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Fixed Point Results in Neutrosophic b- Metric Like Spaces

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Abstract: In this manuscript we introduce the idea of Neutrosophic b-metric like spaces along with numerous significant illustrations. Moreover, we present numerous fixed point results in Neutrosophic b-metric like space and we gave some examples to support our main results.

Keywords: b-metric like spaces; Neutrosophic b-metric like spaces; Fixed point; Contractive map; Unique solution.

1. Introduction

Zadeh [20], laid the foundation for fuzzy mathematics in 1965. Kramosil and Michalek[11] initially brought up the idea of fuzzy metric space and then refitted by George and Veeramani[4]. Harandi [5] imported the perception of metric-like spaces, which nicely broden the concept of metric spaces.b-MetricLike Spaces [$bMLS$] were first discussed by Alghamdiet al.[2] using the idea of metric like spaces. In this manner, Fuzzy Metric Like Space [$FMLS$] were developed by Shukla and Abbas [15] they also developed the concept of metric like space. Park [13] proposed the idea of intuitionistic fuzzy metric spaces.Konwar [10] developed the concept of intuitionistic fuzzy b-metric space . We see [1, 6, 12, 19, 16, 17] for certain required definitions. In the framework of b-metric spaces,Delfani et al. [3] demonstrated several fixed point results.In 1998, Smarandache[18] developed the ideas of neutrosophic logic and Neutrosophic Set [NS]. Kirisci and Simsek[9] founded the concept of Neutrosophic Metric Spaces [NMS]which addresess membership, non-membership and neutralness.

We introduce the notion of Netrosophic$b$-Metric Spaces [$NbMS$]in order to generalise the notion of $NbMS$ and demonstrate several fixed point findings in this framework.We also furnish this work with examples.

2. Preliminaries

Definition 2.1. [2]A $bMLS$ on a set $\mathfrak{S} \neq \emptyset$ is a function $\varphi : \mathfrak{S} \times \mathfrak{S} \to [0, +\infty)$ such that for all $\hbar, \mathfrak{q}, \mathfrak{r} \in \mathfrak{S}$ and $b \geq 1$, if it enjoys the conditions listed below :

1. If $\varphi(\hbar, \mathfrak{q}) = 0$, then $\hbar = \mathfrak{q}$;
2. $\varphi(\hbar, \mathfrak{q}) = \varphi(\mathfrak{q}, \hbar)$;
3. $\varphi(\hbar, \mathfrak{q}) \leq b[\varphi(\hbar, \mathfrak{r}) + \varphi(\mathfrak{r}, \mathfrak{q})]$.

The pair $(\mathfrak{S}, \varphi)$ is said to be $bMLS$.

Example 2.2.[2] Let$\mathfrak{S} = [0, \infty)$. Define $\varphi : \mathfrak{S} \times \mathfrak{S} \to [0, +\infty)$by $\varphi(\hbar, \mathfrak{q}) = (\hbar + \mathfrak{q})^2$. Then $(\mathfrak{S}, \varphi)$ is a $bMLS$ with $b = 1$.

Definition 2.3. [5]A 3-tuple $(\mathfrak{S}, \mathfrak{A}, *)$ is named to be a*FMLS* if $\mathfrak{S} \neq \emptyset$ is a random set, $*$ is a *CTN* and and $\mathfrak{A}$ is a *FS* on $\mathfrak{S} \times \mathfrak{S} \times (0, \infty)$ such that for all $\hbar, \mathfrak{q}, \mathfrak{r} \in \mathfrak{S}$, $\vartheta, \omega > 0$,
$FL1$). $\mathfrak{A}(\hbar, \mathfrak{q}, \vartheta) > 0$;
$FL2$). If $\mathfrak{A}(\hbar, \mathfrak{q}, \vartheta) = 1$, then $\hbar = \mathfrak{q}$;
$FL3$). $\mathfrak{A}(\hbar, \mathfrak{q}, \vartheta) = \mathfrak{A}(\mathfrak{q}, \hbar, \vartheta)$;
$FL4$). $\mathfrak{A}(\hbar, \mathfrak{r}, \vartheta + \omega) \geq \mathfrak{A}(\hbar, \mathfrak{q}, \vartheta) * \mathfrak{A}(\mathfrak{q}, \mathfrak{r}, \omega)$;
$FL5$). $\mathfrak{A}(\hbar, \mathfrak{q}, \cdot): (0, \infty) \to [0,1]$is continuous.

Example 2.4.[15] Let $\mathfrak{S} = \mathbb{R}^+, \rho \in \mathbb{R}^+$and $m > 0$. Define CTN by $\hbar * \mathfrak{q} = \hbar\mathfrak{q}$ and $\mathfrak{A}$ by $\mathfrak{A}(\hbar, \mathfrak{q}, \vartheta) = \frac{\rho\vartheta}{\rho\vartheta + m(\max\{\hbar, \mathfrak{q}\})}$ for all $\hbar, \mathfrak{q} \in \mathfrak{S}, \vartheta > 0$. Then $(\mathfrak{S}, \mathfrak{A}, *)$ is an *FMLS*.

Definition 2.5. Let $\mathfrak{S} \neq \emptyset$. For a sixtuple $(\mathfrak{S}, \mathfrak{A}_{b\ell}, \mathfrak{B}_{b\ell}, \mathfrak{C}_{b\ell}, *, \diamond)$, where $*$ is a *CTN*, $\diamond$ is a *CTCN*, $b \geq 1$ and$\mathfrak{A}_{b\ell}, \mathfrak{B}_{b\ell}, \mathfrak{C}_{b\ell}$ are FS on $\mathfrak{S} \times \mathfrak{S} \times (0, \infty)$, if$(\mathfrak{S}, \mathfrak{A}_{b\ell}, \mathfrak{B}_{b\ell}, \mathfrak{C}_{b\ell}, *, \diamond)$ satisfies the following, for all $\hbar, \mathfrak{q} \in \mathfrak{S}$ and $\vartheta, \omega > 0$,

1. $0 \leq \mathfrak{A}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) \leq 1; 0 \leq \mathfrak{B}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) \leq 1; 0 \leq \mathfrak{C}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) \leq 1$;
2. $\mathfrak{A}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) + \mathfrak{B}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) + \mathfrak{C}_{b}(\hbar, \mathfrak{q}, \vartheta) \leq 3$;
3. $\mathfrak{A}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 1 \Leftrightarrow \hbar = \mathfrak{q}$;
4. $\mathfrak{A}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = \mathfrak{A}_{b\ell}(\mathfrak{q}, \hbar, \vartheta)$;
5. $\mathfrak{A}_{b\ell}(\hbar, \mathfrak{r}, b(\vartheta + \omega)) \geq \mathfrak{A}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) * \mathfrak{A}_{b\ell}(\mathfrak{q}, \mathfrak{r}, \omega)$;
6. $\mathfrak{A}_{b\ell}(\hbar, \mathfrak{q}, .) : [0, \infty) \to [0,1]$ is neutrosophic continuous;
7. $\lim_{\vartheta \to \infty} \mathfrak{A}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 1$;
8. $\mathfrak{B}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 0 \Leftrightarrow \hbar = \mathfrak{q}$;
9. $\mathfrak{B}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = \mathfrak{B}_{b\ell}(\mathfrak{q}, \hbar, \vartheta)$;
10. $\mathfrak{B}_{b\ell}(\hbar, \mathfrak{r}, b(\vartheta + \omega)) \leq \mathfrak{B}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) \diamond \mathfrak{B}_{b\ell}(\mathfrak{q}, \mathfrak{r}, \omega)$;
11. $\mathfrak{B}_{b\ell}(\hbar, \mathfrak{q}, .) : [0, \infty) \to [0,1]$ is neutrosophic continuous;
12. $\lim_{\vartheta \to \infty} \mathfrak{B}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 0$;
13. $\mathfrak{C}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 0 \Leftrightarrow \hbar = \mathfrak{q}$;
14. $\mathfrak{C}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = \mathfrak{C}_{b\ell}(\mathfrak{q}, \hbar, \vartheta)$;
15. $\mathfrak{C}_{b\ell}(\hbar, \mathfrak{r}, b(\vartheta + \omega)) \leq \mathfrak{C}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) \diamond \mathfrak{C}_{b\ell}(\mathfrak{q}, \mathfrak{r}, \omega)$;
16. $\mathfrak{C}_{b\ell}(\hbar, \mathfrak{q}, .) : [0, \infty) \to [0,1]$ is neutrosophic continuous;
17. $\lim_{\vartheta \to \infty} \mathfrak{C}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 0$;
18. $\vartheta < 0$ then $\mathfrak{A}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 0, \mathfrak{B}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 1, \mathfrak{C}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 1$.

Then $(\mathfrak{S}, \mathfrak{A}_{b\ell}, \mathfrak{B}_{b\ell}, \mathfrak{C}_{b\ell}, *, \diamond)$ is called a*NbMLS*.

Remark 2.6. In the above definition, assume that a set $\mathfrak{S}$ is a *NbMLS* with a *CTN*$(*)$ and *CTCN*$(\diamond)$. Then the *NbMLS*$\mathfrak{S}$ does not satisfy (3),(8) and (13) of *NbMS*, that is, the self-distance may not be equal to 1 and 0, i.e., $\mathfrak{A}_{b\ell}(\hbar, \hbar, \vartheta) \neq 1$, $\mathfrak{B}_{b\ell}(\hbar, \hbar, \vartheta) \neq 0$ and $\mathfrak{C}_{b\ell}(\hbar, \hbar, \vartheta) \neq 0$ for all $\vartheta > 0$ or may be for all $\vartheta \in \mathfrak{S}$. But all other conditions are the same.

Example 2.7. Let $\mathfrak{S} = (0, \infty)$. Define a *CTN* by $\mathfrak{u} * \mathfrak{v} = \mathfrak{u}\mathfrak{v}$ and a *CTCN* by $\mathfrak{u} \diamond \mathfrak{v} = \max\{\mathfrak{u}, \mathfrak{v}\}$ and also define $\mathfrak{A}_{b\ell}, \mathfrak{B}_{b\ell}$ and $\mathfrak{C}_{b\ell}$by

$$\mathfrak{A}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = \left[e^{\frac{(\hbar+\mathfrak{q})^2}{\vartheta}}\right]^{-1}, \mathfrak{B}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = 1 - \left[e^{\frac{(\hbar+\mathfrak{q})^2}{\vartheta}}\right]^{-1} \text{and } \mathfrak{C}_{b\ell}(\hbar, \mathfrak{q}, \vartheta) = \left[e^{\frac{(\hbar+\mathfrak{q})^2}{\vartheta}}\right] - 1$$

for all $\hbar, \mathfrak{q} \in \mathfrak{S}, \vartheta > 0$. Then it is a*NbMLS*. But it is not a*NbMS*.

Remark 2.8 The above example shows that *NbMLS* need not be an *NbMS*. Also every *NbMS* must be an *NbMLS*.

The following example shows that *NbMLS* need not be continuous.

Example 2.9 Let $= [0,\infty)$, $\mathfrak{A}_{\ell\ell}(\hbar,\mathrm{q},\vartheta) = e^{-\frac{\varphi(\hbar+\mathrm{q})}{\vartheta}}$, $\mathfrak{B}_{\ell\ell}(\hbar,\mathrm{q},\vartheta) = 1-\left(e^{-\frac{\varphi(\hbar+\mathrm{q})}{\vartheta}}\right)$ and

$\mathfrak{C}_{\ell\ell}(\hbar,\mathrm{q},\vartheta) = e^{\frac{\varphi(\hbar+\mathrm{q})}{\vartheta}} - 1$ for all $\hbar,\mathrm{q}\in\mathfrak{S},\vartheta>0$ and $\varphi(\hbar,\mathrm{q}) = \begin{cases} 0, & \text{if } \hbar=\mathrm{q} \\ 2(\hbar+\mathrm{q})^2, & \text{if } \hbar,\mathrm{q}\in[0,1] \\ \frac{1}{2}(\hbar+\mathrm{q})^2, & \text{otherwise.} \end{cases}$

Define a *CTN* by $\mathfrak{u}*\mathfrak{v}=\mathfrak{u}\mathfrak{v}$ and a *CTCN* by $\mathfrak{u}\diamond\mathfrak{v}=\max\{\mathfrak{u},\mathfrak{v}\}$. Then $(\mathfrak{S},\mathfrak{A}_{\ell\ell},\mathfrak{B}_{\ell\ell},\mathfrak{C}_{\ell\ell},*,\diamond)$ is a *NbMLS* with a coefficient *b*=4. To illustrate the discontinuity, we have

$$\lim_{\ddot{\mathrm{n}}\to\infty}\mathfrak{A}_{\ell\ell}\left(0,\frac{1}{\ddot{\mathrm{n}}},\vartheta\right) = \lim_{\ddot{\mathrm{n}}\to\infty}e^{-2(1-\frac{1}{\ddot{\mathrm{n}}})^2} = \mathrm{e}^{-2} = \mathfrak{A}_{\ell\ell}(0,1,\vartheta)$$

$$\lim_{\ddot{\mathrm{n}}\to\infty}\mathfrak{B}_{\ell\ell}\left(0,\frac{1}{\ddot{\mathrm{n}}},\vartheta\right) = 1-\lim_{\ddot{\mathrm{n}}\to\infty}e^{-2(1-\frac{1}{\ddot{\mathrm{n}}})^2} = 1-\mathrm{e}^{-2} = \mathfrak{B}_{\ell\ell}(0,1,\vartheta) \text{ and}$$

$$\lim_{\ddot{\mathrm{n}}\to\infty}\mathfrak{C}_{\ell\ell}\left(0,\frac{1}{\ddot{\mathrm{n}}},\vartheta\right) = \lim_{\ddot{\mathrm{n}}\to\infty}e^{2(1-\frac{1}{\ddot{\mathrm{n}}})^2}-1 = \mathrm{e}^{2}-1 = \mathfrak{C}_{\ell\ell}(0,1,\vartheta)$$

Since,

$$\lim_{\ddot{\mathrm{n}}\to\infty}\mathfrak{A}_{\ell\ell}\left(1,1-\frac{1}{\ddot{\mathrm{n}}},\vartheta\right) = \lim_{\ddot{\mathrm{n}}\to\infty}e^{-2(2-\frac{1}{\ddot{\mathrm{n}}})^2} = \mathrm{e}^{-8} \neq 1 = \mathfrak{A}_{\ell\ell}(1,1,\vartheta)$$

$$\lim_{\ddot{\mathrm{n}}\to\infty}\mathfrak{B}_{\ell\ell}\left(1,1-\frac{1}{\ddot{\mathrm{n}}},\vartheta\right) = 1-\lim_{\ddot{\mathrm{n}}\to\infty}e^{-2(2-\frac{1}{\ddot{\mathrm{n}}})^2} = 1-\mathrm{e}^{-8} \neq 0 = \mathfrak{B}_{\ell\ell}(1,1,\vartheta)$$

$$\lim_{\ddot{\mathrm{n}}\to\infty}\mathfrak{C}_{\ell\ell}\left(1,1-\frac{1}{\ddot{\mathrm{n}}},\vartheta\right) = \lim_{\ddot{\mathrm{n}}\to\infty}e^{2(2-\frac{1}{\ddot{\mathrm{n}}})^2}-1 = \mathrm{e}^{8}-1 \neq 0 = \mathfrak{C}_{\ell\ell}(0,1,\vartheta)$$

$\mathfrak{A}_{\ell\ell}(\hbar,\mathrm{q},\vartheta)$, $\mathfrak{B}_{\ell\ell}(\hbar,\mathrm{q},\vartheta)$ and $\mathfrak{C}_{\ell\ell}(\hbar,\mathrm{q},\vartheta)$ are not continuous.

Definition 2.10. A sequence $\{\hbar_{\ddot{\mathrm{n}}}\}$ in an *NbMLS*$(\mathfrak{S},\mathfrak{A}_{\ell\ell},\mathfrak{B}_{\ell\ell},\mathfrak{C}_{\ell\ell},*,\diamond)$ is converge to $\hbar\in\mathfrak{S}$ if $\lim_{\ddot{\mathrm{n}}\to\infty}\mathfrak{A}_{\ell\ell}(\hbar_{\ddot{\mathrm{n}}},\hbar,\vartheta) = \mathfrak{A}_{\ell\ell}(\hbar,\hbar,\vartheta)$, $\lim_{\ddot{\mathrm{n}}\to\infty}\mathfrak{B}_{\ell\ell}(\hbar_{\ddot{\mathrm{n}}},\hbar,\vartheta) = \mathfrak{B}_{\ell\ell}(\hbar,\hbar,\vartheta)$ and $\lim_{\ddot{\mathrm{n}}\to\infty}\mathfrak{C}_{\ell\ell}(\hbar_{\ddot{\mathrm{n}}},\hbar,\vartheta) = \mathfrak{C}_{\ell\ell}(\hbar,\hbar,\vartheta)$ for all $\vartheta>0$.

Definition 2.11. A sequence $\{\hbar_{\ddot{\mathrm{n}}}\}$ in an *NbMLS*$(\mathfrak{S},\mathfrak{A}_{\ell\ell},\mathfrak{B}_{\ell\ell},\mathfrak{C}_{\ell\ell},*,\diamond)$ is named to be a Cauchy sequence if $\lim_{\ddot{\mathrm{n}}\to\infty}\mathfrak{A}_{\ell\ell}(\hbar_{\ddot{\mathrm{n}}},\hbar_{\ddot{\mathrm{n}}+p},\vartheta)$, $\lim_{\ddot{\mathrm{n}}\to\infty}\mathfrak{B}_{\ell\ell}(\hbar_{\ddot{\mathrm{n}}},\hbar_{\ddot{\mathrm{n}}+p},\vartheta)$ and $\lim_{\ddot{\mathrm{n}}\to\infty}\mathfrak{C}_{\ell\ell}(\hbar_{\ddot{\mathrm{n}}},\hbar_{\ddot{\mathrm{n}}+p},\vartheta)$ exist and are finite for all $\vartheta\geq0, p\geq1$.

Definition 2.12. An *NbMLS*$(\mathfrak{S},\mathfrak{A}_{\ell\ell},\mathfrak{B}_{\ell\ell},\mathfrak{C}_{\ell\ell},*,\diamond)$ is named to be complete if every Cauchy sequence $\{\hbar_{\ddot{\mathrm{n}}}\}$ in $\mathfrak{S}$ tends to some $\hbar\in\mathfrak{S}$ such that $\lim_{\ddot{\mathrm{n}}\to\infty}\mathfrak{A}_{\ell\ell}(\hbar_{\ddot{\mathrm{n}}},\hbar,\vartheta) = \mathfrak{A}_{\ell\ell}(\hbar,\hbar,\vartheta) = \lim_{\ddot{\mathrm{n}}\to\infty}\mathfrak{A}_{\ell\ell}(\hbar_{\ddot{\mathrm{n}}},\hbar_{\ddot{\mathrm{n}}+p},\vartheta)$
$\lim_{\ddot{\mathrm{n}}\to\infty}\mathfrak{B}_{\ell\ell}(\hbar_{\ddot{\mathrm{n}}},\hbar,\vartheta) = \mathfrak{B}_{\ell\ell}(\hbar,\hbar,\vartheta) = \lim_{\ddot{\mathrm{n}}\to\infty}\mathfrak{B}_{\ell\ell}(\hbar_{\ddot{\mathrm{n}}},\hbar_{\ddot{\mathrm{n}}+p},\vartheta)$ and
$\lim_{\ddot{\mathrm{n}}\to\infty}\mathfrak{C}_{\ell\ell}(\hbar_{\ddot{\mathrm{n}}},\hbar,\vartheta) = \mathfrak{C}_{\ell\ell}(\hbar,\hbar,\vartheta) = \lim_{\ddot{\mathrm{n}}\to\infty}\mathfrak{C}_{\ell\ell}(\hbar_{\ddot{\mathrm{n}}},\hbar_{\ddot{\mathrm{n}}+p},\vartheta)$ for all $\vartheta\geq0, p\geq1$.

3. Main Results.

Theorem 3.1. Let $(\mathfrak{S},\mathfrak{A}_{\ell\ell},\mathfrak{B}_{\ell\ell},\mathfrak{C}_{\ell\ell},*,\diamond)$ be a complete *NbMLS* such that $\lim_{\vartheta\to\infty}\mathfrak{A}_{\ell\ell}(\hbar,\mathrm{q},\vartheta)=1$, $\lim_{\vartheta\to\infty}\mathfrak{B}_{\ell\ell}(\hbar,\mathrm{q},\vartheta)=0$ and $\lim_{\vartheta\to\infty}\mathfrak{C}_{\ell\ell}(\hbar,\mathrm{q},\vartheta)=0$ for all $\hbar,\mathrm{q}\in\mathfrak{S},\vartheta>0$ and $\mathcal{J}:\mathfrak{S}\to\mathfrak{S}$ be a mapping satisfying the conditions

$\mathfrak{A}_{\ell\ell}(\mathcal{J}\hbar,\mathcal{J}\mathrm{q},\tau\vartheta)\geq\mathfrak{A}_{\ell\ell}(\hbar,\mathrm{q},\vartheta)$, $\mathfrak{B}_{\ell\ell}(\mathcal{J}\hbar,\mathcal{J}\mathrm{q},\tau\vartheta)\leq\mathfrak{B}_{\ell\ell}(\hbar,\mathrm{q},\vartheta)$ and $\mathfrak{C}_{\ell\ell}(\mathcal{J}\hbar,\mathcal{J}\mathrm{q},\tau\vartheta)\leq\mathfrak{C}_{\ell\ell}(\hbar,\mathrm{q},\vartheta)$ (3.1.1)

for all $\hbar,\mathrm{q}\in\mathfrak{S},\vartheta>0$, $\tau\in(0,1)$. Then $\mathfrak{S}$ has a unique fixed point $\mathfrak{d}\in\mathfrak{S}$ and $\mathfrak{A}_{\ell\ell}(\mathfrak{d},\mathfrak{d},\vartheta)=1$, $\mathfrak{B}_{\ell\ell}(\mathfrak{d},\mathfrak{d},\vartheta)=0$ and $\mathfrak{C}_{\ell\ell}(\mathfrak{d},\mathfrak{d},\vartheta)=0$ for all $\vartheta>0$.

Proof: Let $(\mathfrak{S},\mathfrak{A}_{\ell\ell},\mathfrak{B}_{\ell\ell},\mathfrak{C}_{\ell\ell},*,\diamond)$ be a complete *NbMLS*. For a given element $\hbar_0\in\mathfrak{S}$, define a sequence $\{\hbar_{\ddot{\mathrm{n}}}\}$ in $\mathfrak{S}$ by

$$\hbar_1=\mathcal{J}\hbar_0, \hbar_2=\mathcal{J}^2\hbar=\mathcal{J}\hbar_1,\ldots,\hbar_{\ddot{\mathrm{n}}}=\mathcal{J}^{\ddot{\mathrm{n}}}\hbar_0=\mathcal{J}\hbar_{\ddot{\mathrm{n}}-1} \text{ for each } \ddot{\mathrm{n}}\in\mathrm{N}$$

If $\hbar_{\ddot{\mathrm{n}}}=\hbar_{\ddot{\mathrm{n}}-1}$ for some $\ddot{\mathrm{n}}\in\mathrm{N}$, then $\hbar_{\ddot{\mathrm{n}}}$ is a fixed point of $\mathcal{J}$. We consider that $\hbar_{\ddot{\mathrm{n}}}\neq\hbar_{\ddot{\mathrm{n}}-1}$ for each $\ddot{\mathrm{n}}\in\mathrm{N}$. For $\vartheta>0$ and $\ddot{\mathrm{n}}\in\mathrm{N}$, we get from (3.1.1) that

$$\mathfrak{A}_{\ell\ell}(\hbar_{\ddot{\mathrm{n}}},\hbar_{\ddot{\mathrm{n}}+1},\vartheta)\geq\mathfrak{A}_{\ell\ell}(\hbar_{\ddot{\mathrm{n}}+1},\hbar_{\ddot{\mathrm{n}}},\tau\vartheta)=\mathfrak{A}_{\ell\ell}(\mathcal{J}\hbar_{\ddot{\mathrm{n}}},\mathcal{J}\hbar_{\ddot{\mathrm{n}}-1},\tau\vartheta)\geq\mathfrak{A}_{\ell\ell}(\hbar_{\ddot{\mathrm{n}}},\hbar_{\ddot{\mathrm{n}}-1},\vartheta)$$

$\mathfrak{B}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\vartheta) \le \mathfrak{B}_{b\ell}(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}},\tau\vartheta) = \mathfrak{B}_{b\ell}(\mathcal{J}\hbar_{\ddot{n}},\mathcal{J}\hbar_{\ddot{n}-1},\tau\vartheta) \le \mathfrak{B}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}-1},\vartheta)$ and

$$\mathfrak{C}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\vartheta) \le \mathfrak{C}_{b\ell}(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}},\tau\vartheta) = \mathfrak{C}_{b\ell}(\mathcal{J}\hbar_{\ddot{n}},\mathcal{J}\hbar_{\ddot{n}-1},\tau\vartheta) \le \mathfrak{C}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}-1},\vartheta)$$

for all $\ddot{n} \in \mathrm{N}$ and $\vartheta > 0$. Thus, by using the above expression we can deduce that

$$\mathfrak{A}_{b\ell}(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}},\vartheta) \ge \mathfrak{A}_{b\ell}(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}},\tau\vartheta) = \mathfrak{A}_{b\ell}(\mathcal{J}\hbar_{\ddot{n}},\mathcal{J}\hbar_{\ddot{n}-1},\tau\vartheta) \ge \mathfrak{A}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}-1},\vartheta) \quad (3.1.2)$$
$$= \mathfrak{A}_{b\ell}(\mathcal{J}\hbar_{\ddot{n}-1},\mathcal{J}\hbar_{\ddot{n}-2},\vartheta) \ge \mathfrak{A}_{b\ell}\left(\hbar_{\ddot{n}-1},\hbar_{\ddot{n}-2},\frac{\vartheta}{\tau}\right) \ge \cdots \ge \mathfrak{A}_{b\ell}\left(\hbar_1,\hbar_0,\frac{\vartheta}{\tau^{\ddot{n}}}\right)$$

$$\mathfrak{B}_{b\ell}(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}},\vartheta) \le \mathfrak{B}_{b\ell}(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}},\tau\vartheta) = \mathfrak{B}_{b\ell}(\mathcal{J}\hbar_{\ddot{n}},\mathcal{J}\hbar_{\ddot{n}-1},\tau\vartheta) \le \mathfrak{B}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}-1},\vartheta) \quad (3.1.3)$$
$$= \mathfrak{B}_{b\ell}(\mathcal{J}\hbar_{\ddot{n}-1},\mathcal{J}\hbar_{\ddot{n}-2},\vartheta) \le \mathfrak{B}_{b\ell}\left(\hbar_{\ddot{n}-1},\hbar_{\ddot{n}-2},\frac{\vartheta}{\tau}\right) \le \cdots \le \mathfrak{B}_{b\ell}\left(\hbar_1,\hbar_0,\frac{\vartheta}{\tau^{\ddot{n}}}\right)$$
and
$$\mathfrak{C}_{b\ell}(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}},\vartheta) \le \mathfrak{C}_{b\ell}(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}},\tau\vartheta) = \mathfrak{C}_{b\ell}(\mathcal{J}\hbar_{\ddot{n}},\mathcal{J}\hbar_{\ddot{n}-1},\tau\vartheta) \le \mathfrak{C}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}-1},\vartheta) \quad (3.1.4)$$
$$= \mathfrak{C}_{b\ell}(\mathcal{J}\hbar_{\ddot{n}-1},\mathcal{J}\hbar_{\ddot{n}-2},\vartheta) \le \mathfrak{C}_{b\ell}\left(\hbar_{\ddot{n}-1},\hbar_{\ddot{n}-2},\frac{\vartheta}{\tau}\right) \le \cdots \le \mathfrak{C}_{b\ell}\left(\hbar_1,\hbar_0,\frac{\vartheta}{\tau^{\ddot{n}}}\right)$$

foreach $\ddot{n} \in \mathrm{N}, p \ge 1$ and $\vartheta > 0$. Thus we have

$$\mathfrak{A}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+p},\vartheta) \ge \mathfrak{A}_{b\ell}\left(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\frac{\vartheta}{b}\right) * \mathfrak{A}_{b\ell}\left(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}+p},\frac{\vartheta}{b}\right)$$
$$\mathfrak{B}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+p},\vartheta) \le \mathfrak{B}_{b\ell}\left(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\frac{\vartheta}{b}\right) \diamond \mathfrak{B}_{b\ell}\left(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}+p},\frac{\vartheta}{b}\right)$$
$$\mathfrak{C}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+p},\vartheta) \le \mathfrak{C}_{b\ell}\left(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\frac{\vartheta}{b}\right) \diamond \mathfrak{C}_{b\ell}\left(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}+p},\frac{\vartheta}{b}\right)$$

Continuing in this way, we get

$$\mathfrak{A}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+p},\vartheta) \ge \mathfrak{A}_{b\ell}\left(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\frac{\vartheta}{b}\right) * \mathfrak{A}_{b\ell}\left(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}+2},\frac{\vartheta}{b^2}\right) * \cdots * \mathfrak{A}_{b\ell}\left(\hbar_{\ddot{n}+p-1},\hbar_{\ddot{n}+p},\frac{\vartheta}{b^{p-1}}\right)$$
$$\mathfrak{B}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+p},\vartheta) \le \mathfrak{B}_{b\ell}\left(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\frac{\vartheta}{b}\right) \diamond \mathfrak{B}_{b\ell}\left(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}+2},\frac{\vartheta}{b^2}\right) \diamond \cdots \diamond \mathfrak{B}_{b\ell}\left(\hbar_{\ddot{n}+p-1},\hbar_{\ddot{n}+p},\frac{\vartheta}{b^{p-1}}\right) \text{and}$$
$$\mathfrak{C}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+p},\vartheta) \le \mathfrak{C}_{b\ell}\left(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\frac{\vartheta}{b}\right) \mathfrak{C}_{b\ell}\left(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}+2},\frac{\vartheta}{b^2}\right) \diamond \cdots \diamond \mathfrak{C}_{b\ell}\left(\hbar_{\ddot{n}+p-1},\hbar_{\ddot{n}+p},\frac{\vartheta}{b^{p-1}}\right)$$

Using (3.1.2), (3.1.3) and (3.1.4) in the above inequality, we have

$$\mathfrak{A}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+p},\vartheta) \ge \mathfrak{A}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b\tau^{\ddot{n}}}\right) * \mathfrak{A}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^2\tau^{\ddot{n}+1}}\right) * \cdots * \mathfrak{A}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^{p-1}\tau^{\ddot{n}+p-1}}\right) \quad (3.1.5)$$
$$\mathfrak{B}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+p},\vartheta) \le \mathfrak{B}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b\tau^{\ddot{n}}}\right) \diamond \mathfrak{B}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^2\tau^{\ddot{n}+1}}\right) \diamond \cdots \diamond \mathfrak{B}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^{p-1}\tau^{\ddot{n}+p-1}}\right) \quad (3.1.6)$$
and
$$\mathfrak{C}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+p},\vartheta) \le \mathfrak{C}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b\tau^{\ddot{n}}}\right) \diamond \mathfrak{C}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^2\tau^{\ddot{n}+1}}\right) \diamond \cdots \diamond \mathfrak{C}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^{p-1}\tau^{\ddot{n}+p-1}}\right) \quad (3.1.7)$$

Here b is a random positive integer.

We know that $\lim_{\vartheta\to\infty}\mathfrak{A}_{b\ell}(\hbar,\mathfrak{q},\vartheta) = 1$, $\lim_{\vartheta\to\infty}\mathfrak{B}_{b\ell}(\hbar,\mathfrak{q},\vartheta) = 0$ and $\lim_{\vartheta\to\infty}\mathfrak{C}_{b\ell}(\hbar,\mathfrak{q},\vartheta) = 0$ for all $\hbar,\mathfrak{q} \in \mathfrak{S}$ and $\vartheta > 0, \tau \in (0,1)$. It follows from (3.1.5) ,(3.1.6) and (3.1.7) that for all $\vartheta > 0, p \ge 1$

$\lim_{\ddot{n}\to\infty}\mathfrak{A}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+p},\vartheta) = 1 * 1 * \cdots * 1 = 1$,

$\lim_{\ddot{n}\to\infty}\mathfrak{B}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+p},\vartheta) = 0 \diamond 0 \diamond \cdots \diamond 0 = 0$ and

$\lim_{\ddot{n}\to\infty}\mathfrak{C}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+p},\vartheta) = 0 \diamond 0 \diamond \cdots \diamond 0 = 0$

Hence $\{\hbar_{\ddot{n}}\}$ is a Cauchy sequence. The completeness of the $NbMLS$ $(\mathfrak{S},\mathfrak{A}_{b\ell},\mathfrak{B}_{b\ell},\mathfrak{C}_{b\ell},*,\diamond)$ ensures that there exists $\mathfrak{d} \in \mathfrak{S}$ such thatfor all $\vartheta > 0, p \ge 1$

$$\lim_{\ddot{n}\to\infty}\mathfrak{A}_{b\ell}(\hbar_{\ddot{n}},\mathfrak{d},\vartheta) = \lim_{\ddot{n}\to\infty}\mathfrak{A}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+p},\vartheta) = \mathfrak{A}_{b\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 1 \quad (3.1.8)$$
$$\lim_{\ddot{n}\to\infty}\mathfrak{B}_{b\ell}(\hbar_{\ddot{n}},\mathfrak{d},\vartheta) = \lim_{\ddot{n}\to\infty}\mathfrak{B}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+p},\vartheta) = \mathfrak{B}_{b\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0 \quad (3.1.9)$$
$$\lim_{\ddot{n}\to\infty}\mathfrak{C}_{b\ell}(\hbar_{\ddot{n}},\mathfrak{d},\vartheta) = \lim_{\ddot{n}\to\infty}\mathfrak{C}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+p},\vartheta) = \mathfrak{C}_{b\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0. \quad (3.1.10)$$

Now, we show that $\mathfrak{d} \in \mathfrak{S}$ is a fixed point of $\mathcal{J}$. We have

$$\mathfrak{A}_{b\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta) \ge \mathfrak{A}_{b\ell}\left(\mathfrak{d},\hbar_{\ddot{n}+1},\frac{\vartheta}{2b}\right) * \mathfrak{A}_{b\ell}\left(\hbar_{\ddot{n}+1},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)$$
$$= \mathfrak{A}_{b\ell}\left(\mathfrak{d},\hbar_{\ddot{n}+1},\frac{\vartheta}{2b}\right) * \mathfrak{A}_{b\ell}\left(\mathcal{J}\hbar_{\ddot{n}},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)$$
$$\ge \mathfrak{A}_{b\ell}\left(\mathfrak{d},\hbar_{\ddot{n}+1},\frac{\vartheta}{2b}\right) * \mathfrak{A}_{b\ell}\left(\hbar_{\ddot{n}},\mathfrak{d},\frac{\vartheta}{2b\tau}\right)$$

$$\begin{aligned}\mathfrak{B}_{\ell\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta) &\leq \mathfrak{B}_{\ell\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)\diamond\mathfrak{B}_{\ell\ell}\left(\hbar_{\ddot{\imath}+1},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)\\ &= \mathfrak{B}_{\ell\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)\diamond\mathfrak{B}_{\ell\ell}\left(\mathcal{J}\hbar_{\ddot{\imath}},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right) \text{ and}\\ &\leq \mathfrak{B}_{\ell\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)\diamond\mathfrak{B}_{\ell\ell}\left(\hbar_{\ddot{\imath}},\mathfrak{d},\frac{\vartheta}{2b\tau}\right)\\ \mathfrak{C}_{\ell\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta) &\leq \mathfrak{C}_{\ell\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)\diamond\mathfrak{C}_{\ell\ell}\left(\hbar_{\ddot{\imath}+1},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)\\ &= \mathfrak{C}_{\ell\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)\diamond\mathfrak{C}_{\ell\ell}\left(\mathcal{J}\hbar_{\ddot{\imath}},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)\\ &\leq \mathfrak{C}_{\ell\ell}\left(\mathfrak{d},\hbar_{\ddot{\imath}+1},\frac{\vartheta}{2b}\right)\diamond\mathfrak{C}_{\ell\ell}\left(\hbar_{\ddot{\imath}},\mathfrak{d},\frac{\vartheta}{2b\tau}\right)\end{aligned}$$

for each$\vartheta > 0$. Taking the limit as $\ddot{\imath} \to +\infty$, and by (3.1.8),(3.1.9) and (3.1.10), we get
$\mathfrak{A}_{\ell\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta) = 1 * 1 = 1and\mathfrak{B}_{\ell\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta) = 0 \diamond 0 = 0$ and $\mathfrak{C}_{\ell\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta) = 0 \diamond 0 = 0$
Therefore, $\mathfrak{d}$ is a fixed point of $\mathcal{J}$and $\mathfrak{A}_{\ell\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 1$,$\mathfrak{B}_{\ell\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0$and $\mathfrak{C}_{\ell\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0$for all $\vartheta > 0$.
Now, we examine the uniqueness of fixed point. For this, assume that $\dot{\mathfrak{h}}$ and $\mathfrak{d}$ are two distinct fixed points of $\mathcal{J}$. Then by (3.1.1), we have

$$\mathfrak{A}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) = \mathfrak{A}_{\ell\ell}(\mathcal{J}\mathfrak{d},\mathcal{J}\dot{\mathfrak{h}},\vartheta) \geq \mathfrak{A}_{\ell\ell}\left(\mathfrak{d},\dot{\mathfrak{h}},\frac{\vartheta}{\tau}\right)$$

$$\mathfrak{B}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) = \mathfrak{B}_{\ell\ell}(\mathcal{J}\mathfrak{d},\mathcal{J}\dot{\mathfrak{h}},\vartheta) \leq \mathfrak{B}_{\ell\ell}\left(\mathfrak{d},\dot{\mathfrak{h}},\frac{\vartheta}{\tau}\right)\text{and}$$

$$\mathfrak{C}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) = \mathfrak{C}_{\ell\ell}(\mathcal{J}\mathfrak{d},\mathcal{J}\dot{\mathfrak{h}},\vartheta) \leq \mathfrak{C}_{\ell\ell}\left(\mathfrak{d},\dot{\mathfrak{h}},\frac{\vartheta}{\tau}\right)$$

for all $\vartheta > 0$. Thus we obtain

$$\mathfrak{A}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) \geq \mathfrak{A}_{\ell\ell}\left(\mathfrak{d},\dot{\mathfrak{h}},\frac{\vartheta}{\tau^{\ddot{\imath}}}\right) \text{ for all } \ddot{\imath} \in N$$

$$\mathfrak{B}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) \leq \mathfrak{B}_{\ell\ell}\left(\mathfrak{d},\dot{\mathfrak{h}},\frac{\vartheta}{\tau^{\ddot{\imath}}}\right)\text{and}\mathfrak{C}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) \leq \mathfrak{C}_{\ell\ell}\left(\mathfrak{d},\dot{\mathfrak{h}},\frac{\vartheta}{\tau^{\ddot{\imath}}}\right)$$

Letting the limit as $\ddot{\imath} \to +\infty$ and using $\lim_{\vartheta\to\infty}\mathfrak{A}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta) = 1$, $\lim_{\vartheta\to\infty}\mathfrak{B}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta) = 0$ and $\lim_{\vartheta\to\infty}\mathfrak{C}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta) = 0$ we get $\mathfrak{d} = \dot{\mathfrak{h}}$. Thus the fixed point is unique.

Example 3.2. Let $\mathfrak{S} = [0,1]$ and the *CTN* and *CTCN*respectively defined by $\mathfrak{u} * \mathfrak{v} = \mathfrak{u}\mathfrak{v}$ and $\mathfrak{u} \diamond \mathfrak{v} = \max\{\mathfrak{u},\mathfrak{v}\}$. Also, $\mathfrak{A}_{\ell\ell}$, $\mathfrak{B}_{\ell\ell}$ and $\mathfrak{C}_{\ell\ell}$are defined by $\mathfrak{u} * \mathfrak{v} = \min\{\mathfrak{u},\mathfrak{v}\}$and $\mathfrak{u} \diamond \mathfrak{v} = \max\{\mathfrak{u},\mathfrak{v}\}$.
$\mathfrak{A}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta) = e^{\frac{-(\hbar+\mathfrak{q})^2}{\vartheta}}$,$\mathfrak{B}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta) = 1 - e^{\frac{-(\hbar+\mathfrak{q})^2}{\vartheta}}$ and $\mathfrak{C}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta) = e^{\frac{(\hbar+\mathfrak{q})^2}{\vartheta}} - 1$ for all $\hbar,\mathfrak{q} \in \mathfrak{S}, \vartheta > 0$.
Then$(\mathfrak{S},\mathfrak{A}_{\ell\ell},\mathfrak{B}_{\ell\ell},\mathfrak{C}_{\ell\ell},*,\diamond)$ is a complete *NMLS*. Define $\mathcal{J}:\mathfrak{S} \to \mathfrak{S}$ by $\mathcal{J}\hbar = \begin{cases} 0, & \hbar \in \left[0,\frac{1}{2}\right] \\ \frac{\hbar}{6}, & \hbar \in \left(\frac{1}{2},1\right] \end{cases}$

Then

$$\lim_{\vartheta\to\infty}\mathfrak{A}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta) = \lim_{\vartheta\to\infty} e^{\frac{-(\hbar+\mathfrak{q})^2}{\vartheta}} = 1 \text{ ,}\lim_{\vartheta\to\infty}\mathfrak{B}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta) = \lim_{\vartheta\to\infty}\left(1 - e^{\frac{-(\hbar+\mathfrak{q})^2}{\vartheta}}\right) = 0\text{and}$$

$$\lim_{\vartheta\to\infty}\mathfrak{C}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta) = \lim_{\vartheta\to\infty}\left(e^{\frac{(\hbar+\mathfrak{q})^2}{\vartheta}} - 1\right) = 0$$

For $\tau \in \left[\frac{1}{2},1\right)$, we have four cases:
Case 1) If $\hbar,\mathfrak{q} \in \left[0,\frac{1}{2}\right]$, then $\mathcal{J}\hbar = \mathcal{J}\mathfrak{q} = 0$.
Case 2) If $\hbar \in \left[0,\frac{1}{2}\right]$and $\mathfrak{q} \in \left(\frac{1}{2},1\right]$, then $\mathcal{J}\hbar = 0$ and $\mathcal{J}\mathfrak{q} = \frac{\mathfrak{q}}{6}$.
Case 3) If $\hbar,\mathfrak{q} \in \left(\frac{1}{2},1\right]$, then $\mathcal{J}\hbar = \frac{\hbar}{6}$ and $\mathcal{J}\mathfrak{q} = \frac{\mathfrak{q}}{6}$.
Case 4) If $\hbar \in \left(\frac{1}{2},1\right]$and $\mathfrak{q} \in \left[0,\frac{1}{2}\right]$, then $\mathcal{J}\hbar = \frac{\hbar}{6}$ and $\mathcal{J}\mathfrak{q} = 0$.
From all four cases, we obtain that
$\mathfrak{A}_{\ell\ell}(\mathcal{J}\hbar,\mathcal{J}\mathfrak{q},\tau\vartheta) \geq \mathfrak{A}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta)$,$\mathfrak{B}_{\ell\ell}(\mathcal{J}\hbar,\mathcal{J}\mathfrak{q},\tau\vartheta) \leq \mathfrak{B}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta)$ and$\mathfrak{C}_{\ell\ell}(\mathcal{J}\hbar,\mathcal{J}\mathfrak{q},\tau\vartheta) \leq \mathfrak{C}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta)$.
Thus all the requirements of Theorem (3.1) are met and 0 is the unique fixed point of $\mathcal{J}$. Also,

$$\mathfrak{A}_{\ell\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = \mathfrak{A}_{\ell\ell}(0,0,\vartheta) = e^0 = 1, \mathfrak{B}_{\ell\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = \mathfrak{B}_{\ell\ell}(0,0,\vartheta) = 1 - e^0 = 0$$

and $\mathfrak{C}_{\ell\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = \mathfrak{C}_{\ell\ell}(0,0,\vartheta) = e^0 - 1 = 0$ for all $\vartheta > 0$.

Definition 3.3. Let$(\mathfrak{S},\mathfrak{A}_{b\ell},\mathfrak{B}_{b\ell},\mathfrak{C}_{b\ell},*,\diamond)$ be an *NbMLS*. A mapping $\mathcal{J}:\mathfrak{S}\to\mathfrak{S}$ is named to be *NbML* contractive if there exists $\rho\in(0,1)$ such that

$$\frac{1}{\mathfrak{A}_{b\ell}(\mathcal{J}\hbar,\mathcal{J}\mathfrak{q},\vartheta)}-1\le\rho\left[\frac{1}{\mathfrak{A}_{b\ell}(\hbar,\mathfrak{q},\vartheta)}-1\right],\ \mathfrak{B}_{b\ell}(\mathcal{J}\hbar,\mathcal{J}\mathfrak{q},\vartheta)\le\rho\mathfrak{B}_{b\ell}(\hbar,\mathfrak{q},\vartheta)\text{ and }\mathfrak{C}_{b\ell}(\mathcal{J}\hbar,\mathcal{J}\mathfrak{q},\vartheta)\le\rho\mathfrak{C}_{b\ell}(\hbar,\mathfrak{q},\vartheta)\quad\text{(I)}$$

for all $\hbar,\mathfrak{q}\in\mathfrak{S}$ and $\vartheta>0$. Here ρ is called the *NbML*contractive constant of $\mathcal{J}$.

Theorem 3.4. Let$(\mathfrak{S},\mathfrak{A}_{b\ell},\mathfrak{B}_{b\ell},\mathfrak{C}_{b\ell},*,\diamond)$ be a complete *NbMLS* and $\mathcal{J}:\mathfrak{S}\to\mathfrak{S}$ be a *NbML* contractive mapping with an *NbML* contractive constant ρ. Then $\mathcal{J}$ has a unique fixed point $\mathfrak{d}\in\mathfrak{S}$ such that $\mathfrak{A}_{b\ell}(\mathfrak{d},\mathfrak{d},\vartheta)=1$, $\mathfrak{B}_{b\ell}(\mathfrak{d},\mathfrak{d},\vartheta)=0$and $\mathfrak{C}_{b\ell}(\mathfrak{d},\mathfrak{d},\vartheta)=0$for all $\vartheta>0$.

Proof: Let$(\mathfrak{S},\mathfrak{A}_{b\ell},\mathfrak{B}_{b\ell},\mathfrak{C}_{b\ell},*,\diamond)$be a complete *NbMLS*. For a given element $\mathfrak{p}_0\in\mathfrak{S}$, define a sequence $\{\hbar_{\ddot{n}}\}$ in $\mathfrak{S}$ by

$$\hbar_1=\mathcal{J}\hbar_0,\hbar_2=\mathcal{J}^2\hbar_0=\mathcal{J}\hbar_1,\dots,\hbar_{\ddot{n}}=\mathcal{J}^{\ddot{n}}\hbar_0=\mathcal{J}\hbar_{\ddot{n}-1}\text{ for all }\ddot{n}\in\mathrm{N}$$

If $\hbar_{\ddot{n}}=\hbar_{\ddot{n}-1}$ for some $\ddot{n}\in\mathrm{N}$, then $\hbar_{\ddot{n}}$ is a fixed point of $\mathcal{J}$. We assume that $\hbar_{\ddot{n}}\ne\hbar_{\ddot{n}-1}$ for all $\ddot{n}\in\mathrm{N}$. For $\vartheta>0$and $\ddot{n}\in\mathrm{N}$, we get from (I) that

$$\frac{1}{\mathfrak{A}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\vartheta)}-1=\frac{1}{\mathfrak{A}_{b\ell}(\mathcal{J}\hbar_{\ddot{n}-1},\mathcal{J}\hbar_{\ddot{n}},\vartheta)}-1\le\rho\left[\frac{1}{\mathfrak{A}_{b\ell}(\hbar_{\ddot{n}-1},\hbar_{\ddot{n}},\vartheta)}-1\right]$$

Then we have

$$\frac{1}{\mathfrak{A}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\vartheta)}\le\frac{\rho}{\mathfrak{A}_{b\ell}(\mathcal{J}\hbar_{\ddot{n}-1},\mathcal{J}\hbar_{\ddot{n}},\vartheta)}+(1-\rho)$$
$$=\frac{\rho}{\mathfrak{A}_{b\ell}(\mathcal{J}\hbar_{\ddot{n}-2},\mathcal{J}\hbar_{\ddot{n}-1},\vartheta)}+(1-\rho)\le\frac{\rho^2}{\mathfrak{A}_{b\ell}(\hbar_{\ddot{n}-2},\hbar_{\ddot{n}-1},\vartheta)}+\rho(1-\rho)+(1-\rho)$$

for all $\vartheta>0$. Continuing in this way, we get

$$\begin{aligned}\frac{1}{\mathfrak{A}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\vartheta)}&\le\frac{\rho^{\ddot{n}}}{\mathfrak{A}_{b\ell}(\hbar_0,\hbar_1,\vartheta)}+\rho^{\ddot{n}-1}(1-\rho)+\rho^{\ddot{n}-2}(1-\rho)+\cdots+\rho(1-\rho)+(1-\rho)\\&\le\frac{\rho^{\ddot{n}}}{\mathfrak{A}_{b\ell}(\hbar_0,\hbar_1,\vartheta)}+(\rho^{\ddot{n}-1}+\rho^{\ddot{n}-2}+\cdots+1)(1-\rho)\\&\le\frac{\rho^{\ddot{n}}}{\mathfrak{A}_{b\ell}(\hbar_0,\hbar_1,\vartheta)}+(1-\rho^{\ddot{n}})\end{aligned}$$

Thus$\frac{1}{\frac{\rho^{\ddot{n}}}{\mathfrak{A}_{b\ell}(\hbar_0,\hbar_1,\vartheta)}+(1-\rho^{\ddot{n}})}\le\mathfrak{A}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\vartheta)$ for all $\vartheta>0,\ddot{n}\in\mathrm{N}$ (3.4.1)

$$\begin{aligned}\mathfrak{B}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\vartheta)&=\mathfrak{B}_{b\ell}(\mathcal{J}\hbar_{\ddot{n}-1},\mathcal{J}\hbar_{\ddot{n}},\vartheta)\le\rho\mathfrak{B}_{b\ell}(\hbar_{\ddot{n}-1},\hbar_{\ddot{n}},\vartheta)=\rho\mathfrak{B}_{b\ell}(\mathcal{J}\hbar_{\ddot{n}-2},\mathcal{J}\hbar_{\ddot{n}-1},\vartheta)\\&\le\rho^2\mathfrak{B}_{b\ell}(\hbar_{\ddot{n}-2},\hbar_{\ddot{n}-1},\vartheta)\le\cdots\le\rho^{\ddot{n}}\mathfrak{B}_{b\ell}(\hbar_0,\hbar_1,\vartheta)\end{aligned}\quad(3.4.2)$$

and

$$\begin{aligned}\mathfrak{C}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\vartheta)&=\mathfrak{C}_{b\ell}(\mathcal{J}\hbar_{\ddot{n}-1},\mathcal{J}\hbar_{\ddot{n}},\vartheta)\le\rho\mathfrak{C}_{b\ell}(\hbar_{\ddot{n}-1},\hbar_{\ddot{n}},\vartheta)=\rho\mathfrak{C}_{b\ell}(\mathcal{J}\hbar_{\ddot{n}-2},\mathcal{J}\hbar_{\ddot{n}-1},\vartheta)\\&\le\rho^2\mathfrak{C}_{b\ell}(\hbar_{\ddot{n}-2},\hbar_{\ddot{n}-1},\vartheta)\le\cdots\le\rho^{\ddot{n}}\mathfrak{C}_{b\ell}(\hbar_0,\hbar_1,\vartheta).\end{aligned}\quad(3.4.3)$$

Now, for $p\ge1$and $\ddot{n}\in\mathrm{N}$, we have

$$\begin{aligned}\mathfrak{A}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\vartheta)&\ge\mathfrak{A}_{b\ell}\left(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\frac{\vartheta}{b}\right)*\mathfrak{A}_{b\ell}\left(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}+p},\frac{\vartheta}{b}\right)\\&\ge\mathfrak{A}_{b\ell}\left(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\frac{\vartheta}{b}\right)*\mathfrak{A}_{b\ell}\left(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}+2},\frac{\vartheta}{b^2}\right)*\mathfrak{A}_{b\ell}\left(\hbar_{\ddot{n}+2},\hbar_{\ddot{n}+p},\frac{\vartheta}{b^2}\right).\end{aligned}$$

Continuing in this way, we get

$$\mathfrak{A}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+p},\vartheta)\ge\mathfrak{A}_{b\ell}\left(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\frac{\vartheta}{b}\right)*\mathfrak{A}_{b\ell}\left(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}+2},\frac{\vartheta}{b^2}\right)*\cdots*\mathfrak{A}_{b\ell}\left(\hbar_{\ddot{n}+p-1},\hbar_{\ddot{n}+p},\frac{\vartheta}{b^{p-1}}\right)$$

$$\begin{aligned}\mathfrak{B}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+p},\vartheta)&\le\mathfrak{B}_{b\ell}\left(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\frac{\vartheta}{b}\right)\diamond\mathfrak{B}_{b\ell}\left(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}+p},\frac{\vartheta}{b}\right)\\&\le\mathfrak{B}_{b\ell}\left(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\frac{\vartheta}{b}\right)\diamond\mathfrak{B}_{b\ell}\left(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}+2},\frac{\vartheta}{b^2}\right)\diamond\mathfrak{B}_{b\ell}\left(\hbar_{\ddot{n}+2},\hbar_{\ddot{n}+p},\frac{\vartheta}{b^2}\right)\end{aligned}$$

Continuing in this way, we get

$$\mathfrak{B}_{b\ell}(\hbar_{\ddot{n}},\hbar_{\ddot{n}+p},\vartheta)\le\mathfrak{B}_{b\ell}\left(\hbar_{\ddot{n}},\hbar_{\ddot{n}+1},\frac{\vartheta}{b}\right)\diamond\mathfrak{B}_{b\ell}\left(\hbar_{\ddot{n}+1},\hbar_{\ddot{n}+2},\frac{\vartheta}{b^2}\right)\diamond\cdots\diamond\mathfrak{B}_{b\ell}\left(\hbar_{\ddot{n}+p-1},\hbar_{\ddot{n}+p},\frac{\vartheta}{b^{p-1}}\right)$$

and

$$\mathfrak{C}_{b\ell}(\hbar_{ӥ},\hbar_{ӥ+p},\vartheta) \leq \mathfrak{C}_{b\ell}\left(\hbar_{ӥ},\hbar_{ӥ+1},\frac{\vartheta}{b}\right) \diamond \mathfrak{C}_{b\ell}\left(\hbar_{ӥ+1},\hbar_{ӥ+p},\frac{\vartheta}{b}\right)$$
$$\leq \mathfrak{C}_{b\ell}\left(\hbar_{ӥ},\hbar_{ӥ+1},\frac{\vartheta}{b}\right) \diamond \mathfrak{C}_{b\ell}\left(\hbar_{ӥ+1},\hbar_{ӥ+2},\frac{\vartheta}{b^2}\right) \diamond \mathfrak{C}_{b\ell}\left(\hbar_{ӥ+2},\hbar_{ӥ+p},\frac{\vartheta}{b^2}\right)$$

Continuing in this way, we get

$$\mathfrak{C}_{b\ell}(\hbar_{ӥ},\hbar_{ӥ+p},\vartheta) \leq \mathfrak{C}_{b\ell}\left(\hbar_{ӥ},\hbar_{ӥ+1},\frac{\vartheta}{b}\right) \diamond \mathfrak{C}_{b\ell}\left(\hbar_{ӥ+1},\hbar_{ӥ+2},\frac{\vartheta}{b^2}\right) \diamond \cdots \diamond \mathfrak{C}_{b\ell}\left(\hbar_{ӥ+p-1},\hbar_{ӥ+p},\frac{\vartheta}{b^{p-1}}\right)$$

Using (3.4.1), (3.4.2) and (3.4.3) in the above inequality, we have

$$\mathfrak{A}_{b\ell}(\hbar_{ӥ},\hbar_{ӥ+p},\vartheta) \geq \frac{1}{\frac{\rho^{ӥ}}{\mathfrak{A}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b}\right)} + (1-\rho^{ӥ})} * \frac{1}{\frac{\rho^{ӥ+1}}{\mathfrak{A}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^2}\right)} + (1-\rho^{ӥ+1})} * \cdots * \frac{1}{\frac{\rho^{ӥ+p-1}}{\mathfrak{A}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^{p-1}}\right)} + (1-\rho^{ӥ+p-1})}$$
$$\geq \frac{1}{\frac{\rho^{ӥ}}{\mathfrak{A}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b}\right)} + 1} * \frac{1}{\mathfrak{A}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^2}\right) + 1} * \cdots * \frac{\rho^{ӥ+p-1}}{\mathfrak{A}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^{p-1}}\right) + 1}$$

$$\mathfrak{B}_{b\ell}(\hbar_{ӥ},\hbar_{ӥ+p},\vartheta) \leq \rho^{ӥ}\mathfrak{B}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b}\right) \diamond \rho^{ӥ+1}\mathfrak{B}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^2}\right) \diamond \cdots \diamond \rho^{ӥ+p-1}\mathfrak{B}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^{p-1}}\right) \text{ and}$$
$$\mathfrak{C}_{b\ell}(\hbar_{ӥ},\hbar_{ӥ+p},\vartheta) \leq \rho^{ӥ}\mathfrak{C}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b}\right) \diamond \rho^{ӥ+1}\mathfrak{C}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^2}\right) \diamond \cdots \diamond \rho^{ӥ+p-1}\mathfrak{C}_{b\ell}\left(\hbar_0,\hbar_1,\frac{\vartheta}{b^{p-1}}\right)$$

Where$b$ is a random positive integer and $\rho \in (0,1)$. So we deduce from the above expression that$\{\hbar_{ӥ}\}$ is a Cauchy sequence in$(\mathfrak{S},\mathfrak{A}_{b\ell},\mathfrak{B}_{b\ell},\mathfrak{C}_{b\ell},*,\diamond)$. By the completeness of$(\mathfrak{S},\mathfrak{A}_{b\ell},\mathfrak{B}_{b\ell},\mathfrak{C}_{b\ell},*,\diamond)$, there is $\mathfrak{d} \in \mathfrak{S}$ such that

$$\lim_{ӥ\to\infty}\mathfrak{A}_{b\ell}(\hbar_{ӥ},\mathfrak{d},\vartheta) = \lim_{ӥ\to\infty}\mathfrak{A}_{b\ell}(\hbar_{ӥ},\hbar_{ӥ+p},\vartheta) = \lim_{ӥ\to\infty}\mathfrak{A}_{b\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 1 \quad (3.4.4)$$
$$\lim_{ӥ\to\infty}\mathfrak{B}_{b\ell}(\hbar_{ӥ},\mathfrak{d},\vartheta) = \lim_{ӥ\to\infty}\mathfrak{B}_{b\ell}(\hbar_{ӥ},\hbar_{ӥ+p},\vartheta) = \lim_{ӥ\to\infty}\mathfrak{B}_{b\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0 \quad (3.4.5)$$
$$\lim_{ӥ\to\infty}\mathfrak{C}_{b\ell}(\hbar_{ӥ},\mathfrak{d},\vartheta) = \lim_{ӥ\to\infty}\mathfrak{C}_{b\ell}(\hbar_{ӥ},\hbar_{ӥ+p},\vartheta) = \lim_{n\to\infty}\mathfrak{C}_{b\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0 \quad (3.4.6)$$

for all $\vartheta > 0, p \geq 1$.

We now establish that $\mathfrak{d}$ is a fixed point for $\mathcal{J}$. We determine this from (I) that

$$\frac{1}{\mathfrak{A}_{b\ell}(\mathcal{J}\hbar_{ӥ},\mathcal{J}\mathfrak{d},\vartheta)} - 1 \leq \rho\left[\frac{1}{\mathfrak{A}_{b\ell}(\hbar_{ӥ},\mathfrak{d},\vartheta)} - 1\right] = \frac{\rho}{\mathfrak{A}_{b\ell}(\hbar_{ӥ},\mathfrak{d},\vartheta)} - \rho,$$
$$\frac{1}{\mathfrak{A}_{b\ell}(\hbar_{ӥ},\mathfrak{d},\vartheta)} + 1 - \rho \leq \mathfrak{A}_{b\ell}(\mathcal{J}\hbar_{ӥ},\mathcal{J}\mathfrak{d},\vartheta).$$

Utilize the above inequality, we get

$$\mathfrak{A}_{b\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta) \geq \mathfrak{A}_{b\ell}\left(\mathfrak{d},\hbar_{ӥ+1},\frac{\vartheta}{2b}\right) * \mathfrak{A}_{b\ell}\left(\hbar_{ӥ+1},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)$$
$$= \mathfrak{A}_{b\ell}\left(\mathfrak{d},\hbar_{ӥ+1},\frac{\vartheta}{2b}\right) * \mathfrak{A}_{b\ell}\left(\mathcal{J}\hbar_{ӥ},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)$$
$$\geq \mathfrak{A}_{b\ell}\left(\mathfrak{d},\hbar_{ӥ+1},\frac{\vartheta}{2b}\right) * \frac{1}{\frac{\rho}{\mathfrak{A}_{b\ell}\left(\hbar_{ӥ},\mathfrak{d},\frac{\vartheta}{2b}\right)} + 1 - \rho}$$

$$\mathfrak{B}_{b\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta) \leq \mathfrak{B}_{b\ell}\left(\mathfrak{d},\hbar_{ӥ+1},\frac{\vartheta}{2b}\right) \diamond \mathfrak{B}_{b\ell}\left(\hbar_{ӥ+1},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right) = \mathfrak{B}_{b\ell}\left(\mathfrak{d},\hbar_{ӥ+1},\frac{\vartheta}{2b}\right) \diamond \mathfrak{B}_{b\ell}\left(\mathcal{J}\hbar_{ӥ},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)$$
$$\leq \mathfrak{B}_{b\ell}\left(\mathfrak{d},\hbar_{ӥ+1},\frac{\vartheta}{2b}\right) \diamond \rho\mathfrak{B}_{b\ell}\left(\hbar_{ӥ},\mathfrak{d},\frac{\vartheta}{2b}\right)$$

$$\mathfrak{C}_{b\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta) \leq \mathfrak{C}_{b\ell}\left(\mathfrak{d},\hbar_{ӥ+1},\frac{\vartheta}{2b}\right) \diamond \mathfrak{C}_{b\ell}\left(\hbar_{ӥ+1},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right) = \mathfrak{C}_{b\ell}\left(\mathfrak{d},\hbar_{ӥ+1},\frac{\vartheta}{2b}\right) \diamond \mathfrak{C}_{b\ell}\left(\mathcal{J}\hbar_{ӥ},\mathcal{J}\mathfrak{d},\frac{\vartheta}{2b}\right)$$
$$\leq \mathfrak{C}_{b\ell}\left(\mathfrak{d},\hbar_{ӥ+1},\frac{\vartheta}{2b}\right) \diamond \rho\mathfrak{C}_{b\ell}\left(\hbar_{ӥ},\mathfrak{d},\frac{\vartheta}{2b}\right)$$

Letting the limit as $ӥ \to \infty$ and applying (3.4.4.), (3.4.5) and (3.4.6) in the above expression, we obtain that $\mathfrak{A}_{b\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta) = 1, \mathfrak{B}_{b\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta) = 0$and $\mathfrak{C}_{b\ell}(\mathfrak{d},\mathcal{J}\mathfrak{d},\vartheta) = 0$,that is, $\mathcal{J}\mathfrak{d} = \mathfrak{d}$. Therefore, $\mathfrak{d}$ is a fixed point of $\mathcal{J}$and $\mathfrak{A}_{b\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 1, \mathfrak{B}_{b\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0$and $\mathfrak{C}_{b\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0$ for all $\vartheta > 0$.

Now we demonstrate the uniqueness of the fixed point $\mathfrak{d}$ of $\mathcal{J}$. Let $\dot{\mathfrak{h}}$ be a fixed point of $\mathcal{J}$different from $\mathfrak{d}$such that $\mathfrak{A}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) \neq 1\mathfrak{B}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) \neq 0$and $\mathfrak{C}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) \neq 0$for some $\vartheta > 0$. It follows from (3.4.1) that

$$\frac{1}{\mathfrak{A}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta)} - 1 = \frac{1}{\mathfrak{A}_{\ell\ell}(\mathcal{J}\mathfrak{d},\mathcal{J}\dot{\mathfrak{h}},\vartheta)} - 1 \le \rho\left[\frac{1}{\mathfrak{A}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta)} - 1\right] < \frac{1}{\mathfrak{A}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta)} - 1$$

$\mathfrak{B}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) = \mathfrak{B}_{\ell\ell}(\mathcal{J}\mathfrak{d},\mathcal{J}\dot{\mathfrak{h}},\vartheta) \le \rho\mathfrak{B}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) < \mathfrak{B}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta)$and
$\mathfrak{C}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) = \mathfrak{C}_{\ell\ell}(\mathcal{J}\mathfrak{d},\mathcal{J}\dot{\mathfrak{h}},\vartheta) \le \rho\mathfrak{C}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) < \mathfrak{C}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta)$this is not possible.
∴we have $\mathfrak{A}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) = 1, \mathfrak{B}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) = 0$and $\mathfrak{C}_{\ell\ell}(\mathfrak{d},\dot{\mathfrak{h}},\vartheta) = 0$for all $\vartheta > 0$, and hence $\mathfrak{d} = \dot{\mathfrak{h}}$.

Corollary 3.5. Let $(\mathfrak{S},\mathfrak{A}_{\ell\ell},\mathfrak{B}_{\ell\ell},\mathfrak{C}_{\ell\ell},*,\diamond)$ be a complete *NbMLS* and $\mathcal{J}:\mathfrak{S}\to\mathfrak{S}$ be a mapping satisfying $\frac{1}{\mathfrak{A}_{\ell\ell}(\mathcal{J}^{\ddot{n}}\hbar,\mathcal{J}^{\ddot{n}}\mathfrak{q},\vartheta)} - 1 \le \rho\left[\frac{1}{\mathfrak{A}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta)} - 1\right], \mathfrak{B}_{\ell\ell}(\mathcal{J}^{\ddot{n}}\hbar,\mathcal{J}^{\ddot{n}}\mathfrak{q},\vartheta) \le \rho\mathfrak{B}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta)$and
$\mathfrak{C}_{\ell\ell}(\mathcal{J}^{\ddot{n}}\hbar,\mathcal{J}^{\ddot{n}}\mathfrak{q},\vartheta) \le \rho\mathfrak{C}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta)$for some $\ddot{n} \in N$ and all $\hbar,\mathfrak{q} \in \mathfrak{S}, \vartheta > 0$, $0 < \rho < 1$. Then $\mathcal{J}$ has a unique fixed point $\mathfrak{d} \in \mathfrak{S}$ moreover$\mathfrak{A}_{\ell\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 1, \mathfrak{B}_{\ell\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0$and $\mathfrak{C}_{\ell\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0$for all $\vartheta > 0$.

Proof: Assume that $\mathfrak{d} \in \mathfrak{S}$ is the unique fixed point of $\mathcal{J}^{\ddot{n}}$ as determined by Theorem (3.4) and $\mathfrak{A}_{\ell\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 1, \mathfrak{B}_{\ell\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0and\mathfrak{C}_{\ell\ell}(\mathfrak{d},\mathfrak{d},\vartheta) = 0$for all $\vartheta > 0$. So$\mathfrak{d}$ is another fixed point of $\mathcal{J}^{\ddot{n}}$ as $\mathcal{J}^{\ddot{n}}(\mathcal{J}\mathfrak{d}) = \mathcal{J}\mathfrak{d}$ and by Theorem(3.4), $\mathcal{J}\mathfrak{d} = \mathfrak{d}$ and so $\mathfrak{d}$ is the unique fixed point, since the unique fixed point of $\mathcal{J}$ is also the unique fixed point of $\mathcal{J}^{\ddot{n}}$.

Example 3.6. Let $\mathfrak{S} = [0,1]$ and the *CTN* and *CTCN* respectively defined by $\mathfrak{u} * \mathfrak{v} = \mathfrak{u}\mathfrak{v}$ and $\mathfrak{u} \diamond \mathfrak{v} = \max\{\mathfrak{u},\mathfrak{v}\}$. Consider $\mathfrak{A}_{\ell\ell}$, $\mathfrak{B}_{\ell\ell}$ and $\mathfrak{C}_{\ell\ell}$by $\mathfrak{A}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta) = e^{\frac{-(\max\{\hbar,\mathfrak{q}\})^2}{\vartheta}}$, $\mathfrak{B}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta) = 1 - e^{\frac{-(\max\{\hbar,\mathfrak{q}\})^2}{\vartheta}}$and $\mathfrak{C}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta) = e^{\frac{(\max\{\hbar,\mathfrak{q}\})^2}{\vartheta}} - 1$
for all $\hbar,\mathfrak{q} \in \mathfrak{S}$ and $\vartheta > 0$. Then $(\mathfrak{S},\mathfrak{A}_{\ell\ell},\mathfrak{B}_{\ell\ell},\mathfrak{C}_{\ell\ell},*,\diamond)$ is a complete *NbMLS*.

Define $\mathcal{J}:\mathfrak{S}\to\mathfrak{S}$ as$\mathcal{J}\hbar = \begin{cases} 0, \hbar = \frac{1}{2} \\ \frac{\hbar}{2}, \hbar \in [0,\frac{1}{2}) \\ \frac{\hbar}{4}, \hbar \in (\frac{1}{2},1] \end{cases}$

Then we have eight cases:
Case (i) If $\hbar = \mathfrak{q} = \frac{1}{2}$, then $\mathcal{J}\hbar = \mathcal{J}\mathfrak{q} = 0$.
Case (ii) If $\hbar = \frac{1}{2}$and $\mathfrak{q} \in \left[0,\frac{1}{2}\right)$ then $\mathcal{J}\hbar = 0$ and $\mathcal{J}\mathfrak{q} = \frac{\mathfrak{q}}{2}$.
Case (iii) If $\hbar = \frac{1}{2}$and $\mathfrak{q} \in (\frac{1}{2},1]$, then $\mathcal{J}\hbar = 0$ and $\mathcal{J}\mathfrak{q} = \frac{\mathfrak{q}}{4}$.
Case (iv) If $\hbar \in \left[0,\frac{1}{2}\right)$and $\mathfrak{q} \in (\frac{1}{2},1]$, then $\mathcal{J}\hbar = \frac{\hbar}{2}$ and $\mathcal{J}\mathfrak{q} = \frac{\mathfrak{q}}{4}$.
Case (v) If $\hbar \in [0,\frac{1}{2})$and $\mathfrak{q} \in [0,\frac{1}{2})$, then $\mathcal{J}\hbar = \frac{\hbar}{2}$ and $\mathcal{J}\mathfrak{q} = \frac{\mathfrak{q}}{2}$.
Case (vi) If $\hbar \in [0,\frac{1}{2})$and $\mathfrak{q} = \frac{1}{2}$, then $\mathcal{J}\hbar = \frac{\hbar}{2}$ and $\mathcal{J}\mathfrak{q} = 0$.
Case (vii) If $\hbar \in \left(\frac{1}{2},1\right]$and $\mathfrak{q} = \frac{1}{2}$, then $\mathcal{J}\hbar = \frac{\hbar}{4}$ and $\mathcal{J}\mathfrak{q} = 0$.
Case (viii) If $\hbar \in \left(\frac{1}{2},1\right]$and $\mathfrak{q} \in (\frac{1}{2},1]$, then $\mathcal{J}\hbar = \frac{\hbar}{4}$ and $\mathcal{J}\mathfrak{q} = \frac{o}{4}$.
NbML contraction is satisfied in all of the above cases:

$$\frac{1}{\mathfrak{A}_{\ell\ell}(\mathcal{J}\hbar,\mathcal{J}\mathfrak{q},\vartheta)} - 1 \le \rho\left[\frac{1}{\mathfrak{A}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta)} - 1\right],$$

$\mathfrak{B}_{\ell\ell}(\mathcal{J}\hbar,\mathcal{J}\mathfrak{q},\vartheta) \le \rho\mathfrak{B}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta)$and

$$\mathfrak{C}_{\ell\ell}(\mathcal{J}\hbar,\mathcal{J}\mathfrak{q},\vartheta) \le \rho\mathfrak{C}_{\ell\ell}(\hbar,\mathfrak{q},\vartheta)$$

with the *NbML* contractive constant $\rho \in \left[\frac{1}{2},1\right)$.
Hence $\mathcal{J}$ is an *NbML* contractive mapping with $\rho \in \left[\frac{1}{2},1\right)$.
All the requirements of Theorem (3.1) have been met.

Moreover, 0 is the unique fixed point of T and $\mathfrak{A}_{\delta\ell}(0,0,\vartheta) = 1$, $\mathfrak{B}_{\delta\ell}(0,0,\vartheta) = 0$and $\mathfrak{C}_{\delta\ell}(0,0,\vartheta) = 0$for all $\vartheta > 0$.

4. Conclusion

In this paper, we develop $NbMLS$ and demonstrate the fixed point theorem in order to demonstrate the unique fixed point in this space. This work is the extended form of fuzzy b-metric like space [6]. We hope that the result proved in this paper will form new connection for those who are working in the $NbMLS$ space and this work opens a new path for researchers in the concerned field.

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