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Pentapartitioned Neutrosophic Subtraction Algebra

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Abstract: This paper aims to define the concepts of Semi-group and Pentapartitioned Neutrosophic Subtraction Algebra. We also examine a few of their fundamental characteristics. Additionally, we provide a few appropriate instances on Pentapartitioned Neutrosophic Subtraction Algebra.

Keywords: Pentapartitioned Neutrosophic Set; Subtraction Algebra, Subtraction Semi-Group.

1. Introduction:

Schein [30] developed the basic principles of Subtraction Algebra in 1992. Afterwards, Zelinka [37] presented the idea of subtraction semi-group. In the year 2004, Kyung et al. [26] presented some notes on subtraction semi-group. Later on, Jun and Kim [23] grounded the notions of ideal in subtraction algebra. In the year 2008, Jun and Kim [24] also studied the concept of prime and irreducible ideals in subtraction algebra. The concepts of Weak Subtraction Algebras were then grounded by Lee et al. [27], who also looked at a technique for creating Weak Subtraction Algebra from a quasi-ordered set. Zadeh [36] proposed the concept of fuzzy sets for the first time in 1963. Later, in 2007, Kim et al. [25] addressed the concept of fuzzy ideals in subtraction algebras. In 1986, Atanassov [3] constructed a new idea of Intuitionistic Fuzzy Set by broadening the idea of Fuzzy Set. Ezhilarasi and Sriram [18] grounded the notion of intuitionistic fuzzy ideals of subtraction algebra. In 1998, Smarandache [31, 32] laid out the concept of Neutrosophic Set (NS) to aid in dealing with unpredictable occurrences with indeterminacy. In 2006, Vasantha Kandasamy and Smarandache [35] came up with neutrosophic algebraic structures in the context of neutrosophic set. In the year 2020, Ibrahim et al. [21] introduced the notions of Neutrosophic Subtraction Algebra and Neutrosophic Subtraction Semi-Group. Mallick and Pramanik [28] recently presented the principles of a Pentapartitioned Neutrosophic Set by broadening the theories of NS, in which any component has a total of five independent components such as: truth, contradiction, ignorance, unknown, and false membership.

In this article, we introduce the notion of Pentapartitioned Neutrosophic Subtraction Algebra and Semi-Group.

This article's remaining portion is structured as follows:

We review some pertinent definitions and findings on semi group and subtraction algebra in section-2. By extending the theory of neutrosophic subtraction algebra, we present the concepts of pentapartitioned neutrosophic subtraction algebra in section-3. We also develop some findings regarding pentapartitioned neutrosophic subtraction algebra. In section-4, we conclude this article.

2. Relevant Definitions and Results:

This section includes a few definitions and results that are fundamental to understanding the drafting of this article's primary outcomes.

Definition 2.1. [26] Let us consider a binary operation '−' on a fixed set $\tilde{\hat{A}}$. Then, the structure ($\tilde{\hat{A}}$, −) is referred to as a subtraction algebra if the following holds:

(i) $\acute{\rho} - (\underset{.}{u} - \acute{\rho}) = \acute{\rho}$;

(ii) $\acute{\rho} - (\acute{\rho} - \underset{.}{u}) = \underset{.}{u} - (\underset{.}{u} - \acute{\rho})$;

(iii) $(\acute{\rho} - \underset{.}{u}) - \tilde{e} = (\acute{\rho} - \tilde{e}) - \underset{.}{u}$; for all $\acute{\rho}, \underset{.}{u}, \tilde{e} \in \tilde{\hat{A}}$.

Remark 2.1. [26] Let us consider a subtraction algebra ($\tilde{\hat{A}}$, −). Then,

(i) $\acute{\rho} - 0 = \acute{\rho}$ and $0 - \acute{\rho} = 0$

(ii) $\acute{\rho} - (\acute{\rho} - \underset{.}{u}) \leq \underset{.}{u}$.

(iii) $\acute{\rho} \leq \underset{.}{u} \Leftrightarrow \acute{\rho} = \underset{.}{u} - w$ for some $w \in \tilde{\hat{A}}$.

(iv) $\acute{\rho} \leq \underset{.}{u} \Rightarrow \acute{\rho} - \tilde{e} \leq \underset{.}{u} - \tilde{e}$ and $\tilde{e} - \underset{.}{u} \leq \tilde{e} - \acute{\rho}$ for all $\tilde{e} \in \tilde{\hat{A}}$.

(v) $\acute{\rho} - (\acute{\rho} - (\acute{\rho} - \underset{.}{u})) = \acute{\rho} - \underset{.}{u}$.

Definition 2.2. [28] Let us consider that $\tilde{\hat{A}}$ be a fixed set. A pentapartitioned neutrosophic set (P-NS) $\tilde{N}$ over $\tilde{\hat{A}}$ is defined by:

$$\tilde{N} = \{(\acute{\rho}, T_{\tilde{N}}(\acute{\rho}), C_{\tilde{N}}(\acute{\rho}), G_{\tilde{N}}(\acute{\rho}), U_{\tilde{N}}(\acute{\rho}), F_{\tilde{N}}(\acute{\rho})): \acute{\rho} \in \tilde{\hat{A}}\},$$

where $T_{\tilde{N}}(\acute{\rho})$, $C_{\tilde{N}}(\acute{\rho})$, $G_{\tilde{N}}(\acute{\rho})$, $U_{\tilde{N}}(\acute{\rho})$, $F_{\tilde{N}}(\acute{\rho})$ ($\in$]0,1[) are the truth, contradiction, ignorance, unknown, falsity membership values of each $\acute{\rho} \in \tilde{\hat{A}}$. So $0 \leq T_{\tilde{N}}(\acute{\rho})+C_{\tilde{N}}(\acute{\rho})+G_{\tilde{N}}(\acute{\rho})+U_{\tilde{N}}(\acute{\rho})+F_{\tilde{N}}(\acute{\rho}) \leq 5$.

The null P-NS (0_{PN}) and the absolute P-NS (1_{PN}) over $\tilde{\hat{A}}$ are defined as follows:

(i) $0_{PN} = \{(\acute{\rho}, 0, 0, 1, 1, 1): \acute{\rho} \in \tilde{\hat{A}}\}$;

(ii) $1_{PN} = \{(\acute{\rho}, 1, 1, 0, 0, 0): \acute{\rho} \in \tilde{\hat{A}}\}$.

Clearly, $0_{PN} \subseteq \tilde{N} \subseteq 1_{PN}$, where $\tilde{N}$ is a P-NS over $\tilde{\hat{A}}$.

Assume that $\tilde{N} = \{(\acute{\rho}, T_{\tilde{N}}(\acute{\rho}), C_{\tilde{N}}(\acute{\rho}), G_{\tilde{N}}(\acute{\rho}), U_{\tilde{N}}(\acute{\rho}), F_{\tilde{N}}(\acute{\rho})): \acute{\rho} \in W\}$ and $\bar{H} = \{(\acute{\rho}, T_H(\acute{\rho}), C_H(\acute{\rho}), G_H(\acute{\rho}), U_H(\acute{\rho}), F_H(\acute{\rho})): \acute{\rho} \in W\}$ be two P-NSs over W. Then,

(i) $\tilde{N} \subseteq \bar{H} \Leftrightarrow T_{\tilde{N}}(\acute{\rho}) \leq T_H(\acute{\rho})$, $C_{\tilde{N}}(\acute{\rho}) \leq C_H(\acute{\rho})$, $G_{\tilde{N}}(\acute{\rho}) \geq G_H(\acute{\rho})$, $U_{\tilde{N}}(\acute{\rho}) \geq U_H(\acute{\rho})$, $F_{\tilde{N}}(\acute{\rho}) \geq F_H(\acute{\rho})$, for all $\acute{\rho} \in W$.

(ii) $\tilde{N} \cap \bar{H} = \{(\acute{\rho}, \min\{T_{\tilde{N}}(\acute{\rho}), T_H(\acute{\rho})\}, \min\{C_{\tilde{N}}(\acute{\rho}), C_H(\acute{\rho})\}, \max\{G_{\tilde{N}}(\acute{\rho}), G_H(\acute{\rho})\}, \max\{U_{\tilde{N}}(\acute{\rho}), U_H(\acute{\rho})\}, \max\{F_{\tilde{N}}(\acute{\rho}), F_H(\acute{\rho})\}): \acute{\rho} \in W\}$.

(iii) $\tilde{N} \cup \bar{H} = \{(\acute{\rho}, \max\{T_{\tilde{N}}(\acute{\rho}), T_H(\acute{\rho})\}, \max\{C_{\tilde{N}}(\acute{\rho}), C_H(\acute{\rho})\}, \min\{G_{\tilde{N}}(\acute{\rho}), G_H(\acute{\rho})\}, \min\{U_{\tilde{N}}(\acute{\rho}), U_H(\acute{\rho})\}, \min\{F_{\tilde{N}}(\acute{\rho}), F_H(\acute{\rho})\}): \acute{\rho} \in W\}$.

(iv) $\tilde{N}^c = \{(\acute{\rho}, F_{\tilde{N}}(\acute{\rho}), U_{\tilde{N}}(\acute{\rho}), 1-G_{\tilde{N}}(\acute{\rho}), C_{\tilde{N}}(\acute{\rho}), T_{\tilde{N}}(\acute{\rho})): \acute{\rho} \in W\}$ and $\bar{H}^c = \{(\acute{\rho}, F_H(\acute{\rho}), U_H(\acute{\rho}), 1-G_H(\acute{\rho}), C_H(\acute{\rho}), T_H(\acute{\rho})): \acute{\rho} \in W\}$.

Definition 2.3. [21] Assume that $\tilde{\hat{A}}$ be a fixed set. A set $\tilde{\hat{A}}(I) = <\tilde{\hat{A}} \cup I>$ generated by $\tilde{\hat{A}}$ and I is referred to as an neutrosophic set. The members of $\tilde{\hat{A}}(I)$ are of the form $(\acute{\rho}, yI)$, where $\acute{\rho}$ and y are elements of $\tilde{\hat{A}}$. I is referred to as an indeterminate and it has the property $I^n = I$ for all positive integer n.

Definition 2.4. [21] Let us consider a classical subtraction algebra ($\tilde{\hat{A}}$, −), and let $\tilde{\hat{A}}(I)$ = <$\tilde{\hat{A}} \cup I$> be a set generated by $\tilde{\hat{A}}$ and I. Consider the neutrosophic algebraic structure ($\tilde{\hat{A}}(I)$, $-_N$) where for all ($\hat{a}$, $\tilde{e}I$), ($\ddot{w}$, $\acute{\rho}I$) ∈ ($\tilde{\hat{A}}(I)$, $-_N$) is defined by ($\hat{a}$, $\tilde{e}I$) $-_N$ ($\ddot{w}$, $\acute{\rho}I$) = ($\hat{a} - \ddot{w}$, $(\tilde{e} - \acute{\rho})I$) ∀$\hat{a}$, $\tilde{e}$, $\ddot{w}$, $\acute{\rho} \in \tilde{\hat{A}}$.
We denote ($\tilde{\hat{A}}(I)$, $-_N$) as a neutrosophic subtraction algebra.
An element $\acute{\rho} \in \tilde{\hat{A}}$ is represented by ($\acute{\rho}$, 0)∈$\tilde{\hat{A}}(I)$ and (0, 0) represents the constant element in $\tilde{\hat{A}}(I)$.
Definition 2.5. [21] Assume that ($\tilde{\hat{A}}(I)$, $-_N$) be a neutrosophic subtraction algebra. Then, a non-empty subset Ħ(I) is referred to as a neutrosophic subtraction sub-algebra of $\tilde{\hat{A}}(I)$ if the following conditions hold:
(i) If Ħ(I) ≠ ∅
(ii) ($\hat{a}$, $\tilde{e}I$) $-_N$ ($\ddot{w}$, $\acute{\rho}I$) ∈ Ħ(I) for all ($\hat{a}$, $\tilde{e}I$),($\ddot{w}$, $\acute{\rho}I$) ∈ Ħ(I).
(iii) Ħ(I) contains a proper subset which is a subtraction algebra.
If Ħ(I) does not contain a proper subset which is a subtraction algebra, then Ħ(I) is referred to as a pseudo neutrosophic subtraction sub-algebra of $\tilde{\hat{A}}(I)$.

3. Pentapartitioned Neutrosophic Subtraction Algebra:

In this section we established the concept of pentapartitioned neutrosophic subtraction algebra on P-N-Ss, and established several results on it in the form of theorems, propositions, etc.

Definition 3.1. Assume that $\tilde{\hat{A}}$ be a fixed set. Then, a set $\tilde{\hat{A}}(P_N)$ = <$\tilde{\hat{A}} \cup C \cup G \cup U$> which is generated by $\tilde{\hat{A}}$ and P_N is referred to as a pentapartitioned neutrosophic set. The members of $\tilde{\hat{A}}(P_N)$ are of the form ($\hat{a}$, $\tilde{e}C$, $\ddot{w}G$, ựU), where $\hat{a}$, $\tilde{e}$, $\ddot{w}$, and ự are the elements of $\tilde{\hat{A}}$. Here, C, G, U are called contradiction, ignorance, unknown and it has the property $C^n = C$, $G^n = G$ and $U^n = U$ for all positive integer n.
Definition 3.2. Assume that ($\tilde{\hat{A}}$, −) be any classical subtraction algebra. Suppose that $\tilde{\hat{A}}(P_N)$ = <$\tilde{\hat{A}} \cup C \cup G \cup U$> be a set generated by $\tilde{\hat{A}}$ and P_N. Consider the pentapartitioned neutrosophic algebraic structure ($\tilde{\hat{A}}(P_N)$, $-_N$) where for all ($\hat{a}$, $\tilde{e}C$, $\ddot{w}G$, ựU), ($\hat{a}_1$, $\tilde{e}_1C$, $\ddot{w}_1G$, ự$_1U$) ∈ ($\tilde{\hat{A}}(I)$, $-_N$) is defined as follows:
($\hat{a}$, $\tilde{e}C$, $\ddot{w}G$, ựU)$-_N$($\hat{a}_1$, $\tilde{e}_1C$, $\ddot{w}_1G$, ự$_1U$) = ($\hat{a} - \hat{a}_1$, $(\tilde{e} - \tilde{e}_1)C$, $(\ddot{w} - \ddot{w}_1)G$, (ự − ự$_1$)U), ∀ $\hat{a}$, $\tilde{e}$, $\ddot{w}$, ự, $\hat{a}_1$, $\tilde{e}_1$, $\ddot{w}_1$, ự$_1$ ∈ $\tilde{\hat{A}}$.
We call ($\tilde{\hat{A}}(P_N)$, $-_N$) a pentapartitioned neutrosophic subtraction algebra.
An element $\acute{\alpha} \in \tilde{\hat{A}}$ is represented by ($\acute{\alpha}$, 0, 0, 0) ∈$\tilde{\hat{A}}(P_N)$ and (0, 0, 0, 0) represents the constant element in $\tilde{\hat{A}}(P_N)$.
Theorem 3.1. Every pentapartitioned neutrosophic subtraction algebra ($\tilde{\hat{A}}(P_N)$, $-_N$) is a subtraction algebra.
Proof. Assume that ($\tilde{\hat{A}}(P_N)$, $-_N$) is a subtraction algebra. Assume that $\acute{\alpha}$ = ($\hat{a}$, $\tilde{e}C$, $\ddot{w}G$, ựU), y = ($\hat{a}_1$, $\tilde{e}_1C$, $\ddot{w}_1G$, ự$_1U$), z = ($\hat{a}_2$, $\tilde{e}_2C$, $\ddot{w}_2G$, ự$_2U$) ∈$\tilde{\hat{A}}(P_N)$, where $\hat{a}$, $\tilde{e}$, $\ddot{w}$, ự, $\hat{a}_1$, $\tilde{e}_1$, $\ddot{w}_1$, ự$_1$, $\hat{a}_2$, $\tilde{e}_2$, $\ddot{w}_2$, ự$_2$ ∈ $\tilde{\hat{A}}$.
(i) We have,
$\acute{\alpha} -_N (y -_N \acute{\alpha})$
= ($\hat{a}$, $\tilde{e}C$, $\ddot{w}G$, ựU)$-_N$(($\hat{a}_1$, $\tilde{e}_1C$, $\ddot{w}_1G$, ự$_1U$) $-_N$ ($\hat{a}$, $\tilde{e}C$, $\ddot{w}G$, ựU))
= ($\hat{a}$, $\tilde{e}C$, $\ddot{w}G$, ựU) $-_N$ (($\hat{a}_1$-$\hat{a}$), ($\tilde{e}_1$-$\tilde{e}$)C, ($\ddot{w}_1$-$\ddot{w}$)G, (ự$_1$-ự)U)
= ($\hat{a} - (\hat{a}_1 - \hat{a})$,($\tilde{e} - (\tilde{e}_1 - \tilde{e})$)$C$, ($\ddot{w} - (\ddot{w}_1 - \ddot{w})$)$G$, (ự − (ự$_1$ − ự))U)
= ($\hat{a}$, $\tilde{e}C$, $\ddot{w}G$, ựU) Since â,ẽ,ẅ,ự∈ $\tilde{\hat{A}}$
= $\acute{\alpha}$
(ii) We have,
$\acute{\alpha} -_N (\acute{\alpha} -_N y)$

$= (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U))$

$= (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (((\hat{a}_1\text{-}\hat{a}), (\tilde{e}_1\text{-}\tilde{e})C, (\ddot{w}_1\text{-}\ddot{w})G, (ự_1\text{-}ự)U))$

$= ((\hat{a} - (\hat{a} - \hat{a}_1)), (\tilde{e} - (\tilde{e} - \tilde{e}_1))C, (\ddot{w} - (\ddot{w} - \ddot{w}_1))G, (ự - (ự - ự_1))U)$

$= ((\hat{a}_1 - (\hat{a}_1 - \hat{a})), (\tilde{e}_1 - (\tilde{e}_1 - \tilde{e}))C, (\ddot{w}_1 - (\ddot{w}_1 - \ddot{w}))G, (ự_1 - (ự_1 - ự))U)$ Since $\hat{a}, \tilde{e}, \ddot{w}, ự, \hat{a}_1, \tilde{e}_1, \ddot{w}_1, ự_1 \in \tilde{\tilde{A}}$.

$= (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N ((\hat{a}_1\text{-}\hat{a}), (\tilde{e}_1\text{-}\tilde{e})C, (\ddot{w}_1\text{-}\ddot{w})G, (ự_1\text{-}ự)U)$

$= (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N ((\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ựU))$

$= y -_N (y -_N \acute{\alpha})$

(iii) We have,

$(\acute{\alpha} -_N y) -_N z$

$= ((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)) -_N (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, ự_2U)$

$= ((\hat{a}\text{-} \hat{a}_1), (\tilde{e}\text{-} \tilde{e}_1)C, (\ddot{w}\text{-} \ddot{w}_1)G, (ự\text{-} ự_1)U) -_N (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, ự_2U)$

$= (((\hat{a} - \hat{a}_1)\text{-}\hat{a}_2), ((\tilde{e} - \tilde{e}_1) \text{-}\tilde{e}_2)C, ((\ddot{w} - \ddot{w}_1) \text{-}\ddot{w}_2)G, ((ự - ự_1) \text{-}ự_2)U)$

$= (((\hat{a} - \hat{a}_2)\text{-}\hat{a}_1), ((\tilde{e} - \tilde{e}_2) - \tilde{e}_1)C, ((\ddot{w} - \ddot{w}_2) - \ddot{w}_1)G, ((ự - ự_2) - ự_1)U)$ Since $\hat{a}, \tilde{e}, \ddot{w}, ự, \hat{a}_1, \tilde{e}_1, \ddot{w}_1, ự_1, \hat{a}_2, \tilde{e}_2, \ddot{w}_2, ự_2 \in \tilde{\tilde{A}}$.

$= ((\hat{a}\text{-} \hat{a}_2), (\tilde{e}\text{-} \tilde{e}_2)C, (\ddot{w}\text{-} \ddot{w}_2)G, (ự\text{-} ự_2)U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)$

$= ((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)) -_N (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, ự_2U)$

$= (\acute{\alpha} -_N z) -_N y$

From Vasantha and Smarandache [30] and Ibrahim et al. [18], we note that, if $\acute{\alpha} \le y$ then we cannot in general say $\acute{\alpha}C \le yC$, $\acute{\alpha}G \le yG$, and $\acute{\alpha}U \le yU$ it may so happen that $\acute{\alpha}C \le yC$, $\acute{\alpha}G \le yG$, and $\acute{\alpha}U \le yU$. Thus, the pentapartitioned neutrosophic order in general needs not to preserve the order. If a set $\tilde{\tilde{A}}$ is ordered under "≤" then the pentapartitioned neutrosophic part of $\langle \tilde{\tilde{A}} \cup C \cup G \cup U \rangle$ may or may not have the preservations of order under ≤; i.e., if $\acute{\alpha} \le y$, $\acute{\alpha}, y \in \tilde{\tilde{A}}$ then $\acute{\alpha}C \le yC$, $\acute{\alpha}G \le yG$, and$\acute{\alpha}U \le yU$ may occur or may not occur. For the work of the partial ordering we consider suppose $\acute{\alpha}C \le yC$, $\acute{\alpha}G \le yG$, and $\acute{\alpha}U \le yU$ occur.

Theorem 3.2. For a pentapartitioned neutrosophic subtraction algebra $(\tilde{\tilde{A}}(P_N), -_N)$, the relation "≤" is a partial order relation on $\tilde{\tilde{A}}(P_N)$.

Proof. Assume that $\acute{\alpha} = (\hat{a}, \tilde{e}C, \ddot{w}G, ựU)$, $\mu = (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)$, $\theta = (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, ự_2U) \in \tilde{\tilde{A}}(P_N)$ with $\hat{a}, \tilde{e}, \ddot{w}, ự, \hat{a}_1, \tilde{e}_1, \ddot{w}_1, ự_1, \hat{a}_2, \tilde{e}_2, \ddot{w}_2, ự_2 \in \tilde{\tilde{A}}$.

(i) Since $\acute{\alpha}\text{-}\acute{\alpha} = (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) = (\hat{a} - \hat{a}, (\tilde{e} - \tilde{e})C, (\ddot{w} - \ddot{w})G, (ự - ự)U) = (0, 0C, 0G, 0U)$ then $\acute{\alpha} \le \acute{\alpha}$.

(ii) Suppose that $\acute{\alpha} \le \mu$ and $\mu \le \acute{\alpha}$.

Then, $(\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) = (0, 0C, 0G, 0U)$ implies $(\hat{a} - \hat{a}_1, (\tilde{e} - \tilde{e}_1)C, (\ddot{w} - \ddot{w}_1)G, (ự - ự_1)U) = (0, 0C, 0G, 0U)$

and $(\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) = (0, 0C, 0G, 0U)$ implies $((\hat{a}_1\text{-}\hat{a}), (\tilde{e}_1\text{-}\tilde{e})C, (\ddot{w}_1\text{-}\ddot{w})G, (ự_1\text{-}ự)U) = (0, 0C, 0G, 0U)$.

Now, we have

$(\hat{a}, \tilde{e}C, \ddot{w}G, ựU) = (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (0, 0C, 0G, 0U)$

$= (\hat{a} - 0, (\tilde{e} - 0)C, (\ddot{w} - 0)G, (ự - 0)U)$

$= ((\hat{a} - (\hat{a} - \hat{a}_1)), (\tilde{e} - (\tilde{e} - \tilde{e}_1))C, (\ddot{w} - (\ddot{w} - \ddot{w}_1))G, (ự - (ự - ự_1))U)$ Since, $\hat{a} - \hat{a}_1 = 0$, $\tilde{e} - \tilde{e}_1 = 0$, $\ddot{w} - \ddot{w}_1 = 0$ and $ự - ự_1 = 0$.

$= ((\hat{a}_1 - (\hat{a}_1 - \hat{a})), (\tilde{e}_1 - (\tilde{e}_1 - \tilde{e}))C, (\ddot{w}_1 - (\ddot{w}_1 - \ddot{w}))G, (ự_1 - (ự_1 - ự))U)$ Since $\tilde{\tilde{A}}$ is a subtraction algebra

$= ((\hat{a}_1 - 0), (\tilde{e}_1 - 0)C, (\ddot{w}_1 - 0)G, (ự_1 - 0)U)$ Since, $\hat{a} - \hat{a}_1 = 0$, $\tilde{e} - \tilde{e}_1 = 0$, $\ddot{w} - \ddot{w}_1 = 0$ and $ự - ự_1 = 0$.

$= (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N (0, 0C, 0G, 0U)$

$= (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)$.

(iii) Let $\acute{\alpha}\leq\mu$ and $\mu\leq\theta$. Therefore, $(\hat{a}, \tilde{e}C, \ddot{w}G, ựU) \leq (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)$ and $(\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) \leq (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, ự_2U)$.

Then, $(\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) = (0, 0C, 0G, 0U)$ implies $(\hat{a} - \hat{a}_1, (\tilde{e} - \tilde{e}_1)C, (\ddot{w} - \ddot{w}_1)G, (ự - ự_1)U) = (0, 0C, 0G, 0U)$

and $(\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, ự_2U) = (0, 0C, 0G, 0U)$ implies $(\hat{a}_1\text{-}\hat{a}_2), (\tilde{e}_1\text{-}\tilde{e}_2)C, (\ddot{w}_1\text{-}\ddot{w}_2)G, (ự_1\text{-}ự_2)U) = (0, 0C, 0G, 0U)$.

Now, we have

$(\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, ự_2U)$

$= (\hat{a} - \hat{a}_2, (\tilde{e} - \tilde{e}_2)C, (\ddot{w} - \ddot{w}_2)G, (ự - ự_2)U)$

$= (\hat{a} - \hat{a}_2, (\tilde{e} - \tilde{e}_2)C, (\ddot{w} - \ddot{w}_2)G, (ự - ự_2)U)$

$= ((\hat{a} - \hat{a}_2)\text{-}0, ((\tilde{e} - \tilde{e}_2)\text{-}0)C, ((\ddot{w} - \ddot{w}_2)\text{-}0)G, ((ự - ự_2)\text{-}0)U)$

$= ((\hat{a} - \hat{a}_2)\text{-}(\hat{a} - \hat{a}_1), ((\tilde{e} - \tilde{e}_2)\text{-}(\tilde{e} - \tilde{e}_1))C, ((\ddot{w} - \ddot{w}_2)\text{-} (\ddot{w} - \ddot{w}_1))G, ((ự - ự_2)\text{-}(ự - ự_1))U)$

$= ((\hat{a} \text{-}(\hat{a} - \hat{a}_1)) - \hat{a}_2), (((\tilde{e} \text{-}(\tilde{e} - \tilde{e}_1)) - \tilde{e}_2))C, (((\ddot{w} \text{-} (\ddot{w} - \ddot{w}_1)) - \ddot{w}_2))G, (((ự \text{-}(ự - ự_1)) - ự_2))U)$

$= ((\hat{a}_1\text{-}(\hat{a}_1 - \hat{a})) - \hat{a}_2), (((\tilde{e}_1\text{-}(\tilde{e}_1 - \tilde{e})) - \tilde{e}_2))C, (((\ddot{w}_1\text{-} (\ddot{w}_1 - \ddot{w})) - \ddot{w}_2))G, (((ự_1\text{-}(ự_1 - ự)) - ự_2))U)$

$= ((\hat{a}_1 - \hat{a}_2)\text{-}(\hat{a}_1 - \hat{a}), ((\tilde{e}_1 - \tilde{e}_2)\text{-} (\tilde{e}_1 - \tilde{e}))C, ((\ddot{w}_1 - \ddot{w}_2)\text{-} (\ddot{w}_1 - \ddot{w}))G, ((ự_1 - ự_2)\text{-}(ự_1 - ự))U)$

$= (0\text{-}(\hat{a}_1 - \hat{a}), (0\text{-} (\tilde{e}_1 - \tilde{e}))C, (0\text{-} (\ddot{w}_1 - \ddot{w}))G, (0\text{-}(ự_1 - ự))U)$

$= (0, 0C, 0G, 0U)$

Hence, $(\hat{a}, \tilde{e}C, \ddot{w}G, ựU) \leq (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, ự_2U)$. Consequently, "$\leq$" is a partial order relation.

Proposition 3.1. Assume that $(\tilde{\hat{A}}(P_N), -_N)$ be a pentapartitioned neutrosophic subtraction algebra . If $(\hat{a}, \tilde{e}C, \ddot{w}G, ựU), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) \in \tilde{\hat{A}}(P_N)$, with $\hat{a}, \tilde{e}, \ddot{w}, ự \in \tilde{\hat{A}}$, then the following are true:

(i) $(\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (0, 0C, 0G, 0U) = (\hat{a}, \tilde{e}C, \ddot{w}G, ựU)$;

(ii) $(0, 0C, 0G, 0U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) = (0, 0C, 0G, 0U)$;

(iii) $((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) = (0, 0C, 0G, 0U)$;

(iv) $((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) = (\hat{a}, \tilde{e}I) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)$;

(v) $((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)) -_N ((\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ựU)) = (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)$;

(vi) $(\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U))) = (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)$;

(vii) $(\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)) \leq (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)$;

(viii) $(\hat{a}, \tilde{e}C, \ddot{w}G, ựU) = (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) \Leftrightarrow (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) = (0, 0C, 0G, 0U)$ *and* $(\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) = (\hat{a}, \tilde{e}C, \ddot{w}G, ựU)$.

Proof. (i) We have, $(\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (0, 0C, 0G, 0U)$

$= (\hat{a} - 0, (\tilde{e} - 0)C, (\tilde{e} - 0)G, (\tilde{e} - 0)U) = (\hat{a} - (\hat{a} - \hat{a}), (\tilde{e} - (\tilde{e} - \tilde{e}))C, (\tilde{e} - (\tilde{e} - \tilde{e}))G, (\tilde{e} - (\tilde{e} - \tilde{e}))U) = (\hat{a}, \tilde{e}C, \ddot{w}G, ựU)$.

(ii) $(0, 0C, 0G, 0U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) = (0\text{-}\hat{a}, (0\text{-}\tilde{e})C, (0\text{-}\ddot{w})G, (0\text{-}ự)U) = (0-(\hat{a}-0), (0-(\tilde{e}-0))C, (0-(\tilde{e}-0))G, (0-(\tilde{e}-0))U) = (0, 0C, 0G, 0U)$.

(iii) We have, $((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ựU)$

$= (\hat{a}\text{-}\hat{a}_1, (\tilde{e}\text{-}\tilde{e}_1)C, (\ddot{w}\text{-}\ddot{w}_1)G, (ự\text{-}ự_1)U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ựU)$

$= ((\hat{a} - \hat{a}_1) - \hat{a}, ((\tilde{e} - \tilde{e}_1) - \tilde{e})C, ((\ddot{w} - \ddot{w}_1) - \ddot{w})G, ((ự - ự_1) - ự)U)$

$= ((\hat{a} - \hat{a}) - \hat{a}_1, ((\tilde{e} - \tilde{e}) - \tilde{e}_1)C, ((\ddot{w} - \ddot{w}) - \ddot{w}_1)G, ((ự - ự) - ự_1)U)$

$= (0\text{-}\hat{a}_1, (0\text{-}\tilde{e}_1)C, (0\text{-}\ddot{w}_1)G, (0\text{-}ự_1)U)$
$= (0, 0C, 0G, 0U)$.

(iv) $((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) = (\hat{a} - \hat{a}_1, (\tilde{e} - \tilde{e}_1)C, (\ddot{w} - \ddot{w}_1)G, (ự - ự_1)U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)$.
$= (((\hat{a} - \hat{a}_1)\text{-}\hat{a}_1), ((\tilde{e} - \tilde{e}_1) - \tilde{e}_1)C, ((\ddot{w} - \ddot{w}_1) - \ddot{w}_1)G, ((ự - ự_1) - ự_1)U)$
$= ((\hat{a} - \hat{a}_1) - (\hat{a}_1 - (\hat{a} - \hat{a}_1)), ((\tilde{e} - \tilde{e}_1) - (\tilde{e}_1 - (\tilde{e} - \tilde{e}_1))C, ((\ddot{w} - \ddot{w}_1) - (\ddot{w}_1 - (\ddot{w} - \ddot{w}_1))G, ((ự - ự_1) - (ự_1 - (ự - ự_1))U$
$= (\hat{a}\text{-}\hat{a}_1, (\tilde{e}\text{-}\tilde{e}_1)C, (\ddot{w}\text{-}\ddot{w}_1)G, (ự\text{-}ự_1)U)$
$= (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)$.

(v) $((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)) -_N ((\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ựU)) = (\hat{a}\text{-}\hat{a}_1, (\tilde{e}\text{-}\tilde{e}_1)C, (\ddot{w}\text{-}\ddot{w}_1)G, (ự\text{-}ự_1)U) -_N ((\hat{a}_1\text{-}\hat{a}), (\tilde{e}_1\text{-}\tilde{e})C, (\ddot{w}_1\text{-}\ddot{w})G, (ự_1\text{-}ự)U) = ((\hat{a}\text{-}\hat{a}_1)\text{-} (\hat{a}_1\text{-}\hat{a}), ((\tilde{e}\text{-}\tilde{e}_1)\text{-} (\tilde{e}_1\text{-}\tilde{e}))C, (\ddot{w}\text{-}\ddot{w}_1)\text{-} (\ddot{w}_1\text{-}\ddot{w}))G, (ự\text{-}ự_1)\text{-} (ự_1\text{-}ự))U) = ((\hat{a} - (\hat{a}_1 - \hat{a})) - \hat{a}_1, ((\tilde{e} - (\tilde{e}_1 - \tilde{e})) - \tilde{e}_1)C, ((\ddot{w} - (\ddot{w}_1 - \ddot{w})) - \ddot{w}_1)G, ((ự - (ự_1 - ự)) - ự_1)I)$
$= (\hat{a} - \hat{a}_1, (\tilde{e} - \tilde{e}_1)C, (\ddot{w} - \ddot{w}_1)G, (ự - ự_1)U)$
$= (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)$.

(vi) $(\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U))) = (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a} - \ddot{w}, (\tilde{e} - ự)I)) = (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a} - \hat{a}_1, (\tilde{e} - \tilde{e}_1)C, (\ddot{w} - \ddot{w}_1)G, (ự - ự_1)U)$.
$= (\hat{a} - (\hat{a} - (\hat{a} - \hat{a}_1)), (\tilde{e} - (\tilde{e} - (\tilde{e} - \tilde{e}_1)))C, (\ddot{w} - (\ddot{w} - (\ddot{w} - \ddot{w}_1)))G, (ự - (ự - (ự - ự_1)))U)$
$= ((\hat{a} - \hat{a}_1) - ((\hat{a} - \hat{a}_1) - \hat{a}), ((\tilde{e} - \tilde{e}_1) - ((\tilde{e} - \tilde{e}_1) - \tilde{e}))C, ((\ddot{w} - \ddot{w}_1) - ((\ddot{w} - \ddot{w}_1) - \ddot{w}))G, ((ự - ự_1) - ((ự - ự_1) - ự))U)$
Since from the properties of $\tilde{\hat{A}}$, if $\hat{a}, \hat{a}_1 \in \tilde{\hat{A}}$ then $(\hat{a} - \hat{a}_1) - \hat{a} = 0$ then we have
$((\hat{a} - \hat{a}_1) - ((\hat{a} - \hat{a}_1) - \hat{a}), ((\tilde{e} - \tilde{e}_1) - ((\tilde{e} - \tilde{e}_1) - \tilde{e}))C, ((\ddot{w} - \ddot{w}_1) - ((\ddot{w} - \ddot{w}_1) - \ddot{w}))G, ((ự - ự_1) - ((ự - ự_1) - ự))U)$
$= ((\hat{a} - \hat{a}_1) - 0, ((\tilde{e} - \tilde{e}_1) - 0)C, ((\ddot{w} - \ddot{w}_1) - 0)G, ((ự - ự_1) - 0)U)$
$= (\hat{a} - \hat{a}_1, (\tilde{e} - \tilde{e}_1)C, (\ddot{w} - \ddot{w}_1)G, (ự - ự_1)U)$
$= (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)$.

(vi) $((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U))) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) = ((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)) = (0, 0C, 0G, 0U) \Rightarrow (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)) \leq (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)$.

(vii) Suppose that $(\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) = (0, 0C, 0G, 0U)$ and $(\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) = (0, 0C, 0G, 0U)$. Then $(\hat{a}, \tilde{e}C, \ddot{w}G, ựU) = (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (0, 0C, 0G, 0U)$.
$= (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U))$
$= (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N ((\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ựU))$
$= (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N (0, 0C, 0G, 0U)$.
$= (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)$.
$\Rightarrow (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) = (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)$.
Conversely, suppose that $(\hat{a}, \tilde{e}C, \ddot{w}G, ựU) = (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)$
Then, $(\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) = (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) = (0, 0C, 0G, 0U)$ and $(\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) = (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) = (0, 0C, 0G, 0U) \Rightarrow (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) = (0, 0C, 0G, 0U)$ and $(\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) = (0, 0C, 0G, 0U)$.

Definition 3.3. Assume that $\tilde{\tilde{A}}_1(P_N)$ and $\tilde{\tilde{A}}_2(P_N)$ be two pentapartitioned neutrosophic subtraction algebra. Then, the direct product of $\tilde{\tilde{A}}_1(P_N)$ and $\tilde{\tilde{A}}_2(P_N)$ is denoted by $\tilde{\tilde{A}}_1(P_N) \times \tilde{\tilde{A}}_2(P_N)$ and defined as follows:

$\tilde{\tilde{A}}_1(P_N) \times \tilde{\tilde{A}}_2(P_N) = \{((\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) : (\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) \in \tilde{\tilde{A}}_1(P_N), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) \in \tilde{\tilde{A}}_2(P_N)\}$.

Proposition 3.2. Assume that $(\tilde{\tilde{A}}_1(P_N), -_N)$ and $(\tilde{\tilde{A}}_2(P_N), -_N)$ be two pentapartitioned neutrosophic subtraction algebra. Then, $(\tilde{\tilde{A}}_1(P_N) \times \tilde{\tilde{A}}_2(P_N), -_N)$ is also a pentapartitioned neutrosophic subtraction algebra.

Proof. Suppose that $\acute{\alpha} = ((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U))$, $\mu = ((\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U), (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U))$, $\theta = ((\hat{a}_4, \tilde{e}_4C, \ddot{w}_4G, \underset{.}{u}_4U), (\hat{a}_5, \tilde{e}_5C, \ddot{w}_5G, \underset{.}{u}_5U)) \in \tilde{\tilde{A}}_1(P_N) \times \tilde{\tilde{A}}_2(P_N)$, for all $\hat{a}_0, \tilde{e}_0, \ddot{w}_0, \underset{.}{u}_0, \hat{a}_2, \tilde{e}_2, \ddot{w}_2, \underset{.}{u}_2, \hat{a}_4, \tilde{e}_4, \ddot{w}_4, \underset{.}{u}_4 \in \tilde{\tilde{A}}_1$ and $\hat{a}_1, \tilde{e}_1, \ddot{w}_1, \underset{.}{u}_1, \hat{a}_3, \tilde{e}_3, \ddot{w}_3, \underset{.}{u}_3, \hat{a}_5, \tilde{e}_5, \ddot{w}_5, \underset{.}{u}_5 \in \tilde{\tilde{A}}_2$. Therefore,

(i) We have,

$\acute{\alpha} -_N (\mu -_N \acute{\alpha})$

$= ((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) -_N [((\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U), (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U)) -_N ((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U))]$

$= ((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) -_N [(\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U) -_N (\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)]$

$= (\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U) -_N ((\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U) -_N (\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U)), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) -_N ((\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) = ((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U))$

$= \acute{\alpha}$.

(ii) We have,

$\acute{\alpha} -_N (\acute{\alpha} -_N \mu)$

$= ((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) -_N [((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) -_N ((\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U), (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U))]$

$= ((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) -_N [((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U) -_N (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U)), ((\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) -_N (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U))]$

$= (\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U) -_N ((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U) -_N (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U)), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) -_N ((\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) -_N (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U))$

$= (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U) -_N ((\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U) -_N (\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U)), (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U) -_N ((\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U))$

$= ((\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U), (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U)) -_N [((\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U), (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U)) -_N ((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U))]$

$= \mu -_N (\mu -_N \acute{\alpha})$.

(iii) We have,

$(\acute{\alpha} -_N \mu) -_N \theta$

$= [((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) -_N ((\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U), (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U))] -_N ((\hat{a}_4, \tilde{e}_4C, \ddot{w}_4G, \underset{.}{u}_4U), (\hat{a}_5, \tilde{e}_5C, \ddot{w}_5G, \underset{.}{u}_5U))$

$= [((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U) -_N (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U)), ((\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) -_N (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U))] -_N ((\hat{a}_4, \tilde{e}_4C, \ddot{w}_4G, \underset{.}{u}_4U), (\hat{a}_5, \tilde{e}_5C, \ddot{w}_5G, \underset{.}{u}_5U))$

$= (((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U) -_N (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U)) -_N (\hat{a}_4, \tilde{e}_4C, \ddot{w}_4G, \underset{.}{u}_4U), ((\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) -_N (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U)) -_N (\hat{a}_5, \tilde{e}_5C, \ddot{w}_5G, \underset{.}{u}_5U))$

= (((â$_0$, ẽ$_0$C, ẅ$_0$G, ṷ$_0$U) $-_N$ (â$_4$, ẽ$_4$C, ẅ$_4$G, ṷ$_4$U))$-_N$ (â$_2$, ẽ$_2$C, ẅ$_2$G, ṷ$_2$U), ((â$_1$, ẽ$_1$C, ẅ$_1$G, ṷ$_1$U) $-_N$ (â$_5$, ẽ$_5$C, ẅ$_5$G, ṷ$_5$U))$-_N$(â$_3$, ẽ$_3$C, ẅ$_3$G, ṷ$_3$U))

= [((â$_0$, ẽ$_0$C, ẅ$_0$G, ṷ$_0$U) $-_N$ (â$_4$, ẽ$_4$C, ẅ$_4$G, ṷ$_4$U)),((â$_1$, ẽ$_1$C, ẅ$_1$G, ṷ$_1$U)$-_N$(â$_5$, ẽ$_5$C, ẅ$_5$G, ṷ$_5$U))] $-_N$ ((â$_2$, ẽ$_2$C, ẅ$_2$G, ṷ$_2$U), (â$_3$, ẽ$_3$C, ẅ$_3$G, ṷ$_3$U))

=[((â$_0$, ẽ$_0$C, ẅ$_0$G, ṷ$_0$U), (â$_1$, ẽ$_1$C, ẅ$_1$G, ṷ$_1$U)) $-_N$ ((â$_4$, ẽ$_4$C, ẅ$_4$G, ṷ$_4$U), (â$_5$, ẽ$_5$C, ẅ$_5$G, ṷ$_5$U))]

$-_N$ ((â$_2$, ẽ$_2$C, ẅ$_2$G, ṷ$_2$U), (â$_3$, ẽ$_3$C, ẅ$_3$G, ṷ$_3$U))

= $(\acute{\alpha} -_N \theta) -_N \mu$.

Proposition 3.3. Assume that Ã̃$_1$(P_N) be a pentapartitioned neutrosophic subtraction algebra. Suppose that A be a classical subtraction algebra. Then, the structure (Ã̃$_1$(P_N)×A,$-_N$) is a pentapartitioned neutrosophic subtraction algebra.

Definition 3.4. Assume that (Ã̃(P_N), $-_N$) be a pentapartitioned neutrosophic subtraction algebra. Then, if the following criteria are met, a non-empty subset *A*(P_N) is referred to as a neutrosophic subtraction sub-algebra of Ã̃(P_N):

(i) A(P_N) ≠∅;

(ii) (â, ẽC, ẅG, ṷU) $-_N$ (â$_1$, ẽ$_1$C, ẅ$_1$G, ṷ$_1$U) ∈ A(P_N), for all (â, ẽC, ẅG, ṷU), (â$_1$, ẽ$_1$C, ẅ$_1$G, ṷ$_1$U)∈ A(P_N);

(iii) A(P_N) contains a proper subset which is a subtraction algebra.

A(P_N) is referred to as a pseudo pentapartitioned neutrosophic subtraction sub-algebra of Ã̃(P_N) if it does not contain a proper subset that is a subtraction algebra.

Definition 3.5. Suppose that (Ã̃, −, *) be any subtraction semi-group, and assume that Ã̃(P_N) = <Ã̃ ∪ C ∪ G ∪ U> be a set generated by Ã̃ and P_N. Then, (Ã̃(P_N), $-_N$, *) is referred to as a pentapartitioned neutrosophic subtraction semi-group. Let (â, ẽC, ẅG, ṷU) and (â$_1$, ẽ$_1$C, ẅ$_1$G, ṷ$_1$U) be any two elements of Ã̃(P_N) with â, ẽ, ẅ, ṷ, â$_1$, ẽ$_1$, ẅ$_1$, ṷ$_1$ ∈ Ã̃. Then we define the following:

(â, ẽC, ẅG, ṷU) $-_N$ (â$_1$, ẽ$_1$C, ẅ$_1$G, ṷ$_1$U) = (â – â$_1$, (ẽ – ẽ$_1$)C, (ẅ – ẅ$_1$)G, (ṷ – ṷ$_1$)U),

and (â, ẽC, ẅG, ṷU) * (â$_1$, ẽ$_1$C, ẅ$_1$G, ṷ$_1$U) = (ââ$_1$, (ẽâ$_1$+âẽ$_1$+ ẽẽ$_1$)C, (ẅâ$_1$+âẅ$_1$+ ẅẅ$_1$)G, (ṷâ$_1$+âṷ$_1$+ ṷṷ$_1$)U).

Definition 3.6. The direct product of two pentapartitioned neutrosophic subtraction semi-groups Ã̃$_1$(P_N) and Ã̃$_2$(P_N) is denoted by Ã̃$_1$(P_N) × Ã̃$_2$(P_N) and defined as follows:

Ã̃$_1$(P_N) × Ã̃$_2$(P_N) = {((â, ẽC, ẅG, ṷU), (â$_1$, ẽ$_1$C, ẅ$_1$G, ṷ$_1$U)) : (â, ẽC, ẅG, ṷU) ∈ Ã̃$_1$(P_N), (â$_1$, ẽ$_1$C, ẅ$_1$G, ṷ$_1$U) ∈ Ã̃$_2$(P_N)}.

Proposition 3.5. Assume that (Ã̃$_1$(P_N), $-_N$, *) and (Ã̃$_2$(P_N), $-_N$, *) be two pentapartitioned neutrosophic subtraction semi-group. Then, (Ã̃$_1$(P_N) × Ã̃$_2$(P_N), $-_N$, *) is also a pentapartitioned neutrosophic subtraction semi-group.

Proposition 3.6. Assume that (Ã̃(P_N), $-_N$, *) be a pentapartitioned neutrosophic subtraction semi-group, and suppose that (A, −, *) be a classical subtraction semi-group. Then, (Ã̃(P_N) × A, $-_N$, *) is a pentapartitioned neutrosophic subtraction semi-group.

4. Conclusions:

The basic ideas of pentapartitioned neutrosophic subtraction semi-group and pentapartitioned neutrosophic subtraction algebra have been examined in this paper. Further, the basic properties of subtraction algebra and subtraction semi-group have been analyzed and established. We believe that numerous fresh investigations can be executed in the years to come by utilizing the concepts of pentapartitioned neutrosophic subtraction semi-group and pentapartitioned neutrosophic subtraction algebra.

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