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A Study on Neutrosophic Zero Rings

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Abstract: Let $N(R, I)$ be a Neutrosopic ring corresponding to the classical ring R and indeterminate I. In this paper, we introduced the Neutrosophic zero rings $N(R, I)^0$ and $N(R^0, I)$ corresponding to the ring R and the zero ring R^0 respectively, and also studied structural properties of these Neutrosophic zero rings. Among many properties, it is shown that $N(R, I) \neq N(R, I)^0$ and $|N(R, I)| = |N(R, I)^0|$. Particularly, we prove that $N(R, I)^0$ is not a Boolean ring and the characteristics of $N(R, I)$ and $N(R, I)^0$ are equal. For every classical ring R, the Neutrosophic zero ring $N(R, I)^0$ is isomorphic to Neutrosophic zero ring $M_2(R, I)^0$ of all 2×2 matrices of the form $\begin{pmatrix} a+bI & -(a+bI) \\ a+bI & -(a+bI) \end{pmatrix}$ with entries from $N(R, I)$. We also find a necessary and sufficient condition for the classical zero rings R^0 and Neutrosophic zero ring $N(R^0, I)$ to be isomorphic under the following actions $r \leftrightarrow \begin{pmatrix} r & -r \\ r & -r \end{pmatrix}$ and $r+sI \leftrightarrow \begin{pmatrix} r+sI & -(r+sI) \\ r+sI & -(r+sI) \end{pmatrix}$.

Keywords: Neutrosophic rings; Neutrosophic zero rings; Neutrosophic square zero matrices; Neutrosophic Boolean rings

1. Introduction

Abstract algebra is largely concerned with the study of abstract sets endowed with one, or, more binary operations along with few axioms. In this paper, we consider one of the basic algebraic structures known as a ring, called a classical ring. A ring $R = (R, +, \cdot)$ is a non-empty set with two binary operations, namely addition (+) and multiplication ($\cdot$) defined on R satisfying some natural axioms, see [1]. A ring $R = (0)$ is called a trivial ring, otherwise R is called nontrivial. A ring R is called commutative if $ab = ba$ for all a and b in R. An element u in R is called a unit if there exists v in R such that $uv = 1 = vu$, where u and v are both multiplicative inverses in R. The set of units of R is denoted by $U(R)$. However, the set $R - U(R)$ is denoted by $Z(R)$ and called zero-divisors of R. For any commutative ring R with unity, we have every non zero elements of R is either unit or, zero divisors. Clearly, $R = U(R) \cup Z(R)$. The Characteristic of R denoted $Char(R)$ is the smallest nonnegative n such that $n \cdot 1 = 0$. If no such n exists then we define the $Char(R) = 0$. Next, a ring R is called cyclic ring if $(R, +)$ is a cyclic group. Every cyclic ring is commutative and these rings have been investigated in [2]. The theory of finite rings occupies a central position in modern mathematics and engineering science. Recently, finite rings play a central role in many research areas such as digital image processing, algebraic coding theory, encryption systems, QUAM signals and linear coding theory; see [4-7].

The notion of zero rings was considered by Buck [2] in 2004. A zero ring R^0 is a triplet $(R^0, +, \cdot)$ where $(R^0, +)$ is an abelian group and $a \cdot b = 0$ for all $a, b \in R^0$. Every zero is a commutative cyclic ring but a cyclic ring need not be a zero ring. For instance, $(Z_6, \oplus, \odot)$ is a cyclic ring but not a zero ring under addition and multiplication modulo 6 .

Neutrosophy is a part of philosophical reasoning, introduced by Smarandache in 1980, which concentrates the origin, nature and extent of neutralities, comparable to their cooperation with particular ideational spectra. Neutrosophy is the premise of Neutrosophic Logic, Neutrosophic likelihood, Neutrosophic set and Neutrosophic realities in [8]. Handling of indeterminacy present in real-world data is introduced in [9, 10] as Neutrosophy. Neutralities and indeterminacies spoken to Neutrosophic Logic have been utilized in the analysis of genuine world and engineering problems. In 2004, the creators Vasantha Kanda Swami and Smarandache presented the ideas of Neutrosophic arithmetical hypothesis and they were utilized in Neutrosophic mathematical structures and build up numerous structures such as Neutrosophic semigroups, groups, rings, fields which are different from classical algebraic structures and are presented and analyzed their application to fuzzy and Neutrosophic models are developed in [11].

Now we begin our attention to the Neutrosophic ring $N(R, I)$, we are considering in this paper. The basic study on Neutrosophic rings was given by Vasantha Kandasamy and Smarandache [11], and there are many interesting properties of Neutrosophic rings available in the literature, see [12-16]. Let I be the indeterminate of the real-world problem with two fundamental properties such as $I^2 = I$ and I^{-1} does not exists. Then generally we define the Neutrosophic set $N(R, I) = \{a + bI : a, b \in R, I^2 = I\}$ which is a nonempty set of Neutrosophic elements $a + bI$ and it is generated by a ring R and indeterminate I under the following Neutrosophic operations.

(1) $(a + bI) + (c + dI) = (a + c) + (b + d)I$ and

(2) $(a + bI)(c + dI) = ac + (ad + bc + bd)I$

for all $a + bI$, $c + dI$ in $N(R, I)$. More specifically, the indeterminate I satisfies the following algebraic properties. (1) $I^2 = I$, (2) $0I = 0$ and $1I = I$ but $I \neq 0,1$, (3) I^{-1} does not exist with respect to Neutrosophic multiplication but $-I = (-1)\mathrm{I}$ exists with respect to Neutrosophic addition such that $\mathrm{I} + (-\mathrm{I}) = 0$ and $-I \neq I$, and (4) $I + I = 2I$ and $I + I \neq I$. Recently, Agboola, Akinola and Oyebola studied further properties of Neutrosophic rings in [13, 14]. In [15-17], Chalapathi and Kiran established relations between units and Neutrosophic units of rings, fields, Neutrosophic rings and Neutrosophic fields. However, we have $|N(R, I)| \geq 4$ for any finite ring R with $|R| > 1$. This clears that $4 \leq |N(R, I)| \leq |R|^2$.

In numerous certifiable circumstances, it is regularly seen that the level of indeterminacy assumes a significant job alongside the fulfillment and disappointment levels of the decision-makers in any decision making process and Internet clients. Because of some uncertainty or dithering, it might important for chiefs to take suppositions from specialists which lead towards a lot of clashing qualities with respect to fulfillment, indeterminacy and dis-fulfillment level of choice makers. So as to feature the previously mentioned understanding, the authors Abdel-Basset et al. [18-20] built up a successful structure which mirrors the truth engaged with any basic decision-making process. In this investigation, a multi-objective nonlinear programming issue has been planned in the assembling framework. Another calculation, Neutrosophic reluctant fluffy programming approach, dependent on single esteemed Neutrosophic reluctant fuzzy decision set has been proposed which contains the idea of indeterminacy reluctant degree alongside truth and lie reluctant degrees of various objectives.

Web of Things associates billions of items and gadgets to outfit a genuine viable open door for the enterprises. Fourth industrial and mechanical upset must guarantee proficient correspondence and work by thinking about the components of expenses and execution. Transition to the fourth industrial and mechanical transformation creates and generates challenges for enterprises. In [21, 22], the authors recognize the fundamental difficulties influencing the change procedure utilizing non-conventional techniques and proposed a hybrid combination between the systematic various leveled process as a Neutrosophic criteria decision-making approach for IoT-based ventures and furthermore Neutrosophic hypothesis to effectively distinguish and deal with the uncertainty and irregularity challenges.

2. Neutrosophic zero rings of rings

In this section, we studied Neutrosophic zero rings of various classical rings and presented their basic properties with many suitable illustrations and examples. First, the language of Neutrosophic element makes it possible to work with indeterminate I and it relationships much as we work with equalities and powers only. Prior to the consideration of Neutrosophic element $a+bI$, the notation $(a+bI)^{-1}$ used for reciprocity relationships but it is not applicable for every element a and b in the classical ring R. So the introduction of a convenient Neutrosophic multiplication notation helped accelerate the development of Neutrosophic theory. For this reason, the Neutrosophic mathematical concepts establish solutions to many problems with indeterminacy.

In working with Neutrosophic multiplications, we will sometimes need to translate them into further Neutrosophic algebraic structures. The following definition is one.

Definition 2.1. Let R be a ring. Then $N(R, I)$ is called a **Neutrosophic zero ring** if the product of any two Neutrosophic elements of $N(R, I)$ is 0, where $0 = 0 + 0I$ is the Neutrosophic additive identity.

For any ring R, there is a Neutrosophic zero ring and is denoted by $N(R, I)^0$. This statement connects the relation $N(R, I) \neq N(R, I)^0$ for every $R \neq (0)$. In particular, if $R = (0)$ then $N(R, I) = (0)$ and $N(R, I)^0 = (0)$. For any ring $R \neq (0)$, the actual construction of Neutrosophic zero rings $N(R, I)^0$ appear below. If R is not a zero ring, then $N(R, I)$ is never a Neutrosophic zero ring. This means that, the only Neutrosophic ring $N(R, I)$ that cannot be described as a Neutrosophic zero ring when R is either finite or infinite. For this reason, the construction of Neutrosophic zero rings depends on the collection Neutrosophic matrices and which are up to Neutrosophic isomorphism. The next definition deals with these constructions.

Definition 2.2. Let $\mathrm{M}_2(R, I)^0$ be the non-empty subset of 2×2 Neutrosophic matrices

$$\mathrm{M}_2(R, I) = \left\{ \begin{pmatrix} a+bI & c+dI \\ e+fI & g+hI \end{pmatrix} : a+bI, c+dI, e+fI, g+hI \in N(R, I) \right\}.$$

Then we define $\mathrm{M}_2(R, I)^0$ as follows

$$\mathrm{M}_2(R,\, I)^0 = \left\{ \begin{pmatrix} a+bI & -(a+bI) \\ a+bI & -(a+bI) \end{pmatrix} : a+bI \in N(R,\, I) \right\}$$

and this collection is called **Neutrosophic square zero matrices**.

Example 2.3. For the ring $Z_2 = \{0,\, 1\}$ under addition and multiplication modulo 2, the Neutrosophic ring and corresponding Neutrosophic square matrices are

$N(Z_{2,}\, I) = \{0,\, 1,\, I,\, 1+I\}$ and $\mathrm{M}_2(\mathrm{Z}_2,\, I)^0 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} I & -I \\ I & -I \end{pmatrix}, \begin{pmatrix} 1+I & -(1+I) \\ 1+I & -(1+I) \end{pmatrix} \right\}$,respectively.

To determine the structure of Neutrosophic zero ring $N(R,\, I)^0$, we must derive a result for determining when an element of $N(R,\, I)^0$ is a Neutrosophic unit, or, Neutrosophic zero divisor. Recall that in a commutative Neutrosophic ring $N(R,\, I)$ a non zero Neutrosophic element $a+bI$ is called a Neutrosophic zero divisor provided there is a non zero Neutrosophic element $c+dI$ in $N(R,\, I)$ such that $(a+bI)(c+dI)=0$. No Neutrosophic element of $N(R,\, I)$ can be both a Neutrosophic unit and Neutrosophic zero divisor, but there are Neutrosophic rings such as $N(\mathrm{Z},\, I)$, $N(\mathrm{Q},\, I)$, $N(\mathrm{R},\, I)$, $N(\mathrm{C},\, I)$ and $N(\mathrm{Z[i]},\, I)$,, with non zero Neutrosophic elements that are neither Neutrosophic units nor Neutrosophic zero divisors, , where Z , Q , R , C and $Z[i]$ are ring of integers, rationals, real numbers, complex numbers, and Gaussian integers, respectively. However, when $N(R,\, I)$ is finite, every non zero Neutrosophic elements of $N(R,\, I)$ is either Neutrosophic unit, or, Neutrosophic zero divisor. In particular, this result is true for $N(\mathrm{Z}_n,\, I)$, $N(\mathrm{Z}_n \times \mathrm{Z}_n,\, I)$, $N(\mathrm{Z}_n[\mathrm{x}]/(\mathrm{x}^n),\, I)$, and $N(\mathrm{Z}_n[\mathrm{i}],\, I)$, where Z_n , $\mathrm{Z}_n \times \mathrm{Z}_n$, $\mathrm{Z}_n[x]/(x^n)$ and $\mathrm{Z}_n[i]$ are finite commutative rings with usual notions under modulo n . We develop this fact in Theorem [2.4]. Since $N(R,\, I)^0 \not\subset N(R,\, I)$ and $N(R,\, I) \not\subset N(R,\, I)^0$, it is not surprising that there is a connection between the Neutrosophic units in the Neutrosophic zero rings.

Theorem 2.4. For any ring R with unity, we have $U(N(R,\, I)^0)$ is empty.

Proof. Assume that $U(N(R,\, I)^0)$ is nonempty. Suppose that $a+bI \in U(N(R,\, I)^0)$. Then there exists some $u+vI$ in $U(N(R,\, I)^0)$ such that $(u+vI)(a+bI)=1$. This implies that $(u+vI)^2(a+bI)^2 = 1^2$, or, it is equivalent to $0=1$ because $(u+vI)^2=0$ and $(a+bI)^2=0$, a contradiction. So our assumption is not true, and hence $U(N(R,\, I)^0) = \phi$.

In general, it is not easy to classify Neutrosophic rings and their corresponding Neutrosophic zero rings by determining their orders. For this reason, we must follow a better approach which is shown below.

Theorem 2.5. For any Neutrosophic ring $N(R,\, I)$, we have

$$N(R,\, I)^0 \cong M_2(R,\, I)^0 .$$

Proof. Let R be any ring. Then there exists $N(R,\, I)$ and $N(R,\, I)^0$. Now we want to show that $N(R,\, I)^0 \cong M_2(R,\, I)^0$. For this, we define a map $f : N(R,\, I)^0 \to M_2(R,\, I)^0$ by the following relation

$$f(a+bI) = \begin{pmatrix} a+bI & -(a+bI) \\ a+bI & -(\mathrm{a}+bI) \end{pmatrix}$$

for every $a+bI \in N(R, I)^0$. If $a+bI \in N(R, I)^0$, then $(a+bI)^2 = (a+bI)\,(a+bI) = 0$. That is, there exists a Neutrosophic matrix $\begin{pmatrix} a+bI & -(a+bI) \\ a+bI & -(a+bI) \end{pmatrix}$ in $M_2(R, I)^0$ such that

$$\begin{pmatrix} a+bI & -(a+bI) \\ a+bI & -(a+bI) \end{pmatrix} = \begin{pmatrix} a+bI & -(a+bI) \\ a+bI & -(a+bI) \end{pmatrix}\begin{pmatrix} a+bI & -(a+bI) \\ a+bI & -(a+bI) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

implying that f makes sense. Therefore f is well defined. Because $f(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $f(I) = \begin{pmatrix} I & -I \\ I & -I \end{pmatrix}$, one can easily verify that f is a Neutrosophic ring homomorphism.

Now, we show that f is one-one and onto. For every two Neutrosophic elements $a+bI$ and $c+dI$ in $N(R, I)^0$, we have

$$f(a+bI) = f(c+dI) \Rightarrow \begin{pmatrix} a+bI & -(a+bI) \\ a+bI & -(a+bI) \end{pmatrix} = \begin{pmatrix} c+dI & -(c+dI) \\ c+dI & -(c+dI) \end{pmatrix} \Rightarrow a+bI = c+dI .$$

Consequently, f is one-one, and also the unique part shows f is surjective. Therefore, f is a Neutrosophic isomorphism from $N(R, I)^0$ onto $M_2(R, I)^0$. Hence, $N(R, I)^0 \cong M_2(R, I)^0$.

Recall that $N(R, I)$ is not equal to $N(R, I)^0$ but the following theorem shows that $N(R, I)$ is equivalent to $N(R, I)^0$, that is we shall show that there is a one-one correspondence between $N(R, I)$ and $N(R, I)^0$.

Theorem 2.6. For any ring R, we have $|N(R, I)| = |N(R, I)^0|$.

Proof. By the Theorem [2.5], we know that $N(R, I)^0 \cong M_2(R, I)^0$. We shall show that $|N(R, I)| = |N(R, I)^0|$. For this, we must show that $|M_2(R, I)^0| = |N(R, I)|$. Define a map $\psi : M_2(R, I)^0 \to N(R, I)$ by the connection

$$\psi\left(\begin{pmatrix} a+bI & -(a+bI) \\ a+bI & -(a+bI) \end{pmatrix}\right) = a+bI$$

for every element $\begin{pmatrix} a+bI & -(a+bI) \\ a+bI & -(a+bI) \end{pmatrix}$ in $M_2(R, I)^0$. Every element $a+bI$ in $\mathrm{N}(R, I)$ has the following form $a+bI = \psi\left(\begin{pmatrix} a+bI & -(a+bI) \\ a+bI & -(a+bI) \end{pmatrix}\right)$ for some $\begin{pmatrix} a+bI & -(a+bI) \\ a+bI & -(a+bI) \end{pmatrix}$ in $M_2(R, I)^0$. Then the map ψ is clearly onto; it is one-one because for every

$A^0 = \begin{pmatrix} a+bI & -(a+bI) \\ a+bI & -(a+bI) \end{pmatrix}$, $B^0 = \begin{pmatrix} c+dI & -(c+dI) \\ c+dI & -(c+dI) \end{pmatrix}$ in $M_2(R, I)^0$, we have

$$\psi\left(A^0\right) = \psi\left(B^0\right) \Rightarrow \psi\left(\begin{pmatrix} a+bI & -(a+bI) \\ a+bI & -(a+bI) \end{pmatrix}\right) = \psi\left(\begin{pmatrix} a+bI & -(a+bI) \\ a+bI & -(a+bI) \end{pmatrix}\right)$$

$$\Rightarrow a+bI = c+dI$$

$$\Rightarrow \begin{pmatrix} a+bI & -(a+bI) \\ a+bI & -(a+bI) \end{pmatrix} = \begin{pmatrix} c+dI & -(c+dI) \\ c+dI & -(c+dI) \end{pmatrix}$$

$$\Rightarrow A^0 = B^0.$$

Therefore, the correspondence $a+bI \leftrightarrow \begin{pmatrix} a+bI & -(a+bI) \\ a+bI & -(a+bI) \end{pmatrix}$ pairs every element in each of two sets $\mathrm{N}(R, I)$ and $M_2(R, I)^0$ with exactly one element of the other set. Hence, $\mathrm{N}(R, I)$ and $M_2(R, I)^0$ contains the same number of elements, and we write this as $|\mathrm{N}(R, I)| = |M_2(R, I)^0|$.

Now because of the Theorem [2.5], we conclude that $|\mathrm{N}(R, I)^0| = |\mathrm{N}(R, I)|$.

This is all somewhat vague; of course, let us look at a concrete example.

Example 2.7. For the ring $Z_2 = \{0, 1\}$, the correspondence ψ from $\mathrm{N}(\mathrm{Z}_2, I)$ onto $M_2(\mathrm{Z}_2, I)^0$ with actions given by the following arrow diagrams:

$$0 \leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ 1 \leftrightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix},\ I \leftrightarrow \begin{pmatrix} I & -I \\ I & -I \end{pmatrix} \text{and}\ 1+I \leftrightarrow \begin{pmatrix} 1+I & -(1+I) \\ 1+I & -(1+I) \end{pmatrix}.$$

These actions illustrate that $|\mathrm{N}(\mathrm{Z}_2, I)| = 4$, $|M_2(\mathrm{Z}_2, I)^0| = 4$, and hence $|\mathrm{N}(\mathrm{Z}_2, I)^0| = 4$. This shows that $|\mathrm{N}(\mathrm{Z}_2, I)| = |\mathrm{N}(\mathrm{Z}_2, I)^0|$ but $\mathrm{N}(\mathrm{Z}_2, I) \neq \mathrm{N}(\mathrm{Z}_2, I)^0$.

We now change focus somewhat take up the study of Neutrosophic isomorphism between $\mathrm{N}(R, I)$ and $\mathrm{N}(R, I)^0$. Particularly we observe that nothing is known of Neutrosophic isomorphism between $\mathrm{N}(R, I)$ and $\mathrm{N}(R, I)^0$. For instance, the Neutrosophic ring $\mathrm{N}(\mathrm{Z}_2, I)$ and Neutrosophic zero ring $\mathrm{N}(\mathrm{Z}_2, I)^0$ are not isomorphic with respect to Neutrosophic isomorphism because $I^2 = I$ in $\mathrm{N}(\mathrm{Z}_2, I)$ but $\begin{pmatrix} I & -I \\ I & -I \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ in $\mathrm{N}(\mathrm{Z}_2, I)^0$. This observation takes place according to Theorem [2.8].

Theorem 2.8. Let R be any non-trivial ring. Then, $\mathrm{N}(R, I)$ is not isomorphic to $\mathrm{N}(R, I)^0$.

Proof. Assume that the element $A^0 = \begin{pmatrix} a+bI & -(a+bI) \\ a+bI & -(a+bI) \end{pmatrix} \neq 0$ in $M_2(R, I)^0$ satisfies the condition $(A^0)^2 = 0$, where $a+bI \neq 0$. Suppose that the Neutrosophic mapping $g : M_2(R, I)^0 \to N(R, I)$ is a Neutrosophic isomorphism. If $a+bI = g(\mathrm{A}^0)$, then

$$(a+bI)^2 = g(A^0)^2 \Rightarrow (a+bI)^2 = g((A^0)^2)$$

$$\Rightarrow (a+bI)^2 = g(0), \text{ since } (A^0)^2 = 0$$

$$\Rightarrow (a+bI)^2 = 0,\ g(0) = 0.$$

But $(a+bI)^2 = 0$ in $N(R, I)$ implies that $a+bI = 0$, giving $A^0 = \begin{pmatrix} a+bI & -(a+bI) \\ a+bI & -(a+bI) \end{pmatrix} = 0$ because g is one-one. This is a contradiction to the fact that $A^0 \neq 0$, so no such isomorphism g can exist between $M_2(R, I)^0$ and $N(R, I)$. But $N(R, I)^0 \cong M_2(R, I)^0$, and hence $\mathrm{N}(R, I)$ is not isomorphic to $\mathrm{N}(R, I)^0$.

Theorem 2.9. Let R be a finite ring with unity. Then, $Char(N(R, I)^0) = Char(R)$.

Proof. Suppose $|R|$ is finite and $1 \in R$. Then, by the definition of the characteristic of a ring,

$$Char(R) = n \Leftrightarrow o(1) = n \text{ in the additive group } (R, +)$$
$$\Leftrightarrow n \cdot 1 = 0 \text{ in the additive group } (R, +)$$
$$\Leftrightarrow n \cdot 1 = 0,\ n \cdot (-1) = 0 \text{ in the additive group } (R, +)$$
$$\Leftrightarrow n \cdot \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} n \cdot 1 & n \cdot (-1) \\ n \cdot 1 & n \cdot (-1) \end{pmatrix} = \begin{pmatrix} & 0 \\ & 0 \end{pmatrix}$$
$$\Leftrightarrow Char(\mathrm{M}_2(R, I)^0) = n$$
$$\Leftrightarrow Char(\mathrm{N}(R, I)^0) = n.$$

A ring R is called Boolean ring if $a^2 = a$ for all a in R. Every finite Boolean ring with unity is isomorphic to the ring $Z_2{}^n$, where $Z_2{}^n$ is the Cartesian product of n copies of the ring $Z_2 = \{0, 1\}$ with respect to addition and multiplication modulo 2. Therefore, $N(Z_2{}^n, I)$ is a Neutrosophic Boolean ring with the property that $|N(Z_2{}^n, I)| = 2^{4n}$. Now we move on to verify that the structure of $N(Z_2{}^n, I)^0$ is Neutrosophic Boolean ring, or, not.

Theorem 2.10. Every Neutrosophic zero ring of a Boolean ring is not a Neutrosophic Boolean ring.

Proof. Suppose $n > 1$ is a positive integer. By the Theorem [2.5], we know that $N(Z_2{}^n, I)^0$ is isomorphic to the Neutrosophic zero ring $\mathrm{M}_2(Z_2{}^n, I)^0$. In anticipation of a contradiction, let us assume that $\mathrm{M}_2(Z_2{}^n, I)^0$ is a Neutrosophic Boolean ring, then for any $\alpha = a + bI \neq 0$ in $N(Z_2{}^n, I)^0$ such that $\begin{pmatrix} \alpha & -\alpha \\ \alpha & -\alpha \end{pmatrix}$ is in $\mathrm{M}_2(Z_2{}^n, I)^0$. Under the condition of Neutrosophic Boolean ring, we have

$$\begin{pmatrix} \alpha & -\alpha \\ \alpha & -\alpha \end{pmatrix}^2 = \begin{pmatrix} \alpha & -\alpha \\ \alpha & -\alpha \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha & -\alpha \\ \alpha & -\alpha \end{pmatrix}\begin{pmatrix} \alpha & -\alpha \\ \alpha & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha & -\alpha \\ \alpha & -\alpha \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} \alpha & -\alpha \\ \alpha & -\alpha \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\Rightarrow \alpha = a + bI = 0.$$

This is not true. Hence, we conclude that every Neutrosophic zero ring of a Boolean ring is not a Neutrosophic Boolean ring.

3. Neutrosophic zero rings of zero rings

This section introduces Neutrosophic Zero rings associated with zero rings. First, we recall that R^0 is a zero ring if the product any two elements in R^0 is zero. If $R^0 \neq (0)$ then clearly $|R^0| \geq 2$ and R^0 is never a field structure. By the Buck's [2] research in 2004, for any ring R with $R \neq R^0$, the zero rings R^0 isomorphic to the zero rings of all 2×2 matrices of the form

$$M_2(R)^0 = \left\{ \begin{pmatrix} r & -r \\ r & -r \end{pmatrix} : r \in R \right\}$$

with the same cardinality of R, that is, $|\mathrm{M}_2(R)^0| = |R|$. For example, the zero ring

$$M_2(\mathrm{Z}_3)^0 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \right\}$$

with an order 3 under usual matrix addition and multiplication of modulo 3. This observation concludes that, if R is not a zero ring then $N(R, I)$ is never a zero ring. However, the following definition gives a concise way of referring to the definition of Neutrosophic zero rings associated with zero rings.

Definition 3.1. If R^0 is a zero ring, then $N(R^0, I) = \{\mathrm{a+bI} : a, b \in R^0\}$ is called **Neutrosophic zero ring** corresponding to the zero ring R^0.

Example 3.2. Suppose that $R^0 = \{0, 3, 6\}$ is a zero ring under addition and multiplication modulo 9. Then

$N(R^0, I) = \{0, 3, 6, 3I, 6I, 3+3I, 3+6I, 6+3I, 6+6I\}$ and

$$N(R^0, I)^0 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & -3 \\ 3 & -3 \end{pmatrix}, \begin{pmatrix} 6 & -6 \\ 6 & -6 \end{pmatrix}, \begin{pmatrix} 3I & -3I \\ 3I & -3I \end{pmatrix}, \begin{pmatrix} 6I & -6I \\ 6I & -6I \end{pmatrix}, \begin{pmatrix} 3+3I & -(3+3I) \\ 3+3I & -(3+3I) \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 3+6I & -(3+6I) \\ 3+3I & -(3+6I) \end{pmatrix}, \begin{pmatrix} 6+3I & -(6+3I) \\ 6+3I & -(6+3I) \end{pmatrix}, \begin{pmatrix} 6+6I & -(6+6I) \\ 6+6I & -(6+6I) \end{pmatrix} \right\}$$

Properties of $N(R^0, I)$.

(1) $N(R^0, I)$ is generated by R^0 and I.

(2) $N(R^0, I)$ is a Neutrosophic square zero ring.

(3) $|N(R^0, I)| = |R^0|^2$.

(4) $N(R^0, I) \neq N(R, I)^0$.

(5) $|N(R^0, I)| = |N(R^0, I)^0|$.

Theorem 3.3. For any finite zero ring R^0, the following equality holds good

$$|N(R^0, I)| = |R^0|^2.$$

Proof. The Cartesian product of R^0 is defined by $R^0 \times R^0 = \{(a, b) : a, b \in R^0\}$. Now define the map $\tau : R^0 \times R^0 \to N(R^0, I)$ by the relation $\tau((a, b)) = a + bI$ for every $(a, b) \in R^0 \times R^0$.

For any two elements (a, b) and (c, d) in the zero ring $R^0 \times R^0$, we have

$\tau((a, b)) = \tau((\mathrm{c}, d)) \Leftrightarrow a + bI = c + dI$

$\Leftrightarrow a = b,\ \mathrm{c} = d$, since $I \neq 0$.

$\Leftrightarrow (a, b) = (\mathrm{c}, d)$.

Thus the mapping τ is a well-defined one-one function. Also τ is onto function, because for any $\alpha \in \tau(R^0 \times R^0)$, there exists $\beta \in R^0 \times R^0$ such that $\alpha = \tau(\beta)$. Therefore, the map $\tau : R^0 \times R^0 \to N(R^0, I)$ is one-one correspondence from $R^0 \times R^0$ onto $N(R^0, I)$, and clear that

$$|N(R^0, I)| = |R^0 \times R^0| = |R^0|^2.$$

Recall that $U(R)$ and $U(N(R, I))$ denotes the set of all units and Neutrosophic units of R and $\mathrm{N}(R, I)$, respectively, see [17]. Note that, if at least one of $U(R)$ and $U(N(R, I))$ is non-empty, then there is nothing to the existence of Neutrosophic zero ring. The next hurdle that stands in our way is to establish that a relation between $U(N(R,\, I))$ and its corresponding Neutrosophic zero ring.

Theorem 3.4. If the set $U(N(R,\, I)) = \phi$, then there is a Neutrosophic zero ring with at least four elements.

Proof. There is no harm in assuming that $|R| > 1$, and automatically $|\mathrm{N}(R,\, I)| \geq 4$ is true. Suppose $U(N(R,\, I)) \neq \phi$. Then there are at least two elements in $U(N(R,\, I))$. If $u + vI$ and $u' + v'I$ are the two distinct elements in $U(N(R,\, I))$, then, bearing in mind that u , u' , v , v' are elements in $U(R)$. As a result, the Neutrosophic product $(u + vI)\ (u' + v'I)$ is given by

$$(u + vI)\ (u' + v'I) = uu' + (uv' + vu' + vv')I .$$

It is never zero because $uu' \in U(R)$. This contraposition proves our result.

Theorem [3.4] indicates that every commutative Neutrosophic zero ring is without unity. For this fact, the following theorem is essential in our paper.

Theorem 3.5. The Neutrosophic ring $N(R,\, I)$ is a Neutrosophic zero ring if and only if R is isomorphic to zero ring. In particular, $R \cong R^0 \Leftrightarrow N(R,\, I) \cong N(R^0,\, I)$.

Proof. Suppose R is isomorphic to a zero ring R^0 . Then there exists a Neutrosophic ring $N(R^0,\, I)$ which is also Neutrosophic zero ring because

$$R^0 \cong M_2(R)^0 \Leftrightarrow N(R^0,\, I) \cong N(M_2(R)^0,\, I)$$

under the following actions

$$r \leftrightarrow \begin{pmatrix} r & -r \\ r & -r \end{pmatrix} \Leftrightarrow r + sI \leftrightarrow \begin{pmatrix} r + sI & -(r + sI) \\ r + sI & -(r + sI) \end{pmatrix}$$

4. Conclusions

In this work, another Neutrosophic Algebraic structure, for the Neutrosophic speculation, in view of the traditional Ring Theory was proposed. This study understands the new structure basis in Neutrosophic hypothesis which builds up another idea for the comparison of two ring structures dependent on the use of the indeterminacy idea and the structural information. The Neutrosophic zero ring structure was characterized utilizing the identical classes of traditional zero rings, to be equipped for choosing any Neutrosophic element of the class. Additionally, we built up a connection between the various zero rings and matrix zero rings R^0 , $M_2(R)^0$,, $M_2(R,\, I)^0$, $N(R^0,\, I)$, $N(R,\, I)^0$ and $N(M_2(R)^0,\, I)$ such as $N(R,\, I)^0 \cong M_2(R,\, I)^0$ and $R^0 \cong M_2(R)^0 \Leftrightarrow N(R^0,\, I) \cong N(M_2(R)^0,\, I)$. The future work will recommend a Neutrosophic square zero elements and Neutrosophic square zero matrices to speak to all Neutrosophic mathematical frameworks, and apply the properties of these frameworks for identifying the total number of Neutrosophic zero subrings and Neutrosophic zero ideals.

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