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N_δ-Closure and N_δ-Interior in Neutrosophic Topological Spaces

K.Damodharan[1] and M.Vigneshwaran[2]

[1]Department of Mathematics, KPR Institute of Engineering and Technology(Autonomous), Coimbatore - 641407, India.; catchmedamo@gmail.com

[2]Department of Mathematics, Kongunadu Arts and Science College(Autonomous), Coimbatore - 641029, India.; vignesh.mat@gmail.com

Abstract. Toplology greatly benefits from the concept of δ-cloure. Its quiet nature to extended its properties in other topological spaces. So, with the concept of quasi-coincidence Ganguly and Saha pioneered and extensively examined the notion of δ-closure within the domain of fuzzy topological spaces.(F_{TS}). The theory of δ-closure in intuitionistic fuzzy topological spaces (IF_{TS}) was further extended by Seok Jong Lee and Yeon Seok Eom. In this work, the notion of N_δ-closure in Neutrosophic Topological Spaces (N_{TS}) is put forward and discussed.

Keywords: neutrosophic; δ-Closure; δ-Interior

1. Introduction

In 1965, Zadeh pioneered the concept of fuzzy sets (F_S). In various areas of our daily lives, uncertainty is handled using this innovative mathematical framework. A membership function with the range of 0 to 1 characterizes a F_S. Over the last few decades, F_S is substantially used and applied in many domains, such as computer vision [45], pattern recognition [22], control [44], and others. Researchers in the fields of engineering [22], social sciences [45], and medical diagnosis [45] have all found this idea to be very useful. There is a tonne of information on F_S theory in [22, 34, 45]. A specific value contained within the unit interval [0,1] indicates the F_S's membership function. There is some hesitation as a result, therefore it's not always true that an element's non-membership function equals 1. In order to clarify this scenario, Atanassov [2] created IF_S in 1986 by including the hesitation degree known as the hesitation margin. The definition of the hesitancy margin is 1 . Thus, a membership function and non-membership function for an intuitionistic fuzzy setIF_S have a range of [0,1] with the additional condition that 0 1. As a result, F_S theory was generalized to include IF_S theory.

Decision-making [20], pattern recognition [33], social sciences [5], medical diagnostics [33], and other domains have all benefited from the application of the IF_S theory. It is impossible for fuzzy sets and IF_S to handle data that is unclear, inconsistent, partial, or uncertain. Consequently, Smarandache (Smarandache, 1999) formulated neutrosophic logic in 1998, drawing inspiration from Neutrosophy, a philosophical paradigm that scrutinizes the origin, composition, and application of neutral elements, alongside their interplay with diverse conceptual spectrums. A Neutrosophic set N_S encompasses three distinct membership functions: 'T' for truth, 'I' for indeterminate membership, and 'F' for falsehood. The 'I' component embodies a notable degree of indeterminacy, a key attribute associated with mediocrity. The theoretical frameworks of classical set theory, fuzzy sets (F_S) theory, intuitionistic fuzzy sets (IF_S) theory, interval-valued fuzzy sets theory, paraconsistent theory, dialetheist theory, paradoxist theory, and tautological theory are all encompassed and extended by the overarching N_S theory. This theoretical construct proves itself to be a robust instrument for grappling with the intricate tapestry of ambiguous and contradictory information that pervades our real-world context. Scholars from a multitude of disciplines have effectively harnessed N_S theory to navigate their respective domains. Notably, Wang et al. (2010) given the application of singular-valued N_S in the realms of science and engineering, providing an additional avenue for describing uncertain, partial, imprecise, and inconsistent data. The correlation coefficient of N_S found its investigation in the works of Hanafy et al. (2012 and 2013), while Ye (2013) explored the correlation coefficient within the context of singular-valued N_S. Further exploration of the correlation coefficient in the interval N_S was undertaken by Broumi and Smaradache (2013). In their discussion of N_{TS}, Salama et al. [27] You can find additional research on the N_S in [21,27,29,37,40–42]. In the decision-making theory [4,39–42], data base [37], medical diagnosis [42], pattern recognition [11,23], and other fields, N_S have been successfully employed.

In the realm of conventional topology, when delving into subjects like H-closed spaces, Katetov's and H-closed extensions, the generalizations of the Stone Weierstrass theorem, and other related topics, the ideas of θ-closure and δ-closure emerge as valuable tools [8,9,24,35,36,43]. Given the substantial importance of these concepts, it becomes almost inevitable to seek their extension into the context of fuzzy topological spaces (F_{TS}). Thus, by harnessing the notion of quasi-coincidence within F_{TS}, Saha and Ganguly introduced and conducted a thorough investigation into the innovative concept of fuzzy δ-closure. [10]. Furthermore, within the context of intuitionistic fuzzy topological spaces (IF_{TS}), extensive research efforts have been directed towards examining the characteristics of continuous mappings and closure operators. [12,17–19,32]. A generalisation of the δ- closure, the idea of δ-closure in IF_{TS} is introduced by Ganguly and Saha [10]. N_{TS} were first introduced in 2012 by Salama and Alblowi [26]. As an advancement beyond the framework of intuitionistic fuzzy topological spaces (IF_{TS}), they introduced the concept of neutrosophic topological spaces (N_{TS}), along with a corresponding neutrosophic set (N_S), which encapsulates the degrees of membership, indeterminacy, and non-membership for each individual element. In 2016, P. Iswarya and Dr. K. Bageerathi [16] contributed to this exploration by proposing the novel concepts of neutrosophic semiopen sets, neutrosophic semiclosed sets, neutrosophic semi-interior, and neutrosophic semi-closure within the context of neutrosophic topological spaces (N_{TS}). In the subsequent year, Parimala M et al(2018). elaborated on some new notions of homeomorphism within the same neutrosophic topological framework (N_{TS}) [25]. This evolutionary trajectory continued into the year 2022, when Shuker Mahmood Khalil delved into the realm of Neutrosophic Delta Generated Per-Continuous Functions in neutrosophic topological spaces (N_{TS}) [30]. Seok Jong Lee and Yeon Seok Eom [43] developed the concepts of δ-closure and δ-Interior in IF_{TS} in 2012. We are extending the aforementioned ideas to N_S in this study. With the help of examples, we discuss some of the fundamental characteristics of $\mathbf{N}_\delta$-Closure and $\mathbf{N}_\delta$-Interior in N_{TS}.

2. Perliminaries

This part of the study gives an insight to the pertinent and basic preparatory operations about N_S's

Definition 2.1. [26] Consider a non-empty fixed set S. A neutrosophic set I (N_S) can be characterized as an entity taking the following structure: $I = \{\langle s, \mu_m(I(s)), \sigma_i(I(s)), \nu_{nm}(I(s))\rangle \forall s \in S\}$, where $\mu_m(I(s))$, $\sigma_i(I(s))$and $\nu_{nm}(I(s))$ represents the degrees of membership function, indeterminacy function and nonmembership function of each element $s \in S$ to the set I.

Remark 2.2. [26] A N_S
$I = \{\langle s, \mu_m(I(s)), \sigma_i(I(s)), \nu_{nm}(I(s))\rangle \forall s \in S\}$ can be represented by an ordered triple $\langle \mu_m(I(s)), \sigma_i(I(s)), \nu_{nm}(I(s))\rangle$ within the interval $]^-0, 1^+[$ defined over the set S.

Definition 2.3. [26] Consider I as a neutrosophic set N_S in the format
$I = \{\langle s, \mu_m(I(s)), \sigma_i(I(s)), \nu_{nm}(I(s))\rangle \forall s \in S\}$, Subsequently, the complement of I, denoted as I^c, can be stipulated as $I^c = \{\langle s, \nu_{nm}(I(s)), \sigma_i(I(s)), \mu_m(I(s))\rangle \forall s \in S\}$

Definition 2.4. [26] Suppose there are two N_Ss with the structure, I and J.

$I = \{\langle s, \mu_m(I(s)), \sigma_i(I(s)), \nu_{nm}(I(s))\rangle \forall s \in S\}$ and
$J = \{\langle s, \mu_m(J(s)), \sigma_i(J(s)), \nu_{nm}(J(s))\rangle \forall s \in S\}$.

Then,

i) Subsets ($I \subseteq J$) may be defined as follows $I \subseteq J$ if and only if
$\mu_m(I(s)) \leq \mu_m(J(s)), \sigma_i(I(s)) \geq \sigma_i(J(s)), \nu_{nm}(I(s)) \geq \nu_{nm}(J(s))$

ii) Subsets $I = J$ if and only if $I \subseteq J$ and $J \subseteq I$

iii) The union of subsets $I \cup J$ can be defined in the following manner:

$$I \cup J = \{s, max\left[\mu_m\left(I\left(s\right), \mu_m\left(J\left(s\right)\right)\right)\right], min\left[\sigma_i\left(I\left(s\right)\right), \sigma_i\left(J\left(s\right)\right)\right], min\left[\nu_{nm}\left(I\left(s\right)\right), \nu_{nm}\left(J\left(s\right)\right)\right] \forall s \in S\},$$

iv) The intersection of subsets $I \cap J$ can be defined in the following manner:

$$I \cap J = \{s, min\left[\mu_m\left(I\left(s\right), \mu_m\left(J\left(s\right)\right)\right)\right], max\left[\sigma_i\left(I\left(s\right)\right), \sigma_i\left(J\left(s\right)\right)\right], max\left[\nu_{nm}\left(I\left(s\right)\right), \nu_{nm}\left(J\left(s\right)\right)\right] \forall s \in S\},$$

Definition 2.5. [26] A N_T $(S, \dot{\tau})$ that meets the axioms listed below

i) $0_N, 1_N \in \dot{\tau}$,

ii) $H_1 \cap H_2 \in \dot{\tau}$ for any $H_1, H_2 \in \dot{\tau}$,

iii) $\cup H_i \in \dot{\tau} \ \ \forall \{H_i : i \in J\} \subseteq \dot{\tau}$ Then the pair $(S, \dot{\tau})$ or simply S is called a N_{TS}.

Definition 2.6. [7] Let I be a N_S contained in a N_{TS}, representing $(S, \dot{\tau})$. Then

i) $Nint\left(I\right) = \bigcup \{J/J$ is aN_{OS}in $(S, \dot{\tau})$ and $J \subseteq I\}$ is termed as the neutrosophic interior of I;

ii) $Ncl\left(I\right) = \bigcap \{J/J$ is aN_{CS}in $(S, \dot{\tau})$ and $J \supseteq I\}$ is termed as the neutrosophic closure of I.;

Theorem 2.7. *[6] For any N_S I in a N_{TS} $(S, \dot{\tau})$, we have*

i) $Ncl\left(I^c\right) = \left(Nint\left(I\right)\right)^c$ *and*

ii) $Nint\left(I^c\right) = \left(Ncl\left(I\right)\right)^c$

Definition 2.8. [15] Let $\upsilon, \omega, \xi \in [0, 1]$ and $\upsilon + \omega + \xi \leq 3$. A neutrosophic point(NP) $s_{(\upsilon,\omega,\xi)}$of S is a NP of S, defined as

$$s_{(\upsilon,\omega,\xi)}(t) = \begin{cases} (\upsilon, \omega, \xi), & \text{if } t = s; \\ (0, 1, 1), & \text{if } t \neq s. \end{cases}$$

In this context, 's' is referred to as the support of $s_{(\upsilon,\omega,\xi)}$ and υ, ω and ξ, respectively. A NP $s_{(\upsilon,\omega,\xi)}$ is considered to be a member of a N_S
$I = \langle \mu_m\left(I\left(s\right)\right), \sigma_i\left(I\left(s\right)\right), \nu_{nm}\left(I\left(s\right)\right)\rangle$ in the set S, shown by $s_{(\upsilon,\omega,\xi)} \in I$ if $\upsilon \leq \mu_m\left(I\left(s\right)\right), \omega \leq \sigma_i\left(I\left(s\right)\right)$ and $\xi \geq \nu_{nm}\left(I\left(s\right)\right)$.

Definition 2.9. [1] Let A be a N_S in a N_{TS} $(S, \dot{\tau})$. A is said to be

i) a neutrosophic semi-open set of S, if there exists a N_{OS} B of S such that $B \leq A \leq cl\left(B\right)$.

ii) a N_{ROS} of S, if $Nint\left(Ncl\left(A\right)\right) = A$. The complement of a N_{ROS} is said to be a N_{RCS}.

3. Neutrosophic δ-Closure and δ-Interior

Definition 3.1. Let $(S, \dot{\tau})$ be a N_{TS}. Let I be a N_S and let $s_{(\upsilon,\omega,\xi)}$ be a NP. $s_{(\upsilon,\omega,\xi)}$ is considered to be neutrosophically quasi-coincident with I [denoted by $s_{(\upsilon,\omega,\xi)}qI$] if $\upsilon + \mu_m(I(s)) > 1; \omega + \sigma_i(I(s)) < 1$ and $\xi + \nu_{nm}(I(s)) < 1$.

Definition 3.2. Let I and J be two N_S's. I is said to be neutrosophic quasi coincident with J [denoted by IqJ] if $\mu_m(I(s)) + \mu_m(J(s)) > 1; \sigma_i(I(s)) + \sigma_i(J(s)) < 1$ and $\nu_{nm}(I(s)) + \nu_{nm}(J(s)) < 1$. The term 'not quasi-coincident' will be abbreviated as $\tilde{q}$.

Proposition 3.3. *Consider two N_S, I and J, and an NP in S, $s_{(\upsilon,\omega,\xi)}$. Then*

i) $I\tilde{q}J^c \Leftrightarrow I \subseteq J$
ii) $IqJ \Leftrightarrow I \nsubseteq J^c$
iii) $s_{(\upsilon,\omega,\xi)} \subseteq I \Leftrightarrow s_{(\upsilon,\omega,\xi)}\tilde{q}I^c$
iv) $s_{(\upsilon,\omega,\xi)}qI \Leftrightarrow s_{(\upsilon,\omega,\xi)} \nsubseteq I^c$

Theorem 3.4. *Let $s_{(\upsilon,\omega,\xi)}$ be a NP in S, and $I = \langle \mu_m(I(s)), \sigma_i(I(s)), \nu_{nm}(I(s)) \rangle$ a N_S in S. Then $s_{(\upsilon,\omega,\xi)} \in Ncl(I)$ if and only if IqN, for any N^q-nhd N of $s_{(\upsilon,\omega,\xi)}$.*

Proof. Consider that $I\tilde{q}N$ exists for every $N \in N^q_\epsilon\left(s_{(\upsilon,\omega,\xi)}\right)$. In this case, $s_{(\upsilon,\omega,\xi)}qG \leq N$ and $G\tilde{q}I$ exist for a set $G \in \dot{\tau}$. since G^c is a N_{CS} and by Proposition 3.3, we have $Ncl(I) \leq G^c$. Also since $s_{(\upsilon,\omega,\xi)} \notin G^c$, we have $s_{(\upsilon,\omega,\xi)} \notin Ncl(I)$. Since, which is contradiction.
Conversely, suppose $s_{(\upsilon,\omega,\xi)} \notin Ncl(I)$. Then, $s_{(\upsilon,\omega,\xi)} \notin V$ and $I \leq V$ exist for a N_{CS} V. Hence by Proposition 3.3, $V^c \in \dot{\tau}$ such that $s_{(\upsilon,\omega,\xi)}qV^c$ and $I\tilde{q}V^c$. Since, which is a contradiction. □

Example 3.5. Consider (X, τ) as a N_{TS} with X as $X = \{p, q, r\}$ and D_1, D_2, D_3, D_4 as N_S's
$D_1 = \left\langle \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}\right), \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2}\right), \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6}\right) \right\rangle$
$D_2 = \left\langle \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6}\right), \left(\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4}\right), \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}\right) \right\rangle$
$D_3 = \left\langle \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6}\right), \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2}\right), \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}\right) \right\rangle$
$D_4 = \left\langle \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}\right), \left(\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4}\right), \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6}\right) \right\rangle$
now the complement of D_1, D_2, D_3, D_4 are
$D_1^c = \left\langle \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6}\right), \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2}\right), \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}\right) \right\rangle$
$D_2^c = \left\langle \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}\right), \left(\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4}\right), \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6}\right) \right\rangle$
$D_3^c = \left\langle \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}\right), \left(\frac{p}{0.3}, \frac{q}{0.2}, \frac{r}{0.2}\right), \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6}\right) \right\rangle$
$D_4^c = \left\langle \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6}\right), \left(\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4}\right), \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}\right) \right\rangle$

Let the neutrosophic point be

$$s_{(\upsilon,\omega,\xi)} = \begin{cases} (0.7, 0.4, 0.3, & \text{if } x = p \\ (0, 1, 1), & \text{if } x \neq p. \end{cases}$$

,

where $D_2 = \left\langle \left(\frac{p}{0.4}, \frac{q}{0.6}, \frac{r}{0.6}\right), \left(\frac{p}{0.5}, \frac{q}{0.4}, \frac{r}{0.4}\right), \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}\right) \right\rangle \rightarrow N_{OS}$
Let $N = \left\langle \left(\frac{p}{0.7}, \frac{q}{0.6}, \frac{r}{0.6}\right), \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.4}\right), \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}\right) \right\rangle$
$D_2 \subseteq N$ therefore N is N^q-nhd N of $s_{(\upsilon,\omega,\xi)}$
Let $I = \left\langle \left(\frac{p}{0.7}, \frac{q}{0.6}, \frac{r}{0.6}\right), \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.4}\right), \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.2}\right) \right\rangle$, also IqN
$\Rightarrow s_{(\upsilon,\omega,\xi)} \in Ncl\,(I)$

In N_{TS}, we put forward the idea of neutrosophic δ-closure.

Definition 3.6. Consider $(S, \dot{\tau})$ as a (N_{TS}). A NP $s_{(\upsilon,\omega,\xi)}$ is said to be a neutroscophic δ-cluster point of a N_S I if AqI for each N^q_{RO}-nhd A of $s_{(\upsilon,\omega,\xi)}$. The set of all neutroscophic δ-cluster point of I is called the neutrosophic δ-closure of I and denoted by $Ncl_\delta\,(I)$. A N_S I is said to be a $N_{\delta-CS}$ if $I = Ncl_\delta\,(I)$. A $N_{\delta-OS}$ is considered to be the exact opposite of a $N_{\delta-CS}$.

Definition 3.7. Given a N_{TS} $(S, \dot{\tau})$, let I be a N_S in S. $Nint_\delta\,(I) = (Ncl_\delta\,(I^c))^c$ is the notation and definition of the neutrosophic δ-interior of I.

Remark 3.8. The following relations can be obtained from the definition above:

i) $Ncl_\delta\,(I^c) = (Nint_\delta\,(I))^c$,
ii) $(Ncl_\delta\,(I))^c = Nint_\delta\,(I^c)$.

Remark 3.9. Let I be a $N_{\delta-OS}$ if and only if $Nint_\delta\,(I) = I$ because I is $N_{\delta-OS}$ if and only if I^c is $N_{\delta-CS}$ if and only if $I^c = Ncl_\delta\,(I^c)$ if and only if $I = (Ncl_\delta\,(I^c))^c = Nint_\delta\,(I)$.

Lemma 3.10. *For any N_{OS} I in a N_{TS} $(S, \dot{\tau})$ such that $s_{(\upsilon,\omega,\xi)}qI$, $Nint\,(Ncl\,(I))$ is a N^q_{RO}-nhd of $s_{(\upsilon,\omega,\xi)}$.*

Proof. Clearly $Nint\,(I) \subseteq Nint\,(Ncl\,(I))$. Since I is a N_{OS}, we have $I = Nint\,(I) \subseteq Nint\,(Ncl\,(I))$. By definition 2.9, $Nint\,(Ncl\,(I))$ is a N_{ROS}. Therefore $Nint\,(Ncl\,(I))$ is a N^q_{RO}-nhd of $s_{(\upsilon,\omega,\xi)}$. □

Corollary 3.11. *I is a N_{CS} if it is a $N_{\delta-CS}$ in N_{TS} $(S, \dot{\tau})$. The Corollary's counterpart is not true. Example 3.18*

Theorem 3.12. *$Ncl\,(I) = Ncl_\delta\,(I)$ exists if I corresponds to N_{OS} in N_{TS} $(S, \dot{\tau})$.*

Proof. Ensuring that $Ncl_{\delta}(I) \subseteq Ncl(I)$ is sufficient. Take any $s_{(\upsilon,\omega,\xi)} \in Ncl_{\delta}(I)$. Suppose that $s_{(\upsilon,\omega,\xi)} \notin Ncl(I)$. By Theorem 3.4, there exists a N^q-nhd G of $s_{(\upsilon,\omega,\xi)}$ such that $G\tilde{q}I$. Since $G\tilde{q}I$, we have $G \subseteq I^c$. Since I^c is a N_{CS}, $Ncl(G) \subseteq Ncl(I^c) = I^c$. Therefore, $Nint(Ncl(G)) \subseteq Nint(I^c) \subseteq I^c$, i.e. $Nint(Ncl(G))\,\tilde{q}I$. By Lemma 3.10, $Nint(Ncl(I))$ is a N^q_{RO}-nhd A of $s_{(\upsilon,\omega,\xi)}$ such that $Nint(Ncl(I))\,\tilde{q}I$. Hence $s_{(\upsilon,\omega,\xi)} \notin Ncl_{\delta}(I)$. □

Theorem 3.13. *In N_{TS}, if P is a semi-open set, then $Ncl(P) = Ncl_{\delta}(P)$.*

Proof. Enough to show that $Ncl_{\delta}(P) \subseteq Ncl(P)$. Take any $s_{(\upsilon,\omega,\xi)} \in Ncl_{\delta}(P)$. Suppose that $s_{(\upsilon,\omega,\xi)} \notin Ncl(P)$. Then there exists a N^q_O-nhd Q of $s_{(\upsilon,\omega,\xi)}$ such that $Q\tilde{q}P$. As per the definition of a semi-open set, there is a N_{OS} R such that $R \subseteq P \subseteq Ncl(R)$. Thus $Q \subseteq P^c \subseteq R^c$. Hence $Ncl(Q) \subseteq Ncl(P^c) \subseteq Ncl(R^c) = R^c$. Also, $Nint(Ncl(Q)) \subseteq Nint(Ncl(P^c)) \subseteq Nint(Ncl(R^c)) = Nint(R^c) \subseteq R^c$, i.e. $Nint(Ncl(Q)) \subseteq R^c$. Therefore $R \subseteq (Nint(Ncl(Q)))^c$. Hence $P \subseteq Ncl(R) \subseteq (Ncl(Nint(Ncl(Q)))^c) = (Nint(Ncl(Q)))^c$ because $(Nint(Ncl(Q)))^c$ is a N_{CS}. Thus $Nint(Ncl(Q))\,\tilde{q}P$. Hence $s_{(\upsilon,\omega,\xi)} \notin Ncl_{\delta}(P)$. □

Theorem 3.14. *Given a N_{TS} $(S,\dot{\tau})$, let I and J be two N_S. Following that, we have the subsequent characteristics:*

i) $Ncl_{\delta}(0_N) = 0_N$
ii) $I \subseteq Ncl_{\delta}(I)$
iii) $I \subseteq J \Rightarrow Ncl_{\delta}(I) \subseteq Ncl_{\delta}(J)$
iv) $Ncl_{\delta}(I) \cup Ncl_{\delta}(J) = Ncl_{\delta}(I \cup J)$
v) $Ncl_{\delta}(I \cap J) \subseteq Ncl_{\delta}(I) \cap Ncl_{\delta}(J)$.

Proof. i) Obvious

ii) Since $I \subseteq Ncl(I) \subseteq Ncl_{\delta}(I), I \subseteq Ncl_{\delta}(I)$.

iii) Let $s_{(\upsilon,\omega,\xi)}$ be a NP in S such that $s_{(\upsilon,\omega,\xi)} \notin Ncl_{\delta}(J)$. Then there is a N^q_{RO}-nhd A of $s_{(\upsilon,\omega,\xi)}$ such that $A\tilde{q}J$. Since $I \subseteq J$, we have $A\tilde{q}I$. Therefore $s_{(\upsilon,\omega,\xi)} \notin Ncl_{\delta}(I)$.

iv) Since $I \subseteq I \cup J, Ncl_{\delta}(I) \subseteq Ncl_{\delta}(I \cup J)$. Similarily, $Ncl_{\delta}(J) \subseteq Ncl_{\delta}(I \cup J)$. Hence $Ncl_{\delta}(I) \cup Ncl_{\delta}(J) \subseteq Ncl_{\delta}(I \cup J)$. Take any $s_{(\upsilon,\omega,\xi)} \in Ncl_{\delta}(I \cup J)$ for evidence that $Ncl_{\delta}(I \cup J) \subseteq Ncl_{\delta}(I) \cup Ncl_{\delta}(J)$. Then for any N^q_{RO}-nhd A of $s_{(\upsilon,\omega,\xi)}$, $Aq(I \cup J)$. Hence, AqI or AqJ. Therefore $s_{(\upsilon,\omega,\xi)} \in Ncl_{\delta}(I)$ or $s_{(\upsilon,\omega,\xi)} \in Ncl_{\delta}(J)$. Hence $s_{(\upsilon,\omega,\xi)} \in Ncl_{\delta}(I) \cup Ncl_{\delta}(J)$.

v) Since $I \cap J \subseteq I, Ncl_{\delta}(I \cap J) \subseteq Ncl_{\delta}(I)$. Similarly, $Ncl_{\delta}(I \cap J) \subseteq Ncl_{\delta}(J)$. Therefore $Ncl_{\delta}(I \cap J) \subseteq Ncl_{\delta}(P) \cap Ncl_{\delta}(J)$.

0.1cm□

Theorem 3.15. *Considering $(S, \dot{\tau})$ to represent a N_{TS}, the following remains true:*

i) *Finite union of $N_{\delta-CS}$ in S is an $N_{\delta-CS}$ in S*
ii) *Arbitrary intersection of $N_{\delta-CS}$s in S is a $N_{\delta-CS}$ in S.*

Proof. i) Let T_1 and T_2 be $N_{\delta-CS}$s. Then $Ncl_\delta(T_1 \cup T_2) = Ncl_\delta(T_1) \cup Ncl_\delta(T_2) = T_1 \cup T_2$. Thus $T_1 \cup T_2$ is a $N_{\delta-CS}$.

ii) Let T_i be a $N_{\delta-CS}$, for each $i \in I$. To show that $Ncl_\delta(\cap T_i) \subseteq \cap T_i$, take any $s_{(\upsilon,\omega,\xi)} \in Ncl_\delta(\cap T_i)$. Suppose that $s_{(\upsilon,\omega,\xi)} \notin \cap T_i$. Then there exists an $i_0 \in I$ such that $s_{(\upsilon,\omega,\xi)} \notin T_{i0}$. Since T_{i0} is a $N_{\delta-CS}$, $s_{(\upsilon,\omega,\xi)} \notin Ncl_\delta(T_{i0})$. Therefore there exists a N^q_{RO}-nhd A of $s_{(\upsilon,\omega,\xi)}$ such that $A\tilde{q}T_{i0}$. Since $A\tilde{q}T_{i0}$ and $\cap T_i \subseteq T_{i0}$, we have $A\tilde{q}(\cap T_{i0})$. Thus $s_{(\upsilon,\omega,\xi)} \notin Ncl_\delta(\cap T_i)$. This is a contradiction. Hence $Ncl_\delta(\cap T_i) \subseteq \cap T_i$.

0.1cm□

Theorem 3.16. *Let R be a N_S in a N_{TS} $(S, \dot{\tau})$, then $Ncl_\delta(R)$ is the intersection of all N_{RCS}s of R or*
$Ncl_\delta(R) = \bigcap\{H/R \subseteq H = Ncl(Nint(H))\}$.

Proof. Suppose that $s_{(\upsilon,\omega,\xi)} \notin \bigcap\{H/R \subseteq H = Ncl(Nint(H))\}$. Then there exists a N_{RCS} H such that $s_{(\upsilon,\omega,\xi)} \notin H$ and $R \subseteq H$. Since $s_{(\upsilon,\omega,\xi)} \notin H$, $s_{(\upsilon,\omega,\xi)}qH^c$. Note that $R \subseteq H$ if and only if $R\tilde{q}H^c$. Thus H^c is a N^q_{RO}-nhd of $s_{(\upsilon,\omega,\xi)}$ such that $R\tilde{q}H^c$. Hence $s_{(\upsilon,\omega,\xi)} \notin Ncl_\delta(R)$.

Let $s_{(\upsilon,\omega,\xi)} \in \bigcap\{H/R \subseteq H = Ncl(Nint(H))\}$. Suppose that $s_{(\upsilon,\omega,\xi)} \notin Ncl_\delta(R)$. Then there exists a N^q_{RO}-nhd I of $s_{(\upsilon,\omega,\xi)}$ such that $R\tilde{q}I$. So, $R \subseteq I^c$. Since $s_{(\upsilon,\omega,\xi)}qI, s_{(\upsilon,\omega,\xi)} \notin I^c$. Therefore there exists a N_{RCS} I^c such that $s_{(\upsilon,\omega,\xi)} \notin I^c$ and $R \subseteq I^c$. Hence $s_{(\upsilon,\omega,\xi)} \notin \bigcap\{H/R \subseteq H = Ncl(Nint(H))\}$. This is a contradiction. Thus $s_{(\upsilon,\omega,\xi)} \in Ncl_\delta(R)$. □

Remark 3.17. From the above theorem, for any N_S R, $Ncl_\delta(R)$ is a N_{CS}. Moreover, $Ncl_\delta(R)$ becomes $N_{\delta-CS}$, which will be shown in the Theorem 3.20.

Example 3.18. Let $S = \{a, b\}$, and R be the N_S defined by
$R = \langle(0.5, 0.3), (0.2, 0.2), (0.3, 0.5)\rangle$ Let $\dot{\tau} = \{0_N, 1_N, R\}$. Then $\dot{\tau}$ is a N_T. Since $Ncl(Nint(R^c)) = Ncl(0_N) = 0_N \neq R^c$, R^c is not a N_{RCS}. Hence 0_N and 1_N are the only regular closed sets. thus $Ncl_\delta(R^c) = \bigcap\{H/R^c \subseteq H = Ncl(Nint(H))\} = 1_N \neq R^c$. Hence R^c is not $N_{\delta-CS}$. Therefore, R^c is a N_{CS} which is not $N_{\delta-CS}$.

Theorem 3.19. *If I is a N_{RCS}, then I is a $N_{\delta-CS}$.*

Proof. Let I be a N_{RCS}. Then $Ncl(Nint(I)) = I$. By Theorem 3.16, $Ncl_\delta(I) = \bigcap\{H/I \subseteq H = Ncl(Nint(H))\} = I$. Thus I is $N_{\delta-CS}$. □

Theorem 3.20. *For any N_S I, $Ncl_\delta(I)$ is a $N_{\delta-CS}$.*

Proof. By Theorem 3.15,3.16,3.19. □

The following properties of neutrosophic δ-interior are the results obtained from neutrosophic δ-closure.

Theorem 3.21. *For a N_{TS} $(S, \dot{\tau})$, let I and J be two N_S. Following that, we have the subsequent characteristics:*

i) $Nint_\delta(1_N) = 1_N$
ii) $Nint_\delta(I) \subseteq I$
iii) $I \subseteq J \Rightarrow Nint_\delta(I) \subseteq Nint_\delta(J)$
iv) $Nint_\delta(I \cap J) = Nint_\delta(I) \cap Nint_\delta(J)$
v) $Nint_\delta(I) \cup Nint_\delta(J) \subseteq Nint_\delta(I \cup J)$.

Theorem 3.22. *Considering $(S, \dot{\tau})$ to represent a N_{TS}, the following remains true:*

i) *Finite intersection of $N_{\delta-OS}$ in S is a $N_{\delta-OS}$ in S*
ii) *Arbitrary union of $N_{\delta-OS}$s in S is a $N_{\delta-OS}$ in S.*

Theorem 3.23. *Given an I of type N_S in the set $(S, \dot{\tau})$, we have $Nint_\delta(I) = \bigcup \{G/Nint(Ncl(G)) = G \subseteq I\}$. It follows that $Nint_\delta(I)$ is a N_{OS}.*

Corollary 3.24. *I is a N_{OS} if and only if I belong to a $N_{\delta-OS}$ in a N_{TS} $(S, \dot{\tau})$.*

Corollary 3.25. *If I is a N_{ROS}, then I is a $N_{\delta-OS}$.*

Corollary 3.26. *For any N_S I, $Nint_\delta(I)$ is a $N_{\delta-OS}$.*

4. Conclusion

This paper covered the fascinating natural subject of $\mathbf{N}_\delta$-Closure and $\mathbf{N}_\delta$-Interior in N_{TS}. It will provide many new opportunities for research into N_{TS}, allowing us to expand on and further analyze the ideas we presented in this paper.

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