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Theory of Distances in NeutroGeometry

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Abstract. NeutroGeometry is one of the most recent approaches to geometry. In NeutroGeometry models, the main condition is to satisfy an axiom, definition, property, operator and so on, that is neither entirely true nor entirely false. When one of these concepts is not satisfied at all it is called AntiGeometry. One of the problems that this new theory has had is the scarcity of models. Another open problem is the definition of angle and distance measurements within the framework of NeutroGeometry. This paper aims to introduce a general theory of distance measures in any NeutroGeometry. We also present an algorithm for distance measurement in real-life problems.

Keywords: NeutroGeometry, path, rectifiable path, single-valued neutrosophic set, Taxicab geometry, Chinese checker metric

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1 Introduction

NeutroGeometries have recently emerged as new proposals by Professor F. Smarandache within the extensive history of geometry. NeutroGeometries are heirs of the neutrosophic vision of the world founded and created by F. Smarandache himself [1, 2]. Neutrosophy is the branch of philosophy that addresses the existence of neutrality, the erroneous, unknown, contradictory, paradoxical, incoherent, inconsistent and neutral, among other concepts of this type [3].

Neutrosophy as a new branch of philosophy has had a favorable impact on some branches of knowledge, such as the emergence of neutrosophic sets as a generalization of fuzzy sets, intuitionistic fuzzy sets and interval-valued fuzzy sets. Also, those logic has led to neutrosophic logic. On the other hand, applications of neutrosophy have appeared in decision-making, digital image processing and statistics, to solve problems in psychology, sociology, economics, physics and mathematics.

NeutroAlgebras appeared before NeutroGeometries, where the degree of indeterminacy is part of some algebraic structure models, besides the degrees of truthfulness and falsehood [4]. However, we must highlight that the study of NeutroGeometries has as a precedent mixed geometries or Smarandachean geometries several decades ago. The definition of a mixed geometry is the acceptance of geometries where at least one axiom is Smarandachely negated in one of the following ways: (a) It is fulfilled by one part of the elements of the structure and is not fulfilled by the rest of them, (b) It is not fulfilled in one way by one part of the elements and is not fulfilled by another part of the elements, but in a different way [5].

That is to say, when an axiom is Smarandachely negated, a geometric structure can exist where the axiom is fulfilled for some elements of the space and not fulfilled for others. For example, a plane that is defined where the fifth Euclidean postulate is satisfied by some straight lines and other lines do not satisfy it. Also, we can have that Euclid's fifth postulate is never fulfilled, but in two different ways: on the one hand, with elements that satisfy the axiom of multiple parallels, and on the other hand with lines and points where there is no parallelism.

This initial historical Smarandache's idea of denying concepts within geometric structure led him to the notion of also including indeterminacy as part of the structure. Thus, NeutroGeometry is understood as the geometric structure where there may be some degree of indeterminacy besides the degrees of truthfulness and falsehood, that is, at least one concept within the structure is partially satisfied by the elements of the structure with some degree of indeterminacy. When one of the concepts, definitions, operations or axioms, among others, is not fulfilled in any case, it is called AntiGeometry. A Smarandache

Geometry is a NeutroGeometry when one axiom is partially false and partially true (and it may also be partially indeterminate), or an AntiGeometry when at least one axiom is totally false.

These more modern geometric approaches recover the primeval emergence of geometry as the branch of mathematics dedicated to the study of real-life physical objects. This first objective had become over time an unreal abstract approximation of physical reality. However, it is obvious that in many cases it is not possible to travel a path through the shortest distance. This is because, in the geometric spaces of the real world, there is indeterminacy, vagueness, uncertainty and so on. This has been Smarandache's main motivation for the creation of NeutroGeometries.

Today, this new theory lacks some points that would make it more solid. Some of them are the scarcity of geometric models where this theory is manifested. Hence, this paper aims to contribute to the solidification of NeutroGeometry introducing a theory of distance within the geometric structures in the neutrosophic framework. However, this is not a simple task. There are multiple circumstances recognized by some authors within the geographical scope, where a region can change in time and space [6]. In the so-called vague regions, one of the elements to consider is geometry [7]. In them, uncertainty is associated with each point, line and region that is studied, where indeterminacy can be due to multiple causes and is not easy to model.

The purpose of vague regions is the use of spatial databases within the geographical scope, such as geographical information systems (GIS), and their application in the modeling of geographical conditions that are uncertain in time and space [8]. They use probabilities, fuzzy sets and supervaluationism to model uncertainty or vagueness [9]. The elements that are components of geography such as forests, plains and mountains cannot be defined exactly in their dimensions, and in everyday life people refer to them vaguely. Neither the relationships between these elements escape the vagueness of natural language for referring to these geographical elements. It is common to use vague terms such as "near", "far" and "small area" rather than precise distance or area measures such as 1 km, 10 km and 3 km^2.

In this paper, however, we propose other objectives, although the motivations are the same. We work directly with geometric properties to solve real problems. Nevertheless, we treat geometric structures beyond the Euclidean terrestrial geometry of GIS. Specifically, we deal with the problem of distances by trajectories in the so-called NeutroGeometries from any geometry applied to any region of the universe.

In classical geometry, the distance between two points is calculated in a segment contained in the geodesic line, where the distance is the shortest one. In NeutroGeometry this is not necessarily true. It depends on the person's knowledge and the obstacles that may be encountered along the way. It is a subjective and cognitive distance, in the sense that it depends on each person - on their knowledge, the historical knowledge about the region and the obstacles that objectively interrupt the path - to determine the shortest path to follow between one point and another. The indeterminacy may be due to the presence of unwanted phenomena such as a conflictive zone, there is a territory dominated by wild animals, there is a river with impassable parts, there are mountains that are difficult to climb and many more examples that we can think of.

It also depends on the orientation; to walk from point A to point B is not always equivalent to walking in the opposite direction. For example, if there is a hill, it is not the same to walk uphill as downhill for the same track. Transiting a river is not the same as navigating with the current or against it.

Also, if we are traveling through a river, it is not efficient to travel a section by boat, then go to land and later continue by boat, only to continue along a straight line. It is better to navigate the river, even if this means deviating at times from the path that coincides with the geodesic line.

They are reasons that motivate this article to propose a general solution to find the shortest distance from one point to another regardless of the geometry in question, whether it is Euclidean or non-Euclidean. This is an important element to consider once a geometric model is established. The other point is the definition of angular measurements, but depending on the geometry in question it is possible to maintain the predefined measurements of angles. However, the problem of finding distances is critical in situations where there are indeterminate or uncertain regions, where the real characteristics of the terrain to be covered may not be known or have degrees of uncertainty.

This paper is divided into section 2 where the preliminary concepts necessary to understand the contribution of this paper to NeutroGeometry theory are explained. Section 3 contains the elements of the theoretical proposal that is the objective of this article. These results are argued with some examples of geometries such as the well-known Taxicab geometry or Chinese checker metric that are cases of the herein proposed theory. Section 4 consists of conclusions.

2. Basic Notions of NeutroGeometry

Smarandache added the prefixes Neutro- and Anti- to certain mathematical structures' names [1, 2, 10-12]. The first of them indicates that the concept presents degrees of truth, falsehood and indeterminacy, and the second one denotes that the concept is not fulfilled in all cases. For example, an Algebra is made up of a set of elements with one or more operations among them that satisfy at least one axiom. NeutroAlgebra satisfies these conditions with a degree of truthfulness, a degree of indeterminacy and a degree of falsity. On the other hand, AntiAlgebra does not satisfy at least one of these conditions at all. So, NeutroAlgebra satisfies its conditions in a triad (t, i, f) of truthfulness, indeterminacy and falseness, where $(t, i, f) \neq (1, 0, 0)$ in at least one of them. All the conditions of classical algebra are satisfied in a triad of $(1, 0, 0)$. AntiAlgebra fulfills that any of its conditions are never satisfied corresponding to the triad $(0, 0, 1)$.

Similarly, NeutroGeometry is made up of elements, generally they are the concepts of "point", "line", "plane", "space" and "hyperspace". There are relationships between them, for example, a point is contained in a line and a line is contained in a plane, and so on. They must satisfy certain axioms. These axioms usually either satisfy or contradict the Euclidean axioms, specifically Euclid's five postulates, with emphasis placed on the fifth postulate.

It is not possible to talk about NeutroGeometry without referring to mixed geometries or Smarandache geometries that emerged several decades ago:

"A Smarandache geometry is a geometry which has at least one Smarandache denied axiom, i.e., an axiom behaves in at least two different ways within the same space, i.e., validated and invalidated, or only invalidated but in multiple distinct ways and a Smarandache n-manifold is an n-manifold that supports a Smarandache geometry." [5]

However, a mixed geometry only takes into account the negation of the concept in a certain sense. NeutroGeometry may deny some components of classical geometry, but it also takes into account situations where there is indeterminacy due to indistinguishability, error, ignorance, paradoxes, contradictions, vagueness, inconsistency and so on.

A mixed geometry may contain some elements that satisfy Euclid's fifth postulate of parallelism, while other elements do not satisfy it. Moreover, a mixed geometry can contradict the fifth postulate, on the one hand, because more than one parallel to a given line passes through an external point and on the other hand because of the absence of parallelism for other elements.

There are real-life situations that cannot be modeled using a mixed geometry. An example illustrated in Smarandache's writings is a river with unknown areas. Due to the lengths of the river compared to the terrestrial measures, we can utilize a geometric Euclidean model. However, it is unrealistic to consider a straight line for measuring the shortest distance between two points within the river. This is because the straight line that joins both points can pass through an unknown area, so we are not sure whether this area is passable or not. Then this can no longer be the shortest line to travel through both points. It is necessary to deviate in some way along another path avoiding this region or otherwise risk exploring an unknown place, see Figure 1.

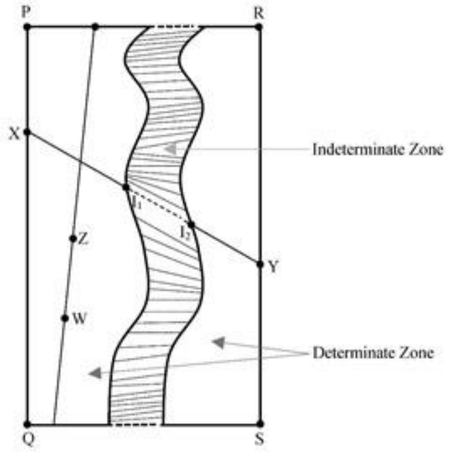

Figure 1: NeutroGeometry example. River with an Indeterminate Zone [1].

In the next section, we propose a theory of distances where there is indeterminacy. Nevertheless, we can consult some references where this topic is discussed, [13-16].

3. Distances in NeutroGeometries

This section contains the details of the proposed theory of distances within the framework of NeutroGeometry.

The first point to consider is that NeutroGeometry must be characterized by the geometry from which it is defined. This can be either Euclidean, hyperbolic, elliptical or mixed geometry. Such a starting geometric structure must have a distance function defined between any two points of the geometric structure. We denote as (G, d_G) this initial geometric space with such a distance. To simplify the results, in this paper we will limit the study to the plane, therefore the geometric structures will only contain points and lines, and the plane is generally defined as a 2-dimensional surface on Euclidean space.

Thus, the two-dimensional coordinate system XOY, where O is the origin together with the distance function forms a metric space. Let us call it (X, d_X) and recall that a metric space is defined from a set of elements and a distance function among them [17].

The other important definition is a *path*, which is a parametric function $p: [0,1] \to X$. That is, it is a function defined on the coordinate system. The paths will be assumed to be rectifiable. In other words, considering the partitions $\mathrm{T} = [t_0 = 0, t_1, t_2, \cdots, t_{n-1}, t_n = 1]$ there is always:

$$L(p) = sup_T\left\{\sum_{i=1}^{n} d\big(p(t_{i-1}), p(t_i)\big)\right\} \tag{1}$$

$L(p)$ is called the *length of* p in (G, d_G), where $\sup_{\mathrm{T}}$ is the supremum over the set of finite partitions T. What is more, let us suppose that we are in the presence of a *path metric space* that is, given $x_1, x_2 \in X$, then:

$$d_G(x_1, x_2) = inf\{L(p) : p \text{ is a rectifiable path from } x_1 \text{ to } x_2\} \tag{2}$$

Where inf is the abbreviation of infimum.

Let us also consider other definitions of paths. For example, each path is oriented when for all partitions $\mathrm{T} = [t_0 = 0, t_1, t_2, \cdots, t_{n-1}, t_n = 1]$ we obtain paths that go from x_1 to x_2, however, we can define $-p(t_i) = p(t_{n-i})$ for obtaining the same path in the opposite direction.

Let us call *subpath* when the path is obtained from p such that for $p(0) = x_1$ and $p(1) = x_2$ the subpath is taken as $q(0) = p(u)$ or $q(1) = p(v)$ for $u > 0$ or $v < 1$. That is, geometrically the subpath is a path contained in the trajectory of the path.

Definition 1. Suppose that besides $L(p)$ we define a *passability function* $\varepsilon(p) \in [0,1]$ for each rectifiable path p. $\varepsilon(p)$ satisfies the following conditions:

i. If $p = \emptyset$ or one of the points $p(t_i)$ is an ideal point, then $\varepsilon(p) = 0$. Let us recall that the ideal points in models of non-Euclidean geometries do not belong to the models, but they are on the border. E.g., the Poincaré Disk model contains the interior points of the unit circle except for the border, thus the circumference of radius 1 is the set of ideal points. So, the distance from one point to an ideal point is infinite.

ii. When $p \equiv x \in X$, that is, the path is formed by a single point, we have $\varepsilon(p) = 1$.

iii. $\forall q$ subpath of p it is fulfilled $\varepsilon(p) \leq \varepsilon(q)$.

Definition 1 does not require that $\varepsilon(p) = \varepsilon(-p)$.

As a consequence of Definition 1, we have the following properties:

1. p, q and r are three paths such that $p(1) = q(0)$. Let us denote by $r = p \cup q$ the operation of obtaining the union of the two paths where $r\left(\frac{t}{2}\right) = p(t)$ and $r\left(\frac{t+1}{2}\right) = q(t)$. Therefore, we have $r(0) = p(0)$, $r(1) = q(1)$, $r\left(\frac{1}{2}\right) = p(1) = q(0)$. In this case $\varepsilon(r) \leq min(\varepsilon(p), \varepsilon(q))$ according to Definition 1 condition (iii).

2. Let p, q and r be three paths equally oriented, such that for certain intervals $\mathrm{I}_1, \mathrm{I}_2 \subset \mathrm{I} = [0,1]$ $r(I) \equiv p(\mathrm{I}_1) \equiv q(\mathrm{I}_2)$. This operation is denoted by $r = p \cap q$. So $max(\varepsilon(p), \varepsilon(q)) \leq \varepsilon(r)$, according to Definition 1 condition (iii) as well.

Definition 2. Given (G, d_G) is a metric space ([17]) in a geometric structure. The *NeutroGeometric distance* between x_1 and x_2 for $x_1, x_2 \in X$ is defined as:

$$d_{NG}(x_1, x_2) = inf\left\{\frac{L(p)}{\varepsilon(p)} : p \text{ is a rectifiable path from } x_1 \text{ to } x_2\right\} \tag{3}$$

Where $\varepsilon(p)$ is the passability function of p.

Some properties of $d_{NG}(x_1, x_2)$ are the following:

1. Fixing p, when the passability function increases, the distance $d_{NG}(x_1, x_2)$ decreases.

2. When $\forall p\ \varepsilon(p) = 1$ then we have $d_{NG}(x_1, x_2) = d_G(x_1, x_2)$. This happens when there is no indeterminacy or obstacle between x_1 and x_2, which is the hypothesis assumed in classical geometries.

3. In general, $d_{NG}(x_1, x_2)$ is not necessarily symmetrical, thus $d_{NG}(x_1, x_2) \neq d_{NG}(x_2, x_1)$.

4. $\forall x \in X\ d_{NG}(x, x) = 0$, since if p is the path that goes from x to itself then since definition we have $\varepsilon(p) = 1$ and $L(p) = 0$.

5. In some recent models, there are cases where $d_G(x_1, x_2) = \infty$ when an ideal point is contained in the path. Then $d_{NG}(x_1, x_2) = \infty$, since from Equation 3 we have $d_{NG}(x_1, x_2) = \frac{\infty}{0} = \infty \cdot \infty = \infty$.

6. When $\varepsilon(p) = 0$, that is, when there is total certainty of the impassibility along the path p, then we have $\frac{L(p)}{\varepsilon(p)} = \infty$ either for $L(p) < \infty$ or $L(p) = \infty$.

$L(p)$ is defined from the characteristics of geometry (G, d_G). However, for the geometric space (NG, d_{NG}) we must define the passability function $\varepsilon(p)$.

For example, for each path it can be determined $\varepsilon(p)$ as the probability that there is transitibility along the path p. Also $\varepsilon(p)$ can be the function of the possibility of traveling along the path p, where this is a possibility measure developed by Dubois and Prade in the field of fuzzy logic [18, 19]. Equivalently, if a multi-agent system is modeled, $\varepsilon(p)$ can be interpreted as the permission that exists to travel the path in the scope of the deontic logic. $\varepsilon(p)$ can be a truth value in fuzzy logic [20], can be a numerical function obtained from the ordered pair of both membership and non-membership values within the intuitionistic fuzzy logic [21], can be an element of an interval-valued fuzzy set [22, 23] or may be a function of a neutrosophic number [24, 25].

In the latter case when there is a neutrosophic valuation formed by the triple $n = (t, i, f)$ we will need to convert n into a real value using the following formula of the Score function [26]:

$$\mathcal{S}(n) = \frac{2+t-i-f}{3} \quad (4)$$

For the neutrosophic order, defined as $n_1 \preccurlyeq n_2$ if and only if $t_1 \leq t_2$, $i_2 \leq i_1$ and $f_2 \leq f_1$, where $n_1 = (t_1, i_1, f_1)$ and $n_2 = (t_2, i_2, f_2)$; then we have $(0,1,1)$ is the smallest value of n and $(1,0,0)$ is the maximum value. Therefore, the score of $(1,0,0)$ corresponds to $\mathcal{S}\big((1,0,0)\big) = 1$ and the score of $(0,1,1)$ corresponds to $\mathcal{S}\big((0,1,1)\big) = 0$.

When we are using intervals, they can be des-neutrosophied converting them into a real single value using [27]:

$$\lambda([a, b]) = \frac{a+b}{2} \quad (5)$$

Also, we can adapt the calculation of the infimum to intervals, and as a final result we obtain an interval containing indeterminacy that can be converted into the so-called neutrosophic number.

Note that $d_{NG}(x_1, x_2)$ is increasing for the length of the path but is decreasing for the passability function of the path. Therefore, the longer the path is, the greater the value of the distance in NeutroGeometry is, both because the length increases and also because the passability function could decrease.

The theory presented so far is a generalization of already known geometries. For example, Minkowski's Taxicab geometry and the distance function that bears this name [28, 29]:

$$\mathrm{d_T(A, B)} = |\mathrm{x_A} - \mathrm{x_B}| + |\mathrm{y_A} - \mathrm{y_B}| \quad (6)$$

Where (x_A, y_A) are the coordinates of A and (x_B, y_B) are the coordinates of B in the Euclidean Cartesian plane.

This geometry is based on Euclidean geometry (G, d_G), where the points, lines and measurements between angles are the usual ones. Only the distance function between two points is changed to the one represented in Equation 6.

It is assumed that the only valid trajectories are those that go in a straight line in angles of 0, $\frac{\pi}{2}$, π or $\frac{3\pi}{2}$ concerning the x-axis. That is, the only possible paths follow trajectories parallel to one of the two coordinate axes. This geometry models the movement of a car in a city where the blocks are square and have the same dimensions. The taxicab can only turn right or left $\frac{\pi}{2}$ radians when it arrives at the corner of the block, see Figure 2.

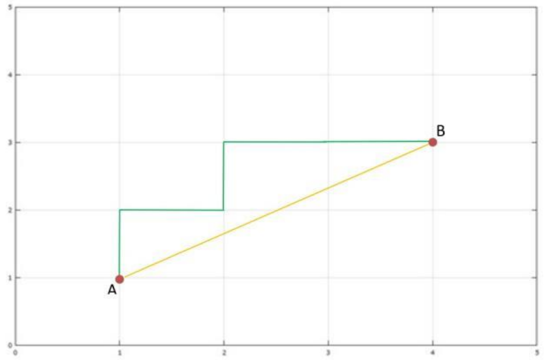

Figure 2: Trajectory from point A to B using paths in the form of broken lines (green lines) in Taxicab geometry. Observe the Euclidean straight line in yellow.

Therefore, paths that contain other types of straight lines are not allowed, thus the passability function is defined by:

$$\varepsilon(p) = \begin{cases} 1, if\ p \text{ is a path containing segments always parallels to one axis} \\ 0, \text{otherwise} \end{cases}$$

So, $\frac{L(p)}{\varepsilon(p)} = \begin{cases} L(p), if\ p \text{ is a path containing segments always parallel to one axis} \\ \infty, \text{otherwise} \end{cases}$

It is easy to check that the previous condition is satisfied only when $d_{NG}(A,B) = d_T(A,B)$. In Figure 2 the path represented with yellow line is therefore not allowed $\frac{L(p)}{\varepsilon(p)} = \infty$. The path of broken lines in green has a length equal to 5 and it can be proven that this is the distance between the two points for this geometry.

This same idea can be extended for other distances inspired by the Taxicab geometry. For example, when we have the following distance:

$$d_C(\mathrm{A},\mathrm{B}) = \max(|\mathrm{x_A} - \mathrm{x_B}|, |\mathrm{y_A} - \mathrm{y_B}|) + (\sqrt{2} - 1)\min(|\mathrm{x_A} - \mathrm{x_B}|, |\mathrm{y_A} - \mathrm{y_B}|) \quad (7)$$

This is known as the *Chinese checker metric*. So, in this geometry it is only allowed movements in straight diagonal lines of $\frac{\pi}{4}$ radians in addition to those already defined in Taxicab geometry [30]. See Figure 3.

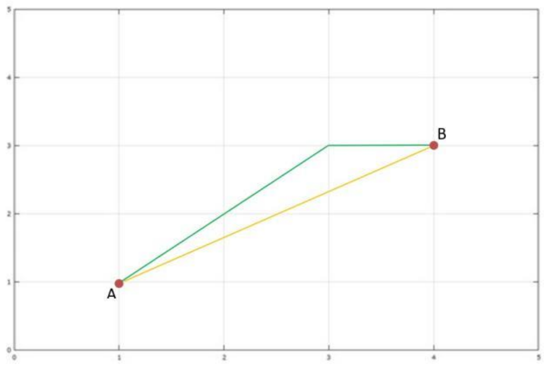

Figure 3: Trajectory from point A to B using paths in the form of lines corresponding to the Chinese checker metric.

In general, it is known that in the cities there are streets with blocks of different geometric shapes, they can be roundabouts, triangular, irregular, as well as square or rectangular, among others. Hence, our theory of distances is more exact, since each of these paths is measurable regardless of the type of shape it presents, see Figure 4.

Figure 4: The streets of a city (in black) have different shapes.

Let us observe that Taxicab or Chinese Checker geometry cannot be applied to measure distances in Figure 4. However, in this city, it is possible to apply Equation 3 because the paths to travel are rectifiable curves in Euclidean geometry as circular and rectilinear paths.

Also, in the case of real traffic, there are additional restrictions. E.g., although ideally we can turn down some streets, this is not always allowed, so we have to comply with traffic laws. These types of real-life situations are modeled thanks to the non-symmetry of the defined distance NG.

Apart from this, an interesting question is why we choose neutrosophy to model the function $\varepsilon(p)$. The neutrosophic theory offers greater possibilities for representing the different types of situations that may arise. In addition, it allows greater accuracy that is achieved at the cost of greater indeterminacy to represent the transitibility from one point to another through a path.

In general, we propose the following operations on the passability functions when there are unions or intersections of paths.

1. $\varepsilon(p \cup q) = min(\varepsilon(p), \varepsilon(q))$,
2. $\varepsilon(p \cap q) = max(\varepsilon(p), \varepsilon(q))$,

A clear limitation is that Equation 3 may be impractical since it requires analyzing all possible paths. In daily life, people make decisions about which path to take according to a good enough decision-making rather than an optimal one. Then, the amount of calculation of possible trajectories is reduced which would be too cumbersome.

The calculation of the path that minimizes the distance between two points in Equation 2 is simplified when we know the type of geometry we are dealing with. So, the path is identified with the geodesic and its equation. For example, in the Euclidean plane, the straight line between the points is the shortest path, and it is also very simple to calculate its algebraic equation. This is not necessarily the case in real life, where we need to know the terrain to get from one point to another throughout the shortest trajectory.

In daily life, human beings use Equation 3 more than Equation 2. Using the theory explained so far is an arduous task to select each path and calculate the shortest distance. In reality, decision-making in daily life is not based on selecting the optimal one, but rather on a "good enough" solution. From the distance defined in Equation 3, it is possible to define an approximate distance based on a finite subset of paths instead of all possible paths.

This leads to the following algorithm shown in Table 1.

Algorithm for deciding on the shortest path

1. Given a region R of the plane, two points A and B are defined as contained in R, and we want to measure the approximate minimum distance to go from A to B.
2. Let $E = \{e_1, e_2, \cdots, e_m\}$ be a group of locals, specialists, geographers, among others who know the terrain to be measured.
3. Γ denotes the subset of possible paths p from A to B, such that it has finite cardinality.
4. Each of the respondents in E is asked based on a scale of 0-10 to rate each $p \in \Gamma$ in terms of:
   (a) Feasibility to go from A to B passing through p.
   (b) Indeterminacy (due to ignorance, indifference, among others) to go from A to B passing through p.
   (c) Impossibility to go from A to B (passing through p).
5. The values of (t, i, f) are obtained in the following way:

$$t = \frac{sum(assessments\ in\ (a))}{10 \cdot m}, i = \frac{sum(assessments\ in\ (b))}{10 \cdot m}, f = \frac{sum(assessments\ in\ (c))}{10 \cdot m}$$

6. It is calculated $\varepsilon(p)$ according to Equation 4 for each of the possible values of (t, i, f). It is also calculated $L'(p)$ as a sufficiently approximate value of the actual length $L(p)$ of the path p. I.e., given $\epsilon > 0$ a prefixed allowed error then $|L'(p) - L(p)| \leq \epsilon$.

7. The approximate distance NG between both points is calculated by Equation 8.

$d'_{NG}(A,B) = inf\left\{\frac{L\prime(p)}{\varepsilon(p)} : p \in \Gamma\right\}$	(8)

Table 1: Algorithm for calculating the shortest approximate distance in NG based on neutrosophic numbers.

In this way, the calculation of a distance is linked to the collective knowledge about the paths to travel, rather than through an inherent single function for each geometry.

Example 1. Suppose we are in a boat, and we want to navigate a river whose bed has irregularities so that the left bank has calm waters and the right bank has turbulent waters in a certain section. However, we wish to go from point A to point B on the right side of the river. Furthermore, the path of the river itself is sinuous, see Figure 5.

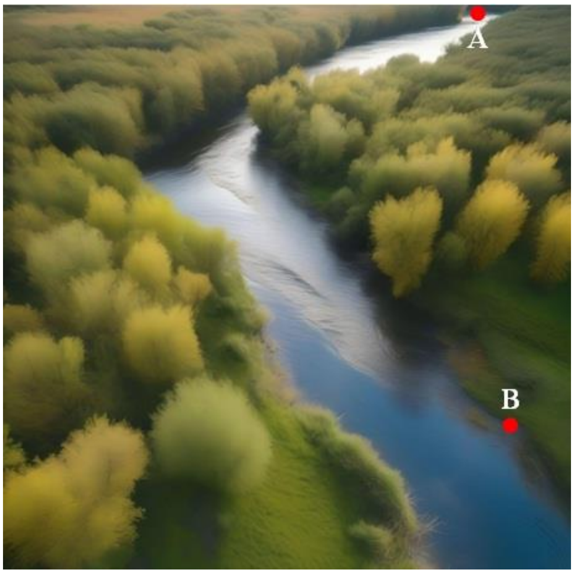

Figure 5: Picture of the river of the example. We need to go from point A to point B. This picture was generated by an AI tool.

That is why the shortest path that passes on the right is impassable and hence $\varepsilon(p_1) = \mathcal{S}\big((0,1,1)\big) = 0$. The path that goes through the center has some areas with a percentage of danger, let us say $\varepsilon(p_2) = \mathcal{S}\big((0.5,0.1,0.4)\big) = 0.66667$. While the path that goes left and then turns right is the longest one and it is completely safe, therefore $\varepsilon(p_3) = \mathcal{S}\big((1,0,0)\big) = 1$, see Figure 6.

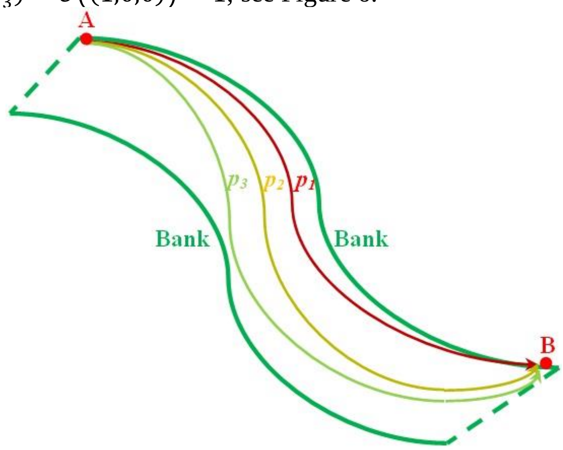

Figure 6: Map of the river showing the three paths, p_1, p_2 and p_3.

Let us suppose that the Euclidean length of each path from A to B is viz., $l(p_1) = 1.5\ Km$, $l(p_2) = 1.9\ Km$ and

$l(p_3) = 2.1\ Km$. To go through the Euclidean straight line, we would have to navigate sections of the river, then go overland in a swampy area, then navigate another section, and so on, which is not practical.

The NG lengths of the three paths in the example are, $l_{NG}(p_1) = \frac{1.5}{0}\ Km = \infty\ Km$, $l_{NG}(p_2) = \frac{1.9}{0.66667}\ Km = 2.85\ Km$, while $l_{NG}(p_3) = \frac{2.1}{1}\ Km = 2.1\ Km$, therefore we prefer the path p_3 to make the crossing, even though it is the longest one according to Euclidean geometry.

That is why the distance NG between A and B is at most 2.1 Km. Note that we have selected a finite number of paths because otherwise, we would have to search for the length of an infinite number of them.

Furthermore, if we have a path p_4 that begins at a point C before A and then continues through p_1, according to our definition of $\varepsilon(\cdot)$ we have $\varepsilon(p_4) \leq \varepsilon(p_1) = 0$, therefore $l_{NG}(p_1) = \infty\ Km$ as well. That is, any path that contains p_1 as a subpath will have infinite length in this geometry.

Another topic of interest is the way that path lengths are calculated. In this paper, we have proposed using the Euclidean arc length, which is a method that can be adapted to other geometries always respecting the definitions of geodesics and lengths, see Figure 7.

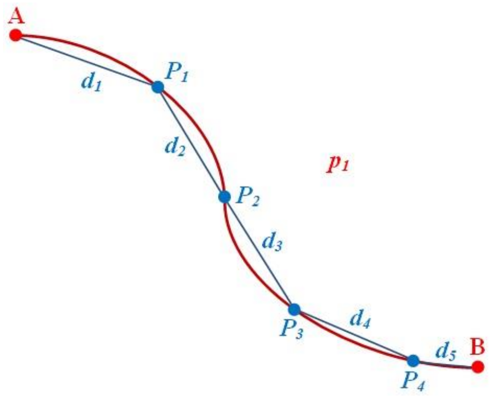

Figure 7: Arc length of the path p_1.

Figure 7 exemplifies the calculus of length of p_1. We select the points $A = P_0, P_1, P_2, P_3, P_4$ and $B = P_5$, all of them are on the curve. Then, we calculate the distances, $d_1 = d_E(A, P_1)$, $d_2 = d_E(P_1, P_2)$, $d_3 = d_E(P_2, P_3)$, $d_4 = d_E(P_3, P_4)$ and $d_5 = d_E(P_4, B)$, where $d_E(\cdot,\cdot)$ is the Euclidean metric. So, the length of p_1 is approximated by $d = \sum_{i=1}^{5} d_i$. For more accuracy, we must use more points on the curve.

In any case, in real life, there are cartographic methods to calculate the length of a path, including satellite spatial vision. Also in certain situations, we can walk along a path at a more or less constant speed and take the transit time from one point to another. Hence, we can approximately calculate the distance (length of the path) by the formula $distance = speed \cdot (time_B - time_A)$.

Conclusion

In this paper, we propose for the first time a theory to measure distance in NeutroGeometry. NeutroGeometry is assumed to be defined from a geometric structure that can be Euclidean or non-Euclidean. Using the distance measure in that base geometric structure, we introduced a distance defined in the NeutroGeometry structure, which depends on a passability function and the length of the path that joins both points. We specify some axioms that the passability function must fulfill. We also verify that the new measure generalizes the cases of classical geometries, as well as other geometries such as Taxicab and Chinese checker metrics. It is more convenient that the passability function depends on neutrosophy and not on other uncertainty models such as fuzzy sets or intuitionistic fuzzy sets models, because a neutrosophic set allows a greater number of possible states of knowledge to be expressed more accurately. Additionally, we propose an algorithm to calculate the approximate distance between two points in the NeutroGeometry plane based on a survey applied to a group of locals, geographers and experts, among others. The theory presented so far, and the proposed algorithm allows us to solve the problem of calculating the distance when there is uncertainty or indeterminacy in a two-dimensional region. These results bring us closer to solving real-life problems than utilizing classical geometries.

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