Neutrosophic Sets and Systems, Vol. 68, 2024
University of New Mexico

The indefinite refined neutrosophic integrals by parts

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Abstract: this article's goal is to present the indefinite refined neutrosophic integrals by parts. All situations where integration by parts can be used are covered, including the use of rotating integrals to solve recurring and non-terminating functions like the product of trigonometric and exponential functions. Furthermore, the Tabular method has been implemented in the computation of the indefinite refined neutrosophic integrals.

Keywords: parts; integrals; neutrosophic; Tabular method; indeterminacy$\boldsymbol{I_1}, \boldsymbol{I_2}$.

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1. Introduction and Preliminaries

To describe a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, and contradiction, Smarandache suggested the neutrosophic Logic as an alternative to the current logics. Smarandache made refined neutrosophic numbers available in the following form: $(a, b_1 I_1, b_2 I_2, \dots, b_n I_n)$ where $a, b_1, b_2, \dots, b_n \in R$ or C [1]

Agboola introduced the concept of refined neutrosophic algebraic structures [2]. Also, the refined neutrosophic rings I was studied in paper [3], where it assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. It then follows logically that: [3]

$$I_1 I_1 = {I_1}^2 = I_1 \qquad (1)$$
$$I_2 I_2 = {I_2}^2 = I_2 \qquad (2)$$
$$I_1 I_2 = I_2 I_1 = I_1 \qquad (3)$$

In addition, there are many papers presenting studies on refined neutrosophic numbers [4-5-6-7-8].
Smarandache discussed neutrosophic indefinite integral (Refined Indeterminacy) [11]
Let $g: \mathbb{R} \to \mathbb{R} \cup \{I_1\} \cup \{I_2\} \cup \{I_3\}$, where I_1, I_2, and I_3 are types of sub indeterminacies,

$$g(x) = 7x - 2I_1 + x^2 I_2 + 4x^3 I_3$$

then:

$$F(x) = \int [7x - 2I_1 + x^2 I_2 + 4x^3 I_3] dx$$

$$= \frac{7x^2}{2} - 2xI_1 + \frac{x^3}{3}I_2 + x^4 I_3 + a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3$$

where a_0, a_1, a_2 and a_3 are real constants.

Additionally, Alhasan gave multiple calculus presentations in which he covered neutrosophic definite and indefinite integrals. Also, he introduced the most significant uses of definite integrals in neutrosophic logic [9-10]. Several studies were also presented in the field of neutro logic in statistics and others [12-13].

2. Main Discussion

2.1 The indefinite refined neutrosophic integration by parts

Let: $f: D_f \subseteq R(I_1, I_2) \to R(I_1, I_2)$ and $g: D_g \subseteq R(I_1, I_2) \to R(I_1, I_2)$

then, for the product rule:

$$\frac{d}{dx}[f(x, I_1, I_2).g(x, I_1, I_2)] = \acute{f}(x, I_1, I_2)g(x, I_1, I_2) + f(x, I_1, I_2)\acute{g}(x, I_1, I_2)$$

integrating both sides of this equation gives us:

$$\int \frac{d}{dx}[f(x, I_1, I_2).g(x, I_1, I_2)] \ dx = \int \acute{f}(x, I_1, I_2)g(x, I_1, I_2) \ dx + \int f(x, I_1, I_2)\acute{g}(x, I_1, I_2) \ dx$$

$$\int f(x, I_1, I_2)\acute{g}(x, I_1, I_2) \ dx = f(x, I_1, I_2).g(x, I_1, I_2) - \int \acute{f}(x, I_1, I_2)g(x, I_1, I_2) \ dx$$

it is usually convenient to write this using the notation:

$$u_N = f(x, I_1, I_2) \quad \Rightarrow \quad du_N = \acute{f}(x, I_1, I_2) \ dx$$

$$dv_N = \acute{g}(x, I_1, I_2) \ dx \quad \Rightarrow \quad v_N = g(x, I_1, I_2)$$

so the neutrosophic integration by parts algorithm becomes

$$\int u_N \ dv_N = u_N.v_N - \int v_N \ du_N$$

There are four cases of the neutrosophic integration by parts:

- state1:

$$\int (a + bI_1 + cI_2)x^n \ e^{(r+sI_1+tI_2)x} \ dx$$

where a, b, c, r, s, t are real numbers, while $I_1, I_2 =$ indeterminacy), $r \neq 0$, $r \neq -t$ and $r \neq -s - t$.

$$u_N = (a + bI_1 + cI_2)x^n \quad \Rightarrow \quad du_N = n(a + bI_1 + cI_2)x^{n-1} \ dx$$

$$dv_N = e^{(r+sI_1+tI_2)x} \ dx \quad \Rightarrow \quad v_N = \frac{1}{r+sI_1+tI_2}e^{(r+sI_1+tI_2)x}$$

$$\int u_N \ dv_N = u_N.v_N - \int v_N \ du_N$$

$$\int (a + bI_1 + cI_2)x^n \ \ e^{(r+sI_1+tI_2)x} \ \ dx = \left(\frac{a + bI_1 + cI_2}{r + sI_1 + tI_2}\right)\left(x^n e^{(r+sI_1+tI_2)x} - \int nx^{n-1} \, e^{(r+sI_1+tI_2)x} dx\right) + C$$

$$= \left(\frac{a}{r} + \left[\frac{rb + bt - as - sc}{(r + t)(r + s + t)}\right] I_1 + \left[\frac{rc - at}{r(r + t)}\right] I_2\right)\left(x^n e^{(r+sI_1+tI_2)x} - \int nx^{n-1} \, e^{(r+sI_1+tI_2)x} dx\right) + C$$

by repeated the integration, then we can find the required integral
where C is an indeterminate real constant (i.e. constant of the form $a_0 + a_1I_1 + a_2I_2$, where a_0, a_1 and a_2 are real numbers, while $I_1, I_2 =$ indeterminacy).

Example 1

Find:

$$\int (2 + 2I_1 + I_2)x \ \ e^{(3+3I_1+2I_2)x} \ \ dx$$

Solution:

$$u_N = (2 + 2I_1 + I_2)x \quad \Rightarrow \quad du_N = (2 + 2I_1 + I_2) \ \ dx$$

$$dv_N = e^{(3+3I_1+2I_2)x} \ \ dx \quad \Rightarrow \quad v_N = \frac{1}{3+3I_1+2I_2} e^{(3+3I_1+2I_2)x}$$

$$\int u_N \ \ dv_N = u_N . v_N - \int v_N \ \ du_N$$

$$\int (2 + 2I_1 + I_2)x \ \ e^{(3+3I_1+2I_2)x} \ \ dx = \left(\frac{2 + 2I_1 + I_2}{3 + 3I_1 + 2I_2}\right)\left(xe^{(3+3I_1+2I_2)x} - \int e^{(3+3I_1+2I_2)x} \, dx\right)$$

$$= \left(\frac{2}{3} + \frac{18 + 12 - 12 - 9}{3(5)(8)} I_1 + \frac{3 - 4}{3(5)} I_2\right)\left(xe^{(3+3I_1+2I_2)x} - \frac{1}{3 + 3I_1 + 2I_2} e^{(3+3I_1+2I_2)x}\right)$$

$$= \left(\frac{2}{3} - \frac{9}{40} I_1 - \frac{1}{15} I_2\right)\left(x - \frac{1}{3} + \frac{3}{40} I_1 + \frac{2}{15} I_2\right) e^{(3+3I_1+2I_2)x} + C$$

- state2:

$$\int (a + bI_1 + cI_2)x^n \ \ sin(r + sI_1 + tI_2)x \ \ dx \quad or \quad \int (a + bI_1 + cI_2)x^n \ \ cos(r + sI_1 + tI_2)x \ \ dx$$

$$u_N = (a + bI_1 + cI_2)x^n \quad \Rightarrow \quad du_N = n(a + bI_1 + cI_2)x^{n-1} \ \ dx$$

$$dv_N = sin(r + sI_1 + tI_2)x \ \ dx \quad \Rightarrow \quad v_N = \frac{-1}{r + sI_1 + tI_2} cos(r + sI_1 + tI_2)x$$

$$\int u_N \ \ dv_N = u_N . v_N - \int v_N \ \ du_N$$

$$\int (a + bI_1 + cI_2)x^n \ \ sin(r + sI_1 + tI_2)x \ \ dx = \left(\frac{a + bI_1 + cI_2}{r + sI_1 + tI_2}\right)\left((a + bI_1 + cI_2)x^n \ \ sin(r + sI_1 + tI_2)x + \int nx^{n-1} \, cos(r + sI_1 + tI_2)x \ \ dx\right) + C$$

$$= \left(\frac{a}{r} + \left[\frac{rb + bt - as - sc}{(r+t)(r+s+t)}\right] I_1 + \left[\frac{rc - at}{r(r+t)}\right] I_2\right)\left((a + bI_1 + cI_2)x^n \ \ sin(r + sI_1 + tI_2)x + \int nx^{n-1} cos(r + sI_1 + tI_2)x \ \ dx\right) + C$$

By repeating the integration, we are able to find the required integral.
We calculate the second integral using the same method:

$$\int (a + bI_1 + cI_2)x^n \ \ cos(r + sI_1 + tI_2)x \ \ dx$$

Example 2
Find:

$$\int (3 + I_1 + 6I_2)x \ \ sin(1 + 2I_1 + 3I_2)x \ \ dx$$

Solution:

$$u_N = (3 + I_1 + 6I_2)x \qquad \Rightarrow \qquad du_N = (3 + I_1 + 6I_2) \ \ dx$$

$$dv_N = sin(1 + 2I_1 + 3I_2)x \ \ dx \qquad \Rightarrow \qquad v_N = \frac{-1}{1 + 2I_1 + 3I_2} cos(1 + 2I_1 + 3I_2)x$$

$$\int u_N \ \ dv_N = u_N . v_N - \int v_N \ \ du_N$$

$$\int (3 + I_1 + 6I_2)x \ \ sin(1 + 2I_1 + 3I_2)x \ \ dx$$

$$= \left(\frac{3 + I_1 + 6I_2}{1 + 2I_1 + 3I_2}\right)\left(-x \ \ cos(1 + 2I_1 + 3I_2)x + \int cos(1 + 2I_1 + 3I_2)x \, dx\right)$$

$$= \left(3 - \frac{7}{12} I_1 - \frac{3}{4} I_2\right)\left(-x \ \ cos(1 + 2I_1 + 3I_2)x + \frac{1}{1 + 2I_1 + 3I_2} \ \ sin(1 + 2I_1 + 3I_2)x\right)$$

$$= \left(3 - \frac{7}{12} I_1 - \frac{3}{4} I_2\right)\left(-x \ \ cos(1 + 2I_1 + 3I_2)x + \left(1 - \frac{1}{12} I_1 - \frac{3}{4} I_2\right) sin(1 + 2I_1 + 3I_2)x\right) + C$$

- state3:

$$\int e^{(a + bI_1 + cI_2)x} \ \ sin(r + sI_1 + tI_2)x \ \ dx \qquad or \qquad \int e^{(a + bI_1 + cI_2)x} \ \ cos(r + sI_1 + tI_2)x \ \ dx$$

$$u_N = e^{(a + bI_1 + cI_2)x} \qquad \Rightarrow \qquad du_N = (a + bI_1 + cI_2) e^{(a + bI_1 + cI_2)x} \ \ dx$$

$$dv_N = sin(r + sI_1 + tI_2)x \ \ dx \qquad \Rightarrow \qquad v_N = \frac{-1}{r + sI_1 + tI_2} cos(r + sI_1 + tI_2)x$$

$$\int u_N \ \ dv_N = u_N . v_N - \int v_N \ \ du_N$$

$$\int e^{(a + bI_1 + cI_2)x} \ \ sin(r + sI_1 + tI_2)x \ \ dx$$

$$= \left(\frac{-1}{r+sI_1+tI_2}\right) e^{(a+bI_1+cI_2)x} \;\; cos(r+sI_1+tI_2)x + \left(\frac{a+bI_1+cI_2}{r+sI_1+tI_2}\right) \int e^{(a+bI_1+cI_2)x} \;\; cos(r+sI_1+tI_2)x \;\; dx \qquad (*)$$

by using integration by parts again to evaluate:

$$\int e^{(a+bI_1+cI_2)x} \;\; cos(r+sI_1+tI_2)x \;\; dx$$

$$u_N = e^{(a+bI_1+cI_2)x} \quad \Longrightarrow \quad du_N = (a+bI_1+cI_2)e^{(a+bI_1+cI_2)x} \;\; dx$$

$$dv_N = cos(r+sI_1+tI_2)x \;\; dx \quad \Longrightarrow \quad v_N = \frac{1}{r+sI_1+tI_2} sin(r+sI_1+tI_2)x$$

$$\int u_N \;\; dv_N = u_N . v_N - \int v_N \;\; du_N$$

$$\int e^{(a+bI_1+cI_2)x} \;\; cos(r+sI_1+tI_2)x \;\; dx$$

$$= \left(\frac{1}{r+sI_1+tI_2}\right) sin(r+sI_1+tI_2)x + \left(\frac{a+bI_1+cI_2}{r+sI_1+tI_2}\right) \int e^{(a+bI_1+cI_2)x} \;\; sin(r+sI_1+tI_2)x \;\; dx$$

by substitution in $(*)$:

$$\int e^{(a+bI_1+cI_2)x} \;\; sin(r+sI_1+tI_2)x \;\; dx$$

$$= \left(\frac{-1}{r+sI_1+tI_2}\right) e^{(a+bI_1+cI_2)x} \;\; cos(r+sI_1+tI_2)x + \left(\frac{a+bI_1+cI_2}{r+sI_1+tI_2}\right) \left[\left(\frac{1}{r+sI_1+tI_2}\right) sin(r+sI_1+tI_2)x + \left(\frac{a+bI_1+cI_2}{r+sI_1+tI_2}\right) \int e^{(a+bI_1+cI_2)x} \;\; sin(r+sI_1+tI_2)x \;\; dx\right]$$

$$= \left(\frac{-1}{r+sI_1+tI_2}\right) e^{(a+bI_1+cI_2)x} \;\; cos(r+sI_1+tI_2)x + \left(\frac{a+bI_1+cI_2}{(r+sI_1+tI_2)^2}\right) sin(r+sI_1+tI_2)x + \left(\frac{a+bI_1+cI_2}{r+sI_1+tI_2}\right)^2 \int e^{(a+bI_1+cI_2)x} \;\; sin(r+sI_1+tI_2)x \;\; dx$$

$$\Longrightarrow \;\; \left(1 - \left(\frac{a+bI_1+cI_2}{r+sI_1+tI_2}\right)^2\right) \int e^{(a+bI_1+cI_2)x} \;\; sin(r+sI_1+tI_2)x \;\; dx$$

$$= \left(\frac{-1}{r+sI_1+tI_2}\right) e^{(a+bI_1+cI_2)x} \;\; cos(r+sI_1+tI_2)x + \left(\frac{a+bI_1+cI_2}{(r+sI_1+tI_2)^2}\right) sin(r+sI_1+tI_2)x$$

$$\Longrightarrow \;\; \int e^{(a+bI_1+cI_2)x} \;\; sin(r+sI_1+tI_2)x \;\; dx$$

$$= \left(\frac{(r+sI_1+tI_2)^2}{(r+sI_1+tI_2)^2-(a+bI_1+cI_2)^2}\right)\left[\left(\frac{-1}{r+sI_1+tI_2}\right)e^{(a+bI_1+cI_2)x}\ cos(r+sI_1+tI_2)x + \left(\frac{a+bI_1+cI_2}{(r+sI_1+tI_2)^2}\right)sin(r+sI_1+tI_2)x + C\right]$$

We calculate the second integral by using the same method:

$$\int e^{(a+bI_1+cI_2)x}\ \ cos(r+sI_1+tI_2)x\ \ dx$$

Example 3

Find:

$$\int e^{(1-I_1+I_2)x}\ \ cos(1+I_1+I_2)x\ \ dx$$

Solution:

$$u_N = e^{(1-I_1+I_2)x} \quad \Longrightarrow \quad du_N = (1-I_1+I_2)e^{(1-I_1+I_2)x}\ \ dx$$

$$dv_N = cos(1+I_1+I_2)x\ \ dx \quad \Longrightarrow \quad v_N = \frac{1}{1+I_1+I_2}sin(1+I_1+I_2)x$$

$$\int u_N\ \ dv_N = u_N.v_N - \int v_N\ \ du_N$$

$$\int e^{(1-I_1+I_2)x}\ \ cos(1+I_1+I_2)x\ \ dx$$

$$= \frac{1}{1+I_1+I_2}e^{(1-I_1+I_2)x}sin(1+I_1+I_2)x - \left(\frac{1-I_1+I_2}{1+I_1+I_2}\right)\int e^{(1-I_1+I_2)x}\ \ sin(1+I_1+I_2)x\ \ dx \quad (*)$$

By using integration by parts again to evaluate:

$$\int e^{(1-I_1+I_2)x}\ \ sin(1+I_1+I_2)x\ \ dx$$

$$u_N = e^{(1-I_1+I_2)x} \quad \Longrightarrow \quad du_N = (1-I_1+I_2)e^{(1-I_1+I_2)x}\ \ dx$$

$$dv_N = sin(1+I_1+I_2)x\ \ dx \quad \Longrightarrow \quad v_N = \frac{-1}{1+I_1+I_2}cos(1+I_1+I_2)x$$

$$\int e^{(1-I_1+I_2)x}\ \ sin(1+I_1+I_2)x\ \ dx$$

$$= \frac{-1}{1+I_1+I_2}e^{(1-I_1+I_2)x}cos(1+I_1+I_2)x + \left(\frac{1-I_1+I_2}{1+I_1+I_2}\right)\int e^{(1-I_1+I_2)x}\ \ cos(1+I_1+I_2)x\,dx$$

by substitution in $(*)$:

$$\int e^{(1-I_1+I_2)x}\ \ cos(1+I_1+I_2)x\ \ dx$$

$$=\frac{1}{1+I_1+I_2}e^{(1-I_1+I_2)x}sin(1+I_1+I_2)x$$
$$-\left(\frac{1-I_1+I_2}{1+I_1+I_2}\right)\left[\frac{-1}{1+I_1+I_2}e^{(1-I_1+I_2)x}cos(1+I_1+I_2)x\right.$$
$$\left.+\left(\frac{1-I_1+I_2}{1+I_1+I_2}\right)\int e^{(1-I_1+I_2)x}\ \ cos(1+I_1+I_2)x\,dx\ \right]$$

$$=\frac{1}{1+I_1+I_2}e^{(1-I_1+I_2)x}sin(1+I_1-2I_2)x+\frac{1-I_1+I_2}{(1+I_1+I_2)^2}e^{(1-I_1+I_2)x}\ \ cos(1+I_1+I_2)x$$
$$-\left(\frac{1-I_1+I_2}{1+I_1+I_2}\right)^2\int e^{(1-I_1+I_2)x}\ \ cos(1+I_1+I_2)x\,dx$$

$$\Rightarrow\left(1+\left(\frac{1-I_1+I_2}{1+I_1+I_2}\right)^2\right)\int e^{(1-I_1+I_2)x}\ \ cos(1+I_1+I_2)x\,dx$$
$$=\frac{1}{1+I_1+I_2}e^{(1-I_1+I_2)x}sin(1+I_1+I_2)x+\frac{1-I_1+I_2}{(1+I_1+I_2)^2}e^{(1-I_1+I_2)x}\ \ cos(1+I_1+I_2)x$$

$$\Rightarrow\left(1+\left(1-\frac{2}{3}I_1\right)^2\right)\int e^{(1-I_1+I_2)x}\ \ cos(1+I_1+I_2)x\,dx$$
$$=\frac{1}{1+I_1+I_2}e^{(1-I_1+I_2)x}sin(1+I_1+I_2)x+\frac{1-I_1+I_2}{1+5I_1+3I_2}e^{(1-I_1+I_2)x}\ \ cos(1+I_1+I_2)x$$

$$\Rightarrow\left(1-\frac{8}{9}I_1\right)\int e^{(1-I_1+I_2)x}\ \ cos(1+I_1+I_2)x\,dx$$
$$=\frac{1}{1+I_1+I_2}e^{(1-I_1+I_2)x}sin(1+I_1+I_2)x+\frac{1-I_1+I_2}{1+5I_1+3I_2}e^{(1-I_1+I_2)x}\ \ cos(1+I_1+I_2)x$$

$$=\frac{1}{1-\frac{8}{9}I_1}\left[\left(1-\frac{1}{6}I_1-\frac{1}{2}I_2\right)e^{(1-I_1+I_2)x}sin(1+I_1+I_2)x\right.$$
$$\left.+\left(1-\frac{7}{18}I_1-\frac{1}{2}I_2\right)e^{(1-I_1+I_2)x}\ \ cos(1+I_1+I_2)x\right]+C$$

$$=(1+8I_1)\left[\left(1-\tfrac{1}{6}I_1-\tfrac{1}{2}I_2\right)e^{(1-I_1+I_2)x}sin(1+I_1+I_2)x+\left(1-\tfrac{7}{18}I_1-\tfrac{1}{2}I_2\right)e^{(1-I_1+I_2)x}\ \ cos(1+I_1+I_2)x\right]+C$$

- state4:

$$\int(a+bI_1+cI_2)x^n\ \ ln(r+sI_1+tI_2)x\ \ dx$$

$$u_N=ln(r+sI_1+tI_2)x\qquad\Rightarrow\qquad du_N=\frac{1}{x}\ \ dx$$

$$dv_N=(a+bI_1+cI_2)x^n\ \ dx\qquad\Rightarrow\qquad v_N=\frac{a+bI_1+cI_2}{n+1}x^{n+1}$$

$$\int u_N \ \ dv_N = u_N . v_N - \int v_N \ \ du_N$$

$$\int (a+bI_1+cI_2)x^n \ \ ln(r+sI_1+tI_2)x \ \ dx$$
$$= \left(\frac{a+bI_1+cI_2}{n+1}\right)x^{n+1}.\,ln(r+sI_1+tI_2)x - \frac{a+bI_1+cI_2}{n+1}\int \frac{1}{x}x^{n+1} \ \ dx$$

$$= \left(\frac{a+bI_1+cI_2}{n+1}\right)x^{n+1}.\,ln(r+sI_1+tI_2)x - \frac{a+bI_1+cI_2}{(n+1)^2}x^{n+1} + C$$

$$= \left(\frac{a+bI_1+cI_2}{n+1}\right)\left[x^{n+1}.\,ln(r+sI_1+tI_2)x - \frac{1}{n+1}x^{n+1}\right] + C$$

Example 4

Find:

$$\int (1+3I_1-2I_2)x \ \ ln(2+I_1+I_2)x \ \ dx$$

Solution:

$$u_N = ln(2+I_1+I_2)x \ \ \Rightarrow \ \ du_N = \frac{1}{x} \ \ dx$$

$$dv_N = (1+3I_1-2I_2)x \ \ dx \qquad \Rightarrow \qquad v_N = \frac{1}{2}(1+3I_1-2I_2)x^2$$

$$\int u_N \ \ dv_N = u_N . v_N - \int v_N \ \ du_N$$

$$\int (1+3I_1-2I_2)x \ \ ln(2+I_1+I_2)x \ \ dx$$
$$= \frac{1}{2}(1+3I_1-2I_2)x^2.\,ln(2+I_1+I_2)x - \frac{1}{2}(1+3I_1-2I_2)\int xdx$$

$$= \frac{1}{2}(1+3I_1-2I_2)x^2 ln(2+I_1+I_2)x - \frac{1}{2}(1+3I_1-2I_2).\frac{x^2}{2}$$

$$= \left(\frac{1}{2}+\frac{3}{2}I_1 - I_2\right)\left[x^2 ln(2+I_1+I_2)x - \frac{1}{2}x^2\right] + C$$

Remark:

To find the following integrals:

1) $\int (a+bI_1+cI_2)x^n \ \ sin^{-1}(r+sI_1+tI_2)x\, dx$

2) $\int (a+bI_1+cI_2)x^n \ \ cos^{-1}(r+sI_1+tI_2)x \ \ dx$

3) $\int (a+bI_1+cI_2)x^n \ \ tan^{-1}(r+sI_1+tI_2)x\, dx$

we are following the same state 4, whereas:

$$u = \quad sin^{-1}(r + sI_1 + tI_2)x \qquad Or \qquad cos^{-1}(r + sI_1 + tI_2)x \qquad Or \qquad tan^{-1}(r + sI_1 + tI_2)x$$

$$and \quad dv = (a + bI_1 + cI_2)x^n \;\; dx$$

Example 5

Find:

$$\int (2 - 2I_1 - I_2)x \;\; tan^{-1}(1 - I_1 + 2I_2)x \;\; dx$$

Solution:

$$u_N = tan^{-1}(1 - I_1 + 2I_2)x \;\; \Rightarrow \quad du_N = \frac{1 - I_1 + 2I_2}{1 + (1 - 5I_1 + 8I_2)x^2} \;\; dx$$

$$dv_N = (2 - 2I_1 - I_2)x \;\; dx \qquad \Rightarrow \qquad v_N = \frac{1}{2}(2 - 2I_1 - I_2)x^2$$

$$\int u_N \;\; dv_N = u_N . v_N - \int v_N \;\; du_N$$

$$\int (2 - 2I_1 - I_2)x \;\; tan^{-1}(1 - I_1 + 2I_2)x \;\; dx$$

$$= \frac{1}{2}(2 - 2I_1 - I_2)x^2 \;\; tan^{-1}(1 - I_1 + 2I_2)x - \frac{1}{2}(2 - 2I_1 - I_2)(1 - I_1 + 2I_2) \int \frac{x^2}{1 + (1 - 5I_1 + 8I_2)x^2} \;\; dx$$

$$= \frac{1}{2}(2 - 2I_1 - I_2)x^2 \;\; tan^{-1}(1 - I_1 + 2I_2)x - \frac{2 - 5I_1 + I_2}{2} \int \left(\frac{1}{1 - 5I_1 + 8I_2} - \frac{1}{1 - 5I_1 + 8I_2} \;\; \frac{1}{1 + (1 - 5I_1 + 8I_2)x^2} \right) \;\; dx$$

$$= \frac{1}{2}(2 - 2I_1 - I_2)x^2 \;\; tan^{-1}(1 - I_1 + 2I_2)x - \frac{2 - 5I_1 + I_2}{2} \left(\frac{1}{1 - 5I_1 + 8I_2} x - \frac{1 - I_1 + 2I_2}{1 - 5I_1 + 8I_2} \quad tan^{-1}(1 - I_1 + 2I_2)x \right) + C$$

$$= \left(1 - I_1 - \frac{1}{2}I_2\right)x^2 \;\; tan^{-1}(1 - I_1 + 2I_2)x - \left(1 - \frac{5}{2}I_1 + \frac{1}{2}I_2\right)\left[\left(1 - \frac{5}{36}I_1 + \frac{8}{9}I_2\right)x - \left(1 - \frac{1}{6}I_1 - \frac{2}{3}I_2\right) tan^{-1}(1 - I_1 + 2I_2)x\right] + C$$

2.2 Tabular method

We use this method to find the integrals by parts in the states 1 and 2, as following:

- Differentiate the polynomial function, and we repeat that until we get to zero.

- Integrate the second function, repeat that, and stop once we reach the zero that resulted from the differentiation.
- Put the derivative products in one column and the integral products in the column that corresponds to it.
- Draw an arrow from each first-column entry to the second-column entry one row below it.
- Beginning with a +, label the arrows with alternating + and − signs.
- Compute the product of the expressions at the tip and tail of each arrow, and then multiply the result by + or −, depending on the arrow's sign.

Example 6

We can find the following integral by using tabular method:

$$\int (1+3I_1-2I_2)x^2\, e^{(2+I_1+I_2)x}\ \ dx$$

Derivation	Integration
$(+)\ \ (1+3I_1-2I_2)x^2$	$e^{(2+I_1+I_2)x}$
$(-)\ \ 2(1+3I_1-2I_2)x$	$\frac{1}{2+I_1+I_2}e^{(2+I_1+I_2)x}$
$(+)\ \ 2(1+3I_1-2I_2)$	$\frac{1}{4+7I_1+5I_2}e^{(2+I_1+I_2)x}$
0	$\frac{1}{16+175I_1+65I_2}e^{(2+I_1+I_2)x}$

hence:

$$\int (1+3I_1-2I_2)x^2\, e^{(2+I_1+I_2)x}\ \ dx$$

$$=\left(\frac{1+3I_1-2I_2}{2+I_1+I_2}\right)x^2e^{(2+I_1+I_2)x}-\left(\frac{4+6I_1-4I_2}{4+7I_1+5I_2}\right)xe^{(2+I_1+I_2)x}+\left(\frac{4+6I_1-4I_2}{16+175I_1+65I_2}\right)e^{(2+I_1+I_2)x}$$

$$=\left(\frac{1}{4}+\frac{5}{12}I_1-\frac{5}{12}I_2\right)x^2e^{(2+I_1+I_2)x}-\left(1+\frac{3}{8}I_1-I_2\right)xe^{(2+I_1+I_2)x}+\left(\frac{1}{4}+\frac{3}{128}I_1-\frac{1}{4}I_2\right)e^{(2+I_1+I_2)x}+C$$

Example 7

We can find the following integral by using tabular method:

$$\int (I_1+I_2)x\,cos(1+I_1+I_2)x\ \ dx$$

derivation	integration
$(+)\ \ (I_1+I_2)x$	$cos(1+I_1+I_2)x$
$(-)\ \ (I_1+I_2)$	$\frac{1}{1+I_1+I_2}sin(1+I_1+I_2)x$
$(+)\ \ \ 0$	$\frac{-1}{1+5I_1+3I_2}cos(1+I_1+I_2)x$

hence:

$$\int (I_1+I_2)x\,cos(1+I_1+I_2)x \quad dx = \frac{I_1+I_2}{1+I_1+I_2}x.sin(1+I_1+I_2)x + \frac{I_1+I_2}{1+5I_1+3I_2}cos(1+I_1+I_2)x$$

$$= \left(\frac{1}{6}I_1+\frac{1}{2}I_2\right)x.sin(1+I_1+I_2)x + \left(-\frac{1}{36}I_1+\frac{1}{4}I_2\right)cos(1+I_1+I_2)x$$

3. Conclusions

This paper is an extension of the papers that were presented on the indefinite refined neutrosophic integrals. The importance of this paper lies in that it presented the indefinite refined neutrosophic integrals by parts and the Tobler method, as we found that applying the Tobler method is easier to calculate the indefinite refined neutrosophic integrals than the indefinite refined neutrosophic integrals by parts for some cases.

Acknowledgments "This study is supported via funding from Prince sattam bin Abdulaziz University project number (PSAU/2024/R/1445)".

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Received: Mar 3, 2024. Accepted: May 28, 2024