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Fuzzy Metric Spaces Of The Two-Fold Fuzzy Algebra

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Abstract:

This paper is dedicated to defining and studying for the first time the concept of fuzzy metric spaces based on two-fold fuzzy algebras, where the elementary properties of this new concept will be studied and presented by many theorems and related examples that explain the validity of our work. Also, many different types of open and closed balls will be discussed, as well as the relationships between these metric substructures.

Keywords: fuzzy metric space, two-fold algebra, open ball, closed ball, torus

Introduction and basic concepts

The applications of neutrosophic sets and fuzzy sets are very wide and open research areas. In the literature, we can find many neutrosophic and fuzzy algebraic structures with deep connection with applied mathematics and number theory [5-11]. The concept of two-fold algebra was presented by Smarandache in [4], where many suggestions for the algebraic structure related to this algebra were defined and presented. This new idea has been used in [1] to study the two-fold algebra based on the standard fuzzy number theoretical system [3].

In [2], Hatip et.al. proposed the two-fold vector space and two-fold algebraic module based on fuzzy mappings, where they have studied the elementary properties of these new generalizations with many interesting examples.

This work is motivated by the modern idea of two-fold algebra, and metric spaces, where we can combine those to different structures in one algebraic structure called two-fold fuzzy metric space. On the other hand, we concentrate on deriving the essential properties and substructures of this new concept.

First, we recall some basic definitions:

Main discussion

Definition:

Let $\mathbb{R}$ be the real field, we define the two-fold fuzzy real algebra as follows:

$\mathbb{R}_{[0,1]} = \{x_a; x \in \mathbb{R}. a \in [0.1]\}$ and $a = \mu(y)$; $y \in \mathbb{R}$ and μ: $\mathbb{R} \to [0,1]$

Definition:

We define the following operations on $\mathbb{R}_{[0.1]}$:

$*: \mathbb{R}_{[0,1]} \times \mathbb{R}_{[0,1]} \to \mathbb{R}_{[0,1]}$ such that:

$x_{\mu(y)} * z_{\mu(t)} = (x+z)_{\mu(ty)}$

$\circ: \mathbb{R}_{[0,1]} \times \mathbb{R}_{[[0,1]} \to \mathbb{R}_{[0,1]}$ such that:

$x_{\mu(y)} \circ z_{\mu(t)} = (x \cdot z)_{\mu(yt)}$

Theorem 1:

Let $\mathbb{R}_{[0.1]}$ be the two-fold fuzzy real algebra, then:

1] $(*,\circ)$ are commutative.

2] $(*,\circ)$ are associative.

3] $(*,\circ)$ have identities.

4] $(*,\circ)$ are anti- inverse in general.

Example:

Take μ: $\mathbb{R} \to [0,1]$; $\mu(x) = \begin{cases} |x| & ; \quad 0 < |x| < 1 \\ \frac{1}{|x|} & ; \quad |x| \geq 1 \\ 0 & ; \quad x = 0 \end{cases}$

Consider $x_{\mu(y)} = 5_{\mu(6)}. z_{\mu(t)} = \left(\frac{1}{2}\right)_{\mu(\frac{1}{3})} \in \mathbb{R}_{[0,1]}$, then:

$x_{\mu(y)} * z_{\mu(t)} = \left(5 + \frac{1}{2}\right)_{\mu\left(\frac{6}{3}\right)} = \left(\frac{11}{2}\right)_{\mu(2)} = \left(\frac{11}{2}\right)_{\frac{1}{2}}.$

$x_{\mu(y)} \circ z_{\mu(t)} = \left(5 \cdot \frac{1}{2}\right)_{\mu\left(\frac{6}{3}\right)} = \left(\frac{5}{2}\right)_{\frac{1}{2}}.$

Definition:

Let $\mathbb{R}_{[0.1]}$ be the two fold fuzzy real algebra, with: μ: $\mathbb{R} \to [0.1]$, then we say that $x_{\mu(y)} \geq z_{\mu(t)}$ if and only if: $\begin{cases} x \geq z \\ \mu(y) \geq \mu(t) \end{cases}$.

Also, $x_{\mu(y)} \geq 0$ if and only if: $\begin{cases} x \geq 0 \\ \mu(y) \geq 0 \end{cases}$

Example:

For μ: $\mathbb{R} \to [0.1]$; $\mu(x) = \begin{cases} x^2 & -1 \leq x \leq 1 \\ \frac{1}{|x|} & |x| > 1 \end{cases}$, and for:

$x_{\mu(y)} = 4_{\mu(\frac{1}{2})}$. $z_{\mu(t)} = (5)_{\mu(4)}$, we can see:

$\begin{cases} x = 4 & \leq z = 5 \\ \mu\left(\frac{1}{2}\right) = \frac{1}{4} & \leq \quad \mu(4) = \frac{1}{4} \end{cases}$. hence $x_{\mu(y)} \leq z_{\mu(t)}$.

Remark:

If $x_{\mu(y)} = z_{\mu(t)}$. then $\begin{cases} x = z \\ \mu(y) = \mu(t) \end{cases}$

Theorem2:

Consider the relation (≤) defined previonsly over $\mathbb{R}_{[0.1]}$, then:

1] $x_{\mu(y)} \leq x_{\mu(y)}$ $for\ \ all\ \ x.y$ $\in \mathbb{R}$.

2] If $x_{\mu(y)} \leq z_{\mu(t)}$ and $z_{\mu(t)} \leq x_{\mu(y)}$. then $x_{\mu(y)} = z_{\mu(t)}$ for all $x.y.z.t$ $\in \mathbb{R}$.

3] If $x_{\mu(y)} \leq z_{\mu(t)}$ and $z_{\mu(t)} \leq N_{\mu(s)}$. then $x_{\mu(y)} \leq N_{\mu(s)}$ for all $x.y.z.N.t.s$ $\in \mathbb{R}$

Remark:

Theorem (2) means that (≤) is a partial order relation on $\mathbb{R}_{[0.1]}$.

Definition:

Let U, V be two non- empty sets, with:

$d = \mathrm{U} \times \mathrm{U} \ \to \mathbb{R}^{+}$. $\mu: \mathrm{V} \times \mathrm{V} \to [0,1]$ such that:

(d) is a metric on U, (μ) is a fuzzy metric on V.

We define the corresponding twofold algebra fuzzy metric as:

$\Delta = \begin{cases} x \\ y \end{cases}$; $x \in U$. $y \in V$, with: d_{μ}: $\Delta \times \Delta \to \mathbb{R}^{+}_{[0.1]}$

Such that:

$d_{\mu}(x_y . z_t) = [d(x.z)]_{\mu(y.t)}$ the mapping (d_{μ}) is called the twofold algebra fuzzy metric.

Theorem 3:

Let $(\Delta . d_\mu)$ be two fold algebra fuzzy metric space defined above, then:

1] $d_\mu(x_a . x_a) = o_\circ$ and $d_\mu(x_a . y_b) \geq o_\circ$

2] $d_\mu(x_a . y_b) = d_\mu(y_b . x_a)$.

3] $d_\mu(x_a . z_c) \leq d_\mu(x_a . y_b) + d_\mu(y_b . z_c)$ for all $x . y . z \in U \; . a . b . c \in V$.

Example:

Take $U = \mathbb{R}$ with $d = \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+ \; ; d(x . y) = \{|x - y|$

And $V = \mathbb{R}$ with $\mu = \mathbb{R} \times \mathbb{R} \rightarrow [0.1] \; ; \mu(a . b) = \begin{cases} \frac{1}{2} & ; \; a \neq b \\ 0 & ; a = b \end{cases}$

We have: $\Delta = \{x_a \; ; x . a \in \mathbb{R}\}$. For example:

$x_a = 3_5 \; . y_b = 4_6$. $d_\mu(x_a . y_b) = (|3 - 4|)_{\mu(5.6)} = 1_{\frac{1}{2}}$.

Definition:

Let $B_d(x . r) = \{y \in U \; ; d(x . y) < r \}$ be an open ball in U, with $x \in U$ as a center and $r \in \mathbb{R}^+$ as a radius.

$\overline{B_d}(x . r) = \{y \in U \; ; d(x . y) \leq r \}$ be the corresponding closed ball, and $T_d(x . r) = \{y \in U \; ; d(x . y) = r \}$ be the corresponding torus.

Also, let $B_\mu(a . t) = \{b \in V \; ; \mu(a . b) < t \}$ be an open ball in V, with $a \in V$ as a center and $t \in [0.1]$ as a radius.

$\overline{B_\mu}(a . t) = \{b \in V \; ; \mu(a . b) \leq t \}$ be the corresponding closed ball, and $T_\mu(a . t) = \{b \in V \; ; \mu(a . b) = t \}$ be the corresponding torus.

We define the following different types of balls in the twofold algebra fuzzy metric spaces:

1] $\Delta_{B_\mu}^{B_d} = \{x_a \in \Delta \; ; x \in B_d \; . \; a \in B_\mu\}$.

2] $\Delta_{\overline{B_\mu}}^{B_d} = \{x_a \in \Delta \; ; x \in B_d \; . \; a \in \overline{B_\mu}\}$.

3] $\Delta_{T_\mu}^{B_d} = \{x_a \in \Delta \; ; x \in B_d . \; a \in T_\mu\}$.

4] $\Delta_{B_\mu}^{\overline{B_d}} = \{x_a \in \Delta \; ; x \in \overline{B_d} \; . \; a \in B_\mu\}$.

5] $\Delta_{\overline{B_\mu}}^{\overline{B_d}} = \{x_a \in \Delta \; ; x \in \overline{B_d} \; . \; a \in \overline{B_\mu}\}$.

6] $\Delta_{T_\mu}^{\overline{B_d}} = \{x_a \in \Delta \; ; x \in \overline{B_d} . \; a \in T_\mu\}$.

7] $\Delta_{B_\mu}^{T_d} = \{x_a \in \Delta \; ; x \in T_d . \; a \in B_\mu\}$.

8] $\Delta_{B_\mu}^{T_d} = \{x_a \in \Delta \ ; x \in T_d. \quad a \in \overline{B_\mu}\}$.

9] $\Delta_{T_\mu}^{T_d} = \{x_a \in \Delta \ ; x \in T_d. \quad a \in T_\mu\}$.

Remark:

$(\Delta. d_\mu)$ has 9 different types of balls.

Theorem4:

Consider $B_d(x.r) = \{y \in U \ ; d(x.y) < r \ \} \subset U. \overline{B_d}(x.r)$ and $T_d(x.r)$.

Consider $B_\mu(a.t) = \{b \in V \ ; \mu(a.b) < t \ \} \subset V. \overline{B_\mu}(a.t)$. and $T_\mu(a.t)$.

Then we have:

1] $\Delta_{B_\mu}^{B_d} \subseteq \Delta_{B_\mu}^{\overline{B_d}} \subseteq \Delta_{\overline{B_\mu}}^{\overline{B_d}}$

2] $\Delta_{B_\mu}^{B_d} \subseteq \Delta_{\overline{B_\mu}}^{B_d} \subseteq \Delta_{\overline{B_\mu}}^{\overline{B_d}}$

3] $\Delta_{T_\mu}^{T_d} \subseteq \Delta_{T_\mu}^{\overline{B_d}} \subseteq \Delta_{\overline{B_\mu}}^{\overline{B_d}}$

4] $\Delta_{T_\mu}^{T_d} \subseteq \Delta_{\overline{B_\mu}}^{T_d} \subseteq \Delta_{\overline{B_\mu}}^{\overline{B_d}}$

5] $\Delta_{B_\mu}^{T_d} \subseteq \Delta_{\overline{B_\mu}}^{T_d}$

6] $\Delta_{T_\mu}^{B_d} \subseteq \Delta_{T_\mu}^{\overline{B_d}}$

Example:

Consider $U = \mathbb{R} \ . V = \mathbb{R}$, with $d = \mathbb{R} \times \mathbb{R} \ \to \mathbb{R}^+ . \mu = \mathbb{R} \times \mathbb{R} \ \to [0.1]$

Such that: $d(x.y) = |x - y|$ and $\mu(a.b) =$

$$\begin{cases} 0 & ; \ a = b \\ \frac{1}{2} & ; \ |a|.|b| \geq 1 \\ \frac{1}{4} & ; \ |a| < 1 \ \ or \ \ |b| < 1 \end{cases}$$

We have for $= 3 \ . r = 1 \ . a = 2 \ . t = \frac{1}{3}$:

$B_d(x.r) = \{y \in \mathbb{R} \ ; |y - 3| < 1 \ \} = \{y \in \mathbb{R} \ ; 2 < y < 4\}$.

$\overline{B_d}(x.r) = \{y \in \mathbb{R} \ ; 2 \leq y \leq 4\}. T_d(x.r) = \{2.4\}$.

Also, $B_\mu(a.t) = \left\{b \in \mathbb{R} \ ; \quad \mu(a.b) < \frac{1}{3} \ \right\} = \{b \in \mathbb{R} \ ; |b| < 1 \ \} \cup \{2\}$.

$\overline{B_\mu}(a.t) = \{b \in \mathbb{R} \ ; \quad |b| < 1 \ \} \cup \{2\} = B_\mu(a.t)$.

$T_\mu(a.t) = \left\{ b \in \mathbb{R} \quad ; \quad \mu(a.b) = \frac{1}{3} \right\} = \emptyset.$

Proof of theorem (1):

1] $x_{\mu(y)} * z_{\mu(t)} = (x+z)_{\mu(yt)} = (z+x)_{\mu(ty)} = z_{\mu(t)} * x_{\mu(y)}.$

Also, $x_{\mu(y)} \circ z_{\mu(t)} = (x \cdot z)_{\mu(yt)} = (z \cdot x)_{\mu(ty)} = z_{\mu(t)} \circ x_{\mu(y)}.$

2] $x_{\mu(a)} * (y_{\mu(b)} * z_{\mu(c)}) = x_{\mu(a)} * (y+z)_{\mu(bc)} = (x+y+z)_{\mu(abc)} = (x+y)_{\mu(ab)} * z_{\mu(c)} = (x_{\mu(a)} * y_{\mu(b)}) * z_{\mu(c)}.$

Also, $x_{\mu(a)} \circ (y_{\mu(b)} \circ z_{\mu(c)}) = x_{\mu(a)} \circ (yz)_{\mu(bc)} = (xyz)_{\mu(abc)} = (xy)_{\mu(ab)} \circ z_{\mu(c)} = (x_{\mu(a)} \circ y_{\mu(b)}) \circ z_{\mu(c)}.$

3] if $x_{\mu(a)} \circ y_{\mu(b)} = x_{\mu(a)}$ .then $\begin{cases} xy = x \\ \mu(ab) = \mu(a) \end{cases}$ for all $x.a \in \mathbb{R}$.

So that: $y = 1. b = 1.$ and the identity of $(\circ)$ is $1_{\mu(1)}$.

If $x_{\mu(a)} * y_{\mu(b)} = x_{\mu(a)}$ .then $\begin{cases} x + y = x \\ \mu(ab) = \mu(a) \end{cases}$ for all $x.a \in \mathbb{R}$.

So that: $\begin{cases} y = 0 \\ b = 1 \end{cases}$. and the identity of $(*)$ is $\circ_1$.

4] if $x_{\mu(a)} * y_{\mu(b)} = \circ_1$ .then $\begin{cases} \mu(ab) = 1 \\ x + y = 0 \end{cases}$.

This implies that (*) is anti- inverse in general, that is because finding (b) for each (a) such that $\mu(ab) = 1$ is depended on μ.

For (○), it can be proved by the same.

Proof of theorem (2):

1] since $\begin{cases} x \le x \\ \mu(y) \le \mu(y) \end{cases}$ .then $x_{\mu(y)} \le x_{\mu(y)}$.

2] $x_{\mu(y)} \le z_{\mu(t)}$ implies that $\begin{cases} x \le z \\ \mu(y) \le \mu(t) \end{cases}$.

$z_{\mu(t)} \le x_{\mu(y)}$ implies that $\begin{cases} z \le x \\ \mu(t) \le \mu(y) \end{cases}$ .thus $\begin{cases} x = z \\ \mu(t) = \mu(y) \end{cases}$ .and $x_{\mu(y)} = z_{\mu(t)}$.

3] Assume that $x_{\mu(y)} \le z_{\mu(t)}$ *and* $z_{\mu(t)} \le N_{\mu(s)}$ .then $\begin{cases} x \le z \le N \\ \mu(y) \le \mu(t) \le \mu(s) \end{cases}$ .thus $x_{\mu(y)} \le N_{\mu(s)}$.

Proof of theorem (3):

1] $d_\mu(x_a . x_a) = [d(x.x)]_{\mu(a.a)} = o_\circ.$

$d_\mu(x_a . y_b) = [d(x.y)]_{\mu(a.b)}$ on the other hand, we have:

$\begin{cases} d(x.y) \geq 0 \\ \mu(a.b) \geq 0 \end{cases}$. hence $d_\mu(x_a.y_b) \geq o_\circ$.

2] $d_\mu(x_a.y_b) = [d(x.y)]_{\mu(a.b)} = [d(y.x)]_{\mu(b.a)} = d_\mu(y_b.x_a)$.

3] We have: $\begin{cases} d(x.z) \leq d(x.y) + d(y.z) \\ \mu(a.c) \leq \mu(a.b) + \mu(b.c) \end{cases}$

Thus: $[d(x.z)]_{\mu(a.c)} \leq [d(x.y)]_{\mu(a.b)} + [d(y.z)]_{\mu(b.c)}$, hence:

$d_\mu(x_a.z_c) \leq d_\mu(x_a.y_b) + d_\mu(y_b.z_c)$.

Proof of theorem (4):

1] Let $y_b \in \Delta_{B_\mu}^{B_d}$ $.then$: $\begin{cases} y \in B_d & \subseteq \overline{B_d} \\ b \in B_\mu & \subseteq \overline{B_\mu} \end{cases}$

Thus $\Delta_{B_\mu}^{B_d} \subseteq \Delta_{B_\mu}^{\overline{B_d}} \subseteq \Delta_{\overline{B_\mu}}^{\overline{B_d}}$.

2] It can be proved by a similar argument of $\boxed{1}$.

3] Let $y_b \in \Delta_{T_\mu}^{T_d}$ $.then$: $\begin{cases} y \in T_d & \subseteq \overline{B_d} \\ b \in T_\mu & \subseteq \overline{B_\mu} \end{cases}$

Thus: $\Delta_{T_\mu}^{T_d} \subseteq \Delta_{T_\mu}^{\overline{B_d}} \subseteq \Delta_{\overline{B_\mu}}^{\overline{B_d}}$.

4] It can be proved by a similar way.

5] Let $y_b \in \Delta_{B_\mu}^{T_d}$ $.then$: $\begin{cases} y \in T_d & \subseteq T_d \\ b \in B_\mu & \subseteq \overline{B_\mu} \end{cases}$

Thus $\Delta_{B_\mu}^{T_d} \subseteq \Delta_{\overline{B_\mu}}^{T_d}$.

6] It can be proved by a similar argument of $\boxed{5}$.

Definition:

Consider $B_d(x.r).\overline{B_d}(x.r).B_\mu(a.t).\overline{B_\mu}(a.t)$, and:

$\sim B_d(x.r) = \{y \in U \;;\quad d(x.y) \geq r\}$.

$\sim \overline{B_d}(x.r) = \{y \in U \;;\quad d(x.y) > r\}$.

$\sim B_\mu(a.t) = \{b \in V \;;\quad \mu(a.b) \geq t\}$.

$\sim \overline{B_\mu}(a.t) = \{b \in V \;;\quad \mu(a.b) > t\}$.

We define:

1] $\Delta_{B_\mu}^{\sim\ B_d} = \{x_a \in \Delta \;;\quad x \in \sim B_d \;.\, a \in B_\mu\}$.

2] $\Delta_{\sim\ B_\mu}^{\sim\ B_d} = \{x_a \in \Delta \;;\quad x \in \sim B_d \;.\, a \in \sim B_\mu\}$.

3] $\Delta^{\sim\,\overline{B_d}}_{B_\mu} = \{x_a \in \Delta \ ; \quad x \in \sim \overline{B_d} \ . a \in B_\mu\}$.

4] $\Delta^{\sim\,\overline{B_d}}_{\sim\,B_\mu} = \{x_a \in \Delta \ ; \quad x \in \sim \overline{B_d} \ . a \in \sim B_\mu\}$.

5] $\Delta^{\sim\,B_d}_{\overline{B_\mu}} = \{x_a \in \Delta \ ; \quad x \in \sim B_d \ . a \in \overline{B_\mu}\}$.

6] $\Delta^{\sim\,B_d}_{\sim\,\overline{B_\mu}} = \{x_a \in \Delta \ ; \quad x \in \sim B_d \ . a \in \sim \overline{B_\mu}\}$.

7] $\Delta^{\sim\,\overline{B_d}}_{B_\mu} = \{x_a \in \Delta \ ; \quad x \in \sim \overline{B_d} \ . a \in B_\mu\}$.

8] $\Delta^{\sim\,\overline{B_d}}_{\sim\,B_\mu} = \{x_a \in \Delta \ ; \quad x \in \sim \overline{B_d} \ . a \in \sim B_\mu\}$.

9] $\Delta^{\sim\,\overline{B_d}}_{\overline{B_\mu}} = \{x_a \in \Delta \ ; \quad x \in \sim \overline{B_d} \ . a \in \overline{B_\mu}\}$.

10] $\Delta^{\sim\,\overline{B_d}}_{\sim\,\overline{B_\mu}} = \{x_a \in \Delta \ ; \quad x \in \sim \overline{B_d} \ . a \in \sim \overline{B_\mu}\}$.

11] $\Delta^{B_d}_{\sim B_\mu} = \{x_a \in \Delta \ ; \quad x \in B_d \ . a \in \sim B_\mu\}$.

12] $\Delta^{B_d}_{\sim\overline{B_\mu}} = \{x_a \in \Delta \ ; \quad x \in B_d \ . a \in \sim \overline{B_\mu}\}$.

13] $\Delta^{\overline{B_d}}_{\sim B_\mu} = \{x_a \in \Delta \ ; \quad x \in \overline{B_d} \ . a \in \sim B_\mu\}$.

14] $\Delta^{\overline{B_d}}_{\sim\overline{B_\mu}} = \{x_a \in \Delta \ ; \quad x \in \overline{B_d} \ . a \in \sim \overline{B_\mu}\}$.

Theorem (5):

1] $\Delta^{\sim\,B_d}_{B_\mu} \cap \Delta^{\sim\,B_d}_{\sim\,B_\mu} = \emptyset$. $\Delta^{\sim\,B_d}_{\overline{B_\mu}} \cap \Delta^{\sim\,B_d}_{\sim\,\overline{B_\mu}} = \emptyset$.

2] $\Delta^{B_d}_{\sim B_\mu} \cap \Delta^{\sim\,B_d}_{\sim\,B_\mu} = \emptyset$. $\Delta^{B_d}_{\sim\overline{B_\mu}} \cap \Delta^{\sim\,B_d}_{\sim\,\overline{B_\mu}} = \emptyset$.

3] $\Delta^{\overline{B_d}}_{\sim B_\mu} \cap \Delta^{\sim\,\overline{B_d}}_{\sim\,B_\mu} = \emptyset$. $\Delta^{\overline{B_d}}_{\sim\overline{B_\mu}} \cap \Delta^{\sim\,\overline{B_d}}_{\sim\,\overline{B_\mu}} = \emptyset$.

4] $\Delta^{\overline{B_d}}_{B_\mu} \cap \Delta^{\overline{B_d}}_{\sim B_\mu} = \emptyset$. $\Delta^{\overline{B_d}}_{\overline{B_\mu}} \cap \Delta^{\overline{B_d}}_{\sim\overline{B_\mu}} = \emptyset$.

5] $\Delta^{\sim\,B_d}_{B_\mu} \subseteq \Delta^{\sim\,B_d}_{\overline{B_\mu}}$. $\Delta^{\sim\,\overline{B_d}}_{B_\mu} \subseteq \Delta^{\sim\,\overline{B_d}}_{\overline{B_\mu}}$.

6] $\Delta^{B_d}_{\sim B_\mu} \subseteq \Delta^{\overline{B_d}}_{\sim B_\mu}$. $\Delta^{B_d}_{\sim B_\mu} \subseteq \Delta^{\overline{B_d}}_{\sim\overline{B_\mu}}$.

7] $\Delta^{\overline{B_d}}_{\sim B_\mu} \cap \Delta^{\sim\,B_d}_{\sim B_\mu} = \Delta^{T_d}_{\sim B_\mu}$. $\Delta^{\overline{B_d}}_{\sim\,\overline{B_\mu}} \cap \Delta^{\sim\,B_d}_{\sim\,\overline{B_\mu}} = \Delta^{T_d}_{\sim\,\overline{B_\mu}}$.

8] $\Delta_{\overline{B_\mu}}^{B_d} \cap \Delta_{\sim B_\mu}^{B_d} = \Delta_{T_\mu}^{B_d}$. $\Delta_{\overline{B_\mu}}^{\overline{B_d}} \cap \Delta_{\sim B_\mu}^{\overline{B_d}} = \Delta_{T_\mu}^{\overline{B_d}}$.

9] $\Delta_{\sim B_\mu}^{\sim B_d} \cap \Delta_{\overline{B_\mu}}^{\overline{B_d}} = \Delta_{T_\mu}^{T_d}$

Proof:

For the proof, we must regard that: $\Delta_y^x \cap \Delta_B^A = \Delta_{y\cap B}^{x\cap A}$ for all $x.A \subseteq U$. $y.B \subseteq V$.

Also, $\Delta_y^x = \emptyset$ if and only if $x = \emptyset$ or $y = \emptyset$.

According to the definitions, we can write:

$$\begin{cases} B_\mu \cap \sim B_\mu = \overline{B_\mu} \cap \sim \overline{B_\mu} = \emptyset \\ B_d \cap \sim B_d = \overline{B_d} \cap \sim \overline{B_d} = \emptyset \end{cases}$$

Thus [1], [2], [3], [4] hold directly.

Also, $\begin{cases} \overline{B_d} \cap \sim B_d = T_d \\ \overline{B_\mu} \cap \sim B_\mu = T_\mu \end{cases}$. thus: [7], [8], [9] hold directly.

On the other hand, we have:

$\begin{cases} B_d \subseteq \overline{B_d} \\ B_\mu \subseteq \overline{B_\mu} \end{cases}$,thus [5], [6] hold directly.

Example:

For $U = V = \mathbb{R}$. $d = \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$. $\mu = \mathbb{R} \times \mathbb{R} \rightarrow [0.1]$. with $d(x.y) = |x - y|$. $\mu(a.b) =$

$$\begin{cases} 0 & ; \ a = b \\ \frac{1}{2} & ; \ |a| \text{and} |b| \geq 1 \\ \frac{1}{4} & ; \ |a| < 1 \text{ or } |b| < 1 \end{cases}$$, we have:

$\sim B_d(3.1) = \{y \in \mathbb{R} \ ; \ |y - 3| \geq 1\} = \{y \in \mathbb{R} \ ; \ y \geq 4 \ or \ y \leq 2\}$,

$\sim \overline{B_d}(3.1) = \{y \in \mathbb{R} \ ; \ y > 4 \ or \ y < 2\}$,

$\sim B_\mu\left(2.\frac{1}{3}\right) = \left\{b \in \mathbb{R} \ ; \ \mu(2.b) \geq \frac{1}{3}\right\} = \{b \in \mathbb{R} \ ; |b| \geq 1 \ \}$,

$\sim \overline{B_\mu}\left(2.\frac{1}{3}\right) = \left\{b \in \mathbb{R} \ ; \ \mu(2.b) > \frac{1}{3}\right\} = \{b \in \mathbb{R} \ ; |b| \geq 1 \ \} = \sim B_\mu\left(2.\frac{1}{3}\right)$.

So that, we have:

$\Delta_{B_\mu}^{B_d} = \{x_a \in \Delta \ ; \ 2 < x < 4 \ . \ |a| < 1 \text{ or } a = 2\}$

$\Delta_{\overline{B_\mu}}^{B_d} = \Delta_{B_\mu}^{B_d}$

$\Delta_{T_\mu}^{B_d} = \emptyset$

$\Delta_{\sim B_\mu}^{B_d} = \{x_a \in \Delta \;;\quad 2 < x < 4 \;.\quad |a| \geq 1\;\}$

$\Delta_{\sim \overline{B_\mu}}^{B_d} = \{x_a \in \Delta \;;\quad 2 < x < 4 \;.\quad |a| \geq 1\;\}$

$\Delta_{B_\mu}^{\overline{B_d}} = \{x_a \in \Delta \;;\quad 2 \leq x \leq 4 \;.\quad |a| < 1\;\}$

$\Delta_{\overline{B_\mu}}^{\overline{B_d}} = \{x_a \in \Delta \;;\quad 2 \leq x \leq 4 \;.\quad |a| < 1\;\}$

$\Delta_{T_\mu}^{\overline{B_d}} = \emptyset$

$\Delta_{\sim B_\mu}^{\overline{B_d}} = \{x_a \in \Delta \;;\quad 2 \leq x \leq 4 \;.\quad |a| \geq 1\;\}$

$\Delta_{\sim \overline{B_\mu}}^{\overline{B_d}} = \{x_a \in \Delta \;;\quad 2 \leq x \leq 4 \;.\quad |a| \geq 1\;\}$

$\Delta_{B_\mu}^{T_d} = \{x_a \in \Delta \;;\quad x \in \{2.4\} \;.\quad |a| < 1 \quad \text{or} \quad a = 2\}$

$\Delta_{\overline{B_\mu}}^{T_d} = \{x_a \in \Delta \;;\quad x \in \{2.4\} \;.\quad |a| < 1 \quad \text{or} \quad a = 2\}$

$\Delta_{T_\mu}^{T_d} = \emptyset$

$\Delta_{\sim B_\mu}^{T_d} = \{x_a \in \Delta \;;\quad x \in \{2.4\} \;.\quad |a| \geq 1\;\}$

$\Delta_{\sim \overline{B_\mu}}^{T_d} = \{x_a \in \Delta \;;\quad x \in \{2.4\} \;.\quad |a| \geq 1\;\}$.

$\Delta_{B_\mu}^{\sim B_d} = \{x_a \in \Delta \;;\quad x \geq 4 \;\text{ or }\; x \leq 2 \;.\quad |a| < 1 \;\text{ or }\; a = 2\}$

$\Delta_{\overline{B_\mu}}^{\sim B_d} = \{x_a \in \Delta \;;\quad x \geq 4 \;\text{ or }\; x \leq 2 \;.\quad |a| < 1 \;\text{ or }\; a = 2\}$

$\Delta_{T_\mu}^{\sim B_d} = \emptyset$

$\Delta_{\sim B_\mu}^{\sim B_d} = \{x_a \in \Delta \;;\quad x \geq 4 \;\text{ or }\; x \leq 2 \;.\quad |a| \geq 1\}$

$\Delta_{\sim \overline{B_\mu}}^{\sim B_d} = \{x_a \in \Delta \;;\quad x \geq 4 \;\text{ or }\; x \leq 2 \;.\quad |a| \geq 1\}$

$\Delta_{B_\mu}^{\sim \overline{B_d}} = \{x_a \in \Delta \;;\quad x > 4 \;\text{ or }\; x > 2 \;.\quad |a| < 1 \quad or \quad a = 2\}$

$\Delta_{\overline{B_\mu}}^{\sim \overline{B_d}} = \{x_a \in \Delta \;;\quad x > 4 \;\text{ or }\; x > 2 \;.\quad |a| < 1 \quad \text{or} \quad a = 2\}$

$\Delta_{T_\mu}^{\sim \overline{B_d}} = \emptyset$

$\Delta_{\sim B_\mu}^{\sim \overline{B_d}} = \{x_a \in \Delta \;;\quad x > 4 \;\text{ or }\; x < 2 \;.\quad |a| \geq 1\;\}$

$\Delta_{\sim\overline{B_\mu}}^{\sim\overline{B_d}} = \{x_a \in \Delta \ ; \quad x > 4 \ \text{ or } \ x < 2 \quad . \quad |a| \geq 1 \ \}.$

Conclusion

In this paper, we defined for the first time the concept of fuzzy metric spaces based on two-fold fuzzy algebras, where the elementary properties of this new concept were studied and presented by many theorems and related examples that explain the validity of this work. Also, many different types of open and closed balls were discussed, as well as the relationships between these metric substructures.

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