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BMBJ-neutrosophic subalgebra in BCI/BCK-algebras

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Abstract: For the first time Smarandache introduced neutrosophic sets which can be used as a mathematical tool for dealing with indeterminate and inconsistent information. the notion of BMBJ-neutrosophic set and subalgebra, as a generalization of a neutrosophic set, is introduced, and it's application to BCI/BCK-algebras is investigated. The concept of BMBJ-neutrosophic subalgebras in BCI/BCK-algebras is introduced, and related properties are investigated. New BMBJ-neutrosophic subalgebra is established by using an BMBJ-neutrosophic subalgebra of a BCI/BCK-algebra. Alos, homomorphic (inverse) image of BMBJ-neutrosophic subalgebra and translation of BMBJ-neutrosophic subalgebra is investigated. At the end, we provided conditions for an BMBJ-neutrosophic set to be an BMBJ-neutrosophic subalgebra.

Keywords: BMBJ-neutrosophic set; BMBJ-neutrosophic subalgebra; BMBJ-neutrosophic S-extension.

1 Introduction

Different types of uncertainties are encountered in some complex system and many fields like biological, behavioural and chemical etc. L.A. Zadeh [33] in 1965 introduced the fuzzy set for the first time to handle uncertainties in many applications. Also K. Atanassov introduced the intuitionistic fuzzy set on the universe X as a generalisation of fuzzy set [6] in 1986. The concept of neutrosophic set is developed by Smarandache ([27], [28] and [29]), and this is a more general platform that extends the notions of classic set like (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set. Neutrosophic set theory is applied to various fields which is referred to the [1], [2], [3], [4], [5] [8], [9], [22] and [24]. Neutrosophic algebraic structures in BCI/BCK-algebras are discussed in the papers [7], [13], [14], [15], [19], [16], [17], [18], [20], [25], [26], [30], [31] and [32].

In this paper, we introduce the notion of BMBJ-neutrosophic sets and subalgebra, as a generalisation of neutrosophic set, and we investigate it's application and related properties it to BCI/BCK-algebras. We provide some characterizations of BMBJ-neutrosophic subalgebra, and by using an BMBJ-neutrosophic subalgebra of a BCI/BCK-algebra, a new BMBJ-neutrosophic subalgebra will be propose. We consider the homomorphic inverse image of BMBJ-neutrosophic subalgebra, and consider translation of BMBJ-neutrosophic subalgebra. At the last step, we provide some conditions for an BMBJ-neutrosophic set to be an BMBJ-neutrosophic subalgebra.

2 Preliminaries

A BCI/BCK-algebra is an important class of logical algebras introduced by K. Iséki (see [11] and [12]) and was extensively investigated by several researchers.

By a *BCI-algebra*, we mean a set X with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra.* Any BCI/BCK-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$
$$(\forall x, y, z \in X)\,(x \leq y\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)\,, \tag{2.2}$$
$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$
$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{2.4}$$

where $x \leq y$ if and only if $x * y = 0$. Any BCI-algebra X satisfies the following conditions (see [10]):

$$(\forall x, y \in X)(x * (x * (x * y)) = x * y), \tag{2.5}$$
$$(\forall x, y \in X)(0 * (x * y) = (0 * x) * (0 * y)). \tag{2.6}$$

A nonempty subset S of a BCI/BCK-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

By an *interval number* we mean a closed subinterval $\tilde{a} = [a^-, a^+]$ of I, where $0 \leq a^- \leq a^+ \leq 1$. Denote by $[I]$ the set of all interval numbers. Let us define what is known as *refined minimum* (briefly, rmin) and *refined maximum* (briefly, rmax) of two elements in $[I]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $[I]$. Consider two interval numbers $\tilde{a}_1 := \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 := \left[a_2^-, a_2^+\right]$. Then

$$\begin{aligned}
&\operatorname{rmin}\{\tilde{a}_1, \tilde{a}_2\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right], \\
&\operatorname{rmax}\{\tilde{a}_1, \tilde{a}_2\} = \left[\max\left\{a_1^-, a_2^-\right\}, \max\left\{a_1^+, a_2^+\right\}\right], \\
&\tilde{a}_1 \succeq \tilde{a}_2\ \Leftrightarrow\ a_1^- \geq a_2^-,\ a_1^+ \geq a_2^+,
\end{aligned}$$

and similarly we may have $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$ (resp. $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$). Let $\tilde{a}_i \in [I]$ where $i \in \Lambda$. We define

$$\operatorname{rinf}_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \operatorname{rsup}_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

Let X be a nonempty set. A function $A : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, an ***IVF set***) in X. Let $[I]^X$ stand for the set of all IVF sets in X. For every $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the *degree* of membership of an element x to A, where $A^- : X \to I$ and $A^+ : X \to I$ are fuzzy sets in X which are called a *lower fuzzy set* and an *upper fuzzy set* in X, respectively. For simplicity, we denote $A = [A^-, A^+]$.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [28]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

We refer the reader to the books [10, 21] for further information regarding BCi/BCK-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

3 BMBJ-neutrosophic structures with applications in BCI/BCK-algebras

Definition 3.1. Let X be a non-empty set. By an *MBJ-neutrosophic set* in X, we mean a structure of the form:

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x)\rangle \mid x \in X\}$$

where M_A and J_A are fuzzy sets in X, which are called a truth membership function and a false membership function, respectively, and $\tilde{B}_A$ is an IVF set in X which is called an indeterminate interval-valued membership function.

For the sake of simplicity, we shall use the symbol $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ for the MBJ-neutrosophic set

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x)\rangle \mid x \in X\}.$$

Definition 3.2. Let X be a BCI/BCK-algebra. An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is called an *BMBJ-neutrosophic subalgebra* of X if it satisfies:

$$(\forall x, y \in X)\left(\begin{array}{l} M_A(x*y) \geq \min\{M_A(x), M_A(y)\}, \\ \tilde{B}_A^-(x*y) \leq \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\}, \\ \tilde{B}_A^+(x*y) \geq \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\}, \\ J_A(x*y) \leq \max\{J_A(x), J_A(y)\}, \\ M_A(x) + \tilde{B}_A^-(x) \leq 1, \tilde{B}_A^+(x) + J_A(x) \geq 1\}. \end{array}\right) \tag{3.1}$$

Example 3.3. Consider a set $X = \{0, a, b, c\}$ with the binary operation $*$ which is given in Table 1. Then

Table 1: Cayley table for the binary operation "$*$"

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

$(X; *, 0)$ is a BCK-algebra (see [21]). Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by Table 2. It is routine to verify that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X.

Table 2: MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$

X	$M_A(x)$	$\tilde{B}_A(x)$	$J_A(x)$
0	0.7	$[0.3, 0.8]$	0.2
a	0.3	$[0.1, 0.5]$	0.6
b	0.1	$[0.3, 0.8]$	0.4
c	0.5	$[0.1, 0.5]$	0.7

In what follows, let X be a BCI/BCK-algebra unless otherwise specified.

Proposition 3.4. *If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X, then $M_A(0) \geq M_A(x)$, $\tilde{B}_A^-(0) \leq \tilde{B}_A^-(x)$, $\tilde{B}_A^+(0) \geq \tilde{B}_A^+(x)$ and $J_A(0) \leq J_A(x)$ for all $x \in X$.*

Proof. For any $x \in X$, we have

$$M_A(0) = M_A(x*x) \geq \min\{M_A(x), M_A(x)\} = M_A(x),$$

$$\tilde{B}_A^-(0) = \tilde{B}_A^-(x*x) \leq \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(x)\} = \tilde{B}_A^-(x),$$

$$\tilde{B}_A^+(0) = \tilde{B}_A^-(x*x) \geq \min\{\tilde{B}_A^+(x), \tilde{B}_A^-(x)\} = \tilde{B}_A^+(x)$$

and

$$J_A(0) = J_A(x * x) \leq \max\{J_A(x), J_A(x)\} = J_A(x).$$

This completes the proof. □

Proposition 3.5. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an BMBJ-neutrosophic subalgebra of X. If there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} M_A(x_n) = 1,\ \lim_{n\to\infty} \tilde{B}_A^-(x_n) = 0,\ \lim_{n\to\infty} \tilde{B}_A^+(x_n) = 1 \text{ and } \lim_{n\to\infty} J_A(x_n) = 0, \tag{3.2}$$

then $M_A(0) = 1$, $\tilde{B}_A^-(0) = 0$, $\tilde{B}_A^+(0) = 1$ and $J_A(0) = 0$.

Proof. Using Proposition 3.4, we know that $M_A(0) \geq M_A(x)$, $\tilde{B}_A^-(0) \leq \tilde{B}_A^-(x)$, $\tilde{B}_A^+(0) \geq \tilde{B}_A^+(x)$ and $J_A(0) \leq J_A(x)$ for all $x \in X$. for every positive integer n. Note that

$$\begin{aligned} 1 \geq M_A(0) \geq \lim_{n\to\infty} M_A(x_n) = 1, \\ 0 \leq \tilde{B}_A^-(0) \leq \lim_{n\to\infty} \tilde{B}_A^-(x_n) = 0, \\ 1 \geq \tilde{B}_A^+(0) \geq \lim_{n\to\infty} \tilde{B}_A^+(x_n) = 1, \\ 0 \leq J_A(0) \leq \lim_{n\to\infty} J_A(x_n) = 0. \end{aligned}$$

Therefore $M_A(0) = 1$, $\tilde{B}_A^-(0) = 0$, $\tilde{B}_A^+(0) = 1$ and $J_A(0) = 0$. □

Theorem 3.6. *Given an BMBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X, if (M_A, J_A) is an intuitionistic fuzzy subalgebra of X, and B_A^- and B_A^+ are fuzzy subalgebras of X, then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X.*

Proof. It is sufficient to show that $\tilde{B}_A$ satisfies the condition

$$(\forall x, y \in X)(\tilde{B}_A^-(x * y) \leq \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\}), \tag{3.3}$$

$$(\forall x, y \in X)(\tilde{B}_A^+(x * y) \geq \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\}). \tag{3.4}$$

For any $x, y \in X$, we get

$$\begin{aligned} \tilde{B}_A(x * y) &= [\tilde{B}_A^-(x * y), \tilde{B}_A^+((x * y)] \\ &\geq [\max \tilde{B}_A^-(x), \tilde{B}_A^-(y)\}, \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\}]. \end{aligned}$$

Therefore $\tilde{B}_A$ satisfies the condition (3.3), and so $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X. □

If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X, then

$$\begin{aligned} [B_A^-(x * y), B_A^+(x * y)] &= \tilde{B}_A(x * y) \succeq \mathrm{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\} \\ &= \mathrm{rmin}\{[B_A^-(x), B_A^+(x), [B_A^-(y), B_A^+(y)]\} \\ &= [\min\{B_A^-(x), B_A^-(y)\}, \min\{B_A^+(x), B_A^+(y)\}] \end{aligned}$$

for all $x, y \in X$. It follows that $B_A^-(x*y) \geq \min\{B_A^-(x), B_A^-(y)\}$ and $B_A^+(x*y) \geq \min\{B_A^+(x), B_A^+(y)\}$. Thus B_A^- and B_A^+ are fuzzy subalgebras of X. But (M_A, J_A) is not an intuitionistic fuzzy subalgebra of X as seen in Example 3.3. This shows that the converse of Theorem 3.6 is not true.

Given an BMBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X, we consider the following sets.

$$\begin{aligned} U(M_A; t) &:= \{x \in X \mid M_A(x) \geq t\}, \\ L(\tilde{B}_A^-; \delta_1) &:= \{x \in X \mid \tilde{B}_A^-(x) \leq \delta_1\}, \\ U(\tilde{B}_A^+; \delta_2) &:= \{x \in X \mid \tilde{B}_A^+(x) \geq \delta_2\}, \\ L(J_A; s) &:= \{x \in X \mid J_A(x) \leq s\} \end{aligned}$$

where $t, s \in [0,1]$ and $[\delta_1, \delta_2] \in [I]$.

Theorem 3.7. *An BMBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is an BMBJ-neutrosophic subalgebra of X if and only if the non-empty sets $U(M_A; t)$, $L(\tilde{B}_A^-; \delta_1)$, $U(\tilde{B}_A^+; \delta_2)$ and $L(J_A; s)$ are subalgebras of X for all $t, \delta_1, \delta_2, \in [0,1]$.*

Proof. Suppose that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X. Let $t, s \in [0,1]$ and $[\delta_1, \delta_2] \in [I]$ be such that $U(M_A; t)$, $L(\tilde{B}_A^-; \delta_1)$, $U(\tilde{B}_A^+; \delta_2)$ and $L(J_A; s)$ are non-empty. For any $x, y, a, b, u, v \in X$, if $x, y \in U(M_A; t)$, $a, b \in L(\tilde{B}_A^-; \delta_1)$, $c, d \in U(\tilde{B}_A^+; \delta_2)$ and $u, v \in L(J_A; s)$, then

$$\begin{aligned} &M_A(x*y) \geq \min\{M_A(x), M_A(y)\} \geq \min\{t, t\} = t, \\ &\tilde{B}_A^-(a*b) \leq \max\{\tilde{B}_A^-(a), \tilde{B}_A^-(b)\} \leq \max\{\delta_1, \delta_1\} = \delta_1, \\ &\tilde{B}_A^+(c*d) \geq \min\{\tilde{B}_A^+(c), \tilde{B}_A^+(d)\} \geq \min\{\delta_2, \delta_2\} = \delta_2, \\ &J_A(u*v) \leq \max\{J_A(u), J_A(v)\} \leq \min\{s, s\} = s, \end{aligned}$$

and so $x*y \in U(M_A; t)$, $a*b \in L(\tilde{B}_A^-; \delta_1)$, $c*d \in U(\tilde{B}_A^+; \delta_2)$ and $u*v \in L(J_A; s)$. Therefore $U(M_A; t)$, $L(\tilde{B}_A^-; \delta_1)$, $U(\tilde{B}_A^+; \delta_2)$ and $L(J_A; s)$ are subalgebras of X.

Conversely, assume that the non-empty sets $U(M_A; t)$, $L(\tilde{B}_A^-; \delta_1)$, $U(\tilde{B}_A^+; \delta_2)$ and $L(J_A; s)$ are subalgebras of X for all $t, s, \delta_1, \delta_2 \in [0,1]$. If $M_A(a_0 * b_0) < \min\{M_A(a_0), M_A(b_0)\}$ for some $a_0, b_0 \in X$, then $a_0, b_0 \in U(M_A; t_0)$ but $a_0 * b_0 \notin U(M_A; t_0)$ for $t_0 := \min\{M_A(a_0), M_A(b_0)\}$. This is a contradiction, and thus $M_A(a*b) \geq \min\{M_A(a), M_A(b)\}$ for all $a, b \in X$. Similarly, we can show that $\tilde{B}_A^-(a*b) \leq \max\{\tilde{B}_A^-(a), \tilde{B}_A^-(b)\}$, $\tilde{B}_A^+(c*d) \geq \min\{\tilde{B}_A^+(c), \tilde{B}_A^+(d)\}$ and $J_A(a*b) \leq \max\{J_A(a), J_A(b)\}$ for all $a, b \in X$. □

Using Proposition 3.4 and Theorem 3.7, we have the following corollary.

Corollary 3.8. *If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X, then the sets $X_{M_A} := \{x \in X \mid M_A(x) = M_A(0)\}$, $X_{\tilde{B}_A^-} := \{x \in X \mid \tilde{B}_A^-(x) = \tilde{B}_A^-(0)\}$, $X_{\tilde{B}_A^+} := \{x \in X \mid \tilde{B}_A^+(x) = \tilde{B}_A^+(0)\}$, and $X_{J_A} := \{x \in X \mid J_A(x) = J_A(0)\}$ are subalgebras of X.*

We say that the subalgebras $U(M_A; t)$, $L(\tilde{B}_A^-; \delta_1)$, $U(\tilde{B}_A^+; \delta_2)$ and $L(J_A; s)$ are *BMBJ-subalgebras* of $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$.

Theorem 3.9. *Every subalgebra of X can be realized as BMBJ-subalgebras of an BMBJ-neutrosophic subalgebra of X.*

Proof. Let K be a subalgebra of X and let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an BMBJ-neutrosophic set in X defined by

$$M_A(x) = \begin{cases} t & \text{if } x \in K, \\ 0 & \text{otherwise,} \end{cases} \quad \tilde{B}_A^-(x) = \begin{cases} \gamma_1 & \text{if } x \in K, \\ 1 & \text{otherwise,} \end{cases} \quad \tilde{B}_A^+(x) = \begin{cases} \gamma_2 & \text{if } x \in K, \\ 0 & \text{otherwise,} \end{cases} \quad J_A(x) = \begin{cases} s & \text{if } x \in K, \\ 1 & \text{otherwise,} \end{cases} \tag{3.5}$$

where $t \in (0, 1]$, $s \in [0, 1)$ and $\gamma_1, \gamma_2 \in (0, 1]$ with $\gamma_1 < \gamma_2$. It is clear that $U(M_A; t) = K$, $L(\tilde{B}_A^-; \gamma_1) = K$, $U(\tilde{B}_A^+; \gamma_2) = K$ and $L(J_A; s) = K$. Let $x, y \in X$. If $x, y \in K$, then $x * y \in K$ and so

$$\begin{aligned} &M_A(x * y) = t = \min\{M_A(x), M_A(y)\} \\ &\tilde{B}_A^-(x * y) = \gamma_1 = \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\}, \\ &\tilde{B}_A^+(x * y) = \gamma_2 = \max\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\}, \\ &J_A(x * y) = s = \max\{J_A(x), J_A(y)\}. \end{aligned}$$

If any one of x and y is contained in K, say $x \in K$, then $M_A(x) = t$, $\tilde{B}_A^-(x) = \gamma_1$, $\tilde{B}_A^+(x) = \gamma_2$, $J_A(x) = s$, $M_A(y) = 0$, $\tilde{B}_A^-(y) = 0$, $\tilde{B}_A^+(y) = 0$ and $J_A(y) = 1$. Hence

$$\begin{aligned} &M_A(x * y) \geq 0 = \min\{t, 0\} = \min\{M_A(x), M_A(y)\} \\ &\tilde{B}_A^-(x * y) \leq 1 = \max\{\gamma_1, 1\} = \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\}, \\ &\tilde{B}_A^+(x * y) \geq 0 = \min\{\gamma_2, 0\} = \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\}, \\ &J_A(x * y) \leq 1 = \max\{s, 1\} = \max\{J_A(x), J_A(y)\}. \end{aligned}$$

If $x, y \notin K$, then $M_A(x) = 0 = M_A(y)$, $\tilde{B}_A^-(x) = 1 = \tilde{B}_A^-(y)$, $\tilde{B}_A^+(x) = 0 = \tilde{B}_A^+(y)$ and $J_A(x) = 1 = J_A(y)$. It follows that

$$\begin{aligned} &M_A(x * y) \geq 0 = \min\{0, 0\} = \min\{M_A(x), M_A(y)\} \\ &\tilde{B}_A^-(x * y) \leq 1 = \max\{1, 1\} = \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\}, \\ &\tilde{B}_A^+(x * y) \geq 0 = \min\{0, 0\} = \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\}, \\ &J_A(x * y) \leq 1 = \max\{1, 1\} = \max\{J_A(x), J_A(y)\}. \end{aligned}$$

Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X. □

Theorem 3.10. *For any non-empty subset K of X, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an BMBJ-neutrosophic set in X which is given in (3.5). If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X, then K is a subalgebra of X.*

Proof. Let $x, y \in K$. Then $M_A(x) = t = M_A(y)$, $\tilde{B}_A^-(x) = \gamma_1 = \tilde{B}_A^-(y)$, $\tilde{B}_A^+(x) = \gamma_2 = \tilde{B}_A^+(y)$ and $J_A(x) = s = J_A(y)$. Thus

$$\begin{aligned} &M_A(x * y) \geq \min\{M_A(x), M_A(y)\} = t, \\ &\tilde{B}_A^-(x * y) \leq \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\} = \gamma_1, \\ &\tilde{B}_A^+(x * y) \geq \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\} = \gamma_2, \\ &J_A(x * y) \leq \max\{J_A(x), J_A(y)\} = s, \end{aligned}$$

and therefore $x * y \in K$. Hence K is a subalgebra of X. □

Using an BMBJ-neutrosophic subalgebra of a BCI-algera, we establish a new BMBJ-neutrosophic subalgebra.

Theorem 3.11. *Given an BMBJ-neutrosophic subalgebra $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ of a BCI-algebra X, let $\mathcal{A}^* = (M_A^*,\, \tilde{B}_A^*,\, J_A^*)$ be an BMBJ-neutrosophic set in X defined by $M_A^*(x) = M_A(0 * x)$, $\tilde{B}_A^*(x) = \tilde{B}_A(0 * x)$ and $J_A^*(x) = J_A(0 * x)$ for all $x \in X$. Then $\mathcal{A}^* = (M_A^*,\, \tilde{B}_A^*,\, J_A^*)$ is an BMBJ-neutrosophic subalgebra of X.*

Proof. Note that $0 * (x * y) = (0 * x) * (0 * y)$ for all $x, y \in X$. We have

$$\begin{aligned} M_A^*(x * y) &= M_A(0 * (x * y)) = M_A((0 * x) * (0 * y)) \\ &\geq \min\{M_A(0 * x), M_A(0 * y)\} \\ &= \min\{M_A^*(x), M_A^*(y)\}, \end{aligned}$$

$$\begin{aligned} (\tilde{B}_A^-)^*(x * y) &= \tilde{B}_A^-(0 * (x * y)) = \tilde{B}_A^-((0 * x) * (0 * y)) \\ &\leq \max\{\tilde{B}_A^-(0 * x), \tilde{B}_A^-(0 * y)\} \\ &= \max\{(\tilde{B}_A^-)^*(x), (\tilde{B}_A^-)^*(y)\} \end{aligned}$$

$$\begin{aligned} (\tilde{B}_A^+)^*(x * y) &= \tilde{B}_A^+(0 * (x * y)) = \tilde{B}_A^+((0 * x) * (0 * y)) \\ &\geq \min\{\tilde{B}_A^+(0 * x), \tilde{B}_A^+(0 * y)\} \\ &= \min(\{\tilde{B}_A^+)^*(x), (\tilde{B}_A^+)^*(y)\}, \end{aligned}$$

and

$$\begin{aligned} J_A^*(x * y) &= J_A(0 * (x * y)) = J_A((0 * x) * (0 * y)) \\ &\leq \max\{J_A(0 * x), J_A(0 * y)\} \\ &= \max\{J_A^*(x), J_A^*(y)\} \end{aligned}$$

for all $x, y \in X$. Therefore $\mathcal{A}^* = (M_A^*,\, \tilde{B}_A^*,\, J_A^*)$ is an BMBJ-neutrosophic subalgebra of X. □

Theorem 3.12. *Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras. If $\mathcal{B} = (M_B,\, \tilde{B}_B,\, J_B)$ is an MBJ-neutrosophic subalgebra of Y, then $f^{-1}(\mathcal{B}) = (f^{-1}(M_B),\, f^{-1}(\tilde{B}_B),\, f^{-1}(J_B))$ is an BMBJ-neutrosophic subalgebra of X, where $f^{-1}(M_B)(x) = M_B(f(x))$, $f^{-1}(\tilde{B}_B)(x) = \tilde{B}_B(f(x))$ and $f^{-1}(J_B)(x) = J_B(f(x))$ for all $x \in X$.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned} f^{-1}(M_B)(x * y) &= M_B(f(x * y)) = M_B(f(x) * f(y)) \\ &\geq \min\{M_B(f(x)), M_B(f(y))\} \\ &= \min\{f^{-1}(M_B)(x), f^{-1}(M_B)(y)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(\tilde{B}_B^-)(x*y) &= \tilde{B}_B^-(f(x*y)) = \tilde{B}_B^-(f(x)*f(y)) \\ &\leq \max\{\tilde{B}_B^-(f(x)), \tilde{B}_B^-(f(y))\} \\ &= \max\{f^{-1}(\tilde{B}_B^-)(x), f^{-1}(\tilde{B}_B^-)(y)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(\tilde{B}_B^+)(x*y) &= \tilde{B}_B^+(f(x*y)) = \tilde{B}_B^+(f(x)*f(y)) \\ &\geq \min\{\tilde{B}_B^+(f(x)), \tilde{B}_B^+(f(y))\} \\ &= \min\{f^{-1}(\tilde{B}_B^+)(x), f^{-1}(\tilde{B}_B^+)(y)\}, \end{aligned}$$

and

$$\begin{aligned} f^{-1}(J_B)(x*y) &= J_B(f(x*y)) = J_B(f(x)*f(y)) \\ &\leq \max\{J_B(f(x)), J_B(f(y))\} \\ &= \max\{f^{-1}(J_B)(x), f^{-1}(J_B)(y)\}. \end{aligned}$$

Hence $f^{-1}(\mathcal{B}) = (f^{-1}(M_B),\, f^{-1}(\tilde{B}_B),\, f^{-1}(J_B))$ is an BMBJ-neutrosophic subalgebra of X. □

Let $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ be an BMBJ-neutrosophic set in a set X. We denote

$$\begin{aligned} &\top := 1 - \sup\{M_A(x) \mid x \in X\}, \\ &\Pi := \inf\{\tilde{B}_B^-(x) \mid x \in X\}. \\ &\pi := 1 - \sup\{\tilde{B}_B^+(x) \mid x \in X\}. \\ &\bot := \inf\{J_A(x) \mid x \in X\}. \end{aligned}$$

For any $p \in [0, \top]$, $a \in [0, \Pi]$, $b \in [0, \pi]$ and $q \in [0, \bot]$, we define $\mathcal{A}^T = (M_A^p,\, \tilde{B}_A^a,\, \tilde{B}_A^b,\, J_A^q)$ by $M_A^p(x) = M_A(x) + p$, $\tilde{B}_A^a(x) = \tilde{B}_A^-(x) + a$, $\tilde{B}_A^b(x) = \tilde{B}_A^+(x) + b$ and $J_A^q(x) = J_A(x) - q$. Then $\mathcal{A}^T = (M_A^p,\, \tilde{B}_A^a,\, \tilde{B}_A^b,\, J_A^q)$ is an BMBJ-neutrosophic set in X, which is called a (p, a, b, q)*-translative BMBJ-neutrosophic set* of $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$.

Theorem 3.13. *If $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ is an BMBJ-neutrosophic subalgebra of X, then the (p, a, b, q)-translative BMBJ-neutrosophic set of $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ is also an BMBJ-neutrosophic subalgebra of X.*

Proof. For any $x, y \in X$, we get

$$\begin{aligned} M_A^p(x*y) &= M_A(x*y) + p \geq \min\{M_A(x), M_A(y)\} + p \\ &= \min\{M_A(x) + p, M_A(y) + p\} = \min\{M_A^p(x), M_A^p(y)\}, \end{aligned}$$

$$\begin{aligned} \tilde{B}_A^a(x*y) &= \tilde{B}_A^-(x*y) + a \leq \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\} + a \\ &= \max\{\tilde{B}_A^-(x) + a, \tilde{B}_A^-(y) + a\} = \max\{\tilde{B}_A^a(x), \tilde{B}_A^a(y)\}, \end{aligned}$$

$$\tilde{B}_A^b(x*y) = \tilde{B}_A^+(x*y) + b \geq \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\} + b$$
$$= \min\{\tilde{B}_A^+(x) + b, \tilde{B}_A^+(y) + b\} = \max\{\tilde{B}_A^b(x), \tilde{B}_A^b(y)\},$$

and

$$J_A^q(x*y) = J_A(x*y) - q \leq \max\{J_A(x), J_A(y)\} - q$$
$$= \max\{J_A(x) - q, J_A(y) - q\} = \max\{J_A^q(x), J_A^q(y)\}.$$

Therefore $\mathcal{A}^T = (M_A^p, \tilde{B}_A^a, \tilde{B}_A^b, J_A^q)$ is an BMBJ-neutrosophic subalgebra of X. □

Theorem 3.14. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an BMBJ-neutrosophic set in X such that its (p, a, b, q)-translative BMBJ-neutrosophic set is an BMBJ-neutrosophic subalgebra of X for $p \in [0, \top]$, $a \in [0, \Pi]$, $b \in [0, \pi]$ and $q \in [0, \perp]$. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X.*

Proof. Assume that $\mathcal{A}^T = (M_A^p, \tilde{B}_A^a, \tilde{B}_A^b, J_A^q)$ is an BMBJ-neutrosophic subalgebra of X for $p \in [0, \top]$, $a \in [0, \Pi]$, $b \in [0, \pi]$ and $q \in [0, \perp]$. Let $x, y \in X$. Then

$$M_A(x*y) + p = M_A^p(x*y) \geq \min\{M_A^p(x), M_A^p(y)\}$$
$$= \min\{M_A(x) + p, M_A(y) + p\}$$
$$= \min\{M_A(x), M_A(y)\} + p,$$

$$\tilde{B}_A^a(x*y) - a = \tilde{B}_A^-(x*y) \leq \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\}$$
$$= \max\{\tilde{B}_A^a(x) - a, \tilde{B}_A^a(y) - a\}$$
$$= \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\} - a.$$

$$\tilde{B}_A^b(x*y) - b = \tilde{B}_A^+(x*y) \geq \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\}$$
$$= \min\{\tilde{B}_A^b(x) - b, \tilde{B}_A^b(y) - b\}$$
$$= \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\} - b.$$

and

$$J_A(x*y) - q = J_A^q(x*y) \leq \max\{J_A^q(x), J_A^q(y)\}$$
$$= \max\{J_A(x) - q, J_A(y) - q\}$$
$$= \max\{J_A(x), J_A(y)\} - q.$$

It follows that $M_A(x*y) \geq \min\{M_A(x), M_A(y)\}$, $\tilde{B}_A^-(x*y) \leq \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\}$, $\tilde{B}_A^+(x*y) \geq \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\}$ and $J_A(x*y) \leq \max\{J_A(x), J_A(y)\}$ for all $x, y \in X$. Hence $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X. □

Definition 3.15. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ and $\mathcal{B} = (M_B, \tilde{B}_B, J_B)$ be BMBJ-neutrosophic sets in X. Then $\mathcal{B} = (M_B, \tilde{B}_B, J_B)$ is called an *BMBJ-neutrosophic S-extension* of $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ if the following assertions are valid.

(1) $M_B(x) \geq M_A(x)$, $\tilde{B}_A^-(x) \leq \tilde{B}_A^-(x)$, $\tilde{B}_A^+(x) \geq \tilde{B}_A^+(x)$ and $J_B(x) \leq J_A(x)$ for all $x \in X$,

(2) If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X, then $\mathcal{B} = (M_B, \tilde{B}_B, J_B)$ is an BMBJ-neutrosophic subalgebra of X.

Theorem 3.16. *Given $p \in [0, \top]$, $a \in [0, \Pi]$, $b \in [0, \pi]$ and $q \in [0, \bot]$, the (p, a, b, q)-translative BMBJ-neutrosophic set $\mathcal{A}^T = (M_A^p, \tilde{B}_A^a, \tilde{B}_A^b, J_A^q)$ of an BMBJ-neutrosophic subalgebra $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic S-extension of $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$.*

Proof. Straightforward. □

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