Neutrosophic Sets and Systems, Vol. 68, 2024
University of New Mexico

The limits of 2- refined neutrosophic

Yaser Ahmad Alhasan[1,*], Mohamed Elghazali Ali Mohieldin Mohamed[2] and Raja Abdullah Abdulfatah[3]

[1]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia.;y.alhasan@psau.edu.sa
[2]Deanship the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj Saudi Arabia.;mo.mohammed@psau.edu.sa
[3]Deanship the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia.; r.abdulfatah@psau.edu.sa

*Corresponding author: y.alhasan@psau.edu.sa

Abstract: this paper aims to present the limits of 2- refined neutrosophic, where we studied the of the neutrosophic factorization method and the neutrosophic rationalization method of the limits of 2- refined neutrosophic, we verified the results of these methods using the L'Hôpital's rule. Also We introduced some special limits and 2- refined neutrosophic trigonometric limits. In addition to clarifying this by solving appropriate numerical examples.

Keywords: indeterminacy; trigonometric; neutrosophic; factorization; limits.

---

1. Introduction and Preliminaries

To describe a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, and contradiction, Smarandache suggested the neutrosophic Logic as an alternative to the current logics. Smarandache made refined neutrosophic numbers available in the following form: $(a, b_1I_1, b_2I_2, \dots, b_nI_n)$ where $a, b_1, b_2, \dots, b_n \in R$ or C [1]

Agboola introduced the concept of refined neutrosophic algebraic structures [2]. Also, the refined neutrosophic rings I was studied in paper [3], where it assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))].

In addition, there are many papers presenting studies on refined neutrosophic numbers [4-5-6-7-8-9-10].

Alhasan and Abdulfatah also presented the division of refined neutrosophic numbers [11], where:

$$\frac{a_1 + b_1I_1 + c_1I_2}{a_2 + b_2I_1 + c_2I_2} \equiv \frac{a_1}{a_2} + \left[\frac{{a_2}^2b_1 + a_2b_1c_2 - a_1a_2b_2 - a_2b_2c_1}{a_2(a_2 + c_2)(a_2 + b_2 + c_2)}\right]I_1 + \left[\frac{a_2c_1 - a_1c_2}{a_2(a_2 + c_2)}\right]I_2$$

where: $a_2 \neq 0$, $a_2 \neq -c_2$ and $a_2 \neq -b_2 - c_2$

This paper addressed many topics, following the introduction and preliminary material presented in the first part, the limits of 2-refined neutrosophic were discussed in the main discussion section. The final section contained the conclusion.

2. Main Discussion

2.1 The neutrosophic factorization method of the limits of 2- refined neutrosophic

Let $\frac{f(x,I_1,I_2)}{g(x,I_1,I_2)}$ is rational 2- refined neutrosophic function, if $f(x,I_1,\ I_2), g(x,I_1,\ I_2)$ contains some common factors, then we can eliminate out the common factors from the numerator and denominator and after that we calculate the limit.

Example 1

Evaluate:

$$\lim_{x\to 3+I_1+I_2} \frac{x-3-I_1-I_2}{x^2-9+9I_1+7I_2}$$

Solution:

$$\lim_{x\to 3+I_1+I_2} \frac{x-3-I_1-I_2}{x^2-9+9I_1+7I_2} = \frac{0}{0}$$

Method1:

$$x^2-9+9I_1+7I_2 = (x+3+I_1+I_2)(x-3-I_1-I_2)$$

$$\lim_{x\to 3+I_1+I_2} \frac{x-3-I_1-I_2}{x^2-9+9I_1+7I_2} = \lim_{x\to 3+I_1+I_2} \frac{x-3-I_1-I_2}{(x+3+I_1+I_2)(x-3-I_1-I_2)}$$

$$= \lim_{x\to 3+I_1+I_2} \frac{1}{x+3+I_1+I_2} = \frac{1}{6+2I_1+2I_2} = \frac{1}{6} - \frac{1}{40}I_1 - \frac{1}{24}I_2$$

Method2:

by using L'Hôpital's rule

$$\Rightarrow \quad \lim_{x\to 3+I_1+I_2} \frac{x-3-I_1-I_2}{x^2-9+9I_1+7I_2} = \lim_{x\to 3+I_1+I_2} \frac{1}{2x}$$

$$= \frac{1}{2(3+I_1+I_2)} = \frac{1}{6+2I_1+2I_2} = \frac{1}{6} - \frac{1}{40}I_1 - \frac{1}{24}I_2$$

2.2 The neutrosophic rationalization method of the limits of 2- refined neutrosophic

Example 2

Evaluate:

$$\lim_{x\to 0+0I_1+0I_2} \frac{\sqrt{1-(1+I_1+2I_2)x} - \sqrt{1+(1+I_1+2I_2)x}}{(2+3I_1-I_2)x}$$

Solution:

$$\lim_{x\to 0+0I_1+0I_2} \frac{\sqrt{1-(1+I_1+2I_2)x} - \sqrt{1+(1+I_1+2I_2)x}}{(2+3I_1-I_2)x} = \frac{0}{0}$$

Method1:

$$\Rightarrow \quad \lim_{x\to 0+0I_1+0I_2} \frac{\sqrt{1-(1+I_1+2I_2)x}-\sqrt{1+(1+I_1+2I_2)x}}{(2+3I_1-I_2)x}$$

$$= \lim_{x\to 0+0I_1+0I_2} \frac{\left(\sqrt{1-(1+I_1+2I_2)x}-\sqrt{1+(1+I_1+2I_2)x}\right)\left(\sqrt{1-(1+I_1+2I_2)x}+\sqrt{1+(1+I_1+2I_2)x}\right)}{(2+3I_1-I_2)x\left(\sqrt{1-(1+I_1+2I_2)x}+\sqrt{1+(1+I_1+2I_2)x}\right)}$$

$$= \lim_{x\to 0+0I_1+0I_2} \frac{1-(1+I_1+2I_2)x-[1+(1+I_1+2I_2)x]}{(2+3I_1-I_2)x\left(\sqrt{1-(1+I_1+2I_2)x}+\sqrt{1+(1+I_1+2I_2)x}\right)}$$

$$= \lim_{x\to 0+0I_1+0I_2} \frac{-(2+2I_1+4I_2)x}{(2+3I_1-I_2)x\left(\sqrt{1-(1+I_1+2I_2)x}+\sqrt{1+(1+I_1+2I_2)x}\right)}$$

$$= \lim_{x\to 0+0I_1+0I_2} \frac{-(2+2I_1+4I_2)}{(2+3I_1-I_2)\left(\sqrt{1-(1+I_1+2I_2)x}+\sqrt{1+(1+I_1+2I_2)x}\right)}$$

$$= \frac{-1-I_1-2I_2}{2+3I_1-I_2} = -\frac{1}{2}+2I_1-\frac{5}{2}I_2$$

Method2:

by using L'Hôpital's rule

$$\Rightarrow \quad \lim_{x\to 0+0I_1+0I_2} \frac{\sqrt{1-(1+I_1+2I_2)x}-\sqrt{1+(1+I_1+2I_2)x}}{(2+3I_1-I_2)x}$$

$$= \lim_{x\to 0+0I_1+0I_2} \frac{\dfrac{-(1+I_1+2I_2)}{2\sqrt{1-(1+I_1+2I_2)x}}-\dfrac{(1+I_1+2I_2)}{2\sqrt{1+(1+I_1+2I_2)x}}}{2+3I_1-I_2}$$

$$= \frac{\dfrac{-(1+I_1+2I_2)}{2\sqrt{1-0}}-\dfrac{(1+I_1+2I_2)}{2\sqrt{1+0}}}{2+3I_1-I_2}$$

$$= \frac{-1-I_1-2I_2}{2+3I_1-I_2} = -\frac{1}{2}+2I_1-\frac{5}{2}I_2$$

Example 3

Evaluate:

$$\lim_{x\to 6+2I_1-3I_2} \frac{1-\sqrt{x-5-2I_1+3I_2}}{x-5-2I_1+3I_2}$$

Solution:

$$\lim_{x\to 6+2I_1-3I_2} \frac{1-\sqrt{x-5-2I_1+3I_2}}{x-5-2I_1+3I_2} = \frac{0}{0}$$

Method1:

$$\Rightarrow \quad \lim_{x \to 6+2I_1-3I_2} \frac{\left(1-\sqrt{x-5-2I_1+3I_2}\right)\left(1+\sqrt{x-5-2I_1+3I_2}\right)}{(x-5-2I_1+3I_2)\left(1+\sqrt{x-5-2I_1+3I_2}\right)}$$

$$= \lim_{x \to 6+2I_1-3I_2} \frac{1-(x-5-2I_1+3I_2)}{(x-5-2I_1+3I_2)\left(1+\sqrt{x-5-2I_1+3I_2}\right)}$$

$$= \lim_{x \to 6+2I_1-3I_2} \frac{-x+5+2I_1-3I_2}{(x-5+I)\left(1+\sqrt{x-5-2I_1+3I_2}\right)}$$

$$= \lim_{x \to 6+2I_1-3I_2} \frac{-(x-5-2I_1+3I_2)}{(x-5-2I_1+3I_2)\left(1+\sqrt{x-5-2I_1+3I_2}\right)}$$

$$= \lim_{x \to 6+2I_1-3I_2} \frac{-1}{\left(1+\sqrt{x-5-2I_1+3I_2}\right)} = \frac{-1}{2}$$

Method2:

by using L'Hôpital's rule

$$\Rightarrow \quad \lim_{x \to 6+2I_1-3I_2} \frac{1-\sqrt{x-5-2I_1+3I_2}}{x-5-2I_1+3I_2}$$

$$= \lim_{x \to 6+2I_1-3I_2} \frac{\frac{-1}{2\sqrt{x-5-2I_1+3I_2}}}{1}$$

$$= \lim_{x \to 6+2I_1-3I_2} \frac{-1}{2\sqrt{x-5-2I_1+3I_2}} = \frac{-1}{2}$$

Example 4

Evaluate:

$$\lim_{x \to a+bI_1+cI_2} \frac{\sqrt{a+bI_1+cI_2+2x}-\sqrt{3x}}{\sqrt{15a+15bI_1+15cI_2+x}-4\sqrt{x}}$$

Solution:

$$\lim_{x \to a+bI_1+cI_2} \frac{\sqrt{a+bI_1+cI_2+2x}-\sqrt{3x}}{\sqrt{15a+15bI_1+15cI_2+x}-4\sqrt{x}} = \frac{0}{0}$$

$$\Rightarrow \quad \lim_{x \to a+bI_1+cI_2} \frac{\sqrt{a+bI_1+cI_2+2x}-\sqrt{3x}}{\sqrt{15a+15bI_1+15cI_2+x}-4\sqrt{x}}$$

$$= \lim_{x \to a+bI_1+cI_2} \frac{\left(\sqrt{a+bI_1+cI_2+2x}-\sqrt{3x}\right)\left(\sqrt{a+bI_1+cI_2+2x}+\sqrt{3x}\right)}{\left(\sqrt{15a+15bI_1+15cI_2+x}-4\sqrt{x}\right)\left(\sqrt{a+bI_1+cI_2+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x \to a+bI_1+cI_2} \frac{a+bI_1+cI_2+2x-3x}{\left(\sqrt{15a+15bI_1+15cI_2+x}-4\sqrt{x}\right)\left(\sqrt{a+bI_1+cI_2+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x \to a+bI_1+cI_2} \frac{a + bI_1 + cI_2 - x}{\left(\sqrt{15a + 15bI_1 + 15cI_2 + x} - 4\sqrt{x}\right)\left(\sqrt{a + bI_1 + cI_2 + 2x} + \sqrt{3x}\right)} = \frac{0}{0}$$

then:

$$\lim_{x \to a+bI_1+cI_2} \frac{a + bI_1 + cI_2 - x}{\left(\sqrt{15a + 15bI_1 + 15cI_2 + x} - 4\sqrt{x}\right)\left(\sqrt{a + bI_1 + cI_2 + 2x} + \sqrt{3x}\right)} \frac{\sqrt{15a + 15bI_1 + 15cI_2 + x} + 4\sqrt{x}}{\sqrt{15a + 15bI_1 + 15cI_2 + x} + 4\sqrt{x}}$$

$$= \lim_{x \to a+bI_1+cI_2} \frac{(a + bI_1 + cI_2 - x)\left(\sqrt{15a + 15bI_1 + 15cI_2 + x} + 4\sqrt{x}\right)}{(15a + 15bI_1 + 15cI_2 + x - 16x)\left(\sqrt{a + bI_1 + cI_2 + 2x} + \sqrt{3x}\right)}$$

$$= \lim_{x \to a+bI_1+cI_2} \frac{(a + bI_1 + cI_2 - x)\left(\sqrt{15a + 15bI_1 + 15cI_2 + x} + 4\sqrt{x}\right)}{15(a + bI_1 + cI_2 - x)\left(\sqrt{a + bI_1 + cI_2 + 2x} + \sqrt{3x}\right)}$$

$$= \lim_{x \to a+bI_1+cI_2} \frac{\left(\sqrt{15a + 15bI_1 + 15cI_2 + x} + 4\sqrt{x}\right)}{15\left(\sqrt{a + bI + 2x} + \sqrt{3x}\right)}$$

$$= \frac{\left(\sqrt{15a + 15bI_1 + 15cI_2 + a + bI_1 + cI_2} + 4\sqrt{a + bI_1 + cI_2}\right)}{15\left(\sqrt{a + bI_1 + cI_2 + 2(a + bI_1 + cI_2)} + \sqrt{3(a + bI_1 + cI_2)}\right)}$$

$$= \frac{\sqrt{16(a + bI_1 + cI_2)} + 2\sqrt{a + bI_1 + cI_2}}{15\left(\sqrt{3(a + bI_1 + cI_2)} + \sqrt{3(a + bI_1 + cI_2)}\right)} = \frac{4\sqrt{a + bI_1 + cI_2}}{30\sqrt{3(a + bI_1 + cI_2)}}$$

$$= \frac{2\sqrt{a + bI_1 + cI_2}}{15\sqrt{3(a + bI_1 + cI_2)}} = \frac{2}{15\sqrt{3}}$$

2.3 2- Refined neutrosophic trigonometric limits

1) $\lim_{x \to 0} sin(a + bI_1 + cI_2)\, x = 0$

2) $\lim_{x \to 0} cos(a + bI_1 + cI_2)\, x = 1$

3) $\lim_{x \to 0} \frac{sin(a + bI_1 + cI_2)x}{x} = a + bI_1 + cI_2$

Proof (3):

Put $(a + bI_1 + cI_2)x = y$ $\Rightarrow$ $x = \frac{1}{a+bI_1+cI_2} y$

When $x \longrightarrow 0$ then: $y \longrightarrow 0$

$$\Rightarrow \quad \lim_{x \to 0} \frac{sin(a + bI_1 + cI_2)\, x}{x} = \lim_{y \to 0} \frac{siny}{\frac{1}{a + bI_1 + cI_2} y}$$

$$= (a + bI_1 + cI_2) \lim_{y \to 0} \frac{siny}{y} = a + bI_1 + cI_2$$

4) $\lim_{x\to 0}\frac{x}{\sin(a+bI_1+cI_2)\,x}=\frac{1}{a+bI_1+cI_2}$

$$=\frac{1}{a}+\left[\frac{-b}{(a+c)(a+b+c)}\right]I_1-\left[\frac{c}{a(a+c)}\right]I_2$$

Where a ,b,c are real coefficients, $a\neq 0$, $a\neq -c$ and $a\neq -b-c$

Proof (4):

Put $(a+bI_1+cI_2)x=y$ $\Rightarrow$ $x=\frac{1}{a+bI_1+cI_2}y$

When $x\longrightarrow 0$ then: $y\longrightarrow 0$

$$\Rightarrow \lim_{x\to 0}\frac{x}{sin(a+bI_1+cI_2)\,x}=\lim_{y\to 0}\frac{\frac{1}{a+bI_1+cI_2}y}{siny}$$

$$=\frac{1}{a+bI_1+cI_2}\lim_{y\to 0}\frac{y}{siny}$$

$$=\frac{1}{a+bI_1+cI_2}$$

$$=\frac{1}{a}+\left[\frac{-b}{(a+c)(a+b+c)}\right]I_1-\left[\frac{c}{a(a+c)}\right]I_2$$

5) $\lim_{x\to 0}\frac{tan(a+bI_1+cI_2)x}{x}=a+bI_1+cI_2$

6) $\lim_{x\to 0}\frac{x}{tan(a+bI_1+cI_2)x}=\frac{1}{a+bI_1+cI_2}=\frac{1}{a}+\left[\frac{-b}{(a+c)(a+b+c)}\right]I_1-\left[\frac{c}{a(a+c)}\right]I_2$

where a ,b,c are real coefficients, $a\neq 0$, $a\neq -c$ and $a\neq -b-c$. We can prove 5 and 6 by the same method in 3, 4

Example 5

1) $\lim_{x\to 0+0I_1+0I_2}\frac{sin(2+I_1+3I_2)\,x}{(1-4I_1+I_2)x}=\frac{2+I_1+3I_2}{1-4I_1+I_2}\lim_{x\to 0+0I_1+0I_2}\frac{sin(2+I_1+3I_2)\,x}{(2+I_1+3I_2)x}$

$$=\frac{2+I_1+3I_2}{1-4I_1+I_2}=2-\frac{11}{2}I_1+\frac{1}{2}I_2$$

2) $\lim_{x\to 0+0I_1+0I_2}\frac{x}{sin(1+5I_1-4I_2)\,x}=\frac{1}{1+5I_1-4I_2}\lim_{x\to 0+0I_1+0I_2}\frac{(1+5I_1-4I_2)x}{sin(1+5I_1-4I_2x}$

$$=\frac{1}{1+5I_1-4I_2}=1+\frac{5}{6}I_1-\frac{4}{3}I_2$$

3) $$\lim_{x\to 0+0I_1+0I_2}\frac{\sin(3+4I_1-4I_2)\,x}{\tan(2-8I_1-4I_2)\,x}=\lim_{x\to 0+0I}\frac{\frac{\sin(3+4I_1-4I_2)\,x}{x}}{\frac{\tan(2-8I_1-4I_2)\,x}{x}}$$

$$=\frac{\lim_{x\to 0+0I_1+0I_2}\frac{\sin(3+4I_1-4I_2)\,x}{x}}{\lim_{x\to 0+0I_1+0I_2}\frac{\tan(2-8I_1-4I_2)\,x}{x}}$$

$$=\frac{3+4I_1-4I_2}{2-8I_1-4I_2}=\frac{3}{2}+4I_1-I_2$$

4) $$\lim_{x\to 0+0I_1+0I_2}\frac{1-\cos(1+4I_1-I_2)\,x}{x^2}=\lim_{x\to 0+0I_1+0I_2}\frac{2sin^2(1+4I_1-I_2)x}{x^2}$$

$$=2\lim_{x\to 0+0I_1+0I_2}\left(\frac{sin(1+4I_1-I_2)x}{x}\right)^2$$

$$=2(1+4I_1-I_2)^2=2+32I_1-2I_2$$

5) $$\lim_{x\to 0+0I_1+0I_2}\frac{(2+I_1+I_2)x-sin(1+I_1+I_2)\,x}{(2+I_1+2I_2)x}=\lim_{x\to 0+0I_1+0I_2}\left(\frac{(2+I_1+I_2)x}{(2+I_1+2I_2)x}-\frac{sin(1+I_1+I_2)\,x}{(2+I_1+2I_2)x}\right)$$

$$=\lim_{x\to 0+0I_1+0I_2}\left(\frac{2+I_1+I_2}{2+I_1+2I_2}-\frac{sin(1+I_1+I_2)x}{(2+I_1+2I_2)x}\right)$$

$$=\frac{2+I_1+I_2}{2+I_1+2I_2}-\left(\frac{1+I_1+I_2}{2+I_1+2I_2}\right)$$

$$=1+\frac{1}{20}I_1-\frac{1}{4}I_2-\left(\frac{1}{2}+\frac{1}{10}I_1+0I_2\right)$$

$$=\frac{1}{2}+\frac{3}{20}I_1-\frac{1}{4}I_2$$

2.4 Some special limits

1) $$\lim_{x\to 0+0I_1+0I_2} e^{(a+bI_1+cI_2)x}=1$$

2) $$\lim_{x\to 0+0I_1+0I_2}\frac{e^{(a+bI_1+cI_2)x}-1}{x}=a+bI_1+cI_2$$

Proof (2):

put $(a+bI_1+cI_2)x=y$ $\Longrightarrow$ $x=\frac{1}{a+bI_1+cI_2}y$

when $x\longrightarrow 0+0I_1+0I_2$ then: $y\longrightarrow 0+0I_1+0I_2$

$$\Longrightarrow \lim_{x\to 0+0I_1+0I_2}\frac{e^{(a+bI_1+cI_2)x}-1}{x}=\lim_{x\to 0+0I_1+0I_2}\frac{e^y-1}{\frac{1}{a+bI_1+cI_2}y}$$

$$= (a + bI_1 + cI_2) \lim_{x\to 0+0I_1+0I_2} \frac{e^y - 1}{y} = (a + bI_1 + cI_2)(1) = a + bI_1 + cI_2$$

3) $\lim_{x\to 0+0I_1+0I_2} \frac{ln(1 + (a + bI_1 + cI_2)x)}{x} = a + bI_1 + cI_2$

Proof (3):

Put $(a + bI_1 + cI_2)x = y \quad \Rightarrow \quad x = \frac{1}{a+bI_1+cI_2} y$

When $x \longrightarrow 0 + 0I_1 + 0I_2$ then: $y \longrightarrow 0 + 0I_1 + 0I_2$

$$\Rightarrow \lim_{x\to 0+0I_1+0I_2} \frac{ln(1 + (a + bI_1 + cI_2)x)}{x} = \lim_{x\to 0+0I_1+0I_2} \lim_{y\to 0+0I} \frac{ln(1 + y)}{\frac{1}{a + bI_1 + cI_2} y}$$

$$= (a + bI_1 + cI_2) \lim_{x\to 0+0I_1+0I_2} \frac{ln(1 + y)}{y} = (a + bI_1 + cI_2)(1)$$

$$= a + bI_1 + cI_2$$

4) $\lim_{x\to 0+0I_1+0I_2} \frac{(a + bI_1 + cI_2)^x - 1}{x} = ln(a + bI_1 + cI_2)$

$$= ln\, a + [ln(a + b + c) - ln(a + c)]I_1 + [ln(a + c) - ln\, a]I_2$$

where: $a > 0$, $a + b > 0$, $a + b + c > 0$

Proof (4):

Put $(a + bI_1 + cI_2)^x - 1 = y \quad \Rightarrow \quad (a + bI_1 + cI_2)^x = y + 1$

$$ln(a + bI_1 + cI_2)^x = ln(1 + y)$$

$$x \; ln(a + bI_1 + cI_2) = ln(1 + y)$$

$$x = \frac{1}{ln(a + bI_1 + cI_2)} ln(1 + y)$$

When $x \longrightarrow 0 + 0I_1 + 0I_2$ then: $y \longrightarrow 0 + 0I_1 + 0I_2$

$$\Rightarrow \lim_{x\to 0+0I_1+0I_2} \frac{(a + bI_1 + cI_2)^x - 1}{x} = \lim_{x\to 0+0I_1+0I_2} \frac{y}{\frac{1}{ln(a + bI_1 + cI_2)} ln(1 + y)}$$

$$= ln(a + bI_1 + cI_2) \lim_{x\to 0+0I_1+0I_2} \frac{y}{ln(1 + y)} = ln(a + bI_1 + cI_2) \;(1) = ln(a + bI_1 + cI_2)$$

$$= ln\, a + [ln(a + b + c) - ln(a + c)]I_1 + [ln(a + c) - ln\, a]I_2$$

Corollary 1

$$\lim_{x\to 0+0I_1+0I_2} \frac{(a+bI_1+cI_2)^x-1}{(r+sI_1+tI_2)^x-1} = \frac{ln(a+bI_1+cI_2)}{ln(r+sI_1+tI_2)}$$

$$= \frac{ln\, a + [ln(a+b+c) - ln(a+c)]I_1 + [ln(a+c) - ln\, a]I_2}{ln\, r + [ln(r+s+t) - ln(r+t)]I_1 + [ln(r+t) - ln\, r]I_2}$$

where: $a > 0$, $a+b > 0$, $a+b+c > 0$ *and* $r > 0$, $r+s > 0$, $r+s+t > 0$

Proof:

$$\lim_{x\to 0+0I_1+0I_2} \frac{\frac{(a+bI_1+cI_2)^x-1}{x}}{\frac{(r+sI_1+tI_2)^x-1}{x}} = \frac{\lim_{x\to 0+0I_1+0I_2} \frac{(a+bI_1+cI_2)^x-1}{x}}{\lim_{x\to 0+0I_1+0I_2} \frac{(r+sI_1+tI_2)^x-1}{x}}$$

$$= \frac{ln(a+bI_1+cI_2)}{ln(r+sI_1+tI_2)}$$

$$= \frac{ln\, a + [ln(a+b+c) - ln(a+c)]I_1 + [ln(a+c) - ln\, a]I_2}{ln\, r + [ln(r+s+t) - ln(r+t)]I_1 + [ln(r+t) - ln\, r]I_2}$$

Example 6

1) $\lim_{x\to 0+0I_1+0I_2} e^{(9+12I_1-15I_2)x} = 1$

2) $\lim_{x\to 0+0I_1+0I_2} \frac{e^{(-5+13I_1-I_2)x}-1}{x} = -5+13I_1-I_2$

3) $\lim_{x\to 0+0I_1+0I_2} \frac{(2+4I_1+7I_2)^x-1}{x} = ln(2+4I_1+7I_2)$

$$= ln\, 2 + [ln\, 13 - ln\, 6]I_1 + [ln\, 9 - ln\, 2]I_2$$

$$= ln\, 2 + [ln\, 13 - ln\, 6]I_1 + [2\, ln\, 3 - ln\, 2]I_2$$

4) $\lim_{x\to 0+0I_1+0I_2} \frac{(1+I_1+I_2)^x-1}{e^{(1+2I_1+3I_2)x}-1} = \lim_{x\to 0+0I_1+0I_2} \frac{\frac{(1+I_1+I_2)^x-1}{x}}{\frac{e^{(1+2I_1+3I_2)x}-1}{x}}$

$$= \frac{\lim_{x\to 0+0I_1+0I_2} \frac{(1+I_1+I_2)^x-1}{x}}{\lim_{x\to 0+0I_1+0I_2} \frac{e^{(1+2I_1+3I_2)x}-1}{x}} = \frac{ln(1+I_1+I_2)}{1+2I_1+3I_2}$$

$$= \left(1 - \frac{1}{12}I_1 - \frac{3}{4}I_2\right)([ln\, 3 - ln\, 2]I_1 + [ln\, 2]I_2)$$

5) $\lim_{x\to 0+0I_1+0I_2} \frac{(2+3I_1+4I_2)^x-1}{(2+I_1+I_2)^x-1} = \frac{ln(2+3I_1+4I_2)}{ln(2+I_1+I_2)} = \frac{ln\, 2 + [ln\, 9 - ln\, 6]I_1 + [ln\, 6 - ln\, 2]I_2}{ln\, 2 + [ln\, 4 - ln\, 3]I_1 + [ln\, 3 - ln\, 2]I_2}$

$$= \frac{ln\, 2 + \left[ln\frac{3}{2}\right]I_1 + [ln\, 3]I_2}{ln\, 2 + \left[ln\frac{4}{3}\right]I_1 + \left[ln\frac{3}{2}\right]I_2}$$

$$= 1 + \left[\frac{ln\,3 + ln\frac{9}{4} - ln\frac{2}{3} - ln\,4}{ln\,3\,.\,ln\,4}\right] I_1 + \left[\frac{ln\,6 - ln\,3}{ln\,2\,.\,ln\,3}\right] I_2$$

$$= 1 + \left[\frac{ln\frac{81}{32}}{ln\,3\,.\,ln\,4}\right] I_1 + \left[\frac{1}{ln\,3}\right] I_2$$

6) $$\lim_{x\to 0+0I_1+0I_2} \frac{(6+5I_1-2I_2)^x - 1}{x} = ln(6+5I_1-2I_2)$$

$$= ln\,6 + [ln\,9 - ln\,4]I_1 + [ln\,4 - ln\,6]I_2$$

$$= ln\,6 + [3\,ln\,2 - 2\,ln\,2]I_1 + [2\,ln\,2 - ln\,6]I_2$$

7) $$\lim_{x\to 0+0I_1+0I_2} \frac{ln(1+(6+6I_1-6I_2)x)}{x} = 6 + 6I_1 - 6I_2$$

8) $$\lim_{x\to 0+0I_1+0I_2} \frac{e^{(3+I_1-I_2)x} - 1}{sin(3+2I_1+I_2)\,x} = \lim_{x\to 0+0I_1+0I_2} \frac{\frac{e^{(3+I_1-I_2)x} - 1}{x}}{\frac{sin(3+2I_1+I_2)\,x}{x}}$$

$$= \lim_{x\to 0+0I_1+0I_2} \frac{\frac{e^{(3+I_1-I_2)x} - 1}{x}}{\frac{sin(3+2I_1+I_2)\,x}{x}} = \frac{\lim_{x\to 0+0I_1+0I_2} \frac{e^{(3+I_1-I_2)x} - 1}{x}}{\lim_{x\to 0+0I_1+0I_2} \frac{sin(3+2I_1+I_2)\,x}{x}}$$

$$= \frac{3+I_1-I_2}{3+2I_1+I_2} = 1 + 0I_1 - \frac{1}{2}I_2$$

9) $$\lim_{x\to 0+0I_1+0I_2} \frac{(7+I_1+4I_2)^x - (6+3I_1+I_2)^x}{x} = \lim_{x\to 0+0I_1+0I_2} \frac{(7+I_1+4I_2)^x - (6+3I_1+I_2)^x - 1 + 1}{x}$$

$$= \lim_{x\to 0+0I_1+0I_2} \frac{(7+I_1+4I_2)^x - 1 - ((6+3I_1+I_2)^x - 1)}{x}$$

$$= \lim_{x\to 0+0I_1+0I_2} \frac{(7+I_1+4I_2)^x - 1}{x} - \lim_{x\to 0+0I_1+0I_2} \frac{(6+3I_1+I_2)^x - 1}{x}$$

$$= ln(7+I_1+4I_2) - ln(6+3I_1+I_2) = ln\left(\frac{7+I_1+4I_2}{6+3I_1+I_2}\right)$$

$$= ln\left(\frac{7}{6} - \frac{13}{35}I_1 + \frac{17}{42}I_2\right) = ln\frac{7}{6} + \left[ln\frac{6}{5} - ln\frac{11}{7}\right] I_1 + \left[ln\frac{11}{7} - ln\frac{7}{6}\right] I_2$$

3. Conclusions

One of the key concepts in calculus is limits. Its focus is on the study of derivation by studying the fundamental ideas of infinitesimal quantities. This was the goal of putting forward the idea of The limits of 2- refined neutrosophic in this paper. Several methods for solving The limits of 2- refined neutrosophic were discussed, in addition to presenting special rules to facilitate finding these limits. Also, We obtained the same results by solving the examples in different ways, such as L'Hôpital's rule.

Acknowledgments: Acknowledgments " This study is supported via funding from Prince sattam bin Abdulaziz University project number (PSAU/2024/R/1445)".

References

[1] Smarandache, F., " (T,I,F)- Neutrosophic Structures", Neutrosophic Sets and Systems, vol 8, pp. 3-10, 2015.

[2] Agboola,A.A.A. "On Refined Neutrosophic Algebraic Structures", Neutrosophic Sets and Systems, vol 10, pp. 99-101, 2015.

[3] Adeleke, E.O., Agboola, A.A.A.,and Smarandache, F., "Refined Neutrosophic Rings I", International Journal of Neutrosophic Science, Vol. 2(2), pp. 77-81. 2020.

[4] Smarandache, F.,"n-Valued Refined Neutrosophic Logic and Its Applications in Physics", Progress in Physics, USA, vol 4, pp. 143-146, 2013.

[5] Vasantha Kandasamy,W.B; Smarandache,F. "Neutrosophic Rings" Hexis, Phoenix, Arizona, 2006, http://fs.gallup.unm.edu/NeutrosophicRings.pdf

[6] Zeina, M., Abobala, M., "On The Refined Neutrosophic Real Analysis Based on Refined Neutrosophic Algebraic AH-Isometry", Neutrosophic Sets and Systems, Volume 54, pp. 158-168, 2023.

[7] Agboola, A.A.A.; Akinola, A.D; Oyebola, O.Y., "Neutrosophic Rings I", Int. J. of Math. Comb., vol 4, pp.1-14, 2011.

[8] Celik, M., and Hatip, A., " On The Refined AH-Isometry And Its Applications In Refined Neutrosophic Surfaces", Galoitica Journal Of Mathematical Structures And Applications, 2022.

[9] Jdid, M., Smarandache, F., and Al Shaqsi, K., "Generating Neutrosophic Random Variables Based Gamma Distribution", Plithogenic Logic and Computation, Vol. 1, pp. 16-24. 2024.

[10] Smarandache, F., "A Refined Neutrosophic Components into Subcomponents with Plausible Applications to Long Term Energy Planning Predominated by Renewable Energy", Plithogenic Logic and Computation, Vol. 1, pp. 45-60. 2024.

[11] Alhasan,Y., and Abdulfatah, R., "Division of refined neutrosophic numbers", Neutrosophic Sets and Systems, Volume 60, pp. 1-5, 2023.

Received: Mar 5, 2024. Accepted: May 28, 2024