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Neutrosophic Linear Diophantine Equations With Two Variables

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Abstract: This paper is devoted to study for the first time the neutrosophic linear Diophantine equations with two variables in the neutrosophic ring of integers $Z(I)$, and refined neutrosophic ring of integers $Z(I_1,I_2)$. This work introduces an algorithm to solve the linear Diophantine equation $AX+BY=C$ in $Z(I)$, $Z(I_1,I_2)$.

Keywords: Neutrosophic ring, refined neutrosophic ring, neutrosophic linear Diophantine equation, refined neutrosophic linear Diophantine equation.

1. Introduction

Neutrosophy is a new kind of logic founded by F. Smarandache to deal with the indeterminacy in nature, mathematics and reality. It plays an interesting role in the progression of algebraic studies. Many neutrosophical algebraic structures were introduced and handled such as neutrosophic groups, neutrosophic rings, refined neutrosophic rings, and n-refined neutrosophic rings. See [1,2,3,4,5,6,8,10,11]. On the other hand neutrosophic sets were used to deal with health care [12], decision making [13], financial goals [14], computer science, and industry [15,16,17,18,20]. Recently, the interesting in neutrosophic number theory has increased. Relationships between neutrosophic rings and refined neutrosophic rings were studied in [1]. Also, some number theoretical concepts were presented in the neutrosophic ring of integers Z(I) such as division, primes and factors [7]. The theory of neutrosophic numbers is concerning with properties of neutrosophic integers, by the same, refined neutrosophic number theory is dealing with the properties of refined neutrosophic integers. One of the most important number theoretical concepts is the concept of linear Diophantine equations, these equations were solved in the case of classical integers [9]. Through this paper, we aim to find an algorithm to solve such equations in the case of neutrosophic integers and refined neutrosophic integers by using classical number theoretical methods, where a relationship between neutrosophic equations and classical equations is described.

This work continues the efforts of establishing neutrosophical number theory. It studies the concept of linear Diophantine equations with two variables with respect to neutrosophic integers and refined neutrosophic integers. We determine the sufficient condition for the solvability of these equations and introduce an algorithm which gives the solution in easy way.

2. Preliminaries

Theorem 2.1: [9]

Let $AX + BY = C$ be a linear Diophantine equation, where $A, B, C \in Z$. Then it is solvable if and only if $\gcd(A, B) | C$. To check the solution's form of classical linear Diophantine equation based on Euclidean division theorem in the ring of integers Z, see [9].

Definition 2.2:[6]

Let (R,+,$\times$) be a ring, R(I)={a+bI ; a,b$\in R$} is called the neutrosophic ring where I is a neutrosophic element with condition $I^2 = I$.

If R=Z, then R(I) is called the neutrosophic ring of integers.

Definition 2.3:[4]

Let (R,+,$\times$) be a ring, (R(I_1, I_2) , + ,$\times$) is called a refined neutrosophic ring generated by R , I_1, I_2.

If R=Z, then (R(I_1, I_2) , + ,$\times$) is called the refined neutrosophic ring of integers.

Definition 2.4: [5]

Let (G,*) be a group . Then the neutrosophic group is generated by G and I under * denoted by N(G)={$< G \cup I >$,*}.

I is called the indeterminate (neutrosophic element) with the property $I^2 = I$.

3. Main results

Definition 3.1:

Let $Z(I) = \{a + bI;\ a, b \in Z\}$ be the neutrosophic ring of integers. The neutrosophic linear Diophantine equation with two variables is defined as follows:

$AX + BY = C;\ A, B, C \in Z(I)$.

Theorem 3.2:

Let $Z(I) = \{a + bI;\ a, b \in Z\}$ be the neutrosophic ring of integers. The neutrosophic linear Diophantine equation $AX + BY = C$ with two variables $X = x_1 + x_2 I, Y = y_1 + y_2 I$, where $A = a_1 + a_2 I, B = b_1 + b_2 I$ is equivalent to the following two classical Diophantine equations:

(1) $a_1 x_1 + b_1 y_1 = c_1$.

(2) $(a_1 + a_2)(x_1 + x_2) + (b_1 + b_2)(y_1 + y_2) = c_1 + c_2$.

Proof:

It is sufficient to show that $AX + BY = C$ implies (1) and (2).

$AX + BY = C$ is equivalent to:

$(a_1 + a_2 I)(x_1 + x_2 I) + (b_1 + b_2 I)(y_1 + y_2 I) = c_1 + c_2 I$, by easy computing we find

$[a_1 x_1 + b_1 y_1] + [a_1 x_2 + a_2 x_1 + a_2 x_2 + b_1 y_2 + b_2 y_1 + b_2 y_2]I = c_1 + c_2 I$, hence

$a_1 x_1 + b_1 y_1 = c_1, and\ a_1 x_2 + a_2 x_1 + a_2 x_2 + b_1 y_2 + b_2 y_1 + b_2 y_2 = c_2$. We can see that we get equation (1). For equation (2) we take

$a_1 x_2 + a_2 x_1 + a_2 x_2 + b_1 y_2 + b_2 y_1 + b_2 y_2 = c_2$, by adding equation (1) to the two sides we obtain

$a_1 x_1 + b_1 y_1 + a_1 x_2 + a_2 x_1 + a_2 x_2 + b_1 y_2 + b_2 y_1 + b_2 y_2 = c_1 + c_2$, which implies equation (2)

$(a_1 + a_2)(x_1 + x_2) + (b_1 + b_2)(y_1 + y_2) = c_1 + c_2$.

The following theorem determines the criteria for the solvability of neutrosophic linear Diophantine equation.

Theorem 3.3:

Let $Z(I) = \{a + bI;\ a, b \in Z\}$ be the neutrosophic ring of integers. The neutrosophic linear Diophantine equation $AX + BY = C$ with two variables $X = x_1 + x_2 I, Y = y_1 + y_2 I$ and $A = a_1 + a_2 I, B = b_1 + b_2 I$ is solvable if and only if $\gcd(a_1, b_1) \mid c_1$, $\gcd(a_1 + a_2, b_1 + b_2) \mid c_1 + c_2$.

Proof:

By Theorem 2.1 , we can solve the neutrosophic linear Diophantine equation by solving (1) and (2).

Equation (1) is solvable if and only if $\gcd(a_1, b_1) \mid c_1$ according to Theorem 2.1 .

Equation (2) is solvable if and only if $\gcd(a_1 + a_2, b_1 + b_2) \mid c_1 + c_2$ according to Theorem 2.1 .

Thus our proof is complete.

Example 3.4:

(a) The neutrosophic Diophantine equation $(2 + 2I)X + (3 + 4I)Y = 5 + 5I$ is solvable, that is because

$\gcd(2,3) \mid 5, and \gcd(4,7) \mid 10.$

(b) The neutrosophic Diophantine equation $(2 + 3I)X + (4 + 5I)Y = 5 + I$ is not solvable, since

$\gcd(2,4) = 2$ does not divide 5.

Now, we describe an algorithm to solve a neutrosophic linear Diophantine equation $AX + BY = C$.

Remark 3.5:

Let $Z(I) = \{a + bI;\ a, b \in Z\}$ be the neutrosophic ring of integers. Consider a neutrosophic linear Diophantine equation $AX + BY = C$ with two variables $X = x_1 + x_2 I, Y = y_1 + y_2 I$ and $A = a_1 + a_2 I, B = b_1 + b_2 I$. To solve this equation follow these steps:

(a) Check the solvability of $AX + BY = C$ by Theorem 3.3 .

(b) Solve $a_1x_1 + b_1y_1 = c_1$.

(c) Solve $(a_1 + a_2)(x_1 + x_2) + (b_1 + b_2)(y_1 + y_2) = c_1 + c_2$.

(d) Compute x_2, y_2.

Example 3.6:

The neutrosophic Diophantine equation $(2 + 2I)X + (3 + 4I)Y = 5 + 5I$ is solvable according to Example 3.4 .

$2x_1 + 3y_1 = 5$ is a classical linear Diophantine equation. It has a solution $x_1 = 4, y_1 = -1$.

$(2 + 2)(x_1 + x_2) + (3 + 4)(y_1 + y_2) = 5 + 5$, i.e $4M + 7N = 10; M = x_1 + x_2, N = y_1 + y_2$. It is a classical linear Diophantine equation with M, N as variables. It has a solution $M = -1, N = 2$.

$x_2 = M - x_1 = -5, y_2 = N - y_1 = 3$, thus the equation $(2 + 2I)X + (3 + 4I)Y = 5 + 5I$ has a solution $X = 4 - 5I, Y = -1 + 3I$.

Definition 3.7:

Let $Z(I_1, I_2) = \{(a, bI_1, cI_2);\ a, b, c \in Z\}$ be the refined neutrosophic ring of integers. The refined neutrosophic linear Diophantine equation with two variables is defined as follows:

$AX + BY = C; A, B, C \in Z(I_1, I_2)$.

Theorem 3.8:

Let $Z(I_1, I_2) = \{(a, bI_1, cI_2);\ a, b, c \in Z\}$ be the refined neutrosophic ring of integers,

$AX + BY = C; A, B, C \in Z(I_1, I_2)$ be a refined neutrosophic linear Diophantine equation, where

$X = (x_0, x_1I_1, x_2I_2), Y = (y_0, y_1I_1, y_2I_2), A = (a_0, a_1I_1, a_2I_2)$,

$B = (b_0, b_1I_1, b_2I_2), C = (c_0, c_1I_1, c_2I_2)$. Then $AX + BY = C$ is equivalent to the following three Diophantine equations:

(1) $a_0x_0 + b_0y_0 = c_0$.

(2) $(a_0 + a_2)(x_0 + x_2) + (b_0 + b_2)(y_0 + y_2) = c_0 + c_2$.

(3) $(a_0 + a_1 + a_2)(x_0 + x_1 + x_2) + (b_0 + b_1 + b_2)(y_0 + y_1 + y_2) = c_0 + c_1 + c_2$.

Proof:

By replacing A, B, C, X, Y we find

$AX = (a_0, a_1I_1, a_2I_2)(x_0, x_1I_1, x_2I_2) =$

$(a_0x_0, [a_0x_1 + a_1x_0 + a_1x_1 + a_1x_2 + a_2x_1]I_1, [a_0x_2 + a_2x_0 + a_2x_2]I_2)$,

$BY = (b_0, b_1I_1, b_2I_2)(y_0, y_1I_1, y_2I_2) =$

$(b_0y_0, [b_0y_1 + b_1y_0 + b_1y_1 + b_1y_2 + b_2y_1]I_1, [b_0y_2 + b_2y_0 + b_2y_2]I_2)$, thus the equation

$AX + BY = C$ implies

(*) $a_0x_0 + b_0y_0 = c_0$. (Equation (1)).

(**) $a_0x_2 + a_2x_0 + a_2x_2 + b_0y_2 + b_2y_0 + b_2y_2 = c_2$.

(***) $a_0x_1 + a_1x_0 + a_1x_1 + a_1x_2 + a_2x_1 + b_0y_1 + b_1y_0 + b_1y_1 + b_1y_2 + b_2y_1 = c_1$.

By adding (*) to (**) we get $(a_0 + a_2)(x_0 + x_2) + (b_0 + b_2)(y_0 + y_2) = c_0 + c_2$. (Equation (2)).

By adding (2) to (***) we get

$(a_0 + a_1 + a_2)(x_0 + x_1 + x_2) + (b_0 + b_1 + b_2)(y_0 + y_1 + y_2) = c_0 + c_1 + c_2$. (Equation (3)).

Theorem 3.9:

Let $Z(I_1, I_2) = \{(a, bI_1, cI_2);\ a, b, c \in Z\}$ be the refined neutrosophic ring of integers,

$AX + BY = C; A, B, C \in Z(I_1, I_2)$ be a refined neutrosophic linear Diophantine equation, where

$X = (x_0, x_1I_1, x_2I_2), Y = (y_0, y_1I_1, y_2I_2), A = (a_0, a_1I_1, a_2I_2)$,

$B = (b_0, b_1I_1, b_2I_2), C = (c_0, c_1I_1, c_2I_2)$. Then $AX + BY = C$ is solvable if and only if:

(a) $\gcd(a_0, b_0) | c_0$.

(b) $\gcd(a_0 + a_2, b_0 + b_2) | c_0 + c_2$.

(c) $\gcd(a_0 + a_1 + a_2, b_0 + b_1 + b_2) | c_0 + c_1 + c_2$.

The proof is similar to that of Theorem 3.3.

Example 3.10:

(a) Consider the refined neutrosophic linear Diophantine equation

$(1,2I_1,3I_2).X + (3,3I_1,8I_2)Y = (2,4I_1,I_2)$, we have

$\gcd(1,3) = 1|2$, $\gcd(1+3,3+8) = \gcd(4,11) = 1|(2+1=3)$,

$\gcd(1+2+3,3+3+8) = \gcd(6,14) = 2$ which does not divide $2+4+1=7$, thus it is not solvable.

(b) Consider the refined neutrosophic linear Diophantine equation

$(1,2I_1,3I_2).X + (3,3I_1,8I_2)Y = (2,4I_1,2I_2)$, we have

$\gcd(1,3) = 1|2$, $\gcd(1+3,3+8) = \gcd(4,11) = 1|(2+2=4)$,

$\gcd(1+2+3,3+3+8) = \gcd(6,14) = 2|\ (2+4+2=8)$. Thus it is solvable.

Remark 3.11:

Let $Z(I_1,I_2) = \{(a,bI_1,cI_2);\ a,b,c \in Z\}$ be the refined neutrosophic ring of integers,

$AX + BY = C; A,B,C \in Z(I_1,I_2)$ be a refined neutrosophic linear Diophantine equation, where

$X = (x_0,x_1I_1,x_2I_2), Y = (y_0,y_1I_1,y_2I_2), A = (a_0,a_1I_1,a_2I_2)$,

$B = (b_0,b_1I_1,b_2I_2), C = (c_0,c_1I_1,c_2I_2)$, we summarize the algorithm of solution as follows:

(a) Check the solvability condition.

(b) Solve $a_0x_0 + b_0y_0 = c_0$.

(c) Solve $(a_0 + a_2)(x_0 + x_2) + (b_0 + b_2)(y_0 + y_2) = c_0 + c_2$.

(d) Compute x_2, y_2.

(e) Solve $(a_0 + a_1 + a_2)(x_0 + x_1 + x_2) + (b_0 + b_1 + b_2)(y_0 + y_1 + y_2) = c_0 + c_1 + c_2$.

(f) Compute x_1, y_1.

Example 3.12:

According to Example 3.10 , we found that $(1,2I_1,3I_2).X + (3,3I_1,8I_2)Y = (2,4I_1,2I_2)$ is solvable.

We consider $x_0 + 3y_0 = 2$. It has a solution $x_0 = -1, y_0 = 1$ We take $(1+3)(x_0 + x_2) + (3+8)(y_0 + y_2) = 2+2$, i.e $4M + 11N = 4; M = x_0 + x_2, and\ N = y_0 + y_2$, it has a solution $M = 1, N = 0$, thus $x_2 = M - x_0 = 2, y_2 = N - y_0 = -1$. The third equation is $(1+2+3)(x_0 + x_1 + x_2) + (3+3+8)(y_0 + y_1 + y_2) = 2+4+2$, i.e

$6S + 14T = 8; S = x_0 + x_1 + x_2, T = y_0 + y_1 + y_2$. It has a solution $S = -1, T = 1$, thus

$x_1 = S - x_0 - x_2 = -2, y_1 = T - y_0 - y_2 = 1$. The solution of

$(1,2I_1,3I_2).X + (3,3I_1,8I_2)Y = (2,4I_1,2I_2)$ is $X = (-1, -2I_1, 2I_2)$, $Y = (1, I_1, -I_2)$.

5. Conclusion

In this paper, we have determined the criteria for the solvability of linear Diophantine equation in the neutrosophic ring of integers and refined neutrosophic rings of integers by finding the relationship between neutrosophic equations and classical equations. Also, we have presented an algorithm which gives a solution of these equations, and constructed some examples to clarify the validity of this work.

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