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Neutrosophic quadruple a-ideals

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Abstract. The notion of neutrosophic quadruple a-ideal is introduced, and related properties are investigated. Relations between a neutrosophic quadruple p-ideal, a neutrosophic quadruple q-ideal, a neutrosophic quadruple a-ideal and a neutrosophic quadruple closed ideal are discussed. Conditions for the neutrosophic quadruple (A, B)-set $N_q(A, B)$ to be a neutrosophic quadruple a-ideal are provided.

Keywords: Neutrosophic quadruple BCK/BCI-number, neutrosophic quadruple BCK/BCI-algebra, neutrosophic quadruple (closed) ideal, neutrosophic quadruple $p(q, a)$-ideal.

1. Introduction

Neutrosophic sets (NSs) proposed by (Smarandache, 1998, 1999, 2002, 2005, 2006, 2010), which is a generalization of fuzzy sets and intuitionistic fuzzy set, is a powerful tool to deal with incomplete, indeterminate and inconsistent information which exist in the real world (see [28–30]). Recently, this concept has been applied more actively to many areas (see [1], [2], [3], [4]). Neutrosophic algebraic structures in BCK/BCI-algebras are discussed in the papers [6–11, 15–18, 20, 23, 27, 32]. Smarandache [31] considered an entry (i.e., a number, an idea, an object etc.) which is represented by a known part (a) and an unknown part (bT, cI, dF) where T, I, F have their usual neutrosophic logic meanings and a, b, c, d are real or complex numbers, and then he introduced the concept of neutrosophic quadruple numbers. Jun et al. [19] used neutrosophic quadruple numbers based on a set, and constructed neutrosophic quadruple BCK/BCI-algebras. They investigated several properties, and considered (closed, positive implicative) ideal in neutrosophic quadruple BCI-algebra. Given subsets A and B of a BCK/BCI-algebra, they considered the set $NQ(A, B)$ which consists of neutrosophic quadruple

BCK/BCI-numbers with a condition. They provided conditions for the set $NQ(A, B)$ to be a (closed, positive implicative) ideal of a neutrosophic quadruple BCK/BCI-algebra. Muhiuddin et al. [24] introduced the concept of implicative neutrosophic quadruple BCK-algebras, and investigated several properties. Muhiuddin et al. [25, 26] discuss neutrosophic quadruple p-ideals and neutrosophic quadruple q-ideals.

In this paper, we consider the neutrosophic quadruple version of an a-ideal in a BCI-algebra. We discuss relations between a neutrosophic quadruple p-ideal, a neutrosophic quadruple q-ideal, a neutrosophic quadruple a-ideal and a neutrosophic quadruple closed ideal. We provide conditions for the neutrosophic quadruple (A, B)-set $N_q(A, B)$ to be a neutrosophic quadruple a-ideal.

2. **Preliminaries**

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [13] and [14]) and was extensively investigated by several researchers.

By a *BCI-algebra*, we mean a set X with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X)$ $(((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)$ $((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)$ $(x * x = 0)$,

(IV) $(\forall x, y \in X)$ $(x * y = 0,\, y * x = 0 \;\Rightarrow\; x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)$ $(0 * x = 0)$,

then X is called a *BCK-algebra.* Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{1}$$

$$(\forall x, y, z \in X)\,(x \leq y \;\Rightarrow\; x * z \leq y * z,\; z * y \leq z * x)\,, \tag{2}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{3}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{4}$$

where $x \leq y$ if and only if $x * y = 0$.

Any BCI-algebra X satisfies the following conditions (see [12]):

$$(\forall x, y \in X)(x * (x * (x * y)) = x * y), \tag{5}$$

$$(\forall x, y \in X)(0 * (x * y) = (0 * x) * (0 * y)), \tag{6}$$

$$(\forall x, y \in X)(0 * (0 * (x * y)) = (0 * y) * (0 * x)). \tag{7}$$

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called

- an *ideal* of X if it satisfies:

$$0 \in I, \tag{8}$$

$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \;\Rightarrow\; x \in I)\,. \tag{9}$$

- a *closed ideal* of X (see [12]) if it is an ideal of X which satisfies:

$$(\forall x \in X)(x \in I \;\Rightarrow\; 0 * x \in I). \tag{10}$$

- a *p-ideal* of X (see [33]) if it satisfies (8) and

$$(\forall x, y, z \in X)(y \in I, (x * z) * (y * z) \in I \;\Rightarrow\; x \in I). \tag{11}$$

- a *q-ideal* of X (see [21]) if it satisfies (8) and

$$(\forall x, y, z \in X)(x * (y * z) \in I,\, y \in I \;\Rightarrow\; x * z \in I). \tag{12}$$

- an *a-ideal* of X (see [21]) if it satisfies (8) and

$$(\forall x, y, z \in X)((x * z) * (0 * y) \in I,\, z \in I \;\Rightarrow\; y * x \in I). \tag{13}$$

Note that a subset of a BCI-algebra is a closed ideal if and only if it is both an ideal and a subalgebra.

Recall that a subset I of a BCI-algebra X is a p-ideal of X if and only if I is an ideal of X which satisfies the following condition:

$$(\forall x \in X)(0 * (0 * x) \in I \;\Rightarrow\; x \in I). \tag{14}$$

We refer the reader to the books [12, 22] for further information regarding BCK/BCI-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

We consider neutrosophic quadruple numbers based on a set instead of real or complex numbers.

Let X be a set. A *neutrosophic quadruple X-number* is an ordered quadruple (a, xT, yI, zF) where $a, x, y, z \in X$ and T, I, F have their usual neutrosophic logic meanings (see [5]).

The set of all neutrosophic quadruple X-numbers is denoted by $N_q(X)$, that is,

$$N_q(X) := \{(a, xT, yI, zF) \mid a, x, y, z \in X\},$$

and it is called the *neutrosophic quadruple set* based on X. If X is a BCK/BCI-algebra, a neutrosophic quadruple X-number is called a *neutrosophic quadruple BCK/BCI-number* and we say that $N_q(X)$ is the *neutrosophic quadruple BCK/BCI-set*.

Let X be a BCK/BCI-algebra. We define a binary operation $\boxdot$ on $N_q(X)$ by

$$(a, xT, yI, zF) \boxdot (b, uT, vI, wF) = (a * b, (x * u)T, (y * v)I, (z * w)F)$$

for all (a, xT, yI, zF), $(b, uT, vI, wF) \in N_q(X)$. Given $a_1, a_2, a_3, a_4 \in X$, the neutrosophic quadruple BCK/BCI-number (a_1, a_2T, a_3I, a_4F) is denoted by $\tilde{a}$, that is,

$$\tilde{a} = (a_1, a_2T, a_3I, a_4F),$$

and the zero neutrosophic quadruple BCK/BCI-number $(0, 0T, 0I, 0F)$ is denoted by $\tilde{0}$, that is,

$$\tilde{0} = (0, 0T, 0I, 0F).$$

Then $(N_q(X); \boxdot, \tilde{0})$ is a BCK/BCI-algebra (see [19]), which is called *neutrosophic quadruple BCK/BCI-algebra*, and it is simply denoted by $N_q(X)$.

We define an order relation "$\ll$" and the equality "$=$" on $N_q(X)$ as follows:

$$\tilde{x} \ll \tilde{y} \Leftrightarrow x_i \leq y_i \text{ for } i = 1, 2, 3, 4,$$
$$\tilde{x} = \tilde{y} \Leftrightarrow x_i = y_i \text{ for } i = 1, 2, 3, 4$$

for all $\tilde{x}, \tilde{y} \in N_q(X)$. It is easy to verify that "$\ll$" is an equivalence relation on $N_q(X)$.

Let X be a BCK/BCI-algebra. Given nonempty subsets A and B of X, consider the set

$$N_q(A, B) := \{(a, xT, yI, zF) \in N_q(X) \mid a, x \in A \;\&\; y, z \in B\},$$

which is called the *neutrosophic quadruple* (A, B)*-set* (briefly, neutrosophic quadruple (A, B)-set).

The set $NQ(A, A)$ is denoted by $N_q(A)$, and it is called the *neutrosophic quadruple* A*-set* (briefly, neutrosophic quadruple A-set).

3. Neutrosophic quadruple a-ideals

Definition 3.1. Given nonempty subsets A and B of X, if the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is an a-ideal of a neutrosophic quadruple BCI-algebra $N_q(X)$, we say $N_q(A, B)$ is a *neutrosophic quadruple* a*-ideal* of $N_q(X)$.

Example 3.2. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the binary operation $*$, which is given in Table 1.
Then the neutrosophic quadruple BCI-algebra $N_q(X)$ has 256 elements. Consider subsets $A = \{0, a\}$ and $B = \{0, b\}$ of X. Then

$$N_q(A, B) = \{\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3, \tilde{\beta}_4, \tilde{\beta}_5, \tilde{\beta}_6, \tilde{\beta}_7, \tilde{\beta}_8, \tilde{\beta}_9, \tilde{\beta}_{10}, \tilde{\beta}_{11}, \tilde{\beta}_{12}, \tilde{\beta}_{13}, \tilde{\beta}_{14}, \tilde{\beta}_{15}\}$$

where

$\tilde{\beta}_0 = (0, 0T, 0I, 0F)$, $\tilde{\beta}_1 = (0, 0T, 0I, bF)$, $\tilde{\beta}_2 = (0, 0T, bI, 0F)$,

TABLE 1. Cayley table for the binary operation "$*$"

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\tilde{\beta}_3 = (0, 0T, bI, bF)$, $\tilde{\beta}_4 = (0, aT, 0I, 0F)$, $\tilde{\beta}_5 = (0, aT, 0I, bF)$,
$\tilde{\beta}_6 = (0, aT, bI, 0F)$, $\tilde{\beta}_7 = (0, aT, bI, bF)$, $\tilde{\beta}_8 = (a, 0T, 0I, 0F)$,
$\tilde{\beta}_9 = (a, 0T, 0I, bF)$, $\tilde{\beta}_{10} = (a, 0T, bI, 0F)$, $\tilde{\beta}_{11} = (a, 0T, bI, bF)$,
$\tilde{\beta}_{12} = (a, aT, 0I, 0F)$, $\tilde{\beta}_{13} = (a, aT, 0I, bF)$,
$\tilde{\beta}_{14} = (a, aT, bI, 0F)$, $\tilde{\beta}_{15} = (a, aT, bI, bF)$.

It is routine to verify that $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.

Proposition 3.3. *For any nonempty subsets A and B of a BCI-algebra X, if the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$, then the following assertions are valid.*

$$(\tilde{x} \boxdot \tilde{z}) \boxdot (\tilde{0} \boxdot \tilde{y}) \in N_q(A, B) \;\Rightarrow\; \tilde{y} \boxdot (\tilde{x} \boxdot \tilde{z}) \in N_q(A, B), \tag{15}$$

$$\tilde{x} \boxdot (\tilde{0} \boxdot \tilde{y}) \in N_q(A, B) \;\Rightarrow\; \tilde{y} \boxdot \tilde{x} \in N_q(A, B) \tag{16}$$

for all $\tilde{x}, \tilde{y}, \tilde{z} \in N_q(X)$.

Proof. Assume that $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$ for any nonempty subsets A and B of a BCI-algebra X. Suppose that $(\tilde{x} \boxdot \tilde{z}) \boxdot (\tilde{0} \boxdot \tilde{y}) \in N_q(A, B)$ for any elements $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ of $N_q(X)$. Then

$$\begin{aligned}
&((\tilde{x} \boxdot \tilde{z}) \boxdot ((\tilde{x} \boxdot \tilde{z}) \boxdot (\tilde{0} \boxdot \tilde{y}))) \boxdot (\tilde{0} \boxdot \tilde{y}) \\
&= ((\tilde{x} \boxdot \tilde{z}) \boxdot (\tilde{0} \boxdot \tilde{y})) \boxdot ((\tilde{x} \boxdot \tilde{z}) \boxdot (\tilde{0} \boxdot \tilde{y})) \\
&= \tilde{0} \in N_q(A, B).
\end{aligned}$$

Since $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$, it follows that $\tilde{y} \boxdot (\tilde{x} \boxdot \tilde{z}) \in N_q(A, B)$. Finally, (16) is induced by taking $\tilde{z} = \tilde{0}$ in (15). □

Lemma 3.4 ([21]). *In a BCI-algebra, every a-ideal is a closed ideal.*

Lemma 3.5 ([19]). *If A and B are closed ideals of a BCI-algebra X, then the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple closed ideal of $N_q(X)$.*

We consider relations between a neutrosophic quadruple a-ideal and a neutrosophic quadruple closed ideal.

Theorem 3.6. *For any nonempty subsets A and B of a BCI-algebra X, if the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$, then it is a neutrosophic quadruple closed ideal of $N_q(X)$.*

Proof. Assume that $N_q(A, B)$ is a neutrosophic quadruple a-ideal of a neutrosophic quadruple BCI-algebra $N_q(X)$ where A and B are nonempty subsets of X. Since $\tilde{0} = (0, 0T, 0I, 0F) \in N_q(A, B)$, we have $0 \in A \cap B$. Let $x, y, z \in X$ be such that $(x * z) * (0 * y) \in A \cap B$ and $z \in A \cap B$. Then $(z, zT, zI, zF) \in N_q(A, B)$ and

$$
\begin{aligned}
&((x, xT, xI, xF) \boxdot (z, zT, zI, zF)) \boxdot (\tilde{0} \boxdot (y, yT, yI, yF)) \\
&= (x * z, (x * z)T, (x * z)I, (x * z)F) \boxdot (0 * y, (0 * y)T, (0 * y)I, (0 * y)F) \\
&= ((x * z) * (0 * y), ((x * z) * (0 * y))T, ((x * z) * (0 * y))I, ((x * z) * (0 * y))F) \\
&\in N_q(A, B).
\end{aligned}
$$

Hence

$$(y * x, (y * x)T, (y * x)I, (y * x)F) = (y, yT, yI, yF) \boxdot (x, xT, xI, xF) \in N_q(A, B),$$

that is, $y * x \in A \cap B$. Therefore A and B are a-ideals of X. Using Lemmas 3.4 and 3.5, $N_q(A, B)$ is a neutrosophic quadruple closed ideal of $N_q(X)$. □

The converse of Theorem 3.6 is not true as seen in the following example.

Example 3.7. Consider a BCI-algebra $X = \{0, 1, a\}$ with the binary operation $*$, which is given in Table 2.

TABLE 2. Cayley table for the binary operation "$*$"

$*$	0	1	a
0	0	0	a
1	1	0	a
a	a	a	0

Then the neutrosophic quadruple BCI-algebra $N_q(X)$ has 81 elements. If we take $A = \{0\}$ and $B = \{0\}$, then

$$N_q(A, B) = \{\tilde{0}\}$$

which is a neutrosophic quadruple closed ideal of $N_q(X)$. But it is not a neutrosophic quadruple a-ideal of $N_q(X)$ because if we take $\tilde{1} = (0, 1T, 1I, 0F) \in N_q(X)$ then

$$(\tilde{0} \boxdot \tilde{0}) \boxdot (\tilde{0} \boxdot \tilde{1}) = \tilde{0} \in N_q(A, B),$$

but $\tilde{1} \boxdot \tilde{0} = \tilde{1} \notin N_q(A, B)$.

We provide conditions for the neutrosophic quadruple (A, B)-set $N_q(A, B)$ to be a neutrosophic quadruple a-ideal.

Theorem 3.8. *If A and B are a-ideals of a BCI-algebra X, then the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.*

Proof. Suppose that A and B are a-ideals of a BCI-algebra X. Obviously, $\tilde{0} \in N_q(A, B)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $N_q(X)$ be such that $(\tilde{x} \boxdot \tilde{z}) \boxdot (\tilde{0} \boxdot \tilde{y}) \in N_q(A, B)$ and $\tilde{z} \in N_q(A, B)$. Then $z_i \in A$, $z_j \in B$ for $i = 1, 2$; $j = 3, 4$, and

$$\begin{aligned}
&(\tilde{x} \boxdot \tilde{z}) \boxdot (\tilde{0} \boxdot \tilde{y}) = ((x_1, x_2T, x_3I, x_4F) \boxdot (z_1, z_2T, z_3I, z_4F)) \boxdot \\
&((0, 0T, 0I, 0F) \boxdot (y_1, y_2T, y_3I, y_4F)) \\
&= (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \boxdot \\
&(0 * y_1, (0 * y_2)T, (0 * y_3)I, (0 * y_4)F) \\
&= ((x_1 * z_1) * (0 * y_1), ((x_2 * z_2) * (0 * y_2))T, \\
&((x_3 * z_3) * (0 * y_3))I, ((x_4 * z_4) * (0 * y_4))F) \\
&\in N_q(A, B),
\end{aligned}$$

that is, $(x_i * z_i) * (0 * y_i) \in A$ and $(x_j * z_j) * (0 * y_j) \in B$ for $i = 1, 2$ and $j = 3, 4$. It follows from (13) that $y_i * x_i \in A$ and $y_j * x_j \in B$ for $i = 1, 2$ and $j = 3, 4$. Thus

$$\tilde{y} \boxdot \tilde{x} = (y_1 * x_1, (y_2 * x_2)T, (y_3 * x_3)I, (y_4 * x_4)F) \in N_q(A, B),$$

and therefore $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$. □

Corollary 3.9. *If A is an a-ideal of a BCI-algebra X, then the neutrosophic quadruple A-set $N_q(A)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.*

Theorem 3.10. *Let A and B be ideals of a BCI-algebra X such that*

$$(\forall x, y \in X)(x * (0 * y) \in A \cap B \;\Rightarrow\; y * x \in A \cap B). \tag{17}$$

Then the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.

Proof. Obviously $\tilde{0} \in N_q(A, B)$. Let $\tilde{x} = (x_1,\ x_2T,\ x_3I,\ x_4F)$, $\tilde{y} = (y_1,\ y_2T,\ y_3I,\ y_4F)$ and $\tilde{z} = (z_1,\ z_2T,\ z_3I,\ z_4F)$ be elements of $N_q(X)$ be such that $(\tilde{x} \boxdot \tilde{z}) \boxdot (\tilde{0} \boxdot \tilde{y}) \in N_q(A, B)$ and $\tilde{z} \in N_q(A, B)$. Then $z_1, z_2 \in A$, $z_3, z_4 \in B$ and

$$\begin{aligned}(\tilde{x} \boxdot \tilde{z}) \boxdot (\tilde{0} \boxdot \tilde{y}) &= ((x_1, x_2T, x_3I, x_4F) \boxdot (z_1, z_2T, z_3I, z_4F)) \boxdot \\ &\quad (\tilde{0} \boxdot (y_1, y_2T, y_3I, y_4F)) \\ &= (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \boxdot \\ &\quad (0 * y_1, (0 * y_2)T, (0 * y_3)I, (0 * y_4)F) \\ &= ((x_1 * z_1) * (0 * y_1), ((x_2 * z_2) * (0 * y_2))T, \\ &\quad ((x_3 * z_3) * (0 * y_3))I, ((x_4 * z_4) * (0 * y_4))F) \\ &\in N_q(A, B),\end{aligned}$$

that is, $(x_i * z_i) * (0 * y_i) \in A$ and $(x_j * z_j) * (0 * y_j) \in B$ for $i = 1, 2$ and $j = 3, 4$. Note that

$$(x_k * (0 * y_k)) * ((x_k * z_k) * (0 * y_k)) \leq x_k * (x_k * z_k) \leq z_k$$

for $k = 1, 2, 3, 4$. Since $z_1, z_2 \in A$ and $z_3, z_4 \in B$, we have $x_i * (0 * y_i) \in A$ and $x_j * (0 * y_j) \in B$ for $i = 1, 2$ and $j = 3, 4$. It follows from (17) that $y_i * x_i \in A$ and $y_j * x_j \in B$ for $i = 1, 2$ and $j = 3, 4$. Hence

$$\begin{aligned}\tilde{y} \boxdot \tilde{x} &= (y_1, y_2T, y_3I, y_4F) \boxdot (x_1, x_2T, x_3I, x_4F) \\ &= (y_1 * x_1, (y_2 * x_2)T, (y_3 * x_3)I, (y_4 * x_4)F) \in N_q(A, B).\end{aligned}$$

Therefore $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$. □

Corollary 3.11. *Let A be an ideal of a BCI-algebra X such that*

$$(\forall x, y \in X)(x * (0 * y) \in A \ \Rightarrow\ y * x \in A). \tag{18}$$

Then the neutrosophic quadruple A-set $N_q(A)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.

Theorem 3.12. *Let A and B be ideals of a BCI-algebra X such that*

$$(\forall x, y, z \in X)((x * z) * (0 * y) \in A \cap B \ \Rightarrow\ y * (x * z) \in A \cap B). \tag{19}$$

Then the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.

Proof. If we put $z = 0$ in (19) and use (1), then we can induce the condition (17). Thus $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$ by Theorem 3.10. □

Corollary 3.13. *Let A be an ideal of a BCI-algebra X such that*

$$(\forall x, y, z \in X)((x * z) * (0 * y) \in A \;\Rightarrow\; y * (x * z) \in A). \tag{20}$$

Then the neutrosophic quadruple A-set $N_q(A)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.

We discuss relations between a neutrosophic quadruple a-ideal, a neutrosophic quadruple p-ideal and a neutrosophic quadruple q-ideal.

Lemma 3.14 ([25])**.** *Let A and B be ideals of X such that*

$$(\forall x \in X)(0 * (0 * x) \in A \text{ (resp., } B) \Rightarrow x \in A \text{ (resp., } B)). \tag{21}$$

Then $N_q(A, B)$ is a neutrosophic quadruple p-ideal of $N_q(X)$.

Theorem 3.15. *For any nonempty subsets A and B of a BCI-algebra X, if the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$, then it is a neutrosophic quadruple p-ideal of $N_q(X)$.*

Proof. Assume that $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$. Then A and B are a-ideals of X (see Proof of Theorem 3.6) and $\tilde{0} \in N_q(A, B)$. For $i = 1, 2$ and $j = 3, 4$, let $x_i, x_j \in X$ be such that $0 * (0 * x_i) \in A$ and $0 * (0 * x_j) \in B$. Then

$$\begin{aligned}
&(\tilde{0} \boxdot \tilde{0}) \boxdot (\tilde{0} \boxdot \tilde{x}) = \tilde{0} \boxdot (\tilde{0} \boxdot \tilde{x}) \\
&= (0 * (0 * x_1), (0 * (0 * x_2))T, (0 * (0 * x_3))I, (0 * (0 * x_4))F) \in N_q(A, B),
\end{aligned}$$

and so

$$\begin{aligned}
(x_1, x_2T, x_3I, x_4F) &= (x_1 * 0, (x_2 * 0)T, (x_3 * 0)I, (x_4 * 0)F) \\
&= (x_1, x_2T, x_3I, x_4F) \boxdot (0, 0T, 0I, 0F) \\
&= \tilde{x} \boxdot \tilde{0} \in N_q(A, B)
\end{aligned}$$

Hence $x_i \in A$ and $x_j \in B$. It follows from Lemma 3.14 that $N_q(A, B)$ is a neutrosophic quadruple p-ideal of $N_q(X)$. □

The following example shows that the converse of Theorem 3.15 is not true in general.

Example 3.16. Consider a BCI-algebra $X = \{0, a, b\}$ with the binary operation $*$, which is given in Table 3.
Then the neutrosophic quadruple BCI-algebra $N_q(X)$ has 81 elements. If we take $A = \{0\}$ and $B = \{0\}$, then $N_q(A, B) = \{\tilde{0}\}$ is a neutrosophic quadruple p-ideal of $N_q(X)$. For two

Table 3. Cayley table for the binary operation "$*$"

$*$	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

elements (a, aT, aI, aF) and (b, bT, bI, bF) of $N_q(X)$, we have

$$\begin{aligned}&((a, aT, aI, aF) \boxdot (0, 0T, 0I, 0F)) \boxdot ((0, 0T, 0I, 0F) \boxdot (b, bT, bI, bF))\\&= (a * 0, (a * 0)T, (a * 0)I, (a * 0)F) \boxdot (0 * b, (0 * b)T, (0 * b)I, (0 * b)F)\\&= (a, aT, aI, aF) \boxdot (a, aT, aI, aF) = \tilde{0} \in N_q(A, B).\end{aligned}$$

But

$$\begin{aligned}(b, bT, bI, bF) \boxdot (a, aT, aI, aF) &= (b * a, (b * a)T, (b * a)I, (b * a)F)\\&= (a, aT, aI, aF) \notin N_q(A, B).\end{aligned}$$

Hence $N_q(A, B)$ is not a neutrosophic quadruple a-ideal of $N_q(X)$.

Lemma 3.17 ([26]). *Let A and B be ideals of a BCI-algebra X such that*

$$(\forall x, y \in X)(x * (0 * y) \in A \cap B \;\Rightarrow\; x * y \in A \cap B). \tag{22}$$

Then the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple q-ideal of $N_q(X)$.

Theorem 3.18. *For any nonempty subsets A and B of a BCI-algebra X, if the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$, then it is a neutrosophic quadruple q-ideal of $N_q(X)$.*

Proof. Assume that $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$. Then A and B are a-ideals of X (see Proof of Theorem 3.6) and $\tilde{0} \in N_q(A, B)$. For $i = 1, 2$ and $j = 3, 4$, let $x_i, y_i, x_j, y_j \in X$ be such that $x_i * (0 * y_i) \in A$ and $x_j * (0 * y_j) \in B$. Since

$$\begin{aligned}&0 * (0 * (y_k * (0 * x_k))) * (x_k * (0 * y_k))\\&= ((0 * (0 * y_k)) * (0 * (0 * (0 * x_k)))) * (x_k * (0 * y_k))\\&= ((0 * (0 * y_k)) * (0 * x_k)) * (x_k * (0 * y_k))\\&\leq (x_k * (0 * y_k)) * (x_k * (0 * y_k)) = 0 \in A \cap B\end{aligned}$$

for $k = 1, 2, 3, 4$, we have $0 * (0 * (y_i * (0 * x_i))) \in A$ and $0 * (0 * (y_j * (0 * x_j))) \in B$. Since every a-ideal is a p-ideal, it follows from (14) that $y_i * (0 * x_i) \in A$ and $y_j * (0 * x_j) \in B$. Thus

$$\begin{aligned}\tilde{y} \boxdot (\tilde{0} \boxdot \tilde{x}) &= (y_1, y_2T, y_3I, y_4F) \boxdot ((0, 0T, 0I, 0F) \boxdot (x_1, x_2T, x_3I, x_4F)) \\ &= (y_1, y_2T, y_3I, y_4F) \boxdot (0 * x_1, (0 * x_2)T, (0 * x_3)I, (0 * x_4)F) \\ &= (y_1 * (0 * x_1), (y_2 * (0 * x_2))T, (y_3 * (0 * x_3))I, (y_4 * (0 * x_4))F) \\ &\in N_q(A, B),\end{aligned}$$

which implies from (16) that

$$\begin{aligned}&(x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \\ &= (x_1, x_2T, x_3I, x_4F) \boxdot (y_1, y_2T, y_3I, y_4F) \\ &= \tilde{x} \boxdot \tilde{y} \in N_q(A, B),\end{aligned}$$

that is, $x_i * y_i \in A$ and $x_j * y_j \in B$ for $i = 1, 2$ and $j = 3, 4$. Using Lemma 3.17, we know that $N_q(A, B)$ is a neutrosophic quadruple q-ideal of $N_q(X)$. □

Corollary 3.19. *For any nonempty subset A of a BCI-algebra X, if the neutrosophic quadruple A-set $N_q(A)$ is a neutrosophic quadruple a-ideal of $N_q(X)$, then it is a neutrosophic quadruple q-ideal of $N_q(X)$.*

Consider the neutrosophic quadruple BCI-algebra $N_q(X)$ in Example 3.7. If we take $A = \{0\}$ and $B = \{0, 1\}$, then $N_q(A, B) = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}\}$, where $\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{1} = (0, 0T, 0I, 1F)$, $\tilde{2} = (0, 0T, 1I, 0F)$ and $\tilde{3} = (0, 0T, 1I, 1F)$, is a neutrosophic quadruple q-ideal of $N_q(X)$. But it is not a neutrosophic quadruple a-ideal of $N_q(X)$ since

$$(\tilde{0} \boxdot \tilde{0}) \boxdot (\tilde{0} \boxdot (1, 0T, 1I, 0F)) = \tilde{0} \in N_q(A, B)$$

and $(1, 0T, 1I, 0F) \boxdot \tilde{0} = (1, 0T, 1I, 0F) \notin N_q(A, B)$. This shows that the converse of Theorem 3.18 is not be true in general.

Lemma 3.20 ([26]). *For any nonempty subsets A and B of a BCI-algebra X, if the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple q-ideal of $N_q(X)$, then it is both a neutrosophic quadruple subalgebra and a neutrosophic quadruple ideal of $N_q(X)$.*

Theorem 3.21. *Given nonempty subsets A and B of a BCI-algebra X, the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$ if and only if $N_q(A, B)$ is both a neutrosophic quadruple p-ideal and a neutrosophic quadruple q-ideal of $N_q(X)$.*

Proof. If $N_q(A,B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$, then $N_q(A,B)$ is both a neutrosophic quadruple p-ideal and a neutrosophic quadruple q-ideal of $N_q(X)$ by Theorems 3.15 and 3.18.

Conversely, suppose that $N_q(A,B)$ is both a neutrosophic quadruple p-ideal and a neutrosophic quadruple q-ideal of $N_q(X)$. Then $N_q(A,B)$ is a neutrosophic quadruple subalgebra of $N_q(X)$ by Lemma 3.20, and A and B are both a p-ideal and a q-ideal of X. For $i = 1,2$ and $j = 3,4$, let $x_i * (0 * y_i) \in A$ and $x_j * (0 * y_j) \in B$ for $x_i, y_i, x_j, y_j \in X$. Then $x_i * y_i \in A$ and $x_j * y_j \in B$ since A and B are q-ideals of X. Recall that

$$\begin{aligned}(0 * (y_k * x_k)) * (x_k * y_k) &= ((0 * y_k) * (0 * x_k)) * (x_k * y_k) \\ &= ((0 * (x_k * y_k)) * y_k) * (0 * x_k) \\ &= (((0 * x_k) * (0 * y_k)) * y_k) * (0 * x_k) \\ &= (0 * (0 * y_k)) * y_k = 0 \in A \cap B\end{aligned}$$

for $k = 1,2,3,4$. Hence $0 * (y_i * x_i) \in A$ and $0 * (y_j * x_j) \in B$, and so $0 * (0 * (y_i * x_i)) \in A$ and $0 * (0 * (y_j * x_j)) \in B$. Since A and B are p-ideals of X, it follows from (14) that $y_i * x_i \in A$ and $y_j * x_j \in B$. Therefore $N_q(A,B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$ by Theorem 3.10. □

Lemma 3.22 ([26]). *Let A, B, I and J be ideals of a BCI-algebra X such that $I \subseteq A$ and $J \subseteq B$. If I and J are q-ideals of X, then the neutrosophic quadruple (A,B)-set $N_q(A,B)$ is a neutrosophic quadruple q-ideal of $N_q(X)$.*

Lemma 3.23 ([]). *If A and B are p-ideals of a BCI-algebra X, then the neutrosophic quadruple (A,B)-set $N_q(A,B)$ is a neutrosophic quadruple p-ideal of $N_q(X)$.*

Theorem 3.24. *Let A, B, I and J be ideals of a BCI-algebra X such that $I \subseteq A$ and $J \subseteq B$. If I and J are a-ideals of X, then the neutrosophic quadruple (A,B)-set $N_q(A,B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.*

Proof. Assume that I and J are a-ideals of X. Then I and J are both p-ideals and q-ideals of X. Thus neutrosophic quadruple (A,B)-set $N_q(A,B)$ is a neutrosophic quadruple q-ideal of $N_q(X)$ by Lemma 3.22. Let $0 * (0 * x) \in A \cap B$ for $x \in X$. Then

$$(0 * (0 * (0 * x))) * (0 * x) = (0 * (0 * x)) * (0 * (0 * x)) = 0 \in I \cap J.$$

Since

$$
\begin{aligned}
&(0*(0*(x*(0*(0*x)))))*((0*(0*(0*x)))*(0*x))\\
&=((0*(0*x))*(0*(0*(0*(0*x)))))*((0*(0*(0*x)))*(0*x))\\
&\leq ((0*(0*(0*x)))*(0*x))*((0*(0*(0*x)))*(0*x))\\
&=0\in I\cap J,
\end{aligned}
$$

it follows that $0*(0*(x*(0*(0*x))))\in I\cap J$. Since I and J are p-ideals of X, we have $x*(0*(0*x))\in I\cap J\subseteq A\cap B$ by (14), and so $x\in A\cap B$. This shows that A and B are p-ideals of X, and thus $N_q(A,B)$ is a neutrosophic quadruple p-ideal of $N_q(X)$ by Lemma 3.23. Therefore $N_q(A,B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$ by Theorem 3.21. □

Corollary 3.25. *Let A and I be ideals of a BCI-algebra X such that $I\subseteq A$. If I is an a-ideal of X, then the neutrosophic quadruple A-set $N_q(A)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.*

Corollary 3.26. *If the zero ideal $\{0\}$ is an a-ideal of a BCI-algebra X, then the neutrosophic quadruple (A,B)-set $N_q(A,B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$ for every ideals A and B of X.*

Theorem 3.27. *Let A,B,I and J be ideals of a BCI-algebra X such that $I\subseteq A$, $J\subseteq B$ and*

$$(\forall x,y\in X)(x*(0*y)\in I\cap J\ \Rightarrow\ y*x\in I\cap J). \tag{23}$$

Then the neutrosophic quadruple (A,B)-set $N_q(A,B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.

Proof. Let $x,y,z\in X$ be such that $(x*z)*(0*y)\in I\cap J$ and $z\in I\cap J$. Note that

$$(x*(0*y))*((x*z)*(0*y))\leq x*(x*z)\leq z\in I\cap J.$$

Hence $x*(0*y)\in I\cap J$, and so $y*x\in I\cap J$ by (23). Thus I and J are a-ideals of X. It follows from Theorem 3.24 that $N_q(A,B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$. □

Corollary 3.28. *Let A and I be ideals of a BCI-algebra X such that $I\subseteq A$ and*

$$(\forall x,y\in X)(x*(0*y)\in I\ \Rightarrow\ y*x\in I). \tag{24}$$

Then the neutrosophic quadruple A-set $N_q(A)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.

Theorem 3.29. *Let A, B, I and J be ideals of a BCI-algebra X such that $I \subseteq A$, $J \subseteq B$ and*

$$(\forall x, y, z \in X)((x * z) * (0 * y) \in I \cap J \;\Rightarrow\; y * (x * z) \in I \cap J). \tag{25}$$

Then the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.

Proof. If we put $z = 0$ in (25) and use (1), then (23) is valid. Therefore $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$ by Theorem 3.27. □

Corollary 3.30. *Let A and I be ideals of a BCI-algebra X such that $I \subseteq A$ and*

$$(\forall x, y, z \in X)((x * z) * (0 * y) \in I \;\Rightarrow\; y * (x * z) \in I). \tag{26}$$

Then the neutrosophic quadruple A-set $N_q(A)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.

4. **Conclusions**

We have applied the notion of neutrosophic quadruple set to an a-ideal in a BCI-algebra. We have introduced the concept of neutrosophic quadruple a-ideal of neutrosophic quadruple BCI-algebras, and have investigated several properties. The notions of neutrosophic quadruple p-ideal, neutrosophic quadruple q-ideal and neutrosophic quadruple closed ideal have been introduced by Smarandache, Muhiuddin, Al-Kenani, Jun, etc. We have discussed relations between a neutrosophic quadruple p-ideal, a neutrosophic quadruple q-ideal, a neutrosophic quadruple a-ideal and a neutrosophic quadruple closed ideal. We have provided conditions for the neutrosophic quadruple (A, B)-set $N_q(A, B)$ to be a neutrosophic quadruple a-ideal. We have shown that every neutrosophic quadruple a-ideal is a neutrosophic quadruple closed ideal, and heve provided example to show that the converse is false. Using the ideas and results of this paper, we will study the structure of various algebraic systems in the future.

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