets applied to ideals in BCK/BCI-algebras

Seok-Zun Song[1], Madad Khan[2], Florentin Smarandache[3], Young Bae Jun[4,*]

[1]Department of Mathematics, Jeju National University, Jeju 63243, Korea. E-mail: szsong@jejunu.ac.kr

[2]Department of Mathematics, COMSATS Institute of Information Technology, Abbottabad, Pakistan. E-mail: madadmath@yahoo.com

[3]Mathematics & Science Department, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA. E-mail: fsmarandache@gmail.com

[4]Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea. E-mail: skywine@gmail.com

*Correspondence: skywine@gmail.com

Abstract: For $i,j,k,l,m,n \in \{1,2,3,4\}$, the notion of $(T(i,j), I(k,l), F(m,n))$-interval neutrosophic ideals in BCK/BCI-algebras is introduced, and their properties and relations are investigated.

Keywords: interval neutrosophic set; interval neutrosophic ideal.

1 Introduction

BCK-algebras entered into mathematics in 1966 through the work of Imai and Iséki [3], and have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. Such algebras generalize Boolean rings as well as Boolean D-posets (= MV-algebras). Also, Iséki introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra (see [4]). The neutrosophic set developed by Smarandache [7, 8, 9] is a formal framework which generalizes the concept of the classic set, fuzzy set [14], interval valued fuzzy set, intuitionistic fuzzy set [1], interval valued intuitionistic fuzzy set and paraconsistent set etc. Neutrosophic set theory is applied to various part, including algebra, topology, control theory, decision making problems, medicines and in many real life problems. Wang et al. [11, 12, 13] presented the concept of interval neutrosophic sets, which is more precise and more fl xible than the single-valued neutrosophic set. An interval-valued neutrosophic set is a generalization of the concept of single-valued neutrosophic set, in which three membership (t, i, f) functions are independent, and their values belong to the unit interval $[0, 1]$. The interval neutrosophic set can represent uncertain, imprecise, incomplete and inconsistent information which exists in real world. Jun et al. [5] discussed interval neutrosophic sets in BCK/BCI-algebras, and introduced the notion of $(T(i,j), I(k,l), F(m,n))$-interval neutrosophic subalgebras in BCK/BCI-algebras for $i,j,k,l,m,n \in \{1,2,3,4\}$. They also introduced the notion of interval neutrosophic length of an interval neutrosophic set, and investigated related properties.

In this article, we apply the notion of interval neutrosophic sets to ideal theory in BCK/BCI-algebras. We introduce the notion of $(T(i,j), I(k,l), F(m,n))$-interval neutrosophic ideals in BCK/BCI-algebras for $i,j,k,l,m,n \in \{1,2,3,4\}$, and investigate their properties and relations.

2 Preliminaries

By a *BCI-algebra* (see [2, 6]) we mean a system $X := (X, *, 0)$ in which the following axioms hold:

(I) $((x * y) * (x * z)) * (z * y) = 0,$

(II) $(x * (x * y)) * y = 0,$

(III) $x * x = 0,$

(IV) $x * y = y * x = 0 \Rightarrow x = y$

for all $x, y, z \in X$. If a BCI-algebra X satisfie $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra* (see [2, 6]).

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* (see [2, 6]) of X if $x * y \in S$ for all $x, y \in S$.

The collection of all BCK-algebras and all BCI-algebras are denoted by $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$, respectively. Also $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$.

We refer the reader to the books [2] and [6] for further information regarding BCK/BCI-algebras.

By a *fuzzy structure* over a nonempty set X we mean an ordered pair (X, ρ) of X and a fuzzy set ρ on X.

Definition 2.1 ([10]). A fuzzy structure (X, μ) over $(X, *, 0) \in \mathcal{B}(X)$ is called a

- *fuzzy ideal* of $(X, *, 0)$ with type 1 (briefl , 1-*fuzzy ideal* of $(X, *, 0)$) if

$$(\forall x \in X)\,(\mu(0) \geq \mu(x)), \tag{2.1}$$
$$(\forall x, y \in X)\,(\mu(x) \geq \min\{\mu(x * y), \mu(y)\}), \tag{2.2}$$

- *fuzzy ideal* of $(X, *, 0)$ with type 2 (briefl , 2-*fuzzy ideal* of $(X, *, 0)$) if

$$(\forall x \in X)\,(\mu(0) \leq \mu(x))\,, \tag{2.3}$$
$$(\forall x, y \in X)\,(\mu(x) \leq \min\{\mu(x * y), \mu(y)\})\,, \tag{2.4}$$

- *fuzzy ideal* of $(X, *, 0)$ with type 3 (briefl , 3-*fuzzy ideal* of $(X, *, 0)$) if it satisfie (2.1) and

$$(\forall x, y \in X)\,(\mu(x) \geq \max\{\mu(x * y), \mu(y)\})\,, \tag{2.5}$$

- *fuzzy ideal* of $(X, *, 0)$ with type 4 (briefl , 4-*fuzzy ideal* of $(X, *, 0)$) if it satisfie (2.3) and

$$(\forall x, y \in X)\,(\mu(x) \leq \max\{\mu(x * y), \mu(y)\})\,. \tag{2.6}$$

Let X be a non-empty set. A neutrosophic set (NS) in X (see [8]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function.

An interval neutrosophic set (INS) A in X is characterized by truth-membership function T_A, indeterminacy membership function I_A and falsity-membership function F_A. For each point x in X, $T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$ (see [12, 13]).

In what follows, let $(X, *, 0) \in \mathcal{B}(X)$ and $\mathcal{P}^*([0, 1])$ be the family of all subintervals of $[0, 1]$ unless otherwise specifie .

Definition 2.2 ([12, 13]). An *interval neutrosophic set* in a nonempty set X is a structure of the form:

$$\mathcal{I} := \{\langle x, \mathcal{I}[T](x), \mathcal{I}[I](x), \mathcal{I}[F](x)\rangle \mid x \in X\}$$

where

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0, 1])$$

which is called *interval truth-membership function*,

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0, 1])$$

which is called *interval indeterminacy-membership function*, and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0, 1])$$

which is called *interval falsity-membership function*.

For the sake of simplicity, we will use the notation $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ for the interval neutrosophic set

$$\mathcal{I} := \{\langle x, \mathcal{I}[T](x), \mathcal{I}[I](x), \mathcal{I}[F](x)\rangle \mid x \in X\}.$$

Given an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X, we consider the following functions (see [5]):

$$\mathcal{I}[T]_{\inf} : X \to [0, 1],\ x \mapsto \inf\{\mathcal{I}[T](x)\}$$
$$\mathcal{I}[I]_{\inf} : X \to [0, 1],\ x \mapsto \inf\{\mathcal{I}[I](x)\}$$
$$\mathcal{I}[F]_{\inf} : X \to [0, 1],\ x \mapsto \inf\{\mathcal{I}[F](x)\}$$

and

$$\mathcal{I}[T]_{\sup} : X \to [0, 1],\ x \mapsto \sup\{\mathcal{I}[T](x)\}$$
$$\mathcal{I}[I]_{\sup} : X \to [0, 1],\ x \mapsto \sup\{\mathcal{I}[I](x)\}$$
$$\mathcal{I}[F]_{\sup} : X \to [0, 1],\ x \mapsto \sup\{\mathcal{I}[F](x)\}.$$

3 Interval neutrosophic ideals

Definition 3.1. For any $i, j, k, l, m, n \in \{1, 2, 3, 4\}$, an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is called a $(T(i, j), I(k, l), F(m, n))$-*interval neutrosophic ideal* of X if the following assertions are valid.

(1) $(X, \mathcal{I}[T]_{\inf})$ is an i-fuzzy ideal of $(X, *, 0)$ and $(X, \mathcal{I}[T]_{\sup})$ is a j-fuzzy ideal of $(X, *, 0)$,

(2) $(X, \mathcal{I}[I]_{\inf})$ is a k-fuzzy ideal of $(X, *, 0)$ and $(X, \mathcal{I}[I]_{\sup})$ is an l-fuzzy ideal of $(X, *, 0)$,

(3) $(X, \mathcal{I}[F]_{\inf})$ is an m-fuzzy ideal of $(X, *, 0)$ and $(X, \mathcal{I}[F]_{\sup})$ is an n-fuzzy ideal of $(X, *, 0)$.

Example 3.2. Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the binary operation $*$ which is given in Table 1 (see [6]).

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	2
3	3	3	3	0

(1) Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ for which $\mathcal{I}[T]$, $\mathcal{I}[I]$ and $\mathcal{I}[F]$ are given as follows:

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0, 1])\ \ x \mapsto \begin{cases} [0.4, 0.6) & \text{if } x = 0, \\ (0.3, 0.6] & \text{if } x = 1, \\ [0.2, 0.7) & \text{if } x = 2, \\ [0.1, 0.8] & \text{if } x = 3, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0, 1])\ \ x \mapsto \begin{cases} [0.5, 0.6) & \text{if } x = 0, \\ (0.4, 0.6) & \text{if } x = 1, \\ [0.2, 0.9] & \text{if } x = 2, \\ [0.5, 0.7) & \text{if } x = 3, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} [0.4, 0.5) & \text{if } x = 0, \\ (0.3, 0.5) & \text{if } x = 1, \\ [0.1, 0.7] & \text{if } x = 2, \\ (0.2, 0.8] & \text{if } x = 3. \end{cases}$$

It is routine to verify that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(1,4)$, $I(1,4)$, $F(1,4))$-interval neutrosophic ideal of $(X, *, 0)$.

(2) Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ for which $\mathcal{I}[T]$, $\mathcal{I}[I]$ and $\mathcal{I}[F]$ are given as follows:

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} [0.1, 0.4) & \text{if } x = 0, \\ (0.2, 0.7] & \text{if } x = 1, \\ [0.3, 0.8] & \text{if } x = 2, \\ [0.4, 0.6) & \text{if } x = 3, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} (0.2, 0.5) & \text{if } x = 0, \\ [0.5, 0.6] & \text{if } x = 1, \\ (0.6, 0.7] & \text{if } x = 2, \\ [0.3, 0.8] & \text{if } x = 3, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} [0.3, 0.4) & \text{if } x = 0, \\ (0.4, 0.7) & \text{if } x = 1, \\ (0.6, 0.8) & \text{if } x = 2, \\ [0.4, 0.6] & \text{if } x = 3. \end{cases}$$

By routine calculations, we know that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(4,4)$, $I(4,4)$, $F(4,4))$-interval neutrosophic ideal of $(X, *, 0)$.

Example 3.3. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the binary operation $*$ which is given in Table 2 (see [6]).

Table 2: Cayley table for the binary operation "$*$"

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ where $\mathcal{I}[T]$, $\mathcal{I}[I]$ and $\mathcal{I}[F]$ are given as follows:

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} [0.33, 0.91) & \text{if } x = 0, \\ (0.72, 0.91) & \text{if } x = a, \\ [0.72, 0.82) & \text{if } x = b, \\ (0.55, 0.82] & \text{if } x = c, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} [0.22, 0.65) & \text{if } x = 0, \\ [0.52, 0.55] & \text{if } x = a, \\ (0.62, 0.65) & \text{if } x = b, \\ [0.62, 0.55) & \text{if } x = c, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} (0.25, 0.63) & \text{if } x = 0, \\ [0.45, 0.63] & \text{if } x = a, \\ (0.35, 0.53] & \text{if } x = b, \\ [0.45, 0.53) & \text{if } x = c. \end{cases}$$

Routine calculations show that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(4,1)$, $I(4,1)$, $F(4,1))$-interval neutrosophic ideal of $(X, *, 0)$. But it is not a $(T(2,1)$, $I(2,1)$, $F(2,1))$-interval neutrosophic ideal of $(X, *, 0)$ since

$$\mathcal{I}[T]_{\inf}(a) = 0.72 > 0.55 = \min\{\mathcal{I}[T]_{\inf}(a * b), \mathcal{I}[T]_{\inf}(b)\},$$

$$\mathcal{I}[I]_{\inf}(b) = 0.62 > 0.52 = \min\{\mathcal{I}[I]_{\inf}(b * c), \mathcal{I}[I]_{\inf}(c)\},$$

and/or

$$\mathcal{I}[F]_{\inf}(c) = 0.45 > 0.35 = \min\{\mathcal{I}[F]_{\inf}(c * a), \mathcal{I}[F]_{\inf}(c)\}.$$

Also, it is not a $(T(4,3)$, $I(4,3)$, $F(4,3))$-interval neutrosophic ideal of $(X, *, 0)$ since

$$\mathcal{I}[T]_{\sup}(c) = 0.82 < 0.91 = \max\{\mathcal{I}[T]_{\inf}(c * b), \mathcal{I}[T]_{\inf}(b)\}$$

and/or

$$\mathcal{I}[F]_{\sup}(b) = 0.35 < 0.62 = \max\{\mathcal{I}[F]_{\inf}(b * a), \mathcal{I}[F]_{\inf}(a)\}.$$

We also know that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is not a $(T(2,3)$, $I(2,3)$, $F(2,3))$-interval neutrosophic ideal of $(X, *, 0)$.

Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in X. We consider the following sets (see [5]):

$$U(\mathcal{I}[T]_\psi; \alpha_I) := \{x \in X \mid \mathcal{I}[T]_\psi(x) \geq \alpha_I\},$$
$$L(\mathcal{I}[T]_\psi; \alpha_S) := \{x \in X \mid \mathcal{I}[T]_\psi(x) \leq \alpha_S\},$$

$$U(\mathcal{I}[I]_\psi; \beta_I) := \{x \in X \mid \mathcal{I}[I]_\psi(x) \geq \beta_I\},$$
$$L(\mathcal{I}[I]_\psi; \beta_S) := \{x \in X \mid \mathcal{I}[I]_\psi(x) \leq \beta_S\},$$

and

$$U(\mathcal{I}[F]_\psi; \gamma_I) := \{x \in X \mid \mathcal{I}[F]_\psi(x) \geq \gamma_I\},$$
$$L(\mathcal{I}[F]_\psi; \gamma_S) := \{x \in X \mid \mathcal{I}[F]_\psi(x) \leq \gamma_S\},$$

where $\psi \in \{\inf, \sup\}$, and α_I, α_S, β_I, β_S, γ_I and γ_S are numbers in $[0, 1]$.

Theorem 3.4. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X, *, 0)$, *we have the following assertions:*

(1) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(1,4), I(1,4), F(1,4))$*-interval neutrosophic ideal of* $(X,*,0)$*, then* $U(\mathcal{I}[T]_{\inf};\alpha_I)$*,* $L(\mathcal{I}[T]_{\sup};\alpha_S)$*,* $U(\mathcal{I}[I]_{\inf};\beta_I)$*,* $L(\mathcal{I}[I]_{\sup};\beta_S)$*,* $U(\mathcal{I}[F]_{\inf};\gamma_I)$ *and* $L(\mathcal{I}[F]_{\sup};\gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*.*

(2) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,1), I(4,1), F(4,1))$*-interval neutrosophic ideal of* $(X,*,0)$*, then* $L(\mathcal{I}[T]_{\inf};\alpha_I)$*,* $U(\mathcal{I}[T]_{\sup};\alpha_S)$*,* $L(\mathcal{I}[I]_{\inf};\beta_I)$*,* $U(\mathcal{I}[I]_{\sup};\beta_S)$*,* $L(\mathcal{I}[F]_{\inf};\gamma_I)$ *and* $U(\mathcal{I}[F]_{\sup};\gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*.*

(3) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(1,1), I(1,1), F(1,1))$*-interval neutrosophic ideal of* $(X,*,0)$*, then* $U(\mathcal{I}[T]_{\inf};\alpha_I)$*,* $U(\mathcal{I}[T]_{\sup};\alpha_S)$*,* $U(\mathcal{I}[I]_{\inf};\beta_I)$*,* $U(\mathcal{I}[I]_{\sup};\beta_S)$*,* $U(\mathcal{I}[F]_{\inf};\gamma_I)$ *and* $U(\mathcal{I}[F]_{\sup};\gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*.*

(4) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,4), I(4,4), F(4,4))$*-interval neutrosophic ideal of* $(X,*,0)$*, then* $L(\mathcal{I}[T]_{\inf};\alpha_I)$*,* $L(\mathcal{I}[T]_{\sup};\alpha_S)$*,* $L(\mathcal{I}[I]_{\inf};\beta_I)$*,* $L(\mathcal{I}[I]_{\sup};\beta_S)$*,* $L(\mathcal{I}[F]_{\inf};\gamma_I)$ *and* $L(\mathcal{I}[F]_{\sup};\gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*.*

Proof. (1) Assume that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(1,4), I(1,4), F(1,4))$-interval neutrosophic ideal of $(X,*,0)$. Then $(X,\mathcal{I}[T]_{\inf})$, $(X,\mathcal{I}[I]_{\inf})$ and $(X,\mathcal{I}[F]_{\inf})$ are 1-fuzzy ideals of X; and $(X,\mathcal{I}[T]_{\sup})$, $(X,\mathcal{I}[I]_{\sup})$ and $(X,\mathcal{I}[F]_{\sup})$ are 4-fuzzy ideals of X. Let $\alpha_I, \alpha_S \in [0,1]$ be such that $U(\mathcal{I}[T]_{\inf};\alpha_I)$ and $L(\mathcal{I}[T]_{\sup};\alpha_S)$ are nonempty. Obviously, $0 \in U(\mathcal{I}[T]_{\inf};\alpha_I)$ and $0 \in L(\mathcal{I}[T]_{\sup};\alpha_S)$. Let $x, y \in X$ be such that $x*y \in U(\mathcal{I}[T]_{\inf};\alpha_I)$ and $y \in U(\mathcal{I}[T]_{\inf};\alpha_I)$. Then $\mathcal{I}[T]_{\inf}(x*y) \geq \alpha_I$ and $\mathcal{I}[T]_{\inf}(y) \geq \alpha_I$, and so

$$\mathcal{I}[T]_{\inf}(x) \geq \min\{\mathcal{I}[T]_{\inf}(x*y), \mathcal{I}[T]_{\inf}(y)\} \geq \alpha_I,$$

that is, $x \in U(\mathcal{I}[T]_{\inf};\alpha_I)$. If $x*y \in L(\mathcal{I}[T]_{\sup};\alpha_S)$ and $y \in L(\mathcal{I}[T]_{\sup};\alpha_S)$, then $\mathcal{I}[T]_{\sup}(x*y) \leq \alpha_S$ and $\mathcal{I}[T]_{\sup}(y) \leq \alpha_S$, which imply that

$$\mathcal{I}[T]_{\sup}(x) \leq \max\{\mathcal{I}[T]_{\sup}(x*y), \mathcal{I}[T]_{\sup}(y)\} \leq \alpha_S,$$

that is, $x \in L(\mathcal{I}[T]_{\sup};\alpha_S)$. Hence $U(\mathcal{I}[T]_{\inf};\alpha_I)$ and $L(\mathcal{I}[T]_{\sup};\alpha_S)$ are ideals of $(X,*,0)$ for all α_I, $\alpha_S \in [0,1]$. Similarly, we can prove that $U(\mathcal{I}[I]_{\inf};\beta_I)$, $L(\mathcal{I}[I]_{\sup};\beta_S)$, $U(\mathcal{I}[F]_{\inf};\gamma_I)$ and $L(\mathcal{I}[F]_{\sup};\gamma_S)$ are either empty or subalgebras of $(X,*,0)$ for all $\beta_I, \beta_S, \gamma_I, \gamma_S \in [0,1]$. By the similarly way to the proof of (1), we can prove that (2), (3) and (4) are true. □

Corollary 3.5. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X,*,0)$*, we have the following assertions:*

(1) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(3,4), I(3,4), F(3,4))$*-interval neutrosophic ideal of* $(X,*,0)$ *or a* $(T(i,2), I(i,2), F(i,2))$*-interval neutrosophic ideal of* $(X,*,0)$ *for* $i \in \{1,3\}$*, then* $U(\mathcal{I}[T]_{\inf};\alpha_I)$*,* $L(\mathcal{I}[T]_{\sup};\alpha_S)$*,* $U(\mathcal{I}[I]_{\inf};\beta_I)$*,* $L(\mathcal{I}[I]_{\sup};\beta_S)$*,* $U(\mathcal{I}[F]_{\inf};\gamma_I)$ *and* $L(\mathcal{I}[F]_{\sup};\gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*.*

(2) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,3), I(4,3), F(4,3))$*-interval neutrosophic ideal of* $(X,*,0)$ *or a* $(T(2,j), I(2,j), F(2,j))$*-interval neutrosophic ideal of* $(X,*,0)$ *for* $j \in \{1,3\}$*, then* $L(\mathcal{I}[T]_{\inf};\alpha_I)$*,* $U(\mathcal{I}[T]_{\sup};\alpha_S)$*,* $L(\mathcal{I}[I]_{\inf};\beta_I)$*,* $U(\mathcal{I}[I]_{\sup};\beta_S)$*,* $L(\mathcal{I}[F]_{\inf};\gamma_I)$ *and* $U(\mathcal{I}[F]_{\sup};\gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*.*

(3) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(3,1), I(3,1), F(3,1))$*-interval neutrosophic ideal of* $(X,*,0)$ *or a* $(T(i,3), I(i,3), F(i,3))$*-interval neutrosophic ideal of* $(X,*,0)$ *for* $i \in \{1,3\}$*, then* $U(\mathcal{I}[T]_{\inf};\alpha_I)$*,* $U(\mathcal{I}[T]_{\sup};\alpha_S)$*,* $U(\mathcal{I}[I]_{\inf};\beta_I)$*,* $U(\mathcal{I}[I]_{\sup};\beta_S)$*,* $U(\mathcal{I}[F]_{\inf};\gamma_I)$ *and* $U(\mathcal{I}[F]_{\sup};\gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*.*

(4) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(2,4), I(2,4), F(2,4))$*-interval neutrosophic ideal of* $(X,*,0)$ *or a* $(T(i,2), I(i,2), F(i,2))$*-interval neutrosophic ideal of* $(X,*,0)$ *for* $i \in \{2,4\}$*, then* $L(\mathcal{I}[T]_{\inf};\alpha_I)$*,* $L(\mathcal{I}[T]_{\sup};\alpha_S)$*,* $L(\mathcal{I}[I]_{\inf};\beta_I)$*,* $L(\mathcal{I}[I]_{\sup};\beta_S)$*,* $L(\mathcal{I}[F]_{\inf};\gamma_I)$ *and* $L(\mathcal{I}[F]_{\sup};\gamma_S)$ *are either empty or ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*.*

Proof. Straightforward since every 3-fuzzy (resp., 2-fuzzy) ideal is a 1-fuzzy (resp., 4-fuzzy) ideal. □

Theorem 3.6. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X,*,0)$*, the following assertions are valid.*

(1) *If* $U(\mathcal{I}[T]_{\inf};\alpha_I)$*,* $L(\mathcal{I}[T]_{\sup};\alpha_S)$*,* $U(\mathcal{I}[I]_{\inf};\beta_I)$*,* $L(\mathcal{I}[I]_{\sup};\beta_S)$*,* $U(\mathcal{I}[F]_{\inf};\gamma_I)$ *and* $L(\mathcal{I}[F]_{\sup};\gamma_S)$ *are nonempty ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*, then* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(1,4), I(1,4), F(1,4))$*-interval neutrosophic ideal of* $(X,*,0)$*.*

(2) *If* $U(\mathcal{I}[T]_{\inf};\alpha_I)$*,* $U(\mathcal{I}[T]_{\sup};\alpha_S)$*,* $U(\mathcal{I}[I]_{\inf};\beta_I)$*,* $U(\mathcal{I}[I]_{\sup};\beta_S)$*,* $U(\mathcal{I}[F]_{\inf};\gamma_I)$ *and* $U(\mathcal{I}[F]_{\sup};\gamma_S)$ *are nonempty ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*, then* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(1,1), I(1,1), F(1,1))$*-interval neutrosophic ideal of* $(X,*,0)$*.*

(3) *If* $L(\mathcal{I}[T]_{\inf};\alpha_I)$*,* $U(\mathcal{I}[T]_{\sup};\alpha_S)$*,* $L(\mathcal{I}[I]_{\inf};\beta_I)$*,* $U(\mathcal{I}[I]_{\sup};\beta_S)$*,* $L(\mathcal{I}[F]_{\inf};\gamma_I)$ *and* $U(\mathcal{I}[F]_{\sup};\gamma_S)$ *are nonempty ideals of* $(X,*,0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*, then* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,1), I(4,1), F(4,1))$*-interval neutrosophic ideal of* $(X,*,0)$*.*

(4) *If* $L(\mathcal{I}[T]_{\inf};\alpha_I)$*,* $L(\mathcal{I}[T]_{\sup};\alpha_S)$*,* $L(\mathcal{I}[I]_{\inf};\beta_I)$*,* $L(\mathcal{I}[I]_{\sup};\beta_S)$*,* $L(\mathcal{I}[F]_{\inf};\gamma_I)$ *and* $L(\mathcal{I}[F]_{\sup};\gamma_S)$ *are*

nonempty ideals of $(X, *, 0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$, *then* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4, 4), I(4, 4), F(4, 4))$*-interval neutrosophic ideal of* $(X, *, 0)$.

Proof. (1) Suppose that $U(\mathcal{I}[T]_{\inf}; \alpha_I)$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)$, $U(\mathcal{I}[I]_{\inf}; \beta_I)$, $L(\mathcal{I}[I]_{\sup}; \beta_S)$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ and $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ are nonempty ideals of $(X, *, 0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$. If $(X, \mathcal{I}[T]_{\inf})$ is not a 1-fuzzy ideal of $(X, *, 0)$, then there exist $x, y \in X$ such that

$$\mathcal{I}[T]_{\inf}(x) < \min\{\mathcal{I}[T]_{\inf}(x * y), \mathcal{I}[T]_{\inf}(y)\}.$$

If we take $\alpha_I = \min\{\mathcal{I}[T]_{\inf}(x*y), \mathcal{I}[T]_{\inf}(y)\}$, then $x*y,\ y \in U(\mathcal{I}[T]_{\inf}; \alpha_I)$ but $x \notin U(\mathcal{I}[T]_{\inf}; \alpha_I)$. This is a contradiction, and so $(X, \mathcal{I}[T]_{\inf})$ is a 1-fuzzy ideal of $(X, *, 0)$. If $(X, \mathcal{I}[T]_{\sup})$ is not a 4-fuzzy ideal of $(X, *, 0)$, then

$$\mathcal{I}[T]_{\sup}(a) > \max\{\mathcal{I}[T]_{\sup}(a * b), \mathcal{I}[T]_{\sup}(b)\}$$

for some $a, b \in X$, and so $a * b,\ b \in L(\mathcal{I}[T]_{\sup}; \alpha_S)$ and $a \notin L(\mathcal{I}[T]_{\sup}; \alpha_S)$ by taking

$$\alpha_S := \max\{\mathcal{I}[T]_{\sup}(a * b), \mathcal{I}[T]_{\sup}(b)\}.$$

This is a contradiction, and therefore $(X, \mathcal{I}[T]_{\sup})$ is a 4-fuzzy ideal of $(X, *, 0)$. Similarly, we can verify that $(X, \mathcal{I}[I]_{\inf})$ is a 1-fuzzy ideal of $(X, *, 0)$ and $(X, \mathcal{I}[I]_{\sup})$ is a 4-fuzzy ideal of $(X, *, 0)$, and $(X, \mathcal{I}[F]_{\inf})$ is a 1-fuzzy ideal of $(X, *, 0)$ and $(X, \mathcal{I}[F]_{\sup})$ is a 4-fuzzy ideal of $(X, *, 0)$. Consequently, $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(1, 4), I(1, 4), F(1, 4))$-interval neutrosophic ideal of $(X, *, 0)$. The assertions (2), (3) and (4) can be proved by the similar way to the proof of (1). □

Theorem 3.7. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X, *, 0)$ *is a* $(T(2, 3), I(2, 3), F(2, 3))$*-interval neutrosophic ideal of* $(X, *, 0)$, *then* $U(\mathcal{I}[T]_{\inf}; \alpha_I)^c$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $U(\mathcal{I}[I]_{\inf}; \beta_I)^c$, $L(\mathcal{I}[I]_{\sup}; \beta_S)^c$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)^c$ *and* $L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$ *are either empty or ideals of* $(X, *, 0)$ *for all* α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.

Proof. Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be a $(T(2, 3), I(2, 3), F(2, 3))$-interval neutrosophic ideal of $(X, *, 0)$. Then

(1) $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 2-fuzzy ideals of $(X, *, 0)$,

(2) $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 3-fuzzy ideals of $(X, *, 0)$.

Let α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$ be such that $U(\mathcal{I}[T]_{\inf}; \alpha_I)^c$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $U(\mathcal{I}[I]_{\inf}; \beta_I)^c$, $L(\mathcal{I}[I]_{\sup}; \beta_S)^c$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)^c$ and $L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$ are nonempty. Then there exist $x, y, z, a, b, d \in X$ such that $x \in U(\mathcal{I}[T]_{\inf}; \alpha_I)^c$, $a \in L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $y \in U(\mathcal{I}[I]_{\inf}; \beta_I)^c$, $b \in L(\mathcal{I}[I]_{\sup}; \beta_S)^c$, $z \in U(\mathcal{I}[F]_{\inf}; \gamma_I)^c$ and $d \in L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$. Hence

$\mathcal{I}[T]_{\inf}(0) \leq \mathcal{I}[T]_{\inf}(x) < \alpha_I$ and $\mathcal{I}[T]_{\sup}(0) \geq \mathcal{I}[T]_{\sup}(a) > \alpha_S$,

$\mathcal{I}[I]_{\inf}(0) \leq \mathcal{I}[I]_{\inf}(y) < \beta_I$ and $\mathcal{I}[I]_{\sup}(0) \geq \mathcal{I}[I]_{\sup}(b) > \beta_S$,

$\mathcal{I}[F]_{\inf}(0) \leq \mathcal{I}[F]_{\inf}(z) < \gamma_I$ and $\mathcal{I}[F]_{\sup}(0) \geq \mathcal{I}[F]_{\sup}(d) > \gamma_S$,

and so $0 \in U(\mathcal{I}[T]_{\inf}; \alpha_I)^c \cap L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $0 \in U(\mathcal{I}[I]_{\inf}; \beta_I)^c \cap L(\mathcal{I}[I]_{\sup}; \beta_S)^c$, and $0 \in U(\mathcal{I}[F]_{\inf}; \gamma_I)^c \cap L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$. Let $x, y \in X$ be such that $x * y \in U(\mathcal{I}[T]_{\inf}; \alpha_I)^c$ and $y \in U(\mathcal{I}[T]_{\inf}; \alpha_I)^c$. Then $\mathcal{I}[T]_{\inf}(x*y) < \alpha_I$ and $\mathcal{I}[T]_{\inf}(y) < \alpha_I$. Hence

$$\mathcal{I}[T]_{\inf}(x) \leq \min\{\mathcal{I}[T]_{\inf}(x * y), \mathcal{I}[T]_{\inf}(y)\} < \alpha_I,$$

and so $x \in U(\mathcal{I}[T]_{\inf}; \alpha_I)^c$. Thus $U(\mathcal{I}[T]_{\inf}; \alpha_I)^c$ is an ideal of $(X, *, 0)$. Similarly, we can verify that

- If $x * y \in L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$ and $y \in L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, then $x \in L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$,
- If $x * y \in U(\mathcal{I}[I]_{\inf}; \beta_I)^c$ and $y \in U(\mathcal{I}[I]_{\inf}; \beta_I)^c$, then $x \in U(\mathcal{I}[I]_{\inf}; \beta_I)^c$,
- If $x * y \in L(\mathcal{I}[I]_{\sup}; \beta_S)^c$ and $y \in L(\mathcal{I}[I]_{\sup}; \beta_S)^c$, then $x \in L(\mathcal{I}[I]_{\sup}; \beta_S)^c$,
- If $x * y \in U(\mathcal{I}[F]_{\inf}; \gamma_I)^c$ and $y \in U(\mathcal{I}[F]_{\inf}; \gamma_I)^c$, then $x \in U(\mathcal{I}[F]_{\inf}; \gamma_I)^c$,
- If $x * y \in L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$ and $y \in L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$, then $x \in L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$.

Therefore $L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $U(\mathcal{I}[I]_{\inf}; \beta_I)^c$, $L(\mathcal{I}[I]_{\sup}; \beta_S)^c$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)^c$ and $L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$ are ideals of $(X, *, 0)$. □

The converse of Theorem 3.7 is not true in general as seen in the following example.

Example 3.8. Consider a BCI-algebra $X = \{0, 1, a, b, c\}$ with the binary operation $*$ which is given in Table 3 (see [6]).

Table 3: Cayley table for the binary operation "$*$"

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ where $\mathcal{I}[T]$, $\mathcal{I}[I]$ and $\mathcal{I}[F]$ are given as follows:

$$\mathcal{I}[T] : X \to \tilde{\mathcal{P}}([0, 1]),\ \ x \mapsto \begin{cases} [0.25, 0.85) & \text{if } x = 0, \\ (0.45, 0.83] & \text{if } x = 1, \\ [0.55, 0.73] & \text{if } x = a, \\ (0.65, 0.73] & \text{if } x = b, \\ [0.65, 0.75) & \text{if } x = c, \end{cases}$$

$$\mathcal{I}[I]: X \to \tilde{\mathcal{P}}([0,1]),\ \ x \mapsto \begin{cases} [0.3, 0.75) & \text{if } x = 0, \\ (0.3, 0.70] & \text{if } x = 1, \\ [0.6, 0.63] & \text{if } x = a, \\ (0.5, 0.63] & \text{if } x = b, \\ [0.6, 0.68) & \text{if } x = c, \end{cases}$$

and

$$\mathcal{I}[F]: X \to \tilde{\mathcal{P}}([0,1]),\ \ x \mapsto \begin{cases} [0.44, 0.9) & \text{if } x = 0, \\ (0.55, 0.9] & \text{if } x = 1, \\ [0.55, 0.7] & \text{if } x = a, \\ (0.66, 0.8] & \text{if } x = b, \\ [0.66, 0.7) & \text{if } x = c. \end{cases}$$

Then

$$U(\mathcal{I}[T]_{\inf}; \alpha_I)^c = \begin{cases} \emptyset & \text{if } \alpha_I \in [0, 0.25], \\ \{0\} & \text{if } \alpha_I \in (0.25, 0.45], \\ \{0, 1\} & \text{if } \alpha_I \in (0.45, 0.55], \\ \{0, 1, a\} & \text{if } \alpha_I \in (0.55, 0.65], \\ X & \text{if } \alpha_I \in (0.65, 1.0], \end{cases}$$

$$L(\mathcal{I}[T]_{\sup}; \alpha_S)^c = \begin{cases} \emptyset & \text{if } \alpha_S \in [0.85, 1.0], \\ \{0\} & \text{if } \alpha_S \in [0.83, 0.85), \\ \{0, 1\} & \text{if } \alpha_S \in [0.75, 0.83), \\ \{0, 1, c\} & \text{if } \alpha_S \in [0.73, 0.75), \\ X & \text{if } \alpha_S \in [0, 0.73), \end{cases}$$

$$U(\mathcal{I}[I]_{\inf}; \beta_I)^c = \begin{cases} \emptyset & \text{if } \beta_I \in [0, 0.3], \\ \{0, 1\} & \text{if } \beta_I \in (0.3, 0.5], \\ \{0, 1, b\} & \text{if } \beta_I \in (0.5, 0.6], \\ X & \text{if } \beta_I \in (0.6, 1.0], \end{cases}$$

$$L(\mathcal{I}[I]_{\sup}; \beta_S)^c = \begin{cases} \emptyset & \text{if } \beta_S \in [0.75, 1.0], \\ \{0\} & \text{if } \beta_S \in [0.70, 0.75), \\ \{0, 1\} & \text{if } \beta_S \in [0.68, 0.70), \\ \{0, 1, c\} & \text{if } \beta_S \in [0.63, 0.68), \\ X & \text{if } \beta_S \in [0, 0.63), \end{cases}$$

$$U(\mathcal{I}[F]_{\inf}; \gamma_I)^c = \begin{cases} \emptyset & \text{if } \gamma_I \in [0, 0.44], \\ \{0\} & \text{if } \gamma_I \in (0.44, 0.55], \\ \{0, 1, a\} & \text{if } \gamma_I \in (0.55, 0.66], \\ X & \text{if } \gamma_I \in (0.66, 1.0], \end{cases}$$

$$L(\mathcal{I}[F]_{\sup}; \gamma_S)^c = \begin{cases} \emptyset & \text{if } \gamma_S \in [0.9, 1.0], \\ \{0, 1\} & \text{if } \gamma_S \in [0.8, 0.9), \\ \{0, 1, b\} & \text{if } \gamma_S \in [0.7, 0.8), \\ X & \text{if } \gamma_S \in [0, 0.7). \end{cases}$$

Hence the nonempty sets $U(\mathcal{I}[T]_{\inf}; \alpha_I)^c$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $U(\mathcal{I}[I]_{\inf}; \beta_I)^c$, $L(\mathcal{I}[I]_{\sup}; \beta_S)^c$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)^c$ and $L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$ are ideals of $(X, *, 0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$. But $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is not a $(T(2,3), I(2,3), F(2,3))$-interval neutrosophic ideal of $(X, *, 0)$ since

$$\mathcal{I}[T]_{\inf}(c) = 0.65 > 0.55 = \min\{\mathcal{I}[T]_{\inf}(c * a), \mathcal{I}[T]_{\inf}(a)\},$$

$$\mathcal{I}[T]_{\sup}(a) = 0.73 < 0.75 = \max\{\mathcal{I}[T]_{\sup}(a * c), \mathcal{I}[T]_{\sup}(c)\},$$

$$\mathcal{I}[I]_{\inf}(c) = 0.6 > 0.5 = \min\{\mathcal{I}[I]_{\inf}(c * a), \mathcal{I}[I]_{\inf}(a)\},$$

$$\mathcal{I}[I]_{\sup}(a) = 0.63 < 0.68 = \max\{\mathcal{I}[I]_{\sup}(a * c), \mathcal{I}[I]_{\sup}(c)\},$$

$$\mathcal{I}[F]_{\inf}(c) = 0.66 > 0.55 = \min\{\mathcal{I}[F]_{\inf}(c * a), \mathcal{I}[F]_{\inf}(a)\},$$

and/or

$$\mathcal{I}[F]_{\sup}(a) = 0.7 < 0.8 = \max\{\mathcal{I}[F]_{\sup}(a * c), \mathcal{I}[F]_{\sup}(c)\}.$$

Using the similar way to the proof of Theorem 3.7, we have the following theorems.

Theorem 3.9. *Given an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in $(X, *, 0)$, we have the following assertions:*

(1) *If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(2,2), I(2,2), F(2,2))$-interval neutrosophic ideal of $(X, *, 0)$, then $U(\mathcal{I}[T]_{\inf}; \alpha_I)^c$, $U(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $U(\mathcal{I}[I]_{\inf}; \beta_I)^c$, $U(\mathcal{I}[I]_{\sup}; \beta_S)^c$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)^c$ and $U(\mathcal{I}[F]_{\sup}; \gamma_S)^c$ are either empty or ideals of $(X, *, 0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.*

(2) *If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(3,2), I(3,2), F(3,2))$-interval neutrosophic ideal of $(X, *, 0)$, then $L(\mathcal{I}[T]_{\inf}; \alpha_I)^c$, $U(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $L(\mathcal{I}[I]_{\inf}; \beta_I)^c$, $U(\mathcal{I}[I]_{\sup}; \beta_S)^c$, $L(\mathcal{I}[F]_{\inf}; \gamma_I)^c$ and $U(\mathcal{I}[F]_{\sup}; \gamma_S)^c$ are either empty or ideals of $(X, *, 0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.*

(3) *If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(3,3), I(3,3), F(3,3))$-interval neutrosophic ideal of $(X, *, 0)$, then $L(\mathcal{I}[T]_{\inf}; \alpha_I)^c$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $L(\mathcal{I}[I]_{\inf}; \beta_I)^c$, $L(\mathcal{I}[I]_{\sup}; \beta_S)^c$, $L(\mathcal{I}[F]_{\inf}; \gamma_I)^c$ and $L(\mathcal{I}[F]_{\sup}; \gamma_S)^c$ are either empty or ideals of $(X, *, 0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.*

Using the similar way to the proofs of Theorems 3.4 and 3.7, we have the following theorem.

Theorem 3.10. *Given an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in $(X, *, 0)$, we have the following assertions:*

(1) *If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(1,2), I(1,2), F(1,2))$-interval neutrosophic ideal of $(X, *, 0)$, then $U(\mathcal{I}[T]_{\inf}; \alpha_I)$, $U(\mathcal{I}[T]_{\sup}; \alpha_S)^c$, $U(\mathcal{I}[I]_{\inf}; \beta_I)$, $U(\mathcal{I}[I]_{\sup}; \beta_S)^c$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ and $U(\mathcal{I}[F]_{\sup}; \gamma_S)^c$ are either empty or ideals of $(X, *, 0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.*

(2) If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(1,3), I(1,3), F(1,3))$-interval neutrosophic ideal of $(X,*,0)$, then $U(\mathcal{I}[T]_{\inf};\alpha_I)$, $L(\mathcal{I}[T]_{\sup};\alpha_S)^c$, $U(\mathcal{I}[I]_{\inf};\beta_I)$, $L(\mathcal{I}[I]_{\sup};\beta_S)^c$, $U(\mathcal{I}[F]_{\inf};\gamma_I)$ and $L(\mathcal{I}[F]_{\sup};\gamma_S)^c$ are either empty or ideals of $(X,*,0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$.

(3) If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(2,1), I(2,1), F(2,1))$-interval neutrosophic ideal of $(X,*,0)$, then $U(\mathcal{I}[T]_{\inf};\alpha_I)^c$, $U(\mathcal{I}[T]_{\sup};\alpha_S)$, $U(\mathcal{I}[I]_{\inf};\beta_I)^c$, $U(\mathcal{I}[I]_{\sup};\beta_S)$, $U(\mathcal{I}[F]_{\inf};\gamma_I)^c$ and $U(\mathcal{I}[F]_{\sup};\gamma_S)$ are either empty or ideals of $(X,*,0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$.

(4) If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(3,1), I(3,1), F(3,1))$-interval neutrosophic ideal of $(X,*,0)$, then $L(\mathcal{I}[T]_{\inf};\alpha_I)^c$, $U(\mathcal{I}[T]_{\sup};\alpha_S)$, $L(\mathcal{I}[I]_{\inf};\beta_I)^c$, $U(\mathcal{I}[I]_{\sup};\beta_S)$, $L(\mathcal{I}[F]_{\inf};\gamma_I)^c$ and $U(\mathcal{I}[F]_{\sup};\gamma_S)$ are either empty or ideals of $(X,*,0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$.

(5) If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(2,4), I(2,4), F(2,4))$-interval neutrosophic ideal of $(X,*,0)$, then $U(\mathcal{I}[T]_{\inf};\alpha_I)^c$, $L(\mathcal{I}[T]_{\sup};\alpha_S)$, $U(\mathcal{I}[I]_{\inf};\beta_I)^c$, $L(\mathcal{I}[I]_{\sup};\beta_S)$, $U(\mathcal{I}[F]_{\inf};\gamma_I)^c$ and $L(\mathcal{I}[F]_{\sup};\gamma_S)$ are either empty or ideals of $(X,*,0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$.

(6) If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(3,4), I(3,4), F(3,4))$-interval neutrosophic ideal of $(X,*,0)$, then $L(\mathcal{I}[T]_{\inf};\alpha_I)^c$, $L(\mathcal{I}[T]_{\sup};\alpha_S)$, $L(\mathcal{I}[I]_{\inf};\beta_I)^c$, $L(\mathcal{I}[I]_{\sup};\beta_S)$, $L(\mathcal{I}[F]_{\inf};\gamma_I)^c$ and $L(\mathcal{I}[F]_{\sup};\gamma_S)$ are either empty or ideals of $(X,*,0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$.

(7) If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(4,2), I(4,2), F(4,2))$-interval neutrosophic ideal of $(X,*,0)$, then $L(\mathcal{I}[T]_{\inf};\alpha_I)$, $U(\mathcal{I}[T]_{\sup};\alpha_S)^c$, $L(\mathcal{I}[I]_{\inf};\beta_I)$, $U(\mathcal{I}[I]_{\sup};\beta_S)^c$, $L(\mathcal{I}[F]_{\inf};\gamma_I)$ and $U(\mathcal{I}[F]_{\sup};\gamma_S)^c$ are either empty or ideals of $(X,*,0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$.

(8) If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(4,3), I(4,3), F(4,3))$-interval neutrosophic ideal of $(X,*,0)$, then $L(\mathcal{I}[T]_{\inf};\alpha_I)$, $L(\mathcal{I}[T]_{\sup};\alpha_S)^c$, $L(\mathcal{I}[I]_{\inf};\beta_I)$, $L(\mathcal{I}[I]_{\sup};\beta_S)^c$, $L(\mathcal{I}[F]_{\inf};\gamma_I)$ and $L(\mathcal{I}[F]_{\sup};\gamma_S)^c$ are either empty or ideals of $(X,*,0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$.

Proposition 3.11. *Every $(T(1,4), I(1,4), F(1,4))$-interval neutrosophic ideal $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ of $(X,*,0)$ satisfies*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \geq \mathcal{I}[T]_{\inf}(y) \\ \mathcal{I}[T]_{\sup}(x) \leq \mathcal{I}[T]_{\sup}(y) \\ \mathcal{I}[I]_{\inf}(x) \geq \mathcal{I}[I]_{\inf}(y) \\ \mathcal{I}[I]_{\sup}(x) \leq \mathcal{I}[I]_{\sup}(y) \\ \mathcal{I}[F]_{\inf}(x) \geq \mathcal{I}[F]_{\inf}(y) \\ \mathcal{I}[F]_{\sup}(x) \leq \mathcal{I}[F]_{\sup}(y) \end{cases} \tag{3.1}$$

for all $x, y \in X$ with $x \leq y$.

Proof. If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(1,4), I(1,4), F(1,4))$-interval neutrosophic ideal of $(X,*,0)$, then $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 1-fuzzy ideals of $(X,*,0)$, and $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 4-fuzzy ideals of $(X,*,0)$. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and so

$$\begin{aligned} \mathcal{I}[T]_{\inf}(x) &\geq \min\{\mathcal{I}[T]_{\inf}(x*y), \mathcal{I}[T]_{\inf}(y)\} \\ &= \min\{\mathcal{I}[T]_{\inf}(0), \mathcal{I}[T]_{\inf}(y)\} = \mathcal{I}[T]_{\inf}(y), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[T]_{\sup}(x) &\leq \max\{\mathcal{I}[T]_{\sup}(x*y), \mathcal{I}[T]_{\sup}(y)\} \\ &= \max\{\mathcal{I}[T]_{\sup}(0), \mathcal{I}[T]_{\sup}(y)\} = \mathcal{I}[T]_{\sup}(y), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[I]_{\inf}(x) &\geq \min\{\mathcal{I}[I]_{\inf}(x*y), \mathcal{I}[I]_{\inf}(y)\} \\ &= \min\{\mathcal{I}[I]_{\inf}(0), \mathcal{I}[I]_{\inf}(y)\} = \mathcal{I}[I]_{\inf}(y), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[I]_{\sup}(x) &\leq \max\{\mathcal{I}[I]_{\sup}(x*y), \mathcal{I}[I]_{\sup}(y)\} \\ &= \max\{\mathcal{I}[I]_{\sup}(0), \mathcal{I}[I]_{\sup}(y)\} = \mathcal{I}[I]_{\sup}(y), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[F]_{\inf}(x) &\geq \min\{\mathcal{I}[F]_{\inf}(x*y), \mathcal{I}[F]_{\inf}(y)\} \\ &= \min\{\mathcal{I}[F]_{\inf}(0), \mathcal{I}[F]_{\inf}(y)\} = \mathcal{I}[F]_{\inf}(y), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[F]_{\sup}(x) &\leq \max\{\mathcal{I}[F]_{\sup}(x*y), \mathcal{I}[F]_{\sup}(y)\} \\ &= \max\{\mathcal{I}[F]_{\sup}(0), \mathcal{I}[F]_{\sup}(y)\} = \mathcal{I}[F]_{\sup}(y). \end{aligned}$$

This completes the proof. □

Using the similar way to the proof of Proposition 3.11, we have the following proposition.

Proposition 3.12. *Given an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in $(X,*,0)$, we have the following assertions:*

(1) *If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(1,1), I(1,1), F(1,1))$-*

interval neutrosophic ideal of $(X,*,0)$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \geq \mathcal{I}[T]_{\inf}(y) \\ \mathcal{I}[T]_{\sup}(x) \geq \mathcal{I}[T]_{\sup}(y) \\ \mathcal{I}[I]_{\inf}(x) \geq \mathcal{I}[I]_{\inf}(y) \\ \mathcal{I}[I]_{\sup}(x) \geq \mathcal{I}[I]_{\sup}(y) \\ \mathcal{I}[F]_{\inf}(x) \geq \mathcal{I}[F]_{\inf}(y) \\ \mathcal{I}[F]_{\sup}(x) \geq \mathcal{I}[F]_{\sup}(y) \end{cases} \tag{3.2}$$

for all $x, y \in X$ *with* $x \leq y$.

(2) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,1), I(4,1), F(4,1))$*-interval neutrosophic ideal of* $(X,*,0)$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \leq \mathcal{I}[T]_{\inf}(y) \\ \mathcal{I}[T]_{\sup}(x) \geq \mathcal{I}[T]_{\sup}(y) \\ \mathcal{I}[I]_{\inf}(x) \leq \mathcal{I}[I]_{\inf}(y) \\ \mathcal{I}[I]_{\sup}(x) \geq \mathcal{I}[I]_{\sup}(y) \\ \mathcal{I}[F]_{\inf}(x) \leq \mathcal{I}[F]_{\inf}(y) \\ \mathcal{I}[F]_{\sup}(x) \geq \mathcal{I}[F]_{\sup}(y) \end{cases} \tag{3.3}$$

for all $x, y \in X$ *with* $x \leq y$.

(2) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,4), I(4,4), F(4,4))$*-interval neutrosophic ideal of* $(X,*,0)$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \leq \mathcal{I}[T]_{\inf}(y) \\ \mathcal{I}[T]_{\sup}(x) \leq \mathcal{I}[T]_{\sup}(y) \\ \mathcal{I}[I]_{\inf}(x) \leq \mathcal{I}[I]_{\inf}(y) \\ \mathcal{I}[I]_{\sup}(x) \leq \mathcal{I}[I]_{\sup}(y) \\ \mathcal{I}[F]_{\inf}(x) \leq \mathcal{I}[F]_{\inf}(y) \\ \mathcal{I}[F]_{\sup}(x) \leq \mathcal{I}[F]_{\sup}(y) \end{cases} \tag{3.4}$$

for all $x, y \in X$ *with* $x \leq y$.

Proposition 3.13. *For every* $(i,j) \in \{(2,2),(2,3),(3,2),(3,3)\}$*, Every* $(T(i,j), I(i,j), F(i,j))$*-interval neutrosophic ideal* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *of* $(X,*,0)$ *satisfies*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_{\inf}(0) \\ \mathcal{I}[T]_{\sup}(x) = \mathcal{I}[T]_{\sup}(0) \\ \mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_{\inf}(0) \\ \mathcal{I}[I]_{\sup}(x) = \mathcal{I}[I]_{\sup}(0) \\ \mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_{\inf}(0) \\ \mathcal{I}[F]_{\sup}(x) = \mathcal{I}[F]_{\sup}(0) \end{cases} \tag{3.5}$$

for all $x, y \in X$ *with* $x \leq y$.

Proof. Assume that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(2,3), I(2,3), F(2,3))$-interval neutrosophic ideal of $(X,*,0)$. Then $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 2-fuzzy ideals of $(X,*,0)$, and $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 3-fuzzy ideals of $(X,*,0)$. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and thus

$$\begin{aligned} \mathcal{I}[T]_{\inf}(x) &\leq \min\{\mathcal{I}[T]_{\inf}(x*y), \mathcal{I}[T]_{\inf}(y)\} \\ &= \min\{\mathcal{I}[T]_{\inf}(0), \mathcal{I}[T]_{\inf}(y)\} = \mathcal{I}[T]_{\inf}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[T]_{\sup}(x) &\geq \max\{\mathcal{I}[T]_{\sup}(x*y), \mathcal{I}[T]_{\sup}(y)\} \\ &= \max\{\mathcal{I}[T]_{\sup}(0), \mathcal{I}[T]_{\sup}(y)\} = \mathcal{I}[T]_{\inf}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[I]_{\inf}(x) &\leq \min\{\mathcal{I}[I]_{\inf}(x*y), \mathcal{I}[I]_{\inf}(y)\} \\ &= \min\{\mathcal{I}[I]_{\inf}(0), \mathcal{I}[I]_{\inf}(y)\} = \mathcal{I}[I]_{\inf}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[I]_{\sup}(x) &\geq \max\{\mathcal{I}[I]_{\sup}(x*y), \mathcal{I}[I]_{\sup}(y)\} \\ &= \max\{\mathcal{I}[I]_{\sup}(0), \mathcal{I}[I]_{\sup}(y)\} = \mathcal{I}[I]_{\inf}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[F]_{\inf}(x) &\leq \min\{\mathcal{I}[F]_{\inf}(x*y), \mathcal{I}[F]_{\inf}(y)\} \\ &= \min\{\mathcal{I}[F]_{\inf}(0), \mathcal{I}[F]_{\inf}(y)\} = \mathcal{I}[F]_{\inf}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[F]_{\sup}(x) &\geq \max\{\mathcal{I}[F]_{\sup}(x*y), \mathcal{I}[F]_{\sup}(y)\} \\ &= \max\{\mathcal{I}[F]_{\sup}(0), \mathcal{I}[F]_{\sup}(y)\} = \mathcal{I}[F]_{\inf}(0). \end{aligned}$$

It follows that $\mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_{\inf}(0)$, $\mathcal{I}[T]_{\sup}(x) = \mathcal{I}[T]_{\sup}(0)$, $\mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_{\inf}(0)$, $\mathcal{I}[I]_{\sup}(x) = \mathcal{I}[I]_{\sup}(0)$, $\mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_{\inf}(0)$ and $\mathcal{I}[F]_{\sup}(x) = \mathcal{I}[F]_{\sup}(0)$ for all $x, y \in X$ with $x \leq y$. Similarly, we can verify that (3.5) is true for $(i,j) \in \{(2,2),(3,2),(3,3)\}$. □

Using the similar way to the proof of Propositions 3.11 and 3.13, we have the following proposition.

Proposition 3.14. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X,*,0)$*, we have the following assertions:*

(1) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(1,j), I(1,j), F(1,j))$*-interval neutrosophic ideal of* $(X,*,0)$ *for* $j \in \{2,3\}$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \geq \mathcal{I}[T]_{\inf}(y) \\ \mathcal{I}[T]_{\sup}(x) = \mathcal{I}[T]_{\sup}(0) \\ \mathcal{I}[I]_{\inf}(x) \geq \mathcal{I}[I]_{\inf}(y) \\ \mathcal{I}[I]_{\sup}(x) = \mathcal{I}[I]_{\sup}(0) \\ \mathcal{I}[F]_{\inf}(x) \geq \mathcal{I}[F]_{\inf}(y) \\ \mathcal{I}[F]_{\sup}(x) = \mathcal{I}[F]_{\sup}(0) \end{cases} \tag{3.6}$$

for all $x, y \in X$ *with* $x \leq y$.

(2) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(i,1), I(i,1), F(i,1))$*-*

interval neutrosophic ideal of $(X, *, 0)$ *for* $i \in \{2, 3\}$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_{\inf}(0) \\ \mathcal{I}[T]_{\sup}(x) \geq \mathcal{I}[T]_{\sup}(y) \\ \mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_{\inf}(0) \\ \mathcal{I}[I]_{\sup}(x) \geq \mathcal{I}[I]_{\sup}(y) \\ \mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_{\inf}(0) \\ \mathcal{I}[F]_{\sup}(x) \geq \mathcal{I}[F]_{\sup}(y) \end{cases} \tag{3.7}$$

for all $x, y \in X$ *with* $x \leq y$*.*

(3) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(i,4),\ I(i,4),\ F(i,4))$*-interval neutrosophic ideal of* $(X, *, 0)$ *for* $i \in \{2, 3\}$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_{\inf}(0) \\ \mathcal{I}[T]_{\sup}(x) \leq \mathcal{I}[T]_{\sup}(y) \\ \mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_{\inf}(0) \\ \mathcal{I}[I]_{\sup}(x) \leq \mathcal{I}[I]_{\sup}(y) \\ \mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_{\inf}(0) \\ \mathcal{I}[F]_{\sup}(x) \leq \mathcal{I}[F]_{\sup}(y) \end{cases} \tag{3.8}$$

for all $x, y \in X$ *with* $x \leq y$*.*

(4) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,j),\ I(4,j),\ F(4,j))$*-interval neutrosophic ideal of* $(X, *, 0)$ *for* $j \in \{2, 3\}$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \leq \mathcal{I}[T]_{\inf}(y) \\ \mathcal{I}[T]_{\sup}(x) = \mathcal{I}[T]_{\sup}(0) \\ \mathcal{I}[I]_{\inf}(x) \leq \mathcal{I}[I]_{\inf}(y) \\ \mathcal{I}[I]_{\sup}(x) = \mathcal{I}[I]_{\sup}(0) \\ \mathcal{I}[F]_{\inf}(x) \leq \mathcal{I}[F]_{\inf}(y) \\ \mathcal{I}[F]_{\sup}(x) = \mathcal{I}[F]_{\sup}(0) \end{cases} \tag{3.9}$$

for all $x, y \in X$ *with* $x \leq y$*.*

Proposition 3.15. *Every* $(T(1,4),\ I(1,4),\ F(1,4))$*-interval neutrosophic ideal* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *of* $(X, *, 0)$ *satisfies*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \geq \min\{\mathcal{I}[T]_{\inf}(y), \mathcal{I}[T]_{\inf}(z)\} \\ \mathcal{I}[T]_{\sup}(x) \leq \max\{\mathcal{I}[T]_{\sup}(y), \mathcal{I}[T]_{\sup}(z)\} \\ \mathcal{I}[I]_{\inf}(x) \geq \min\{\mathcal{I}[I]_{\inf}(y), \mathcal{I}[I]_{\inf}(z)\} \\ \mathcal{I}[I]_{\sup}(x) \leq \max\{\mathcal{I}[I]_{\sup}(y), \mathcal{I}[I]_{\sup}(z)\} \\ \mathcal{I}[F]_{\inf}(x) \geq \min\{\mathcal{I}[F]_{\inf}(y), \mathcal{I}[F]_{\inf}(z)\} \\ \mathcal{I}[F]_{\sup}(x) \leq \max\{\mathcal{I}[F]_{\sup}(y), \mathcal{I}[F]_{\sup}(z)\} \end{cases} \tag{3.10}$$

for all $x, y, z \in X$ *with* $x * y \leq z$*.*

Proof. Let $x, y, z \in X$ be such that $x*y \leq z$. Then $(x*y)*z = 0$, and so

$$\begin{aligned} \mathcal{I}[T]_{\inf}(x) &\geq \min\{\mathcal{I}[T]_{\inf}(x*y), \mathcal{I}[T]_{\inf}(y)\} \\ &\geq \min\{\min\{\mathcal{I}[T]_{\inf}((x*y)*z), \mathcal{I}[T]_{\inf}(z)\}, \\ &\qquad \mathcal{I}[T]_{\inf}(y)\} \\ &= \min\{\min\{\mathcal{I}[T]_{\inf}(0), \mathcal{I}[T]_{\inf}(z)\}, \mathcal{I}[T]_{\inf}(y)\} \\ &= \min\{\mathcal{I}[T]_{\inf}(y), \mathcal{I}[T]_{\inf}(z)\}, \end{aligned}$$

$$\begin{aligned} \mathcal{I}[T]_{\sup}(x) &\leq \max\{\mathcal{I}[T]_{\sup}(x*y), \mathcal{I}[T]_{\sup}(y)\} \\ &\leq \max\{\max\{\mathcal{I}[T]_{\sup}((x*y)*z), \mathcal{I}[T]_{\sup}(z)\}, \\ &\qquad \mathcal{I}[T]_{\sup}(y)\} \\ &= \max\{\max\{\mathcal{I}[T]_{\sup}(0), \mathcal{I}[T]_{\sup}(z)\}, \mathcal{I}[T]_{\sup}(y)\} \\ &= \max\{\mathcal{I}[T]_{\sup}(y), \mathcal{I}[T]_{\sup}(z)\}, \end{aligned}$$

$$\begin{aligned} \mathcal{I}[I]_{\inf}(x) &\geq \min\{\mathcal{I}[I]_{\inf}(x*y), \mathcal{I}[I]_{\inf}(y)\} \\ &\geq \min\{\min\{\mathcal{I}[I]_{\inf}((x*y)*z), \mathcal{I}[I]_{\inf}(z)\}, \\ &\qquad \mathcal{I}[I]_{\inf}(y)\} \\ &= \min\{\min\{\mathcal{I}[I]_{\inf}(0), \mathcal{I}[I]_{\inf}(z)\}, \mathcal{I}[I]_{\inf}(y)\} \\ &= \min\{\mathcal{I}[I]_{\inf}(y), \mathcal{I}[I]_{\inf}(z)\}, \end{aligned}$$

$$\begin{aligned} \mathcal{I}[I]_{\sup}(x) &\leq \max\{\mathcal{I}[I]_{\sup}(x*y), \mathcal{I}[I]_{\sup}(y)\} \\ &\leq \max\{\max\{\mathcal{I}[I]_{\sup}((x*y)*z), \mathcal{I}[I]_{\sup}(z)\}, \\ &\qquad \mathcal{I}[I]_{\sup}(y)\} \\ &= \max\{\max\{\mathcal{I}[I]_{\sup}(0), \mathcal{I}[I]_{\sup}(z)\}, \mathcal{I}[I]_{\sup}(y)\} \\ &= \max\{\mathcal{I}[I]_{\sup}(y), \mathcal{I}[I]_{\sup}(z)\}, \end{aligned}$$

$$\begin{aligned} \mathcal{I}[F]_{\inf}(x) &\geq \min\{\mathcal{I}[F]_{\inf}(x*y), \mathcal{I}[F]_{\inf}(y)\} \\ &\geq \min\{\min\{\mathcal{I}[F]_{\inf}((x*y)*z), \mathcal{I}[F]_{\inf}(z)\}, \\ &\qquad \mathcal{I}[F]_{\inf}(y)\} \\ &= \min\{\min\{\mathcal{I}[F]_{\inf}(0), \mathcal{I}[F]_{\inf}(z)\}, \mathcal{I}[F]_{\inf}(y)\} \\ &= \min\{\mathcal{I}[F]_{\inf}(y), \mathcal{I}[F]_{\inf}(z)\}, \end{aligned}$$

$$\begin{aligned} \mathcal{I}[F]_{\sup}(x) &\leq \max\{\mathcal{I}[F]_{\sup}(x*y), \mathcal{I}[F]_{\sup}(y)\} \\ &\leq \max\{\max\{\mathcal{I}[F]_{\sup}((x*y)*z), \mathcal{I}[F]_{\sup}(z)\}, \\ &\qquad \mathcal{I}[F]_{\sup}(y)\} \\ &= \max\{\max\{\mathcal{I}[F]_{\sup}(0), \mathcal{I}[F]_{\sup}(z)\}, \mathcal{I}[F]_{\sup}(y)\} \\ &= \max\{\mathcal{I}[F]_{\sup}(y), \mathcal{I}[F]_{\sup}(z)\}. \end{aligned}$$

This completes the proof. □

Using the similar way to the proof of Proposition 3.15, we have the following proposition.

Proposition 3.16. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X, *, 0)$*, we have the following assertions:*

(1) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(1,1),\ I(1,1),\ F(1,1))$-

interval neutrosophic ideal of $(X, *, 0)$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \geq \min\{\mathcal{I}[T]_{\inf}(y), \mathcal{I}[T]_{\inf}(z)\} \\ \mathcal{I}[T]_{\sup}(x) \geq \max\{\mathcal{I}[T]_{\sup}(y), \mathcal{I}[T]_{\sup}(z)\} \\ \mathcal{I}[I]_{\inf}(x) \geq \min\{\mathcal{I}[I]_{\inf}(y), \mathcal{I}[I]_{\inf}(z)\} \\ \mathcal{I}[I]_{\sup}(x) \geq \max\{\mathcal{I}[I]_{\sup}(y), \mathcal{I}[I]_{\sup}(z)\} \\ \mathcal{I}[F]_{\inf}(x) \geq \min\{\mathcal{I}[F]_{\inf}(y), \mathcal{I}[F]_{\inf}(z)\} \\ \mathcal{I}[F]_{\sup}(x) \geq \max\{\mathcal{I}[F]_{\sup}(y), \mathcal{I}[F]_{\sup}(z)\} \end{cases}$$

for all $x, y, z \in X$ *with* $x * y \leq z$.

(2) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,1), I(4,1), F(4,1))$*-interval neutrosophic ideal of* $(X, *, 0)$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \leq \min\{\mathcal{I}[T]_{\inf}(y), \mathcal{I}[T]_{\inf}(z)\} \\ \mathcal{I}[T]_{\sup}(x) \geq \max\{\mathcal{I}[T]_{\sup}(y), \mathcal{I}[T]_{\sup}(z)\} \\ \mathcal{I}[I]_{\inf}(x) \leq \min\{\mathcal{I}[I]_{\inf}(y), \mathcal{I}[I]_{\inf}(z)\} \\ \mathcal{I}[I]_{\sup}(x) \geq \max\{\mathcal{I}[I]_{\sup}(y), \mathcal{I}[I]_{\sup}(z)\} \\ \mathcal{I}[F]_{\inf}(x) \leq \min\{\mathcal{I}[F]_{\inf}(y), \mathcal{I}[F]_{\inf}(z)\} \\ \mathcal{I}[F]_{\sup}(x) \geq \max\{\mathcal{I}[F]_{\sup}(y), \mathcal{I}[F]_{\sup}(z)\} \end{cases}$$

for all $x, y, z \in X$ *with* $x * y \leq z$.

(3) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,4), I(4,4), F(4,4))$*-interval neutrosophic ideal of* $(X, *, 0)$*, then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \leq \min\{\mathcal{I}[T]_{\inf}(y), \mathcal{I}[T]_{\inf}(z)\} \\ \mathcal{I}[T]_{\sup}(x) \leq \max\{\mathcal{I}[T]_{\sup}(y), \mathcal{I}[T]_{\sup}(z)\} \\ \mathcal{I}[I]_{\inf}(x) \leq \min\{\mathcal{I}[I]_{\inf}(y), \mathcal{I}[I]_{\inf}(z)\} \\ \mathcal{I}[I]_{\sup}(x) \leq \max\{\mathcal{I}[I]_{\sup}(y), \mathcal{I}[I]_{\sup}(z)\} \\ \mathcal{I}[F]_{\inf}(x) \leq \min\{\mathcal{I}[F]_{\inf}(y), \mathcal{I}[F]_{\inf}(z)\} \\ \mathcal{I}[F]_{\sup}(x) \leq \max\{\mathcal{I}[F]_{\sup}(y), \mathcal{I}[F]_{\sup}(z)\} \end{cases}$$

for all $x, y, z \in X$ *with* $x * y \leq z$.

Proposition 3.17. *For every* $(i, j) \in \{(2,2), (2,3), (3,2), (3,3)\}$*, Every* $(T(i,j), I(i,j), F(i,j))$*-interval neutrosophic ideal* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *of* $(X, *, 0)$ *satisfies*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_{\inf}(0) \\ \mathcal{I}[T]_{\sup}(x) = \mathcal{I}[T]_{\sup}(0) \\ \mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_{\inf}(0) \\ \mathcal{I}[I]_{\sup}(x) = \mathcal{I}[I]_{\sup}(0) \\ \mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_{\inf}(0) \\ \mathcal{I}[F]_{\sup}(x) = \mathcal{I}[F]_{\sup}(0) \end{cases} \tag{3.11}$$

for all $x, y, z \in X$ *with* $x * y \leq z$.

Proof. Assume that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(2,3), I(2,3), F(2,3))$-interval neutrosophic ideal of $(X, *, 0)$. Then $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 2-fuzzy ideals of $(X, *, 0)$, and $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 3-fuzzy ideals of $(X, *, 0)$. Let $x, y, z \in X$ be such that $x * y \leq z$. Then $(x * y) * z = 0$, and thus

$$\begin{aligned} \mathcal{I}[T]_{\inf}(x) &\leq \min\{\mathcal{I}[T]_{\inf}(x * y), \mathcal{I}[T]_{\inf}(y)\} \\ &\leq \min\{\min\{\mathcal{I}[T]_{\inf}((x * y) * z), \mathcal{I}[T]_{\inf}(z)\}, \\ &\qquad \mathcal{I}[T]_{\inf}(y)\} \\ &= \min\{\min\{\mathcal{I}[T]_{\inf}(0), \mathcal{I}[T]_{\inf}(z)\}, \mathcal{I}[T]_{\inf}(y)\} \\ &= \mathcal{I}[T]_{\inf}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[T]_{\sup}(x) &\geq \max\{\mathcal{I}[T]_{\sup}(x * y), \mathcal{I}[T]_{\sup}(y)\} \\ &\geq \max\{\max\{\mathcal{I}[T]_{\sup}((x * y) * z), \mathcal{I}[T]_{\sup}(z)\}, \\ &\qquad \mathcal{I}[T]_{\sup}(y)\} \\ &= \max\{\max\{\mathcal{I}[T]_{\sup}(0), \mathcal{I}[T]_{\sup}(z)\}, \mathcal{I}[T]_{\sup}(y)\} \\ &= \mathcal{I}[T]_{\sup}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[I]_{\inf}(x) &\leq \min\{\mathcal{I}[I]_{\inf}(x * y), \mathcal{I}[I]_{\inf}(y)\} \\ &\leq \min\{\min\{\mathcal{I}[I]_{\inf}((x * y) * z), \mathcal{I}[I]_{\inf}(z)\}, \\ &\qquad \mathcal{I}[I]_{\inf}(y)\} \\ &= \min\{\min\{\mathcal{I}[I]_{\inf}(0), \mathcal{I}[I]_{\inf}(z)\}, \mathcal{I}[I]_{\inf}(y)\} \\ &= \mathcal{I}[I]_{\inf}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[I]_{\sup}(x) &\geq \max\{\mathcal{I}[I]_{\sup}(x * y), \mathcal{I}[I]_{\sup}(y)\} \\ &\geq \max\{\max\{\mathcal{I}[I]_{\sup}((x * y) * z), \mathcal{I}[I]_{\sup}(z)\}, \\ &\qquad \mathcal{I}[I]_{\sup}(y)\} \\ &= \max\{\max\{\mathcal{I}[I]_{\sup}(0), \mathcal{I}[I]_{\sup}(z)\}, \mathcal{I}[I]_{\sup}(y)\} \\ &= \mathcal{I}[I]_{\sup}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[F]_{\inf}(x) &\leq \min\{\mathcal{I}[F]_{\inf}(x * y), \mathcal{I}[F]_{\inf}(y)\} \\ &\leq \min\{\min\{\mathcal{I}[F]_{\inf}((x * y) * z), \mathcal{I}[F]_{\inf}(z)\}, \\ &\qquad \mathcal{I}[F]_{\inf}(y)\} \\ &= \min\{\min\{\mathcal{I}[F]_{\inf}(0), \mathcal{I}[F]_{\inf}(z)\}, \mathcal{I}[F]_{\inf}(y)\} \\ &= \mathcal{I}[F]_{\inf}(0), \end{aligned}$$

$$\begin{aligned} \mathcal{I}[F]_{\sup}(x) &\geq \max\{\mathcal{I}[F]_{\sup}(x * y), \mathcal{I}[F]_{\sup}(y)\} \\ &\geq \max\{\max\{\mathcal{I}[F]_{\sup}((x * y) * z), \mathcal{I}[F]_{\sup}(z)\}, \\ &\qquad \mathcal{I}[F]_{\sup}(y)\} \\ &= \max\{\max\{\mathcal{I}[F]_{\sup}(0), \mathcal{I}[F]_{\sup}(z)\}, \mathcal{I}[F]_{\sup}(y)\} \\ &= \mathcal{I}[F]_{\sup}(0). \end{aligned}$$

Since $\mathcal{I}[T]_{\inf}(0) \leq \mathcal{I}[T]_{\inf}(x)$, $\mathcal{I}[T]_{\sup}(0) \geq \mathcal{I}[T]_{\sup}(x)$, $\mathcal{I}[I]_{\inf}(0) \leq \mathcal{I}[I]_{\inf}(x)$, $\mathcal{I}[I]_{\sup}(0) \geq \mathcal{I}[I]_{\sup}(x)$, $\mathcal{I}[F]_{\inf}(0) \leq \mathcal{I}[F]_{\inf}(x)$ and $\mathcal{I}[F]_{\sup}(0) \geq \mathcal{I}[F]_{\sup}(x)$, it follows that $\mathcal{I}[T]_{\inf}(0) = \mathcal{I}[T]_{\inf}(x)$, $\mathcal{I}[T]_{\sup}(0) = \mathcal{I}[T]_{\sup}(x)$, $\mathcal{I}[I]_{\inf}(0) = \mathcal{I}[I]_{\inf}(x)$, $\mathcal{I}[I]_{\sup}(0) = \mathcal{I}[I]_{\sup}(x)$, $\mathcal{I}[F]_{\inf}(0) =$

$\mathcal{I}[F]_{\inf}(x)$ and $\mathcal{I}[F]_{\sup}(0) = \mathcal{I}[F]_{\sup}(x)$. Similarly, we can verify that (3.11) is true for $(i, j) \in \{(2, 2), (3, 2), (3, 3)\}$. □

Using the similar way to the proof of Propositions 3.15 and 3.17, we have the following proposition.

Proposition 3.18. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X, *, 0)$, *we have the following assertions:*

(1) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(1, j), I(1, j), F(1, j))$-*interval neutrosophic ideal of* $(X, *, 0)$ *for* $j \in \{2, 3\}$, *then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \geq \min\{\mathcal{I}[T]_{\inf}(y), \mathcal{I}[T]_{\inf}(z)\} \\ \mathcal{I}[T]_{\sup}(x) = \mathcal{I}[T]_{\sup}(0) \\ \mathcal{I}[I]_{\inf}(x) \geq \min\{\mathcal{I}[I]_{\inf}(y), \mathcal{I}[I]_{\inf}(z)\} \\ \mathcal{I}[I]_{\sup}(x) = \mathcal{I}[I]_{\sup}(0) \\ \mathcal{I}[F]_{\inf}(x) \geq \min\{\mathcal{I}[F]_{\inf}(y), \mathcal{I}[F]_{\inf}(z)\} \\ \mathcal{I}[F]_{\sup}(x) = \mathcal{I}[F]_{\sup}(0) \end{cases}$$

for all $x, y, z \in X$ *with* $x * y \leq z$.

(2) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(i, 1), I(i, 1), F(i, 1))$-*interval neutrosophic ideal of* $(X, *, 0)$ *for* $i \in \{2, 3\}$, *then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_{\inf}(0) \\ \mathcal{I}[T]_{\sup}(x) \geq \min\{\mathcal{I}[T]_{\sup}(y), \mathcal{I}[T]_{\sup}(z)\} \\ \mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_{\inf}(0) \\ \mathcal{I}[I]_{\sup}(x) \geq \min\{\mathcal{I}[I]_{\sup}(y), \mathcal{I}[I]_{\sup}(z)\} \\ \mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_{\inf}(0) \\ \mathcal{I}[F]_{\sup}(x) \geq \min\{\mathcal{I}[F]_{\sup}(y), \mathcal{I}[F]_{\sup}(z)\} \end{cases}$$

for all $x, y, z \in X$ *with* $x * y \leq z$.

(3) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(i, 4), I(i, 4), F(i, 4))$-*interval neutrosophic ideal of* $(X, *, 0)$ *for* $i \in \{2, 3\}$, *then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_{\inf}(0) \\ \mathcal{I}[T]_{\sup}(x) \leq \max\{\mathcal{I}[T]_{\sup}(y), \mathcal{I}[T]_{\sup}(z)\} \\ \mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_{\inf}(0) \\ \mathcal{I}[I]_{\sup}(x) \leq \max\{\mathcal{I}[I]_{\sup}(y), \mathcal{I}[I]_{\sup}(z)\} \\ \mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_{\inf}(0) \\ \mathcal{I}[F]_{\sup}(x) \leq \max\{\mathcal{I}[F]_{\sup}(y), \mathcal{I}[F]_{\sup}(z)\} \end{cases}$$

for all $x, y, z \in X$ *with* $x * y \leq z$.

(4) *If* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4, j), I(4, j), F(4, j))$-*interval neutrosophic ideal of* $(X, *, 0)$ *for* $j \in \{2, 3\}$, *then*

$$\begin{cases} \mathcal{I}[T]_{\inf}(x) \leq \max\{\mathcal{I}[T]_{\inf}(y), \mathcal{I}[T]_{\inf}(z)\} \\ \mathcal{I}[T]_{\sup}(x) = \mathcal{I}[T]_{\sup}(0) \\ \mathcal{I}[I]_{\inf}(x) \leq \max\{\mathcal{I}[I]_{\inf}(y), \mathcal{I}[I]_{\inf}(z)\} \\ \mathcal{I}[I]_{\sup}(x) = \mathcal{I}[I]_{\sup}(0) \\ \mathcal{I}[F]_{\inf}(x) \leq \max\{\mathcal{I}[F]_{\inf}(y), \mathcal{I}[F]_{\inf}(z)\} \\ \mathcal{I}[F]_{\sup}(x) = \mathcal{I}[F]_{\sup}(0) \end{cases}$$

for all $x, y, z \in X$ *with* $x * y \leq z$.

4 Acknowledgments

The firs author, S.Z. Song, was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2016R1D1A1B02006812).

References

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.

[2] Y.S. Huang, BCI-algebra, Science Press, Beijing, 2006.

[3] Y. Imai and K. Iséki, On axiom systems of propositional calculi, Proc. Japan Acad. 42 (1966) 19–21.

[4] K. Iséki, An algebra related with a propositional calculus, Proc. Japan Acad. 42 (1966) 26–29.

[5] Y.B. Jun, S.J. Kim and F. Smarandache, Interval neutrosophic sets with applications in BCK/BCI-algebras, New Mathematics & Natural Computation (submitted).

[6] J. Meng and Y.B. Jun, BCK-algebras, Kyungmoon Sa Co., Seoul, 1994.

[7] F. Smarandache, Neutrosophy, Neutrosophic Probability, Set, and Logic, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998. http://fs.gallup.unm.edu/eBook-neutrosophics6.pdf (last edition online).

[8] F. Smarandache, A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability, (American Reserch Press, Rehoboth, NM, 1999).

[9] F. Smarandache, Neutrosophic set-a generalization of the intuitionistic fuzzy set, Int. J. Pure Appl. Math. 24(3) (2005) 287–297.

[10] S.Z. Song, S.J. Kim and Y.B. Jun, Hyperfuzzy ideals in BCK/BCI-algebras, Mathematics 2017, 5, 81.

[11] H. Wang, F. Smarandache, Y.Q. Zhang and R. Sunderraman, Interval Neutrosophic Sets and Logic: Theory and Applications in Computing, Hexis; Neutrosophic book series, No. 5, 2005.

[12] H. Wang, F. Smarandache, Y.Q. Zhang and R. Sunderraman, Interval Neutrosophic Sets and Logic: Theory and Applications in Computing, (Hexis, Phoenix, Ariz, USA, 2005).

[13] H. Wang, Y. Zhang and R. Sunderraman, Truth-value based interval neutrosophic sets, Granular Computing, 2005 IEEE International Conference, 1(2005) 274–277 DOI: 10.1109/GRC.2005.1547284.

[14] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

Received: October 9, 2017. Accepted: October 27, 2017.