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On the Isotopy of some Varieties of Fenyves Quasi Neutrosophic Triplet Loop (Fenyves BCI-algebras)

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Abstract: Neutrosophy theory has found application in health sciences in recent years. There is the need to develop neutrosophic algebraic systems which are good and appropriate for studying and understanding the effects of diseases and their possible treatments. In order to achieve this, special types of quasi neutrosophic loops and their isotopy needed to be introduced for this purpose. Fenyves BCI-algebras are BCI-algebras (special types of quasi neutrosophic loops) that satisfy the 60 Bol-Moufang identities. In this paper, the isotopy of BCI-algebras are studied. Neccessary and sufficient conditions for a groupoid isotope of a BCI-algebra to be a BCI-algebra are established. It is shown that p-semisimplicity, quasi-associativity and BCK-algebra are invariant under isotopies which are determined by some regular permutation groups. Furthermore, the isotopy of both the 46 associative and 14 non-associative Fenyves BCI-algebras are also studied. It is shown that for BCI-alegbras, associativity is isotopic invariant. Hence, the following set of Fenyves BCI algebras (F_i-algebras) are invariant under any isotopy: $i \in \{1,2,4,6,7,9,10,11,12,13,14,15,16,17,18,20,22,23,24,25,26,27,28,30,31,32,33,34,35,36,37,38,40,41,43,44,45,47,48,49,50,51,53,57,58,60\}$. It is shown that the following sets of non-associative Fenyves BCI algebras (F_i-algebras) are invariant under isotopies which are determined by some regular permutation groups: $i \in \{3,5,8,19,21,29,39,42,46,52,55,56,59\}, \{56\}, \{8,19,29,39,46,59\}$. In conclusion, this is the isotopic study of 120 particular types of the 540 varieties of Fenyves quasi neutrosophic triplet loops (FQNTLs) which were recently discovered, wherein the 14 non-associative Fenyves BCI-algebras do not necessarily have the Iseki's conditions (S). Importantly, applying these results, the initial (old, sick or healthy) state of a person can be represented by a type of Fenyves BCI-algebra, while the Fenyves BCI-algebra isotope will represent the final (new, healthy or sick) state of the person as a result of the prescribed medical treatment, which the isotopism represents. The isotopism is a measure of the change from the old state of body condition to the new state.

Keywords: BCI-algebra; quasi neutrosophic loops; Fenyves identities; Bol-Moufang Type

1. Introduction

The prevalence and spread of diseases among inhabitants of the world, especially tropical regions has raised serious concerns among scientists. In this work, we embarked on an algebraic way of representing the effects of diseases on the health of the people. This is based on the philosophy of representing disease-victim(s) by algebraic structures. These structures represent the state of health before the "invasion" by organisms which cause disease(s). The transformation of the body by these diseases is represented by the isotopisms which form the crux of the study. The isotopisms transform a hitherto healthy person to somebody with health challenges. Other researchers who have worked on neutrosophy theory and its applications to medicine and other fields include Abdel-Basset et al. [1], [2], [3], [4].

1.1. BCI-algebra and BCK-algebra

BCK-algebras and BCI-algebras are abbreviated as two B-algebras. The former was raised in 1966 by Imai and Iseki [16], Japanese mathematicians, and the latter was put forward in the same year by Iseki [17]. The two algebras originated from two different sources: set theory and propositional calculi.

There are some systems which contain the only implicational functor among logical functors, such as the system of weak positive implicational calculus, BCK-system and BCI-system. Undoubtedly, there are common properties among those systems. We know that there are close relationships between the notions of the set difference in set theory and the implication functor in logical systems. For example, we have the following simple inclusion relations in set theory:

$$(A - B) - (A - C) \subseteq C - B, \qquad A - (A - B) \subseteq B.$$

These are similar to the propositional formulas in propositional calculi:

$$(p \to q) \to ((q \to r) \to (p \to r)), \qquad p \to ((p \to q) \to q),$$

which raise the following questions: What are the most essential and fundamental properties of these relationships? Can we formulate a general algebra from the above consideration? How do we find an axiomatic system to establish a good theory of general algebras? Answering these questions, K.Iseki formulated two kinds of B-algebras, in which BCI-algebras are of wider class than BCK-algebras. Their names are taken from BCK and BCI systems in combinatory logic.

BCI-Algebras are very interesting algebraic structures that have generated wide interest among pure mathematicians. In fact, since late 1970s, much attention has been paid to the study of BCI and BCK algebras. In particular, the participation in the research of polish mathematicians Tadeusz Traczyk and Andrzej Wronski as well as Australian mathematician William H. Cornish and so on, is really making this branch of algebra to develop rapidly. Many interesting and important results are discovered continuously. Now, the theory of BCI-algebras has been widely spread to many areas such as general theory which includes congruences, quotient algebras, BCI-Homomorphisms, direct sums and direct products, commutative BCK-algebras, positive implicative and implicative BCK-algebras, derivations of BCI-algebras, and ideal theory of BCI-algebras ([16], [18], [14], [41], [50]).

1.2. BCI-algebra and the Fenyves Identities

We shall now discuss BCI-algebras in relation to Fenyves identities.

Definition 1 *A triple* $(X,*,0)$ *is called a BCI-algebra if the following conditions are satisfied for any* $x, y, z \in X$*:*

1. $((x * y) * (x * z)) * (z * y) = 0$;
2. $x * 0 = x$;
3. $x * y = 0$ and $y * x = 0 \Rightarrow x = y$.

We call the binary operation $*$ on X multiplication, and the constant 0 in X the zero element of X. We often write X instead of $(X,*,0)$ for a BCI-algebra in brevity. Juxtaposition xy shall be at times used for $x*y$ and will have preference over $*$ i.e. $xy*z=(x*y)*z$.

Example 1 *Let S be a set. Let 2^S be the power set of S, $-$ the set difference and $\emptyset$ for the empty set. Then $(2^S,-,\emptyset)$ is a BCI-algebra.*

Example 2 *Suppose $(G,\cdot,e)$ is an abelian group with e as the identity element. Define a binary operation $*$ on G by putting $x*y=xy^{-1}$. Then $(G,*,e)$ is a BCI-algebra.*

Example 3 $(\mathbb{Z},-,0)$ *and* $(\mathbb{R}-\{0\},\div,1)$ *are BCI-algebras.*

Example 4 *Let S be a set. Let 2^S be the power set of S, Δ the symmetric difference and $\emptyset$ the empty set. Then $(2^S,\Delta,\emptyset)$ is a BCI-algebra.*

The following theorems give necessary and sufficient conditions for the existence of a BCI-algebra.

Theorem 1 *(Yisheng [51])*

Let X be a non-empty set, $*$ a binary operation on X and 0 a constant element of X. Then $(X,*,0)$ is a BCI- algebra if and only if the following conditions hold:

1. $((x*y)*(x*z))*(z*y)=0$;
2. $(x*(x*y))*y=0$;
3. $x*x=0$;
4. $x*y=0$ and $y*x=0$ imply $x=y$.

Definition 2 *A BCI- algebra $(X,*,0)$ is called a BCK-algebra if $0*x=0$ for all $x\in X$.*

Definition 3 *(Jaiyéolá et al. [36])*

A BCI- algebra $(X,*,0)$ is called a Fenyves BCI-algebra if it satisfies an identity of Bol-Moufang type. The identities of Bol-Moufang type are given below:

$F_1: xy*zx=(xy*z)x$ $F_2: xy*zx=(x*yz)x$ (Moufang identity) $F_3: xy*zx=x(y*zx)$

$F_4: xy*zx=x(yz*x)$ (Moufang identity) $F_5: (xy*z)x=(x*yz)x$ $F_6: (xy*z)x=x(y*zx)$ (extra identity)

$F_7: (xy*z)x=x(yz*x)$ $F_8: (x*yz)x=x(y*zx)$ $F_9: (x*yz)x=x(yz*x)$ $F_{10}: x(y*zx)=x(yz*x)$

$F_{11}: xy\cdot xz=(xy*x)z$ $F_{12}: xy*xz=(x*yx)z$ $F_{13}: xy*xz=x(yx*z)$ (extra identity)

$F_{14}: xy*xz=x(y*xz)$ $F_{15}: (xy*x)z=(x*yx)z$ $F_{16}: (xy*x)z=x(yx*z)$

$F_{17}: (xy*x)z=x(y*xz)$ (Moufang identity) $F_{18}: (x*yx)z=x(yx*z)$

$F_{19}: (x*yx)z=x(y*xz)$ (left Bol identity) $F_{20}: x(yx*z)=x(y*xz)$ $F_{21}: yx*zx=(yx*z)x$

$F_{22}: yx*zx=(y*xz)x$ (extra identity) $F_{23}: yx*zx=y(xz*x)$ $F_{24}: yx*zx=y(x*zx)$

$F_{25}: (yx*z)x=(y*xz)x$ $F_{26}: (yx*z)x=y(xz*x)$ (right Bol identity)

$F_{27}: (yx*z)x=y(x*zx)$ (Moufang identity) $F_{28}: (y*xz)x=y(xz*x)$ $F_{29}: (y*xz)x=y(x*zx)$

$F_{30}: y(xz*x)=y(x*zx)$ $F_{31}: yx*xz=(yx*x)z$ $F_{32}: yx*xz=(y*xx)z$ $F_{33}: yx*xz=y(xx*z)$

$F_{34}: yx*xz=y(x*xz)$ $F_{35}: (yx*x)z=(y*xx)z$ $F_{36}: (yx*x)z=y(xx*z)$ (RC identity)

$F_{37}: (yx*x)z=y(x*xz)$ (C-identity) $F_{38}: (y*xx)z=y(xx*z)$ $F_{39}: (y*xx)z=y(x*xz)$ (LC identity)

$F_{40}: y(xx*z)=y(x*xz)$ $F_{41}: xx*yz=(x*xy)z$ (LC identity) $F_{42}: xx*yz=(xx*y)z$

$F_{43}: xx*yz=x(x*yz)$ $F_{44}: xx*yz=x(xy*z)$ $F_{45}: (x*xy)z=(xx*y)z$

$F_{46}: (x*xy)z=x(x*yz)$ (LC identity) $F_{47}: (x*xy)z=x(xy*z)$ $F_{48}: (xx*y)z=x(x*yz)$ (LC identity)

$F_{49}: (xx*y)z=x(xy*z)$ $F_{50}: x(x*yz)=x(xy*z)$ $F_{51}: yz*xx=(yz*x)x$ $F_{52}: yz*xx=(y*zx)x$

$F_{53}: yz*xx=y(zx*x)$ (RC identity) $F_{54}: yz*xx=y(z*xx)$ $F_{55}: (yz*x)x=(y*zx)x$

$F_{56}: (yz*x)x=y(zx*x)$ (RC identity) $F_{57}: (yz*x)x=y(z*xx)$ (RC identity)

$F_{58}: (y * zx)x = y(zx * x)$ $F_{59}: (y * zx)x = y(z * xx)$ $F_{60}: y(zx * x) = y(z * xx)$

The identities of Bol-Moufang type are sixty in number based on Fenyves [12], [13]. The identities of Bol-Moufang type were investigated in BCI-algebras by Jaiyéolá et al. [36], thereby leading to the study of the sixty varieties of Fenyves BCI -algebras, as well as their holomorphic study in Ilojide et al. [15]. Here are some examples.

Example 5 *Let us assume the BCI-algebra $(G,*,e)$ in Example 2. Then $(G,*,e)$ is an F_8-algebra, F_{19}-algebra, F_{29}-algebra, F_{39}-algebra, F_{46}-algebra, F_{52}-algebra, F_{54}-algebra, F_{59}-algebra.*

Example 6 *Let us assume the BCI-algebra $(2^S,-,\emptyset)$ in Example 1. Then $(2^S,-,\emptyset)$ is an F_3-algebra, F_5-algebra, F_{21}-algebra, F_{29}-algebra, F_{42}-algebra, F_{46}-algebra, F_{54}-algebra and F_{55}-algebra.*

Example 7 *The BCI-algebra $(2^S,\Delta,\emptyset)$ in Example 4 is associative.*

Example 8 *By considering the direct product of the BCI-algebras $(G,*,e)$ and $(2^S,-,\emptyset)$ of Example 2 and Example 1 respectively, we have a BCI-algebra $(G\times 2^S,(*,-),(e,\emptyset))$ which is a F_{29}-algebra and a F_{46}-algebra.*

Remark 1 *The direct product of two or more BCI-algebras which are F_i-algebras will give a BCI-algebra which is an F_i-algebra for distinct i's.*

Definition 4 *A BCI-algebra $(X,*,0)$ is called associative if $(x*y)*z = x*(y*z)$ for all $x,y,z\in X$.*

Definition 5 *A BCI-algebra $(X,*,0)$ is called p-semisimple if $0*(0*x) = x$ for all $x\in X$.*

Theorem 2 *(Yisheng [51]) Suppose that $(X,*,0)$ is a BCI-algebra. Define a binary relation $\leq$ on X by which $x\leq y$ if and only if $x*y=0$ for any $x,y\in X$. Then $(X,\leq)$ is a partially ordered set with 0 as a minimal element(meaning that $x\leq 0$ implies $x=0$ for any $x\in X$).*

Definition 6 *A BCI-algebra $(X,*,0)$ is called quasi-associative if $(x*y)*z\leq x*(y*z)$ for all $x,y,z\in X$.*

The following theorems give equivalent conditions for associativity, quasi-associativity and p-semisimplicity in a BCI-algebra:

Theorem 3 *(Yisheng [51])*

Given a BCI-algebra X, the following are equivalent $x,y,z\in X$:

1. X is associative.
2. $0*x=x$.
3. $x*y=y*x\ \forall\, x,y\in X$.

Theorem 4 *(Yisheng [51])*

Let X be a BCI-algebra. Then the following conditions are equivalent for any $x,y,z,u\in X$:

1. X is p-semisimple
2. $(x*y)*(z*u) = (x*z)*(y*u)$.
3. $0*(y*x) = x*y$.
4. $(x*y)*(x*z) = z*y$.
5. $z*x = z*y$ implies $x=y$. (the left cancellation law)
6. $x*y=0$ implies $x=y$.

Theorem 5 *(Yisheng [51])*

Given a BCI-algebra X, the following are equivalent for all $x,y\in X$:

1. X is quasi-associative.

2. $x * (0 * y) = 0$ implies $x * y = 0$.
3. $0 * x = 0 * (0 * x)$.
4. $(0 * x) * x = 0$.

Theorem 6 *(Yisheng [51])*
A triple $(X, *, 0)$ is a BCI-algebra if and only if there is a partial ordering $\leq$ on X such that the following conditions hold for any $x, y, z \in X$:

1. $(x * y) * (x * z) \leq z * y$;
2. $x * (x * y) \leq y$;
3. $x * y = 0$ if and only if $x \leq y$.

Theorem 7 *(Yisheng [51])*
Let X be a BCI-algebra. X is p-semisimple if and only if one of the following conditions holds for any $x, y, z \in X$:

1. $x * z = y * z$ implies $x = y$. (the right cancellation law)
2. $(y * x) * (z * x) = y * z$.
3. $(x * y) * (x * z) = 0 * (y * z)$.

Theorem 8 *(Yisheng [51]) Suppose that $(X, *, 0)$ is a BCI-algebra. X is associative if and only if X is p-semisimple and X is quasi-associative.*

Theorem 9 *(Yisheng [51]) Suppose that $(X, *, 0)$ is a BCI-algebra. Then for all $x, y, z \in X$:*

1. $(x * y) * z = (x * z) * y$.
2. $x \geq y$ implies $0 * x = 0 * y$.

Remark 2 *In Theorem 8, quasi-associativity in BCI-algebra plays a similar role which weak associativity (i.e. the F_i identities) plays in quasigroup and loop theory.*

1.3. Isotopy and Autotopy in Quasigroups and Loops

We now move on to quasigroups and loops, their isotopy and autotopy.

Definition 7 *Let L be a non-empty set. Define a binary operation $(\cdot)$ on L . If $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If in a groupoid $(L, \cdot)$, the equations:*

$$a \cdot x = b \qquad and \qquad y \cdot a = b$$

have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. If in a quasigroup $(L, \cdot)$, there exists a unique element e called the identity element such that for all $x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop.

Remark 3 *For a groupoid $(G, \cdot)$, $R_x: G \to G$, the right translation is defined by $yR_x = y \cdot x$ and $L_x: G \to G$, the left translation is defined by $yL_x = x \cdot y$ for all $x, y \in G$. This mappings are not necessarily bijections. But for a quasigroup, they are.*

Consider $(G, \cdot)$ and $(H, \circ)$ being two groupoids (quasigroups, loops). Let A, B and C be three bijective mappings, that map G onto H. The triple $\alpha = (A, B, C)$ is called an *isotopism* of $(G, \cdot)$ onto $(H, \circ)$, written as

$$(G, \cdot) \xrightarrow{(A,B,C)} (H, \circ) \text{ if } xA \circ yB = (x \cdot y)C \forall x, y \in G.$$

So, $(H, \circ)$ is called a groupoid (quasigroup, loop) *isotope* of $(G, \cdot)$.

If $C = I$ is the identity map on G so that $H = G$, then the triple $\alpha = (A, B, I)$ is called a *principal isotopism* of $(G,\cdot)$ onto $(G,\circ)$ and $(G,\circ)$ is called a *principal isotope* of $(G,\cdot)$. Eventually, the equation of relationship now becomes

$$x \cdot y = xA \circ yB \forall x, y \in G$$

which is easier to work with. But if $A = R_g$ and $B = L_f$ where $f, g \in G$, the relationship now becomes

$$x \cdot y = xR_g \circ yL_f \forall x, y \in G.$$

With this new form, the triple $\alpha = (R_g, L_f, I)$ is called an f, g*-principal isotopism* of $(G,\cdot)$ onto $(G,\circ)$, f and g are called *translation elements* of G or at times written in the pair form (g, f), while $(G,\circ)$ is called an f, g*-principal isotope* of $(G,\cdot)$.

The following theorem shows that the principal isotopes of a groupoid account for all its isotopes.

Theorem 10 *(Pflugfelder [43])*
If $(G,\cdot)$ and $(H,\circ)$ are isotopic groupoids, then $(H,\circ)$ is isomorphic to some principal isotope $(G, å)$ of $(G,\cdot)$.

Let $(X,*,0)$ be a BCI-algebra and let $x + y = x * (0 * x)$. A groupoid $(X, +)$ is called an associated groupoid of $(X,*,0)$. Based on Theorem 2, Corollaries 3, 4 and 5 of Dudek [9], $x * y = x - y = x + (-y) \Leftrightarrow (x * y)I = xI + yJ$ where $J: x \mapsto -x$. so, we have

Lemma 1 *A BCI-algebra* $(X,*,0)$ *is a quasigroup if and only if there exists an abelian group* $(X, +, 0)$ *such that* $(X, +, 0) \xrightarrow{(I,I,J)} (X,*,0)$.

According to Dudek [9], the variety of all BCI-algebras that are quasigroups (BCI-quasigroups) is selected from the quasivariety of all BCI-algebra by any of the following equivalent laws:

(i) p-semi simplicity law: $0 * (0 * x) = x$
(ii) Semi left inverse property: $x * (x * y) = y$ (SLIP)
(iii) Medial law: $(x * y) * (z * u) = (x * z) * (y * u)$
(iv) $(x * y) * (x * z) = (z * y)$
(v) $0 * (x * z) = z * x$
(vi) $(x * y) * (z * x) = (x * z) * (y * x)$
(vii) $[(x * y) * z] * [(x * u) * y] = (u * z)$

Thus, following Lemma 1, it can further be said that the variety of all BCI-algebras that are quasigroups is determined by abelian group under the isotopy (I, I, J) where J is the inverse mapping on the abelian group.

Dudek [11] showed that a BCI-algebra with the medial law obeys the SLIP and further showed in Dudek [10] that every BCI-algebra that obeys the SLIP has the Iseki's condition (S)-[19] and form a variety characterized with an associated abelian group.

In Theorem 10, if $(G,\cdot) = (H,\circ)$, then the triple $\alpha = (A, B, C)$ of bijections on $(G,\cdot)$ is called an autotopism of the groupoid (quasigroup, loop) $(G,\cdot)$. Such triples form a group $AUT(G,\cdot)$ called the autotopism group of $(G,\cdot)$. Furthermore, if $A = B = C$, then A is called an automorphism of the groupoid (quasigroup, loop) $(G,\cdot)$. Such bijections form a group $AUM(G,\cdot)$ called the automorphism group of $(G,\cdot)$.

The group of all permutation on G is called the permutation group of G and denoted by $SYM(G)$.

1. $U \in SYM(G)$ is called autotopic if there exists $(U,V,W) \in AUT(G,\cdot)$; the set of all such mappings forms a group $\Sigma(G,\cdot)$.
2. $U \in SYM(G)$ is called λ-regular if there exists $(U,I,U) \in AUT(G,\cdot)$; the set of all such mappings forms a group $\Lambda(G,\cdot) \leq \Sigma(G,\cdot)$.
3. $U \in SYM(G)$ is called ρ-regular if there exists $(I,U,U) \in AUT(G,\cdot)$; the set of all such mappings forms a group $\mathcal{P}(G,\cdot) \leq SYM(G)$.
4. $U \in SYM(G)$ is called μ-regular if there exists $U' \in SYM(G)$ such that $(U,U'^{-1},I) \in AUT(G,\cdot)$. U' is called the adjoint of U. The set of all μ-regular mappings forms a group $\Phi(G,\cdot) \leq \Sigma(G,\cdot)$. The set of all adjoint mapping forms a group $\Psi(G,\cdot) \leq SYM(G)$. Whenever $U' = U$, then U is said to be μ-regular and self adjoint.

1.4. Quasigroup, Loop and their Universality

In recent past, and up to the present time, identities of Bol-Moufang type have been studied on the platform of groupoids, quasigroups and loops by Fenyves [12], Phillips and Vojtěchovský, P. [44], [45], [46], Jaiyeola [20], Robinson [47], Burn [6], [7], [8], Kinyon and Kunen [40] and by several other authors to mention a few. Fenyves [13], Kinyon and Kunen [40], and Phillips and Vojtěchovský [46] found some of these identities to be equivalent to associativity in quasigroups and loops (i.e. groups), and others to describe weak associative laws such as extra, Bol, Moufang, central, flexible laws in quasigroups and loops. These results are tabularly summarised in Jaiyéolá et al. [36].

Loops such as Bol loops, Moufang loops, central loops and extra loops are the most popular loops of Bol-Moufang type whose isotopic invariance (universality) has been considered. Some others are flexible loops, F-quasigroups, totally symmetric quasigroups(TSQ), distributive quasigroups, weak inverse property loops(WIPLs), cross inverse property loops(CIPLs), semi-automorphic inverse property loops(SAIPLs) and inverse property loops(IPLs). As shown in Pflugfelder [43], a left(right) inverse property loop is universal if and only if it is a left(right) Bol loop, so an IPL is universal if and only if it is a Moufang loop. Kepka et. al. [37], [38], [39] solved the Belousov problem concerning the universality of F-quasigroup which has been open since 1967. The universality of WIPLs and CIPLs has been addressed by Osborn [42] and Artzy [5] respectively while the universality of elasticity(flexibility) was studied by Syrbu [49]. Jaiyéolá [20], [22], Jaiyéolá and Adéníran [26], [27], [28] studied the universality of central loops while Jaiyéolá [23], [21], [24], [25], Jaiyéolá and Adéníran [29], [31], [30], [32], and Jaiyéolá et al. [33] studied the universality Osborn loops.

1.5. Some Existing Results on Fenyves BCI-algebras

Jaiyéolá et al. [36] investigated Fenyves identities on the platform of BCI-algebras. They classified the Fenyves BCI-algebras into 46 associative and 14 non-associative types and showed that some Fenyves identities played the role of quasi-associativity, vis-a-vis Theorem 8 in

BCI-algebras. Their work clarified the relationship between a BCI-algebra, a quasigroup and a loop. Some of their results are stated below.

Theorem 11 *(Jaiyéolá et al. [36])*

1. A BCI algebra X is a quasigroup if and only if it is p-semisimple.
2. A BCI algebra X is a loop if and only if it is associative.
3. An associative BCI algebra X is a Boolean group.

Theorem 12 *(Jaiyéolá et al. [36])*

Let $(X,*,0)$ be a BCI-algebra. If X is any of the following Fenyves BCI-algebras, then X is associative.

1. F_1-algebra 2. F_2-algebra 3. F_4-algebra 4. F_6-algebra 5. F_7-algebra 6. F_9-algebra 7. F_{10}-algebra 8. F_{11}-algebra 9. F_{12}-algebra 10. F_{13}-algebra 11. F_{14}-algebra 12. F_{15}-algebra 13. F_{16}-algebra 14. F_{17}-algebra 15. F_{18}-algebra 16. F_{20}-algebra 17. F_{22}-algebra 18. F_{23}-algebra 19. F_{24}-algebra 20. F_{25}-algebra 21. F_{26}-algebra 22. F_{27}-algebra 23. F_{28}-algebra 24. F_{30}-algebra 25. F_{31}-algebra 26. F_{32}-algebra 27. F_{33}-algebra 28. F_{34}-algebra 29. F_{35}-algebra 30. F_{36}-algebra 31. F_{37}-algebra 32. F_{38}-algebra 33. F_{40}-algebra 34. F_{41}-algebra 35. F_{43}-algebra 36. F_{44}-algebra 37. F_{45}-algebra 38. F_{47}-algebra 39. F_{48}-algebra 40. F_{49}-algebra 41. F_{50}-algebra 42. F_{51}-algebra 43. F_{53}-algebra 44. F_{57}-algebra 45. F_{58}-algebra 46. F_{60}-algebra.

Remark 4 *All other F_i's which are not mentioned in Theorem 12 were found to be non-associative. Every BCI-algebra is naturally an F_{54} BCI-algebra. A BCI-algebra that obeys any of the F_i's in Theorem 12 is a Boolean group by Theorem 11(3), hence isomorphic to its associated groupoid (the abelian group in Lemma 1).*

Zhang et al. [52] introduced quasi-neutrosophic triplet loops (QNTLs) which is made up of nine main types (cf. Definition 9 of Jaiyéolá et al. [36]). BCI-algebra belong to the class of three of these nine main types of QNTLs: (r-r)-QNT, (r-l)-QNTL and (r-lr)-QNTL. Therefore, any F_i BCI-algebra, $1 \leq i \leq 60$ belongs to at least one of the following varieties of Fenyves quasi neutrosophic triplet loops: (r-r)-FQNTL, (r-l)-FQNTL and (r-lr)-FQNTL. Any associative QNTL is called a quasi neutrosophic triplet group (QNTG).

The variety of quasi neutrosophic triplet loop is a generalization of neutrosophic triplet group (NTG) which was originally introduced by Smarandache and Ali [48]. New results and developments on neutrosophic triplet groups and neutrosophic triplet loop have been reported by Zhang et al. [52], [54], [55], [53], and Smarandache and Jaiyéolá [34], [35].

1.6. Motivation, Problem Statement, Aims and Objectives, Methodology

In this current paper, the isotopy of BCI-algebras is the main focus of this study (an extension of the work in Jaiyéolá et al. [36]). Necessary and sufficient conditions for a groupoid isotope of a BCI-algebra to be a BCI-algebra will be established. It will be shown that p-semisimplicity, quasi-associativity and BCK-algebra are invariant under isotopies which are determined by some regular permutation groups. Furthermore, the isotopy of both the 46 associative and 14 non-associative Fenyves BCI-algebras will also be studied. This is with the view of showing that there exist some other laws aside (i) to (vii) in subsection 1.3 which can be used to select some other varieties of BCI-algebra (e.g. F_i BCI-algebras, which are not necessarily quasigroups) from the quasivariety of all BCI-algebras. Furthermore, this will mean that such varieties of BCI-algebra (which are not necessarily quasigroups) can be determined by another structure under an isotopy which differs from (I,I,J). Consequently, the 14 non-associative Fenyves BCI-algebras do not necessarily have the Iseki's conditions (S) based on the results in Theorem 14 of Jaiyéolá et al. [36].

2. Main Results

2.1. Regular Bijections of BCI-Algebras

We need the following results on regular bijections of BCI-algebras.

Lemma 2 *Let $(G,\cdot,0)$ be a BCI-algebra with $\delta, U \in SYM(G)$. Then the following hold:*

1. δ is λ-regular $\Leftrightarrow \delta R_x = R_x\delta \Leftrightarrow L_{x\delta} = L_x\delta$ for all $x \in G$.
2. δ is ρ-regular $\Leftrightarrow \delta L_x = L_x\delta \Leftrightarrow R_{x\delta} = R_x\delta$ for all $x \in G$.
3. δ is μ-regular and self-adjoint $\Leftrightarrow \delta R_x = R_{x\delta} \Leftrightarrow L_{x\delta} = \delta L_x$ for all $x \in G$.
4. If U is λ-regular, then $L_{0U} = L_0U$, $xU \cdot x = 0U$ for all $x \in G$.
5. If U is ρ-regular, then $U = R_{0U}$, $0 \cdot 0U = 0U$, $UL_0 = L_0U$.
6. If U is μ-regular and self-adjoint, then $0U \cdot 0U^{-1} = 0$, $UR_{0U^{-1}} = I$, $L_{0U} = UL_0$.
7. If U is autotopic, then there exist $V, W \in SYM(G)$ such that $U^{-1}W = R_{0V}$, $VL_{0U} = L_0W$, $xU \cdot xV = 0W$ for all $x \in G$.

Proof.

1. δ is λ-regular $\Leftrightarrow (\delta, I, \delta) \in$ AUT $(G,\cdot) \Leftrightarrow y\delta \cdot xI = (y \cdot x)\delta \Leftrightarrow y\delta R_x = yR_x\delta \Leftrightarrow \delta R_x = R_x\delta \Leftrightarrow y\delta R_x = yR_x\delta \Leftrightarrow y\delta \cdot x = (y \cdot x)\delta \Leftrightarrow xL_{y\delta} = xL_y\delta \Leftrightarrow L_{y\delta} = L_y\delta$.
2. δ is ρ-regular $\Leftrightarrow (I, \delta, \delta) \in$ AUT $(G,\cdot) \Leftrightarrow xI \cdot y\delta = (x \cdot y)\delta \Leftrightarrow y\delta L_x = yL_x\delta \Leftrightarrow \delta L_x = L_x\delta \Leftrightarrow y\delta L_x = yL_x\delta \Leftrightarrow x \cdot y\delta = (x \cdot y)\delta \Leftrightarrow xR_{y\delta} = xR_y\delta \Leftrightarrow R_{y\delta} = R_y\delta$.
3. δ is μ-regular with adjoint $\delta' = \delta \Leftrightarrow (\delta, \delta'^{-1}, I) \in$ AUT $(G,\cdot) \Leftrightarrow x\delta \cdot y\delta'^{-1} = (x \cdot y)I \Leftrightarrow x\delta \cdot y\delta\delta^{-1} = x \cdot y\delta$ (by replacing y by $y\delta$) $\Leftrightarrow x\delta \cdot y = x \cdot y\delta \Leftrightarrow x\delta R_y = xR_{y\delta} \Leftrightarrow \delta R_y = R_{y\delta} \Leftrightarrow x\delta R_y = xR_{y\delta} \Leftrightarrow x\delta \cdot y = x \cdot y\delta \Leftrightarrow yL_{x\delta} = y\delta L_x \Leftrightarrow L_{x\delta} = \delta L_x$.
4. If U is λ-regular, then $xU \cdot y = (xy)U$. Put $x = 0$ in this, then you have $L_{0U} = L_0U$. Putting $y = x$, we have $xU \cdot x = 0U$.
5. If U is ρ-regular, then $x \cdot yU = (xy)U$. Put $y = 0$, then you get $U = R_{0U}$. Putting $x = y = 0$, we have $0 \cdot 0U = 0U$. Substituting $x = 0$, we get $UL_0 = L_0U$.
6. If U is μ-regular with adjoint $U' = U$, then $x \cdot yU^{-1} = x \cdot y$. Put $x = y = 0$ to get $0U \cdot 0U^{-1} = 0$. Put $y = 0$ to get $UR_{0U^{-1}} = I$. Put $x = 0$ to get $L_{0U} = UL_0$.
7. If U is autotopic, then there exist $V, W \in SYM(G)$ such that $xU \cdot yV = x \cdot y$. Putting $y = 0$, we get $U^{-1}W = R_{0V}$. Substituting $x = 0$, we have $VL_{0U} = L_0W$. Substituting $y = x$, we get $xU \cdot xV = 0W$.

2.2. Quasi Neutrosophic Triplet Loop Isotopes of BCI-Algebras

We now present results on isotopy of BCI-algebras.

Theorem 13 *Let $(G,\cdot,0) \xrightarrow{(\delta,\varepsilon,I)} (G,*)$ where $(G,\cdot,0)$ is a BCI-algebra and $(G,*)$ is a groupoid.*

1. Let $\varepsilon^{-1}\delta = \delta^{-1}\varepsilon$. Then, $(G,*,0)$ is a (r-r)-quasi NTL or (r-l)-quasi NTL or (r-rl)-quasi NTL if

and only if $\delta = \varepsilon$ and $\delta = R_{0\varepsilon^{-1}}$ (i.e. $\exists g \in G \ni \delta = R_g; g = 0\varepsilon^{-1}$).

2. $(G,*,0)$ is a BCI-algebra if and only if the following hold:
   a. $\delta = R_{0\varepsilon^{-1}}$ ($\exists g \in G \ni \delta = R_g; g = 0\varepsilon^{-1}$);
   b. $\delta = \varepsilon$;
   c. $[(x \cdot y) * (x \cdot z)] * (z \cdot y) = 0$.

Proof.

1. $(G,*,0)$ is a (r-r)-quasi NTL or (r-l)-quasi NTL or (r-rl)-quasi NTL if and only if $x * 0 = x$ and $x * x = 0$.
   a. $x * 0 = x \Leftrightarrow (x\delta^{-1} \cdot 0\varepsilon^{-1})I = x \Leftrightarrow x\delta^{-1}R_{0\varepsilon^{-1}} = x \Leftrightarrow \delta^{-1}R_{0\varepsilon^{-1}} = I \Leftrightarrow \delta = R_{0\varepsilon^{-1}}$.
   b. $x * x = 0 \Leftrightarrow x\delta^{-1} \cdot x\varepsilon^{-1} = 0 = x^2$. Replace x by $x\varepsilon^{-1}\delta$ to get $x * x = 0 \Leftrightarrow x\varepsilon^{-1}\delta\delta^{-1} \cdot x\varepsilon^{-1}\delta\varepsilon^{-1} = (x\varepsilon^{-1}\delta)^2 \Leftrightarrow x\varepsilon^{-1} \cdot x\varepsilon^{-1}\delta\varepsilon^{-1} = 0 \Leftrightarrow x\varepsilon^{-1} \cdot x\delta^{-1} = 0$. So, $x\delta^{-1} \cdot x\varepsilon^{-1} = 0$ and $x\varepsilon^{-1} \cdot x\delta^{-1} = 0$ implies that $x\delta^{-1} = x\varepsilon^{-1} \Leftrightarrow \delta = \varepsilon$.

2. For the forward, we shall assume that $(G,\cdot,0) \xrightarrow{(\delta,\varepsilon,I)} (G,*)$ and $(G,*,0)$ is a BCI-algebra.
   a. As above in 1, $x * 0 = x \Leftrightarrow \delta = R_{0\varepsilon^{-1}}$.
   b. Let $x * y = 0$ and $y * x = 0$, and so $x\delta^{-1} \cdot y\varepsilon^{-1} = 0$ and $y\delta^{-1} \cdot x\varepsilon^{-1} = 0$ respectively. The equation $y\delta^{-1} \cdot x\varepsilon^{-1} = 0$ can be re-written as $y\delta^{-1} \cdot x\varepsilon^{-1} = y^2$. Now, replacing y by $y\varepsilon^{-1}\delta$ to get $y\varepsilon^{-1}\delta\delta^{-1} \cdot x\varepsilon^{-1} = (y\varepsilon^{-1}\delta)^2 \Rightarrow y\varepsilon^{-1} \cdot x\varepsilon^{-1} = 0 \Rightarrow y\varepsilon^{-1} \cdot x\varepsilon^{-1} = x^2$. Furthermore, x by $x\delta^{-1}\varepsilon$ to get $y\varepsilon^{-1} \cdot x\delta^{-1}\varepsilon\varepsilon^{-1} = (x\delta^{-1}\varepsilon)^2 \Rightarrow y\varepsilon^{-1} \cdot x\delta^{-1} = 0$.

   Thus, we have shown that $x\delta^{-1} \cdot y\varepsilon^{-1} = 0$ and $y\varepsilon^{-1} \cdot x\delta^{-1} = 0$. Recall that $x \cdot y = 0$ and $y \cdot x = 0$ imply that $x = y$. So, $x\delta^{-1} = y\varepsilon^{-1} \Rightarrow \delta = \varepsilon$.
   c. $[(x * y) * (x * z)] * (z * y) = 0 \Leftrightarrow [(x\delta^{-1} \cdot y\varepsilon^{-1})\delta^{-1} \cdot (x\delta^{-1} \cdot z\varepsilon^{-1})\varepsilon^{-1}]\delta^{-1} \cdot [(z\delta^{-1} \cdot y\varepsilon^{-1})]\varepsilon^{-1} = 0$. Replace $x\delta^{-1}$ by x, $y\varepsilon^{-1}$ by y, and $z\varepsilon^{-1}$ by z to get $[(x \cdot y)\delta^{-1} \cdot (x \cdot z)\varepsilon^{-1}]\delta^{-1} \cdot [z\varepsilon\delta^{-1} \cdot y]\varepsilon^{-1} = 0 \Rightarrow [(x \cdot y) * (x \cdot z)]\delta^{-1} \cdot [z\varepsilon\delta^{-1} \cdot y]\varepsilon^{-1} = 0 \Rightarrow [(x \cdot y) * (x \cdot z)] * [z\varepsilon\delta^{-1} \cdot y] = 0 \Rightarrow [(x \cdot y) * (x \cdot z)] * [z \cdot y] = 0$.

   For the converse: we shall assume (a), (b) and (c). Following directly the reverse of 2(a), $x * 0 = x$. Since $\delta = \varepsilon$, then $x * y = 0 \Rightarrow x\delta^{-1} \cdot y\varepsilon^{-1} = 0$ and $y * x = 0 \Rightarrow y\delta^{-1} \cdot x\varepsilon^{-1} = 0$ which means that $x\delta^{-1} \cdot y\delta^{-1} = 0$ and $y\delta^{-1} \cdot x\delta^{-1} = 0$ imply $x = y$. Since $\delta = \varepsilon$, then (c) can be reversed to get $[(x * y) * (x * z)] * (z * y) = 0$. $\therefore$ $(G,*,0)$ is a BCI-algebra.

Corollary 1 *Let* $(G,\cdot,0) \xrightarrow{(R_g,R_g,I)} (G,*)$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(G,*)$ *is a groupoid.*

1. $(G,*,0)$ is a (r-r)-quasi NTL, (r-l)-quasi NTL and (r-rl)-quasi NTL.
2. $(G,*,0)$ is a BCI-algebra if and only if $[(x \cdot y) * (x \cdot z)] * (z \cdot y) = 0$ holds.

Proof. We shall use Theorem 13. 1 and 2 are true because $R_g = R_{0R_g^{-1}}$ since $g = 0R_g^{-1} \Leftrightarrow g^2 = 0$, which is true in the BCI-algebra $(G,\cdot,0)$.

Theorem 14 *Let* $(G,\cdot,0) \xrightarrow{(A,B,C)} (H,\diamond)$ *such that* $0C = 0'$, *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(H,\diamond)$ *is a groupoid.*

1. Let $A^{-1}B = B^{-1}A$, then $(H,\diamond,0')$ is a (r-r)-quasi NTL or (r-l)-quasi NTL or (r-rl)-quasi NTL if and only if $A = B$ and $A = R_{0'B^{-1}}C$ (i.e. $\exists g \in G \ni A = R_gC, g =$

$0'B^{-1}$).

2. $(H,\diamond,0')$ is a BCI-algebra if and only if the following hold:
   a. $A = R_{0'B^{-1}}C$ $(\exists g \in G \ni A = R_gC, g = 0'B^{-1})$;
   b. $A = B$;
   c. $[(x \diamond y) \diamond (x \diamond z)] \diamond (z \diamond y) = 0'$.

Proof. We make use of Theorem 13. Theorem 10 shall be applied in here as follows: $(G,*)$ is a principal isotope of $(G,\cdot)$ such that $(G,*) \overset{C}{\cong} (H,\diamond)$.

a. is true $\Leftrightarrow AC^{-1} = R_{0(BC^{-1})^{-1}} \Leftrightarrow AC^{-1} = R_{0CB^{-1}} \Leftrightarrow A = R_{0'B^{-1}}C$.

b. is true $\Leftrightarrow AC^{-1} = BC^{-1} \Leftrightarrow A = B$.

c. $[(x \cdot y) * (x \cdot z)] * (z \cdot y) = 0 \Leftrightarrow \{[(x \cdot y) * (x \cdot z)] * (z \cdot y)\}C = 0C \Leftrightarrow [(x \cdot y) * (x \cdot z)]C \diamond (z \cdot y)C = 0' \Leftrightarrow [(x \cdot y)C \diamond (x \cdot z)C] \diamond (z \cdot y)C = 0' \Leftrightarrow [(xA \diamond yB) \diamond (xA \diamond zB)] \diamond (zA \diamond yB) = 0'$.

Replace xA by x, yB by y, and zB by z to get $[(x \diamond y) \diamond (x \diamond z)] \diamond (zB^{-1}A \diamond y) = 0' \Leftrightarrow [(x \diamond y) \diamond (x \diamond z)] \diamond (z \diamond y) = 0'$.

Corollary 2 *Let* $(G,\cdot,0) \xrightarrow{(R_gC,R_gC,C)} (H,\diamond)$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(H,\diamond)$ *is a groupoid. Let* $0C = 0'$, *then*

1. $(H,\diamond,0')$ is a (r-r)-quasi NTL, (r-l)-quasi NTL and (r-rl)-quasi NTL.
2. $(H,\diamond,0')$ is a BCI-algebra if and only if $[(x \diamond y) \diamond (x \diamond z)] \diamond (z \diamond y) = 0'$ holds.

Proof. We shall use Theorem 14. 1 and 2 are true because $R_gC = R_{0'(R_gC)^{-1}}C$ since $g = 0'(R_gC)^{-1} \Leftrightarrow g = 0'C^{-1}R_g^{-1} \Leftrightarrow g = 0R_g^{-1} \Leftrightarrow g^2 = 0$, which is true in the BCI-algebra $(G,\cdot,0)$.

2.3. *Isotopy of [p-semisimple, quasi-associative] BCI-Algebras and BCK-Algebras*

Isotopy of p-semisimple, quasi-associative BCI-algebras and BCK-Algebras is presented.

Theorem 15 *Let* $(G,\cdot,0) \xrightarrow{(\delta,\varepsilon,I)} (G,*,0)$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(G,*,0)$ *is a BCI-algebra. Under any of the following conditions:*

1. $0\delta = 0, \delta \in \mathcal{P}(G,*)$ and $|\delta| = 2$ (i.e. $\delta^2 = I$);
2. $\delta \in \Phi(G,*)$ with $\delta' = \delta \in \Psi(G,*)$ and $|\delta| = 2$;

$(G,\cdot,0)$ is p-semisimple if and only if $(G,*,0)$ is p-semisimple.

Proof. By Theorem 13, $\delta = \varepsilon$.

1. $(G,\cdot,0)$ is p-semisimple if and only if $0 \cdot (0 \cdot x) = x \Leftrightarrow L_0^2 = I$. $(G,\cdot,0)$ is p-semisimple if and only if $0\delta * (0\delta * x\delta)\delta = x \Leftrightarrow 0 * (0 * x\delta)\delta = x \Leftrightarrow 0 * (0 * x)\delta = x\delta \Leftrightarrow \mathbb{L}_0\delta\mathbb{L}_0 = \delta$. Following 2. of Lemma 2, $(G,\cdot,0)$ is p-semisimple if and only if $\mathbb{L}_0^2 = I \Leftrightarrow (G,*,0)$ is p-semisimple.
2. $(G,\cdot,0)$ is p-semisimple if and only if $(x \cdot y) \cdot (x \cdot z) = z \cdot y \Leftrightarrow L_xL_{x\cdot y} = R_y$. $(G,\cdot,0)$ is p-semisimple if and only if $(x\delta * y\varepsilon)\delta * (x\delta * z\varepsilon)\varepsilon = z\delta * y\varepsilon \Leftrightarrow (x * y)\delta * (x * z)\delta = z * y \Leftrightarrow \mathbb{L}_x\delta\mathbb{L}_{(x*y)\delta} = \mathbb{R}_y$.

Following 3. of Lemma 2, $(G,\cdot,0)$ is p-semisimple if and only if $\mathbb{L}_x\delta^2\mathbb{L}_{(x*y)}=\mathbb{R}_y \Leftrightarrow \mathbb{L}_x\mathbb{L}_{(x*y)}=$
$\mathbb{R}_y \Leftrightarrow (G,*,0)$ is p-semisimple.

Corollary 3 *Let* $(G,\cdot,0)\xrightarrow{(A,B,C)}(H,\diamond,0')$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(H,\diamond,0')$ *is a BCI-algebra, and* $(G,*)$ *is a principal isotope of* $(G,\cdot)$. *Under any of the following conditions:*

1. $0C=0A, AC^{-1}\in\mathcal{P}(G,*)$ and $CA^{-1}C=A$;
2. $AC^{-1}\in\Phi(G,*)$ with $(AC^{-1})'=AC^{-1}\in\Psi(G,*)$ and $CA^{-1}C=A$;

$(G,\cdot,0)$ is p-semisimple if and only if $(H,\diamond,0')$ is p-semisimple.
Proof. Use the Theorem 15.

Theorem 16 *Let* $(G,\cdot,0)\xrightarrow{(\delta,\varepsilon,I)}(G,*,0)$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(G,*,0)$ *is a BCI-algebra such that* $0\delta=0$. $(G,\cdot,0)$ *is a BCK-algebra if and only if* $(G,*,0)$ *is a BCK-algebra.*

Proof. $(G,\cdot,0)$ is a BCK-algebra if and only if $0\cdot x=0\Leftrightarrow 0\delta*x\varepsilon=0\Leftrightarrow 0*x\delta=0\Leftrightarrow 0*x=0$ if and only if $(G,*,0)$ is a BCK-algebra.

Corollary 4 *Let* $(G,\cdot,0)\xrightarrow{(A,B,C)}(H,\diamond,0')$ *where* $(G,\cdot,0)$ *is a zero-cancellative BCI-algebra and* $(H,\diamond,0')$ *is a BCI-algebra such that* $0C=0A=0'$. $(G,\cdot,0)$ *is a BCK-algebra if and only if* $(H,\diamond,0')$ *is a BCK-algebra.*

Proof. Use the Theorem 16.

Theorem 17 *Let* $(G,\cdot,0,\leq)\xrightarrow{(\delta,\varepsilon,I)}(G,*,0,\lessapprox)$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(G,*,0)$ *is a BCI-algebra. Under any of the following conditions:*

1. $\delta\in\mathcal{P}(G,*)\cap\Lambda(G,*)$;
2. $\delta\in\mathcal{P}(G,*)\cap\Phi(G,*)$ with $\delta'=\delta\in\Psi(G,*)$;
3. $\delta\in\Lambda(G,*)\cap\Phi(G,*)$ with $\delta'=\delta\in\Psi(G,*)$;

$(G,\cdot,0)$ is quasi-associative if and only if $(G,*,0)$ is quasi-associative.

Proof. In the light of Theorem 2, we shall adopt the following representation for any two self maps A and B on G: $A\leq B\Leftrightarrow xA\leq xB$ and $A\lessapprox B\Leftrightarrow xA\lessapprox xB$ for all $x\in G$. Recall that by Theorem 2, $x\cdot y=0\Leftrightarrow x\leq y$ and $x*y=0\Leftrightarrow x\lessapprox y$. So, $x\leq y\Leftrightarrow x\cdot y=0\Leftrightarrow x\delta*y\varepsilon=0\Leftrightarrow x\delta\lessapprox y\varepsilon$. Hence, $x\leq y\Leftrightarrow x\delta\lessapprox y\varepsilon$. Note that by Theorem 13, $\delta=\varepsilon$.

1. $(G,\cdot,0)$ is quasi-associative if and only if $(x\cdot y)\cdot z\leq x\cdot(y\cdot z)\Leftrightarrow(x\delta*y\varepsilon)\delta*z\varepsilon\leq x\delta*(y\delta*z\varepsilon)\varepsilon\Leftrightarrow(x*y)\delta*z\leq x*(y*z)\varepsilon\Leftrightarrow\mathbb{R}_y\delta\mathbb{R}_z\leq\mathbb{R}_{(y*z)\delta}$.

Following 1. and 2. of Lemma 2, $(G,\cdot,0)$ is quasi-associative if and only if $\delta\mathbb{R}_y\mathbb{R}_z\leq\delta\mathbb{R}_{y*z}\Leftrightarrow(x\delta*y)*z\leq x\delta*(y*z)\Leftrightarrow(x*y)*z\leq x*(y*z)\Leftrightarrow[(x*y)*z]\cdot[x*(y*z)]=0\Leftrightarrow[(x*y)*z]\delta*[x*(y*z)]\varepsilon=0\Leftrightarrow[(x*y)*z\delta]*[x*(y*z\varepsilon)]=0\Leftrightarrow[(x*y)*z]*[x*(y*z)]=0\Leftrightarrow[(x*y)*z]\lessapprox[x*(y*z)]$ if and only if $(G,*,0)$ is quasi-associative.

2. By Lemma 2, $\delta\in\mathcal{P}(G,*)\cap\Lambda(G,*)\Leftrightarrow\delta\in\mathcal{P}(G,*)\cap\Phi(G,*)$ with $\delta'=\delta\in\Psi(G,*)$. Hence, the

conclusion follows by 1.

3. By Lemma 2, $\delta \in \mathcal{P}(G,*) \cap \Lambda(G,*) \Leftrightarrow \delta \in \Lambda(G,*) \cap \Phi(G,*)$ with $\delta' = \delta \in \Psi(G,*)$. Hence, the conclusion follows by 1.

Corollary 5 *Let* $(G,\cdot,0) \xrightarrow{(A,B,C)} (H,\diamond,0')$ *where* $(G,\cdot,0)$ *is a BCI-algebra,* $(H,\diamond,0')$ *is a BCI-algebra and* $(G,*)$ *is a principal isotope of* $(G,\cdot)$ *with* $0C = 0'$. *Under any of the following conditions:*

1. $AC^{-1} \in \mathcal{P}(G,*) \cap \Lambda(G,*)$;
2. $AC^{-1} \in \mathcal{P}(G,*) \cap \Phi(G,*)$ with $(AC^{-1})' = AC^{-1} \in \Psi(G,*)$;
3. $AC^{-1} \in \Lambda(G,*) \cap \Phi(G,*)$ with $(AC^{-1})' = AC^{-1} \in \Psi(G,*)$;

$(G,\cdot,0)$ is quasi-associative if and only if $(H,\diamond,0')$ is quasi-associative.

Proof. Use the Theorem 5.

2.4. Isotopy of Associative Fenyves BCI-Algebras

Isotopy of associative Fenyves BCI-algebras is presented. The set $Centrum(G,\cdot)$ of a groupoid $(G,\cdot)$ is defined as $Centrum(G,\cdot) = \{x \in G: xy = yx \forall y \in G\}$.

Theorem 18 *Let* $(G,\cdot,0) \xrightarrow{(\alpha,\alpha,I)} (G,*,0)$ *where* $(G,\cdot,0)$ *and* $(G,*,0)$ *are BCI-algebras.* $(G,*,0)$ *is associative if and only if* $0\alpha^{-1} \in Centrum(G,\cdot)$.

Proof. $0 * x = x \Leftrightarrow 0\alpha^{-1} \cdot x\alpha^{-1} = x \Leftrightarrow \alpha = L_{0\alpha^{-1}} \Leftrightarrow R_{0\alpha^{-1}} = L_{0\alpha^{-1}} \Leftrightarrow 0\alpha^{-1} \in Centrum(G,\cdot)$.

Corollary 6 *Let* $(G,\cdot,0) \xrightarrow{(A,A,C)} (H,\diamond,0')$ *where* $(G,\cdot,0)$ *and* $(H,\diamond,0')$ *are BCI-algebras.* $(H,\diamond,0')$ *is associative if and only if* $0CA^{-1} \in Centrum(G,\cdot)$.

Proof. Use Theorem 18.

Corollary 7 *Let* $(G,\cdot,0) \xrightarrow{(\alpha,\alpha,I)} (G,*,0)$ *where* $(G,\cdot,0)$ *and* $(G,*,0)$ *are BCI-algebras.* $(G,*,0)$ *is an* F_i*-algebra if and only if* $0\alpha^{-1} \in Centrum(G,\cdot)$ *for* $i = 1,2,4,6,7,9,10,11,12,13,14,15,16,17,18,20,22, 23,24,25,26,27,28,30,31,32,33,34,35,36, 37,38,40,41,43,44,45,47,48,49,50,51,53,57,58,60$.

Proof. This follows by Theorem 18 and Theorem 12.

Corollary 8 *Let* $(G,\cdot,0) \xrightarrow{(A,A,C)} (H,\diamond,0')$ *where* $(G,\cdot,0)$ *and* $(H,\diamond,0')$ *are BCI-algebras.* $(H,\diamond,0')$ *is an* F_i*-algebra if and only if* $0CA^{-1} \in Centrum(G,\cdot)$ *for* $i = 1,2,4,6,7,9,10,11,12,13,14,15,16,17,18,20,22, 23,24,25,26,27,28,30,31,32,33,34,35,36, 37,38,40,41,43,44,45,47,48,49,50,51,53,57,58,60$.

Proof. This follows by Corollary 6 and Theorem 12.

Theorem 19 *Let* $(G,\cdot,0) \xrightarrow{(\delta,\varepsilon,I)} (G,*,0)$ *where* $(G,\cdot,0)$ *BCI-algebra and* $(G,*,0)$ *is a BCI-algebra. Then* $(G,\cdot,0)$ *is associative if and only if* $(G,*,0)$ *is associative.*

Proof. $(G,\cdot,0)$ is associative if and only if $x\cdot y = y\cdot x \Leftrightarrow x\delta * y\varepsilon = y\delta * x\varepsilon \Leftrightarrow x*y = y*x \Leftrightarrow (G,*,0)$ is associative.

Corollary 9 *Let* $(G,\cdot,0) \xrightarrow{(A,B,C)} (H,\diamond,0')$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(H,\diamond,0')$ *is a BCI-algebra. Then* $(G,\cdot,0)$ *is associative if and only if* $(H,\diamond,0')$ *is associative.*

Proof. This follows from Theorem 19.

Corollary 10 *Let* $(G,\cdot,0) \xrightarrow{(\delta,\varepsilon,I)} (G,*,0)$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(G,*,0)$ *is a BCI-algebra. Then* $(G,\cdot,0)$ *is an* F_i *-algebra if and only if* $(G,*,0)$ *is an* F_i *-algebra,* $i =$ 1,2,4,6,7,9,10,11,12,13,14,15,16,17,18,20,22,23,24,25,26,27,28,30,31,32,33,34,35,36,37,38,40,41, 43,44,45,47,48,49,50,51,53,57,58,60.

Proof. This follows from Theorem 12 and Theorem 19.

Corollary 11 *Let* $(G,\cdot,0) \xrightarrow{(A,B,C)} (H,\diamond,0')$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(H,\diamond,0')$ *is a BCI-algebra. Then* $(G,\cdot,0)$ *is an* F_i *-algebra if and only if* $(H,\diamond,0')$ *is an* F_i *-algebra,* $i =$ 1,2,4,6,7,9,10,11,12,13,14,15,16,17,18,20,22,23,24,25,26,27,28,30,31,32,33,34,35,36,37,38,40,41, 43,44,45,47,48,49,50,51,53,57,58,60.

Proof. This follows from Theorem 12 and Corollary 9.

Remark 5 *Note that those* F_i *identities which are not in Corollary 11, do not necessarily imply associativity in BCI-algebra, hence, they need some isotopic conditions for isotopic invariance. The next subsection addresses this.*

2.5. Isotopy of Non-Associative Fenyves BCI-Algebras

Isotopy of non-associative Fenyves BCI-algebras is presented.

Theorem 20 *Let* $(G,\cdot,0) \xrightarrow{(\delta,\varepsilon,I)} (G,*,0)$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(G,*,0)$ *is a BCI-algebra such that any of the following is true:*

1. .$\delta \in \mathcal{P}(G,*) \cap \Lambda(G,*)$;
2. $\delta \in \mathcal{P}(G,*) \cap \Phi(G,*)$ with $\delta' = \delta \in \Psi(G,*)$;
3. $\delta \in \Lambda(G,*) \cap \Phi(G,*)$ with $\delta' = \delta \in \Psi(G,*)$.

Then, $(G,\cdot,0)$ is an F_i -algebra if and only if $(G,*,0)$ is an F_i -algebra; where $i=3,5,8,19,21,29,39,42,46,52,55,56,59$.

Proof. By Lemma 2, $\delta\in\mathcal{P}(G,*)\cap\Lambda(G,*)\Leftrightarrow\delta\in\mathcal{P}(G,*)\cap\Phi(G,*)$ with $\delta'=\delta\in\Psi(G,*)\Leftrightarrow\delta\in\Lambda(G,*)\cap\Phi(G,*)$ with $\delta'=\delta\in\Psi(G,*)$. By Theorem 13, $\delta=\varepsilon$. The arguments of the proof is based on condition 1.

$(G,\cdot,0)$ is an F_3-algebra if and only if $(x\cdot y)\cdot(z\cdot x)=x\cdot[y\cdot(z\cdot x)]\Leftrightarrow(x\delta*y\varepsilon)\delta*(z\delta*x\varepsilon)\varepsilon=x\delta*[y\delta*(z\delta*x\varepsilon)\varepsilon]\varepsilon\Leftrightarrow(x*y)\delta*(z*x)\varepsilon=x*[y*(z*x)\varepsilon]\varepsilon\Leftrightarrow y\mathbb{L}_x\delta\mathbb{R}_{(z*x)\varepsilon}=y\mathbb{R}_{(z*x)\varepsilon}\varepsilon\mathbb{L}_x\Leftrightarrow \mathbb{L}_x\delta\mathbb{R}_{(z*x)}\varepsilon=\mathbb{R}_{(z*x)}\varepsilon^2\mathbb{L}_x\Leftrightarrow y\mathbb{L}_x\mathbb{R}_{(z*x)}=y\mathbb{R}_{(z*x)}\mathbb{L}_x\Leftrightarrow[(x*y)*(z*x)=x*[y*(z*x)]\Leftrightarrow(G,*,0)$ is an F_3-algebra.

$(G,\cdot,0)$ is an F_5-algebra if and only if $[(x\cdot y)\cdot z)]x=[x\cdot(y\cdot z)]x\Leftrightarrow[(x*y)\delta*z]\delta*x=[x*(y*z)\varepsilon]\delta*x\Leftrightarrow y\mathbb{R}_z\varepsilon\mathbb{L}_x\delta\mathbb{R}_x=y\mathbb{L}_x\delta\mathbb{R}_z\delta\mathbb{R}_x\Leftrightarrow\mathbb{R}_z\varepsilon\mathbb{L}_x\delta\mathbb{R}_x=\mathbb{L}_x\delta\mathbb{R}_z\delta\mathbb{R}_x\Leftrightarrow\mathbb{R}_z\mathbb{L}_x\varepsilon\delta\mathbb{R}_x=\mathbb{L}_x\mathbb{R}_z\delta^2\mathbb{R}_x\Leftrightarrow \mathbb{R}_z\mathbb{L}_x\mathbb{R}_x=\mathbb{L}_x\mathbb{R}_z\mathbb{R}_x\Leftrightarrow y\mathbb{R}_z\mathbb{L}_x\mathbb{R}_x=y\mathbb{L}_x\mathbb{R}_z\mathbb{R}_x\Leftrightarrow[x*(y*z)]*x=[(x*y)*z]*x\Leftrightarrow(G,*,0)$ is an F_5-algebra.

$(G,\cdot,0)$ is an F_8-algebra if and only if $[x\cdot(y\cdot z)]\cdot x=x\cdot[y\cdot(z\cdot x)]\Leftrightarrow[x\delta*(y\delta*z\varepsilon)\varepsilon]\delta*x\varepsilon=x\delta*[y\delta*(z\delta*x\varepsilon)\varepsilon]\varepsilon\Leftrightarrow[x*(y*z)\varepsilon]\delta*x=x*[y*(z*x)\varepsilon]\varepsilon\Leftrightarrow y\mathbb{R}_z\varepsilon\mathbb{L}_x\delta\mathbb{R}_x=y\mathbb{R}_{(z*x)\varepsilon}\varepsilon\mathbb{L}_x\Leftrightarrow \mathbb{R}_z\mathbb{L}_x\varepsilon\delta\mathbb{R}_x=\mathbb{R}_{(z*x)}\varepsilon^2\mathbb{L}_x\Leftrightarrow\mathbb{R}_z\mathbb{L}_x\mathbb{R}_x=\mathbb{R}_{(z*x)}\mathbb{L}_x\Leftrightarrow[x*(y*z)]*x=x*[y*(z*x)]\Leftrightarrow(G,*,0)$ is an F_8-algebra

$(G,\cdot,0)$ is an F_{19}-algebra if and only if $[x\cdot(y\cdot x)]\cdot z=x\cdot[y\cdot(x\cdot z)]\Leftrightarrow[x\delta*(y\delta*x\varepsilon)\varepsilon]\delta*z\varepsilon=x\delta*[y\delta*(x\delta*z\varepsilon)\varepsilon]\varepsilon\Leftrightarrow[x*(y*x)\varepsilon]\delta*z\varepsilon=x*[y*(x*z)\varepsilon]\varepsilon\Leftrightarrow y\mathbb{R}_x\varepsilon\mathbb{L}_x\delta\mathbb{R}_z=y\mathbb{R}_{(x*z)\varepsilon}\varepsilon\mathbb{R}_x\Leftrightarrow \mathbb{R}_x\mathbb{L}_x\varepsilon\delta\mathbb{R}_z=\mathbb{R}_{(x*z)}\varepsilon^2\mathbb{R}_x\Leftrightarrow\mathbb{R}_x\mathbb{L}_x\mathbb{R}_z=\mathbb{R}_{(x*z)}\mathbb{R}_x\Leftrightarrow[x*(y*x)]*z=x*[y*(x*z)]\Leftrightarrow(G,*,0)$ is an F_{19}-algebra.

$(G,\cdot,0)$ is an F_{21}-algebra if and only if $[(y\cdot x)\cdot(z\cdot x)]=[(y\cdot x)\cdot z]\cdot x\Leftrightarrow(y\delta*x\varepsilon)\delta*(z\delta*x\varepsilon)\varepsilon=[(y\delta*x\varepsilon)\delta*z\varepsilon]\delta*x\varepsilon\Leftrightarrow(y*x)\delta*(z*x)\varepsilon=[(y*x)\delta*z]\delta*x\Leftrightarrow z\mathbb{L}_{y\mathbb{R}_x\delta}\delta\mathbb{R}_x=z\mathbb{R}_x\delta\mathbb{L}_{y\mathbb{R}_x\delta}\Leftrightarrow \mathbb{L}_{y\mathbb{R}_x\delta}\mathbb{R}_x=\mathbb{R}_x\mathbb{L}_{y\mathbb{R}_x\delta}\Leftrightarrow\mathbb{L}_{y\delta\mathbb{R}_x}\mathbb{R}_x=\mathbb{R}_x\mathbb{L}_{y\delta\mathbb{R}_x}\Leftrightarrow z\mathbb{L}_{y\mathbb{R}_x}\mathbb{R}_x=z\mathbb{R}_x\mathbb{L}_{y\mathbb{R}_x}\Leftrightarrow[(y*x)*(z*x)]=[(y*x)*z]*x\Leftrightarrow(G,*,0)$ is an F_{21}-algebra.

$(G,\cdot,0)$ is an F_{29}-algebra if and only if $[y\cdot(x\cdot z)]\cdot x=y\cdot[x\cdot(z\cdot x)]\Leftrightarrow[y\delta*(x\delta*z\varepsilon)\varepsilon]\delta*x\varepsilon=y\delta*[x\delta*(z\delta*x\varepsilon)\varepsilon]\varepsilon\Leftrightarrow[y*(x*z)\varepsilon]\delta*x=y*[x*(z*x)\varepsilon]\varepsilon\Leftrightarrow z\mathbb{L}_x\varepsilon\mathbb{L}_y\delta\mathbb{R}_x=z\mathbb{R}_x\varepsilon\mathbb{L}_x\varepsilon\mathbb{L}_y\Leftrightarrow \mathbb{L}_x\mathbb{L}_y\varepsilon\delta\mathbb{R}_x=z\mathbb{R}_x\mathbb{L}_x\varepsilon^2\mathbb{L}_y\Leftrightarrow\mathbb{L}_x\mathbb{L}_y\mathbb{R}_x=z\mathbb{R}_x\mathbb{L}_x\mathbb{L}_y\Leftrightarrow[y*(x*z)]*x=y*[x*(z*x)]\Leftrightarrow(G,*,0)$ is an F_{29}-algebra.

$(G,\cdot,0)$ is an F_{39}-algebra if and only if $[y\cdot(x\cdot x)]\cdot z=y\cdot[x\cdot(x\cdot z)]\Leftrightarrow[y\delta*(x\delta*x\varepsilon)\varepsilon]\delta*z\varepsilon=y\delta*[x\delta*(x\delta*z\varepsilon)\varepsilon]\varepsilon\Leftrightarrow[y*(x*x)\varepsilon]\delta*z=y*[x*(x*z)\varepsilon]\varepsilon\Leftrightarrow z\mathbb{L}_{[y*(x*x)\varepsilon]\delta}=z\mathbb{L}_x\varepsilon\mathbb{L}_x\varepsilon\mathbb{L}_y\Leftrightarrow$

$\mathbb{L}_{[y*(x*x)\varepsilon\delta]}=\mathbb{L}_x^2\varepsilon^2\mathbb{L}_y \Leftrightarrow \mathbb{L}_{[y*(x*x)]}=\mathbb{L}_x^2\mathbb{L}_y \Leftrightarrow [y*(x*x)]*z=y*[x*(x*z)] \Leftrightarrow (G,*,0)$ is an F_{39}-algebra.

$(G,\cdot,0)$ is an F_{42}-algebra if and only if $(x\cdot x)\cdot(y\cdot z)=[(x\cdot x)\cdot y]\cdot z \Leftrightarrow 0\delta*(y*z)\varepsilon=(0\delta*y)\delta*z \Leftrightarrow y\mathbb{R}_z\varepsilon\mathbb{L}_{0\delta}=y\mathbb{L}_{0\delta}\delta\mathbb{R}_z \Leftrightarrow y\mathbb{R}_z\varepsilon\mathbb{L}_0\delta=y\mathbb{L}_0\delta\delta\mathbb{R}_z \Leftrightarrow y\mathbb{R}_z\mathbb{L}_0\varepsilon\delta=y\mathbb{L}_0\mathbb{R}_z \Leftrightarrow y\mathbb{R}_z\mathbb{L}_0=y\mathbb{L}_0\mathbb{R}_z \Leftrightarrow 0*(y*z)=(0*y)*z \Leftrightarrow (G,*,0)$ is an F_{42}-algebra.

$(G,\cdot,0)$ is an F_{46}-algebra if and only if $[x\cdot(x\cdot y)]\cdot z=x\cdot[x\cdot(y\cdot z)] \Leftrightarrow [x\delta*(x\delta*y\varepsilon)\varepsilon]\delta*z\varepsilon=x\delta*[x\delta*(y\delta*z\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [x*(x*y)\varepsilon]\delta*z=x*[x*(y*z)\varepsilon]\varepsilon \Leftrightarrow y\mathbb{L}_x\varepsilon\mathbb{L}_x\delta\mathbb{R}_z=y\mathbb{R}_z\varepsilon\mathbb{L}_x\varepsilon\mathbb{L}_z \Leftrightarrow \mathbb{L}_x\mathbb{L}_x\varepsilon\delta\mathbb{R}_z=\mathbb{R}_z\mathbb{L}_x\varepsilon^2\mathbb{L}_z \Leftrightarrow [x*(x*y)]*z=x*[x*(y*z)] \Leftrightarrow (G,*,0)$ is an F_{46}-algebra.

$(G,\cdot,0)$ is an F_{52}-algebra if and only if $(y\cdot z)\cdot(x\cdot x)=[(y\cdot z)\cdot x]\cdot x \Leftrightarrow (y\delta*z\varepsilon)\delta*(x\delta*x\varepsilon)\varepsilon=[(y\delta*z\varepsilon)\delta*x\varepsilon]\delta*x\varepsilon \Leftrightarrow (y*z)\delta*(x*x)\varepsilon=[(y*z)\delta*x]\delta*x \Leftrightarrow y\mathbb{R}_z\delta\mathbb{R}_{(x*x)\varepsilon}=y\mathbb{R}_z\delta\mathbb{R}_x\delta\mathbb{R}_x \Leftrightarrow \mathbb{R}_z\mathbb{R}_{(x*x)}\varepsilon\delta=\mathbb{R}_z\mathbb{R}_x\delta^2\mathbb{R}_x \Leftrightarrow \mathbb{R}_z\mathbb{R}_{(x*x)}=\mathbb{R}_z\mathbb{R}_x^2 \Leftrightarrow (y*z)*(x*x)=[(y*z)*x]*x \Leftrightarrow (G,*,0)$ is an F_{52}-algebra.

$(G,\cdot,0)$ is an F_{55}-algebra if and only if $[(y\cdot z)\cdot x]x=[y\cdot(z\cdot x)]\cdot x \Leftrightarrow [(y*z)\delta*x]\delta*x=[y*(z*x)\varepsilon]\delta*x \Leftrightarrow z\mathbb{L}_y\delta\mathbb{R}_x\delta\mathbb{R}_x=z\mathbb{R}_x\varepsilon\mathbb{L}_y\delta\mathbb{R}_x=z\mathbb{R}_x\varepsilon\mathbb{L}_y\delta\mathbb{R}_x \Leftrightarrow z\mathbb{L}_y\mathbb{R}_x\delta\delta\mathbb{R}_x=z\mathbb{R}_x\mathbb{L}_y\varepsilon\delta\mathbb{R}_x=z\mathbb{R}_x\varepsilon\mathbb{L}_y\delta\mathbb{R}_x \Leftrightarrow z\mathbb{L}_y\mathbb{R}_x\mathbb{R}_x=z\mathbb{R}_x\mathbb{L}_y\mathbb{R}_x=z\mathbb{R}_x\varepsilon\mathbb{L}_y\delta\mathbb{R}_x \Leftrightarrow [(y*z)*x]*x=[y*(z*x)]*x \Leftrightarrow (G,*,0)$ is an F_{55}-algebra.

$(G,\cdot,0)$ is an F_{56}-algebra if and only if $[(y\cdot z)\cdot x]\cdot x=y\cdot[(z\cdot x)\cdot x] \Leftrightarrow [(y\delta*z\varepsilon)\delta*x\varepsilon]\delta*x\varepsilon=y\delta*[(z\delta*x\varepsilon)\delta*x\varepsilon]\varepsilon \Leftrightarrow [(y*z)\delta*x]\delta*x=y*[(z*x)\delta*x]\varepsilon \Leftrightarrow z\mathbb{L}_y\delta\mathbb{R}_x\delta\mathbb{R}_x=z\mathbb{R}_x\delta\mathbb{R}_x\varepsilon\mathbb{L}_y \Leftrightarrow \mathbb{L}_y\delta\mathbb{R}_x\delta\mathbb{R}_x=\mathbb{R}_x\delta\mathbb{R}_x\varepsilon\mathbb{L}_y \Leftrightarrow \mathbb{L}_y\mathbb{R}_x\delta^2\mathbb{R}_x=\mathbb{R}_x\mathbb{R}_x\delta\varepsilon\mathbb{L}_y \Leftrightarrow \mathbb{L}_y\mathbb{R}_x\mathbb{R}_x=\mathbb{R}_x\mathbb{R}_x\mathbb{L}_y \Leftrightarrow z\mathbb{L}_y\mathbb{R}_x\mathbb{R}_x=z\mathbb{R}_x\mathbb{R}_x\mathbb{L}_y \Leftrightarrow [(y*z)*x]*x=y*[(z*x)*x] \Leftrightarrow (G,*,0)$ is an F_{56}-algebra.

$(G,\cdot,0)$ is an F_{59}-algebra if and only if $[y\cdot(z\cdot x)]\cdot x=y\cdot[z\cdot(x\cdot x)] \Leftrightarrow [y\delta*(z\delta*x\varepsilon)\varepsilon]\delta*x\varepsilon=y\delta*[z\delta*(x\delta*x\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [y*(z*x)\varepsilon]\delta*x=y*[z*(x*x)\varepsilon]\varepsilon \Leftrightarrow y\mathbb{R}_{(z*x)\varepsilon}\delta\mathbb{R}_x=y\mathbb{R}_{[z*(x*x)\varepsilon]\varepsilon} \Leftrightarrow \mathbb{R}_{(z*x)}\varepsilon\delta\mathbb{R}_x=\mathbb{R}_{[z*(x*x)]}\varepsilon^2 \Leftrightarrow \mathbb{R}_{(z*x)}\mathbb{R}_x=\mathbb{R}_{[z*(x*x)]} \Leftrightarrow [y*(z*x)]*x=y*[z*(x*x)] \Leftrightarrow (G,*,0)$ is an F_{59}-algebra.

Corollary 12 *Let* $(G,\cdot,0)\xrightarrow{(A,B,C)}(H,\diamond,0')$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(H,\diamond,0')$ *is a BCI-algebra such that any of the following is true:*

1. $AC^{-1}\in\mathcal{P}(G,*)\cap\Lambda(G,*)$;
2. $AC^{-1}\in\mathcal{P}(G,*)\cap\Phi(G,*)$ with $\delta'=\delta\in\Psi(G,*)$;
3. $AC^{-1}\in\Lambda(G,*)\cap\Phi(G,*)$ with $(AC^{-1})'=AC^{-1}\in\Psi(G,*)$;

where $(G,*)$ is a principal isotope of $(G,\cdot)$ with $0C=0'$. Then $(G,\cdot,0)$ is an F_i-algebra if and only if $(H,\diamond,0')$ is an F_i-algebra; where $i=3,5,8,19,21,29,39,42,46,52,55,56,59$.

Proof. This follows from Theorem 20 and Theorem 14.

Theorem 21 *Let* $(G,\cdot,0) \xrightarrow{(\delta,\varepsilon,I)} (G,*,0)$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(G,*,0)$ *is a BCI-algebra such that* $\delta \in \Lambda(G,*)$ *and* $|\delta| = 2$. *Then* $(G,\cdot,0)$ *is an* F_{56}*-algebra if and only if* $(G,*,0)$ *is an* F_{56}*-algebra.*

Proof. By Theorem 13, $\delta = \varepsilon$.
$(G,\cdot,0)$ is an F_{56}-algebra if and only if $[(y\cdot z)\cdot x]\cdot x = y\cdot[(z\cdot x)\cdot x] \Leftrightarrow [(y*z)\delta*x]\delta*x = y*[(z*x)\delta*x]\varepsilon \Leftrightarrow z\mathbb{L}_y\delta\mathbb{R}_x\delta\mathbb{R}_x = z\mathbb{R}_x\delta\mathbb{R}_x\varepsilon\mathbb{L}_y \Leftrightarrow z\mathbb{L}_y\mathbb{R}_x\delta\delta\mathbb{R}_x = z\mathbb{R}_x\mathbb{R}_x\delta\varepsilon\mathbb{L}_y \Leftrightarrow z\mathbb{L}_y\mathbb{R}_x\mathbb{R}_x = z\mathbb{R}_x\mathbb{R}_x\mathbb{L}_y \Leftrightarrow [(y*z)*x]*x = y*[(z*x)*x] \Leftrightarrow (G,*,0)$ is an F_{56}-algebra.

Corollary 13 *Let* $(G,\cdot,0) \xrightarrow{(A,B,C)} (H,\diamond,0')$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(H,\diamond,0')$ *is a BCI-algebra such that* $AC^{-1} \in \Lambda(G,*)$ *and* $|AC^{-1}| = 2$. *Then* $(G,\cdot,0)$ *is an* F_{56}*-algebra if and only if* $(H,\diamond,0')$ *is an* F_{56}*-algebra.*

Proof. This follows from Theorem 21 and Theorem 14.

Theorem 22 *Let* $(G,\cdot,0) \xrightarrow{(\delta,\varepsilon,I)} (G,*,0)$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(G,*,0)$ *is a BCI-algebra such that* $\delta \in \mathcal{P}(G,*)$ *and* $|\delta| = 2$. *Then* $(G,\cdot,0)$ *is an* F_i*-algebra if and only if* $(G,*,0)$ *is an* F_i*-algebra; where* $i = 8,19,29,39,46,59$.

Proof. By Theorem 13, $\delta = \varepsilon$.
$(G,\cdot,0)$ is an F_8-algebra if and only if $[x\cdot(y\cdot z)]\cdot x = x\cdot[y\cdot(z\cdot x)] \Leftrightarrow [x\delta*(y\delta*z\varepsilon)\varepsilon]\delta*x\varepsilon = x\delta*[y\delta*(z\delta*x\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [x*(y*z)\varepsilon]\delta*x = x*[y*(z*x)\varepsilon]\varepsilon \Leftrightarrow y\mathbb{R}_z\varepsilon\mathbb{L}_x\delta\mathbb{R}_x = y\mathbb{R}_{(z*x)\varepsilon}\varepsilon\mathbb{L}_x \Leftrightarrow \mathbb{R}_z\mathbb{L}_x\varepsilon\delta\mathbb{R}_x = \mathbb{R}_{(z*x)}\varepsilon^2\mathbb{L}_x \Leftrightarrow \mathbb{R}_z\mathbb{L}_x\mathbb{R}_x = \mathbb{R}_{(z*x)}\mathbb{L}_x \Leftrightarrow [x*(y*z)]*x = x*[y*(z*x)] \Leftrightarrow (G,*,0)$ is an F_8-algebra.

$(G,\cdot,0)$ is an F_{19}-algebra if and only if $[x\cdot(y\cdot x)]\cdot z = x\cdot[y\cdot(x\cdot z)] \Leftrightarrow [x\delta*(y\delta*x\varepsilon)\varepsilon]\delta*z\varepsilon = x\delta*[y\delta*(x\delta*z\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [x*(y*x)\varepsilon]\delta*z\varepsilon = x*[y*(x*z)\varepsilon]\varepsilon \Leftrightarrow y\mathbb{R}_x\varepsilon\mathbb{L}_x\delta\mathbb{R}_z = y\mathbb{R}_{(x*z)\varepsilon}\varepsilon\mathbb{R}_x \Leftrightarrow \mathbb{R}_x\mathbb{L}_x\varepsilon\delta\mathbb{R}_z = \mathbb{R}_{(x*z)}\varepsilon^2\mathbb{R}_x \Leftrightarrow \mathbb{R}_x\mathbb{L}_x\mathbb{R}_z = \mathbb{R}_{x*z)}\mathbb{R}_x \Leftrightarrow [x*(y*x)]*z = x*[y*(x*z)] \Leftrightarrow (G,*,0)$ is an F_{19}-algebra.

$(G,\cdot,0)$ is an F_{29}-algebra if and only if $[y\cdot(x\cdot z)]\cdot x = y\cdot[x\cdot(z\cdot x)] \Leftrightarrow [y\delta*(x\delta*z\varepsilon)\varepsilon]\delta*x\varepsilon = y\delta*[x\delta*(z\delta*x\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [y*(x*z)\varepsilon]\delta*x = y*[x*(z*x)\varepsilon]\varepsilon \Leftrightarrow z\mathbb{L}_x\varepsilon\mathbb{L}_y\delta\mathbb{R}_x = z\mathbb{R}_x\varepsilon\mathbb{L}_x\varepsilon\mathbb{L}_y \Leftrightarrow \mathbb{L}_x\mathbb{L}_y\varepsilon\delta\mathbb{R}_x = z\mathbb{R}_x\mathbb{L}_x\varepsilon^2\mathbb{L}_y \Leftrightarrow \mathbb{L}_x\mathbb{L}_y\mathbb{R}_x = z\mathbb{R}_x\mathbb{L}_x\mathbb{L}_y \Leftrightarrow [y*(x*z)]*x = y*[x*(z*x)] \Leftrightarrow (G,*,0)$ is an F_{29}-algebra.

$(G,\cdot,0)$ is an F_{39}-algebra if and only if $[y\cdot(x\cdot x)]\cdot z = y\cdot[x\cdot(x\cdot z)] \Leftrightarrow [y\delta*(x\delta*x\varepsilon)\varepsilon]\delta*z\varepsilon = y\delta*[x\delta*(x\delta*z\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [y*(x*x)\varepsilon]\delta*z = y*[x*(x*z)\varepsilon]\varepsilon \Leftrightarrow z\mathbb{L}_{[y*(x*x)\varepsilon]\delta} = z\mathbb{L}_x\varepsilon\mathbb{L}_x\varepsilon\mathbb{L}_y \Leftrightarrow$

$\mathbb{L}_{[y*(x*x)\varepsilon\delta]} = \mathbb{L}_x^2\varepsilon^2\mathbb{L}_y \Leftrightarrow \mathbb{L}_{[y*(x*x)]} = \mathbb{L}_x^2\mathbb{L}_y \Leftrightarrow [y*(x*x)]*z = y*[x*(x*z)] \Leftrightarrow (G,*,0)$ is an F_{39}-algebra.

$(G,\cdot,0)$ is an F_{46}-algebra if and only if $[x\cdot(x\cdot y)]\cdot z = x\cdot[x\cdot(y\cdot z)] \Leftrightarrow [x\delta*(x\delta*y\varepsilon)\varepsilon]\delta*z\varepsilon = x\delta*[x\delta*(y\delta*z\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [x*(x*y)\varepsilon]\delta*z = x*[x*(y*z)\varepsilon]\varepsilon \Leftrightarrow y\mathbb{L}_x\varepsilon\mathbb{L}_x\delta\mathbb{R}_z = y\mathbb{R}_z\varepsilon\mathbb{L}_x\varepsilon\mathbb{L}_z \Leftrightarrow \mathbb{L}_x\mathbb{L}_x\varepsilon\delta\mathbb{R}_z = \mathbb{R}_z\mathbb{L}_x\varepsilon^2\mathbb{L}_z \Leftrightarrow [x*(x*y)]*z = x*[x*(y*z)] \Leftrightarrow (G,\cdot,0)$ is an F_{46}-algebra.

$(G,\cdot,0)$ is an F_{59}-algebra if and only if $[y\cdot(z\cdot x)]\cdot x = y\cdot[z\cdot(x\cdot x)] \Leftrightarrow [y\delta*(z\delta*x\varepsilon)\varepsilon]\delta*x\varepsilon = y\delta*[z\delta*(x\delta*x\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [y*(z*x)\varepsilon]\delta*x = y*[z*(x*x)\varepsilon]\varepsilon \Leftrightarrow y\mathbb{R}_{(z*x)\varepsilon}\delta\mathbb{R}_x = y\mathbb{R}_{[z*(x*x)\varepsilon]\varepsilon} \Leftrightarrow \mathbb{R}_{(z*x)}\varepsilon\delta\mathbb{R}_x = \mathbb{R}_{[z*(x*x)]}\varepsilon^2 \Leftrightarrow \mathbb{R}_{(z*x)}\mathbb{R}_x = \mathbb{R}_{[z*(x*x)]} \Leftrightarrow [y*(z*x)]*x = y*[z*(x*x)] \Leftrightarrow (G,*,0)$ is an F_{59}-algebra.

Corollary 14 *Let* $(G,\cdot,0) \xrightarrow{(A,B,C)} (H,\diamond,0')$ *be an isotopism; where* $(G,\cdot,0)$ *is a BCI-algebra and* $(H,\diamond,0')$ *is a BCI-algebra such that* $AC^{-1} \in \mathcal{P}(G,*)$ *and* $|AC^{-1}| = 2$, *where* $(G,*)$ *is a principal isotope of* $(G,\cdot)$ *with* $0C = 0'$. *Then,* $(G,\cdot,0)$ *is an* F_i*-algebra if and only if* $(H,\diamond,0')$ *is an* F_i*-algebra; where* $i = 8,19,29,39,46,59$.
Proof. This follows from Theorem 22 and Theorem 14.

Remark 6 *Note that those* F_i *identities which do not appear in Corollaries 12,13,14 will trivially obey these corollaries because they imply associativity in BCI-algebra with no condition(s) placed on the isotopy.*

3. Summary, Conclusion and Future Studies

We shall now highlight the theoretical and practical implications of this research, discuss our research findings, highlight practical advantages and research limitations, and then suggest some future studies.

Comparing the characterization of the permutation in the isotopy for the isotopic invariance of quasi-associativity (a measure of weak associativity) in Theorem 17 and the characterization of the permutation in the isotopy for the isotopic invariance of the 13 non-associative F_i algebras in Theorem 20, the three are the same. This is a new contribution to the fact that isotopy in BCI-algebras and quasi-associativity can be measured with 14 non-associative F_i identities.

In loop theory, all the 30 Fenyves identities that are equivalent to associativity are isotopic invariant for any isotopy and some of the other 30 Fenyves identities that are non-associative (e.g. Moufang, Bol, Extra) are also isotopic invariant for any isotopy, while the others (e.g. LC, RC, C) are not. From our results in this work, all the 46 F_i identities that are equivalent to associativity in BCI-algebras are isotopic invariant for any isotopy, while for the 14 Fenyves identities that are non-associative in BCI-algebras; they are isotopic invariant for special isotopies including some well known identities (e.g. left Bol, LC and RC). Thus, it can be concluded that the isotopy of Fenyves identities that are non-associative in BCI-algebras is of better advantage over Fenyves identities that are non-associative in loops. But, there is limitation on the isotopy of all the 46 F_i identities that are equivalent to associativity in BCI-algebras.

Those 46 Fenyves identities that are equivalent to associativity in BCI-algebras as well as F_{54} which of course are isotopic invariant under any isotopy are denoted by √ in the fourth and fifth columns of Table 1 and Table 2. While the 13 Fenyves identities that are equivalent to associativity in BCI-algebras excluding F_{54} which are isotopic invariant under special isotopies are identified by the symbol '‡' in the fourth and fifth columns of Table 1 and Table 2. Theoretically and practically, this research implies the isotopic study of 120 particular types of the 540 varieties of Fenyves quasi neutrosophic triplet loops (FQNTLs) discovered in Jaiyéolá et al. [36] (cf. Figure 1).

For future studies, based on the philosophy of representing disease-victim(s) by neutrosophic algebraic structures, some of the 14 Fenyves identities that are non-associative in BCI-algebras (quasi neutrosophic loops) can be judiciously selected with good and appropriate choice of special isotopies for which such are isotopic invariant in order to study and understand the effects of diseases and possible treatment of a patient.

FENYVES IDENTITY	$F_i \equiv ASS$ IN A LOOP	F_i ISO INVAR IN A LOOP	F_i ISO INVAR IN BCI ALG	$F_i + BCI \Rightarrow ASS$
F_1	√		√	√
F_2		√	√	√
F_3	√		‡	‡
F_4		√	√	√
F_5	√		‡	‡
F_6		√	√	√
F_7	√		√	√
F_8	√		‡	‡
F_9		√	√	√
F_{10}	√		√	√
F_{11}	√		√	√
F_{12}	√		√	√
F_{13}		√	√	√
F_{14}	√		√	√
F_{15}		√	√	√
F_{16}	√		√	√
F_{17}		√	√	√
F_{18}	√		√	√
F_{19}		√	‡	‡
F_{20}	√		√	√
F_{21}	√		‡	‡
F_{22}		√	√	√
F_{23}	√		√	√
F_{24}	√		√	√
F_{25}	√		√	√
F_{26}		√	√	√
F_{27}		√	√	√
F_{28}	√		√	√
F_{29}	√		‡	‡
F_{30}		√	√	√
F_{31}	√		√	√
F_{32}	√		√	√
F_{33}	√		√	√
F_{34}	√		√	√
F_{35}		√	√	√

Table 1: Characterization of the Isotopy of Fenyves Identities in Loops and BCI-Algebras

FENYVES IDENTITY	$F_i \equiv ASS$ IN A LOOP	F_i ISO INVAR IN A LOOP	F_i ISO INVAR IN BCI ALG	$F_i + BCI \Rightarrow ASS$
F_{36}		√	√	√
F_{37}		√	√	√
F_{38}		√	√	√
F_{39}		√	‡	‡
F_{40}		√	√	√
F_{41}		√	√	√
F_{42}		√	‡	‡
F_{43}		√	√	√
F_{44}	√		√	√
F_{45}		√	√	√
F_{46}		√	‡	‡
F_{47}	√		√	√
F_{48}		√	√	√
F_{49}	√		√	√
F_{50}	√		√	√
F_{51}		√	√	√
F_{52}	√		‡	‡
F_{53}		√	√	√
F_{54}		√	√	‡
F_{55}	√		‡	‡
F_{56}		√	‡	‡
F_{57}		√	√	√
F_{58}	√		√	√
F_{59}	√		‡	‡
F_{60}		√	√	√

Table 2: Characterization of the Isotopy of Fenyves Identities in Loops and BCI-Algebras

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