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# MAGDM Technique Based on Linguistic Neutrosophic Tangent Dombi Aggregation Operators and Its Application in Selecting New Energy Design Schemes

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**Abstract:** In linguistic decision theory, multi-attribute group decision-making (MAGDM) of linguistic neutrosophic numbers (LNNs) is one of the crucial research topics. Existing LNN aggregation algorithms do not consider trigonometric periodicity. In this case, they cannot perform MAGDM problems including periodic/multitemporal applications in LNN scenarios. To fill this gap, this article aims to present a MAGDM technique using the LNN tangent Dombi aggregation operators (TDAOs) for addressing MAGDM issues including multitemporal/periodic applications in LNN scenarios. First, we present the linguistic tangent Dombi t-norm and t-conorm and the LNN tangent Dombi operation laws. Second, we propose the LNN tangent Dombi weighted average and geometric operators and investigate their properties. Third, a MAGDM technique is built based on the two presented operators to tackle MAGDM problems in a LNN scenario. Lastly, the built MAGDM technique is applied to a choice of new energy design schemes in Ningbo City, China, and then its efficiency and appropriateness are verified by comparing with the existing MAGDM technique in the LNN scenario.

**Keywords:** linguistic neutrosophic number; linguistic tangent Dombi operation; linguistic neutrosophic number tangent Dombi aggregation operator; group decision making; new energy design scheme

## 1. Introduction

In multi-attribute (group) decision-making (MADM/MAGDM) problems, neutrosophic MADM methods have obtained wild applications [1-3]. Then, linguistic information can be adapted to human cognition and expression forms. Therefore, decision information is given in the form of linguistic or uncertain linguistic variables in MADM/MAGDM problems such as supply chain management, medical diagnosis, and risk investment. In the evaluation process, decision makers often provide qualitative information (e.g., bad, good, or other natural language forms) directly when evaluating decision objects such as the overall quality of people, performance of equipment, engineering projects, business partners. Linguistic decision theory and methods are not only of great academic value, but also have broad application prospects. Under linguistic scenarios, many scholars have proposed various MADM/MAGDM methods to tackle linguistic MADM/MAGDM issues [4–9]. In uncertain linguistic scenarios, some researchers also presented uncertain linguistic MADM/MAGDM models to carry out uncertain linguistic MADM/MAGDM applications [10–15]. Based on true and false linguistic variables, some researchers [16–20] developed linguistic intuitionistic fuzzy numbers (LIFNs) and some LIFN aggregation operators to perform MADM/MAGDM issues in LIFN scenarios. Concerning true, indeterminate, and false linguistic variables, many scholars have introduced the linguistic neutrosophic number (LNN) aggregation operators (AOs) [21], the LNN Bonferroni mean AOs [22], LNN Hamy mean AOs [23], the LNN Hamacher AOs [24], the LNN power Heronian AOs [25], the LNN Einstein AO [26], the LNN Maclaurin symmetric mean AOs [27], the LNN Muirhead mean AO [28], and the LNN Dombi AOs [29], and then developed their MADM/MAGDM approaches in LNN scenarios. However, all these AOs and MADM/MAGDM approaches lack trigonometric AOs and periodic MADM/MAGDM capability in different linguistic scenarios.

Regarding the trigonometric AOs and periodic MADM issues of single-valued neutrosophic numbers (SVNNs), some researchers [30–32] have proposed the trigonometric AOs and trigonometric Dombi AOs of SVNNs and their MADM techniques to handle periodic/multitemporal MADM/MAGDM issues. However, all these existing SVNN trigonometric AOs and MADM/MAGDM techniques cannot deal with MADM/MAGDM issues under a linguistic neutrosophic scenario.

As mentioned above, neither the linguistic trigonometric operations and AOs nor their MADM/MAGDM methods/applications are present in the existing literature, which shows the obvious research gap. Since LNN contains a more universal linguistic framework including true, false and indeterminate linguistic variables in uncertainty, the existing linguistic value and LIFN can be viewed as special cases of LNN. Furthermore, Dombi operations show the merit of the flexible operations due to an adjusting parameter. Motivated by the trigonometric Dombi AOs of SVNNs [32], this study aims to develop a novel MAGDM technique using the LNN tangent Dombi AOs and apply it to a choice issue of new energy design schemes (NEDSs) to fill the research gap.

In this study, we first propose the linguistic tangent Dombi t-norm and t-conorm and the LNN tangent Dombi operation laws (LNNTDOLs). Next, we propose the LNN tangent Dombi weighted average and geometric (LNNTDWA and LNNTDWG) operators in terms of the LNNTDOLs. Based on the proposed LNNTDWA and LNNTDWG operators, a MAGDM technique is developed in a LNN scenario. Finally, the developed MAGDM technique is applied to a choice case of NEDSs. By comparing with the existing MAGDM technique, the efficiency and appropriateness of the developed MAGDM technique is investigated in the LNN scenario.

Generally, this original study mainly provides the following novel contributions:

- The novel linguistic tangent Dombi t-norm and t-conorm and LNNTDOLs are proposed to obtain the multitemporal/periodic operations in the LNN scenario.
- The LNNTDWA and LNNTDWG operators are presented to yield the periodic aggregation tools for MAGDM modeling in the LNN scenario.
- The MAGDM technique using the proposed LNNTDWA and LNNTDWG operators is built based on the multitemporal/periodic requirements to effectively solve the choice issue of NEDSs and in LNN scenarios.

The rest of the article is constructed as follows. Section 2 reviews some basic knowledge of LNNs. Section 3 presents the linguistic tangent Dombi t-norm and t-conorm and LNNTDOLs. Section 4 proposes the LNNTDWA and LNNTDWG operators based on LNNTDOLs. In Section 5, a MAGDM technique is built by the proposed LNNTDWA and LNNTDWG operators in a LNN scenario. In Section 6, the built MAGDM technique is applied in a choice case of NEDSs, and then its appropriateness and validity are examined by comparing it with the existing relative MAGDM technique in the LNN scenario. In Section 7 some conclusions and future work are mentioned.

## 2. Some Basic Knowledge of LNNs

Set $\Psi_\beta = \{\beta_1, \beta_2, \ldots, \beta_g\}$ as a set of linguistic terms (SLT) with odd cardinality $g$+1. Then LNN is defined as $\beta_L = \langle\beta_{Lt}, \beta_{Lu}, \beta_{Lf}\rangle$ [18] over $\Psi_\beta$ for $\beta_{Lt}, \beta_{Lu}, \beta_{Lf} \in \Psi_\beta$, where $\beta_{Lt}$, $\beta_{Lu}$, and $\beta_{Lf}$ are the true, indeterminate, and false linguistic variables, respectively.

If two LNNs are $\beta_{L(1)} = <\beta_{Lt(1)}, \beta_{Lu(1)}, \beta_{Lf(1)}>$ and $\beta_{L(2)} = <\beta_{Lt(2)}, \beta_{Lu(2)}, \beta_{Lf(2)}>$ over $\Psi_\beta$ and $\eta > 0$, then their Dombi operation laws [29] are introduced below:

$$(1)\ \ \beta_{L(1)} \oplus_D \beta_{L(2)} = \left\langle \beta_{g\left\{1-\frac{1}{1+\left\{\left(\frac{Lt(1)/g}{1-Lt(1)/g}\right)^{\varepsilon}+\left(\frac{Lt(2)/g}{1-Lt(2)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{\frac{1}{1+\left\{\left(\frac{1-Lu(1)/g}{Lu(1)/g}\right)^{\varepsilon}+\left(\frac{1-Lu(2)/g}{Lu(2)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{\frac{1}{1+\left\{\left(\frac{1-Lf(1)/g}{Lf(1)/g}\right)^{\varepsilon}+\left(\frac{1-Lf(2)/g}{Lf(2)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle;$$

$$(2)\ \ \beta_{L(1)} \otimes_D \beta_{L(2)} = \left\langle \beta_{g\left\{\frac{1}{1+\left\{\left(\frac{1-Lt(1)/g}{Lt(1)/g}\right)^{\varepsilon}+\left(\frac{1-Lt(2)/4g}{Lt(2)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{1-\frac{1}{1+\left\{\left(\frac{Lu(1)/g}{1-Lu(1)/g}\right)^{\varepsilon}+\left(\frac{Lu(2)/g}{1-Lu(2)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{1-\frac{1}{1+\left\{\left(\frac{Lf(1)/g}{1-Lf(1)/g}\right)^{\varepsilon}+\left(\frac{Lf(2)/g}{1-Lf(2)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle;$$

$$(3)\ \ \eta\beta_{L(1)} = \left\langle \beta_{g\left\{1-\frac{1}{1+\left\{\eta\left(\frac{Lt(1)/g}{1-Lt(1)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{\frac{1}{1+\left\{\eta\left(\frac{1-Lu(1)/g}{Lu(1)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{\frac{1}{1+\left\{\eta\left(\frac{1-Lf(1)/g}{Lf(1)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle;$$

$$(4)\ \ \beta_{L(1)}^{\eta} = \left\langle \beta_{g\left\{\frac{1}{1+\left\{\eta\left(\frac{1-Lt(1)/g}{Lt(1)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{1-\frac{1}{1+\left\{\eta\left(\frac{Lu(1)/g}{1-Lu(1)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{1-\frac{1}{1+\left\{\eta\left(\frac{Lf(1)/g}{1-Lf(1)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle.$$

If $\beta_{L(j)} = <\beta_{Lt(j)}, \beta_{Lu(j)}, \beta_{Lf(j)}>$ ($j = 1, 2, \ldots, e$) are $e$ LNNs with their weights $\eta_j$ and $\sum_{j=1}^{e}\eta_j = 1$, then the LNN Dombi weighted average and geometric (LNNDWA and LNNDWG) operators [29] are introduced below:

$$LNNDWA(\beta_{L(1)}, \beta_{L(2)}, \ldots, \beta_{L(e)}) = \mathop{\oplus_D}_{j=1}^{e} \eta_j \beta_{L(j)} = \left\langle \beta_{g\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{Lt(j)/g}{1-Lt(j)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-Lu(j)/g}{Lu(j)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-Lf(j)/g)}{Lf(j)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle. \quad (1)$$

$$LNNDWG(\beta_{L(1)}, \beta_{L(2)}, \ldots, \beta_{L(e)}) = \mathop{\otimes_D}_{j=1}^{e} \beta_{L(j)}^{\eta_j} = \left\langle \beta_{g\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-Lt(j)/g}{Lt(j)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{Lu(j)/g}{1-Lu(j)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{g\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{Lf(j)/g}{1-Lf(j)/g}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle. \quad (2)$$

Then, the score and accuracy equations of $\beta_{L(j)} = <\beta_{Lt(j)}, \beta_{Lu(j)}, \beta_{Lf(j)}>$ are introduced below [21]:

$$D(\beta_{L(j)}) = (2g + Lt(j) - Lu(j) - Lf(j))/(3g) \text{ for } D(\beta_{L(j)}) \in [0, 1]; \tag{3}$$

$$E(\beta_{L(j)}) = (Lt(j) - Lf(j))/g \text{ for } E(\beta_{L(j)}) \in [-1, 1]. \tag{4}$$

Comparative laws of two LNNs $\beta_{L(1)}$ and $\beta_{L(2)}$ are introduced below [21]:

(i) $\beta_{L(1)} > \beta_{L(2)}$ if $D(\beta_{L(1)}) > D(\beta_{L(2)})$;

(ii) $\beta_{L(1)} > \beta_{L(2)}$ if $D(\beta_{L(1)}) = D(\beta_{L(2)})$ and $E(\beta_{L(1)}) > E(\beta_{L(2)})$;

(iii) $\beta_{L(1)} = \beta_{L(2)}$ if $D(\beta_{L(1)}) = D(\beta_{L(2)})$ and $E(\beta_{L(1)}) = E(\beta_{L(2)})$.

## 3. Linguistic Tangent Dombi Operations and LNNTDOLs

This part proposes the linguistic tangent Dombi operations (LTDOs) and LNNTDOLs.

First, we propose LTDOs below.

**Definition 1.** Let $\beta_{Lt(1)}$ and $\beta_{Lt(2)}$ be two linguistic values on the SLT $\Psi_\beta = \{\beta_0, \beta_1, \ldots, \beta_g\}$ and $\varepsilon \geq 1$. Then the linguistic tangent Dombi t-norm and t-conorm of $\beta_{Lt(1)}$ and $\beta_{Lt(2)}$ are defined, respectively, by the LTDOs:

$$T_{TD}(\beta_{Lt(1)}, \beta_{Lt(2)}) = \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(k\pi+Lt(1)\pi/(4g))}{\tan(k\pi+Lt(1)\pi/(4g))}\right)^{\varepsilon}+\left(\frac{1-\tan(k\pi+Lt(2)\pi/(4g))}{\tan(k\pi+Lt(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} = \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(Lt(1)\pi/(4g))}{\tan(Lt(1)\pi/(4g))}\right)^{\varepsilon}+\left(\frac{1-\tan(Lt(2)\pi/(4g))}{\tan(Lt(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \tag{5}$$

$$T_{SD}(\beta_{Lt(1)}, \beta_{Lt(2)}) = \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(k\pi+Lt(1)\pi/4g)}{1-\tan(k\pi+Lt(1)\pi/4g)}\right)^{\varepsilon}+\left(\frac{\tan(k\pi+Lt(2)\pi/(4g))}{1-\tan(k\pi+Lt(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} = \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(Lt(1)\pi/(4g))}{1-\tan(Lt(1)\pi/(4g))}\right)^{\varepsilon}+\left(\frac{\tan(Lt(2)\pi/(4g))}{1-\tan(Lt(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}. \tag{6}$$

Especially when $\beta_{Lt(1)} = \beta_0$ or $\beta_g$ and $\beta_{Lt(2)} = \beta_0$ or $\beta_g$ in Eqs. (5) and (6), let $T_{TD}(\beta_0, \beta_0) = T_{TD}(\beta_0, \beta_g) = T_{TD}(\beta_g, \beta_0) = \beta_0$, $T_{TD}(\beta_g, \beta_g) = \beta_g$, $T_{SD}(\beta_0, \beta_g) = T_{SD}(\beta_g, \beta_0) = T_{SD}(\beta_g, \beta_g) = \beta_g$ and $T_{SD}(\beta_0, \beta_0) = \beta_0$. Clearly, the LTDOs of Eqs. (5) and (6) imply $k\pi$ periodicity for $k = 1, 2, \ldots, n$. According to Eqs. (5) and (6), LNNTDOLs are proposed below.

**Definition 2.** Let $\beta_{L(1)} = \langle\beta_{Lt(1)}, \beta_{Lu(1)}, \beta_{Lf(1)}\rangle$ and $\beta_{L(2)} = \langle\beta_{Lt(2)}, \beta_{Lu(2)}, \beta_{Lf(2)}\rangle$ be two LNNs, $\eta > 0$, and $\varepsilon \geq 1$. Then, the LNNTDOLs are defined below:

$$(1)\quad \beta_{L(1)} \oplus_{TD} \beta_{L(2)} = \left\langle \begin{array}{l} \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(Lt(1)\pi/(4g))}{1-\tan(Lt(1)\pi/(4g))}\right)^{\varepsilon}+\left(\frac{\sin(Lt(2)\pi/(4g))}{1-\sin(Lt(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(Lu(1)\pi/(4g))}{\tan(Lu(1)\pi/(4g))}\right)^{\varepsilon}+\left(\frac{1-\tan(Lu(2)\pi/(4g))}{\tan(Lu(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(Lf(1)\pi/(4g))}{\tan(Lf(1)\pi/(4g))}\right)^{\varepsilon}+\left(\frac{1-\tan(Lf(2)\pi/(4g))}{\tan(Lf(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \end{array} \right\rangle;$$

$$(2)\ \beta_{L(1)} \otimes_{TD} \beta_{L(2)} = \left\langle \begin{array}{l} \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(Lt(1)\pi/(4g))}{\tan(Lt(1)\pi/(4g))}\right)^{\varepsilon}+\left(\frac{1-\tan(Lt(2)\pi/(4g))}{\tan(Lt(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(Lu(1)\pi/(4g))}{1-\tan(Lu(1)\pi/(4g))}\right)^{\varepsilon}+\left(\frac{\tan(Lu(2)\pi/(4g))}{1-\tan(Lu(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(Lf(1)\pi/(4g))}{1-\tan(Lf(1)\pi/(4g))}\right)^{\varepsilon}+\left(\frac{\tan(Lf(2)\pi/(4g))}{1-\tan(Lf(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \end{array} \right\rangle;$$

$$(3)\ \eta\beta_{L(1)} = \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\eta\left(\frac{\tan(Lt(1)\pi/(4g))}{1-\tan(Lt(1)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\eta\left(\frac{1-\tan(Lu(1)\pi/(4g))}{\tan(Lu(1)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\eta\left(\frac{1-\tan(Lf(1)\pi/(4g))}{\tan(Lf(1)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle;$$

$$(4)\ \beta_{L(1)}^{\eta} = \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\eta\left(\frac{1-\tan(Lt(1)\pi/(4g))}{\tan(Lt(1)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\eta\left(\frac{\tan(Lu(1)\pi/(4g))}{1-\tan(Lu(1)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\eta\left(\frac{\tan(Lf(1)\pi/(4g))}{1-\tan(Lf(1)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle.$$

**Example 1.** Set two LNNs on the SLT $\Psi_\beta = \{\beta_0, \beta_1, \ldots, \beta_8\}$ as $\beta_{L(1)} = <\beta_7, \beta_3, \beta_2>$ and $\beta_{L(2)} = <\beta_7, \beta_2, \beta_4>$ with $\eta = 0.8$ and $\varepsilon = 2$. Then, using the TDOLs (1)-(4) in Definition 2, their operations are shown below:

$$\beta_{L(1)} \oplus_{TD} \beta_{L(2)} = \left\langle \begin{array}{l} \beta_{\frac{32}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(7\pi/32)}{1-\tan(7\pi/32)}\right)^{2}+\left(\frac{\tan(7\pi/32)}{1-\tan(7\pi/32)}\right)^{2}\right\}^{1/2}}\right\}}, \\ \beta_{\frac{32}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(3\pi/32)}{\tan(3\pi/32)}\right)^{2}+\left(\frac{1-\tan(2\pi/32)}{\tan(2\pi/32)}\right)^{2}\right\}^{1/2}}\right\}}, \\ \beta_{\frac{32}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(2\pi/32)}{\tan(2\pi/32)}\right)^{2}+\left(\frac{1-\tan(4\pi/32)}{\tan(4\pi/32)}\right)^{2}\right\}^{1/2}}\right\}} \end{array} \right\rangle = \left\langle \beta_{7.7626}, \beta_{0.865}, \beta_{1.0043} \right\rangle;$$

$$\beta_{L(1)} \otimes_{TD} \beta_{L(2)} = \left\langle \begin{array}{l} \beta_{\frac{32}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(7\pi/32)}{\tan(7\pi/32)}\right)^2+\left(\frac{1-\tan(7\pi/32)}{\tan(7\pi/32)}\right)^2\right\}^{1/2}}\right\}}, \\ \beta_{\frac{32}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(3\pi/32)}{1-\tan(3\pi/32)}\right)^2+\left(\frac{\tan(2\pi/32)}{1-\tan(2\pi/32)}\right)^2\right\}^{1/2}}\right\}}, \\ \beta_{\frac{32}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(2\pi/32)}{1-\tan(2\pi/32)}\right)^2+\left(\frac{\tan(4\pi/32)}{1-\tan(4\pi/32)}\right)^2\right\}^{1/2}}\right\}} \end{array} \right\rangle = \langle \beta_{7.7626}, \beta_{1.1321}, \beta_{2.1985} \rangle;$$

$$\eta\beta_{L(1)} = \left\langle \beta_{\frac{32}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{0.8\left(\frac{\tan(7\pi/32)}{1-\tan(7\pi/32)}\right)^2\right\}^{1/2}}\right\}}, \beta_{\frac{32}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{0.8\left(\frac{1-\tan(3\pi/32)}{\tan(3\pi/32)}\right)^2\right\}^{1/2}}\right\}}, \beta_{\frac{32}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{0.8\left(\frac{1-\tan(2\pi/32)}{\tan(2\pi/32)}\right)^2\right\}^{1/2}}\right\}} \right\rangle;$$
$$= \langle \beta_{7.4269}, \beta_{3.1692}, \beta_{1.3523} \rangle$$

$$\beta_{L(1)}^{\eta} = \left\langle \beta_{\frac{32}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{0.8\left(\frac{1-\tan(7\pi/32)}{\tan(7\pi/32)}\right)^2\right\}^{1/2}}\right\}}, \beta_{\frac{32}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{0.8\left(\frac{\tan(3\pi/32)}{1-\tan(3\pi/32)}\right)^2\right\}^{1/2}}\right\}}, \beta_{\frac{32}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{0.8\left(\frac{\tan(2\pi/32)}{1-\tan(2\pi/32)}\right)^2\right\}^{1/2}}\right\}} \right\rangle.$$
$$= \langle \beta_{7.9037}, \beta_{0.7169}, \beta_{0.2451} \rangle$$

It is clear that the above LTDO results are still LNNs.

## 4. Tangent Dombi AOs of LNNs

According to the LNNTDOLs in Definition 2, the LNNTDWA and LNNTDWG operators of LNNs are built, and then their properties are investigated in this section.

**Definition 3.** Let $\beta_{L(j)} = <\beta_{Lt(j)}, \beta_{Lu(j)}, \beta_{Lf(j)}>$ ($j$ = 1, 2, …, $e$) be $e$ LNNs with their weight vector $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_e)$ for $\eta_j \in [0, 1]$ and $\sum_{j=1}^{e}\eta_j = 1$. Thus, the LNNTDWA and LNNTDWG operators of LNNs are defined below:

$$LNNTDWA(\beta_{L(1)}, \beta_{L(2)}, ..., \beta_{L(e)}) = \overset{e}{\underset{j=1}{\oplus_{TD}}} \eta_j \beta_{L(j)}, \tag{7}$$

$$LNNTDWG(\beta_{L(1)}, \beta_{L(2)}, ..., \beta_{L(e)}) = \overset{e}{\underset{j=1}{\otimes_{TD}}} \beta_{L(j)}^{\eta_j}. \tag{8}$$

**Theorem 1.** Let $\beta_{L(j)} = <\beta_{Lt(j)}, \beta_{Lu(j)}, \beta_{Lf(j)}>$ ($j = 1, 2, \ldots, e$) be $e$ LNNs with the weight vector $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_e)$ with $\eta_j \in [0, 1]$ and $\sum_{j=1}^{e}\eta_j = 1$. Then, the LNNTDWA operator can obtain the aggregated LNN by the following equation:

$$LNNTDWA(\beta_{L(1)}, \beta_{L(2)}, \ldots, \beta_{L(e)}) = \mathop{\oplus_{TD}}_{j=1}^{e} \eta_j \beta_{L(j)}$$

$$= \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\tan(Lt(j)\pi/(4g))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle. \quad (9)$$

**Proof**:

Here, Theorem 1 can be verified by mathematical induction.

Set $e = 2$. Based on the LNNTDOLs in Definition 2, we can give the result:

$$LNNTDWA(\beta_{L(1)}, \beta_{L(2)}) = \beta_{L(1)} \oplus_{TD} \beta_{L(2)}$$

$$= \left\langle \begin{array}{l} \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\eta_1\left(\frac{\tan(Lt(1)\pi/(4g))}{1-\tan(Lt(1)\pi/(4g))}\right)^{\varepsilon}+\eta_2\left(\frac{\tan(Lt(2)\pi/(4g))}{1-\tan(Lt(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\eta_1\left(\frac{1-\tan(Lu(1)\pi/(4g))}{\tan(Lu(1)\pi/(4g))}\right)^{\varepsilon}+\eta_2\left(\frac{1-\tan(Lu(2)\pi/(4g))}{\tan(Lu(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\eta_1\left(\frac{1-\tan(Lf(1)\pi/(4g))}{\tan(Lf(1)\pi/(4g))}\right)^{\varepsilon}+\eta_2\left(\frac{1-\tan(Lf(2)\pi/(4g))}{\tan(Lf(2)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \end{array} \right\rangle = \left\langle \begin{array}{l} \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{2}\eta_j\left(\frac{\tan(Lt(j)\pi/(4g))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{2}\eta_j\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{2}\eta_j\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \end{array} \right\rangle. \quad (10)$$

Set $e = n$. Eq. (9) can keep the equation:

$$LNNTDWA(\beta_{L(1)}, \beta_{L(2)}, \ldots, \beta_{L(n)}) = \mathop{\oplus_{TD}}_{j=1}^{n} \eta_j \beta_{L(j)}$$

$$= \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{n}\eta_j\left(\frac{\tan(Lt(j)\pi/(4g))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{n}\eta_j\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{n}\eta_j\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle. \quad (11)$$

Set $e = n + 1$. Based on Eqs. (10) and (11), there is the result:

$$LNNTDWA(\beta_{L(1)},\beta_{L(2)},...,\beta_{L(n)},\beta_{L(n+1)}) = \mathop{\oplus_{TD}}_{j=1}^{n} \eta_j \beta_{L(j)} \oplus_{TD} \eta_{n+1}\beta_{L(n+1)}$$

$$= \left\langle \begin{matrix} \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{n}\eta_j\left(\frac{\tan(Lt(j)\pi/(4g))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{n}\eta_j\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{k=1}^{n}\tau_k\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \end{matrix} \right\rangle \oplus_{TD} \eta_{n+1}\beta_{L(n+1)} = \left\langle \begin{matrix} \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{n+1}\eta_j\left(\frac{\tan(Lt(j)\pi/(4g))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{n+1}\eta_j\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{n+1}\eta_j\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \end{matrix} \right\rangle.$$

Therefore, Eq. (9) exists for any $e$. Thus, Theorem 1 is true. ■

**Theorem 2.** The LNNTDWA operator of Eq. (9) contains the properties below:

(i) Reducibility: When $\eta_j = 1/e$ ($j$ = 1, 2, …, $e$), the operation result of Eq. (9) is

$$LNNTDWA(\beta_{L(1)},\beta_{L(2)},...,\beta_{L(e)}) = \mathop{\oplus_{TD}}_{j=1}^{e} \eta_j \beta_{L(j)}$$

$$= \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\frac{1}{e}\left(\frac{\tan(Lt(j)\pi/(4g))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\frac{1}{e}\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\frac{1}{e}\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle.$$

(ii) Idempotency: Set $\beta_{L(j)} = <\beta_{Lt(j)}, \beta_{Lu(j)}, \beta_{Lf(j)}>$ ($j$ = 1, 2, ..., $e$) as $e$ LNNs. When $\beta_{L(j)} = \beta_L$ ($j$ = 1, 2, …, $e$), $LNNTDWA(\beta_{L(1)}, \beta_{L(2)},…, \beta_{L(e)}) = \beta_L$.

(iii) Commutativity: Set $(\beta'_{L(1)}, \beta'_{L(2)}, ..., \beta'_{L(e)})$ as any permutation of $(\beta_{L(1)}, \beta_{L(2)}, …, \beta_{L(e)})$. Then $LNNTDWA(\beta'_{L(1)}, \beta'_{L(2)}, ..., \beta'_{L(e)}) = LNNTDWA\left(\beta_{L(1)},\beta_{L(2)},\ldots,\beta_{L(e)}\right)$.

(iv) Boundedness: Set the minimum and maximum LNNs as $\beta_{L\min} = \left\langle \min_j(\beta_{Lt(j)}),\ \max_j(\beta_{Lu(j)}),\ \max_j(\beta_{Lf(j)})\right\rangle$ and $\beta_{L\max} = \left\langle \max_j(\beta_{Lt(j)}),\ \min_j(\beta_{Lu(j)}),\ \min_j(\beta_{Lf(j)})\right\rangle$. Then $\beta_{L\min} \le LNNTDWA(\beta_{L(1)}, \beta_{L(2)}, …, \beta_{L(e)}) \le \beta_{L\max}$.

**Proof:**

(i) The property of Eq. (9) is obvious.

(ii) Using Eq. (9) for $\beta_{L(j)} = <\beta_{Lt(j)}, \beta_{Lu(j)}, \beta_{Lf(j)}> = \beta_L = <\beta_{Lt}, \beta_{Lu}, \beta_{Lf}>$ ($j$ = 1, 2, ..., $e$), the result of Eq. (9) is given below:

$$LNNTDWA(\beta_{L(1)},\beta_{L(2)},...,\beta_{L(e)})=\mathop{\oplus_{TD}}_{j=1}^{e}\eta_j\beta_{L(j)}$$

$$=\left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\tan(Lt(j)\pi/(4g))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}},\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}},\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}\right\rangle$$

$$=\left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(Lt\pi/(4g))}{1-\tan(Lt\pi/(4g))}\right)^{\varepsilon}\sum_{j=1}^{e}\eta_j\right\}^{1/\varepsilon}}\right\}},\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(Lu\pi/(4g))}{\tan(Lu\pi/(4g))}\right)^{\varepsilon}\sum_{j=1}^{e}\eta_j\right\}^{1/\varepsilon}}\right\}},\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(Lu\pi/(4g))}{\tan(Lu\pi/(4g))}\right)^{\varepsilon}\sum_{j=1}^{e}\eta_j\right\}^{1/\varepsilon}}\right\}}\right\rangle$$

$$=\left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\left(\frac{\tan(Lt\pi/(4g))}{1-\tan(Lt\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}},\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(Lu\pi/(4g))}{\tan(Lu\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}},\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\left(\frac{1-\tan(Lf\pi/(4g))}{\tan(Lf\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}\right\rangle$$

$$=\left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\frac{\tan(Lt\pi/(4g))}{1-\tan(Lt\pi/(4g))}}\right\}},\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\frac{1-\tan(Lu\pi/(4g))}{\tan(Lu\pi/(4g))}}\right\}},\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\frac{1-\tan(Lf\pi/(4g))}{\tan(Lf\pi/(4g))}}\right\}}\right\rangle=\left\langle \beta_{Lt},\beta_{Lu},\beta_{Lf}\right\rangle=\beta_L.$$

Subsequently, $LNNTDWA(\beta_{L(1)}, \beta_{L(2)},\ldots, \beta_{L(e)}) = \beta_L$ exists.

(iii) It is clear that this feature is true.

(iv) Since $\min_j(\beta_{Lt(j)})\le\beta_{Lt(j)}\le\max_j(\beta_{Lt(j)})$, $\min_j(\beta_{Lu(j)})\le\beta_{Lu(j)}\le\max_j(\beta_{Lu(j)})$, and $\min_j(\beta_{Lf(j)})\le\beta_{Lf(j)}\le\max_j(\beta_{Lf(j)})$, there are the inequalities:

$$1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\min_j(\tan(Lt(j)\pi/(4g)))}{1-\min_j(\tan(Lt(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\le 1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{(\tan(Lt(j)\pi/(4g))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\le 1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\max_j(\tan(Lt(j)\pi/(4g)))}{1-\max_j(\tan(Lt(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}},$$

$$\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\min_j(\tan(Lu(j)\pi/(4g)))}{\min_j(\tan(Lu(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\le\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\le\frac{1}{1+\left\{\sum_{j=1}^{e}\beta_j\left(\frac{1-\max_j(\tan(Lu(j)\pi/(4g)))}{\max_j(\tan(Lu(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}},$$

$$\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\min_j(\tan(Lf(j)\pi/(4g)))}{\min_j(\tan(Lf(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\le\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\le\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\max_j(\tan(Lf(j)\pi/(4g)))}{\max_j(\tan(Lf(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}.$$

Then, we also have the result:

$$\beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\min_j(\tan(Lt(j)\pi/(4g)))}{1-\min_j(\tan(Lt(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \le \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{(\tan(Lt(j)\pi/(4g))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \le \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\max_j(\tan(Lt(j)\pi/(4g)))}{1-\max_j(\tan(Lt(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}},$$

$$\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\min_j(\tan(Lu(j)\pi/(4g)))}{\min_j(\tan(Lu(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \le \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \le \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\max_j(\tan(Lu(j)\pi/(4g)))}{\max_j(\tan(Lu(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}},$$

$$\beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\min_j(\tan(Lf(j)\pi/(4g)))}{\min_j(\tan(Lf(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \le \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{k=1}^{e}\tau_k\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \le \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\max_j(\tan(Lf(j)\pi/(4g)))}{\max_j(\tan(Lf(j)\pi/(4g)))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}.$$

According to the property (ii), we have the inequalities:

$$\min_j(\beta_{Lt(j)}) \le \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{(\tan(Lt(j)\pi/(4g))}{1-\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \le \max_j(\beta_{Lt(j)})'$$

$$\min_j(\beta_{Lu(j)}) \le \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lu(j)\pi/(4g))}{\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \le \max_j(\beta_{Lu(j)k})'$$

$$\min_j(\beta_{Lf(j)}) \le \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lf(j)\pi/(4g))}{\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \le \max_j(\beta_{Lf(j)}).$$

Thus, $\beta_{L\min} \le LNNTDWA(\beta_{L(1)}, \beta_{L(2)}, \ldots, \beta_{L(e)}) \le \beta_{L\max}$ can also held.

Based on the proof of the above properties, Theorem 2 is true. ■

**Theorem 3.** Let $\beta_{L(j)} = <\beta_{Lt(j)}, \beta_{Lu(j)}, \beta_{Lf(j)}>$ for $j$ = 1, 2, …, $e$ be $e$ LNNs with the weight vector $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_e)$, subject to $\eta_j \in [0, 1]$ and $\sum_{j=1}^{e}\eta_j = 1$. Then, the LNNTDWG operator can obtain the aggregated LNN by the following equation:

$$LNNTDWG(\beta_{L(1)},\beta_{L(2)},...,\beta_{L(e)}) = \underset{j=1}{\overset{h}{\otimes}}{}_{TD}\ \beta_{L(j)}^{\eta_j}$$

$$= \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lt(j)\pi/(4g))}{\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\tan(Lu(j)\pi/(4g))}{1-\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\tan(Lf(j)\pi/(4g))}{1-\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle. \quad (12)$$

A similar proof of Theorem 1 can also be used for Theorem 3 (omitted).

**Theorem 4.** The LNNTDWG operator of Eq. (12) has the properties:

(i) Reducibility: When $\eta_j = 1/e$ ($j$ = 1, 2, …, $e$), Eq. (12) has the result:

$$LNNTDWG(\beta_{L(1)},\beta_{L(2)},...,\beta_{L(e)}) = \underset{j=1}{\overset{e}{\otimes}}{}_{TD}\ \beta_{L(j)}^{\eta_j}$$

$$= \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\frac{1}{e}\left(\frac{1-\tan(Lt(j)\pi/(4g))}{\tan(Lt(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\frac{1}{e}\left(\frac{\tan(Lu(j)\pi/(4g))}{1-\tan(Lu(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\frac{1}{e}\left(\frac{\tan(Lf(j)\pi/(4g))}{1-\tan(Lf(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle.$$

(ii) Idempotency: Set $\beta_{L(j)} = <\beta_{Lt(j)}, \beta_{Lu(j)}, \beta_{Lf(j)}>$ ($j$ = 1, 2, ..., $e$) as $e$ LNNs. If $\beta_{L(j)} = \beta_L$ ($j$ = 1, 2, …, $e$), then $LNNTDWG(\beta_{L(1)}, \beta_{L(2)},\ldots, \beta_{L(e)}) = \beta_L$.

(iii) Commutativity: Set $(\beta'_{L(1)}, \beta'_{L(2)}, ..., \beta'_{L(e)})$ as any permutation of $(\beta_{L(1)}, \beta_{L(2)}, \ldots, \beta_{L(e)})$. Then $LNNTDWG\,(\beta'_{L(1)}, \beta'_{L(2)}, ..., \beta'_{L(e)}) = LNNTDWG\ \left(\beta_{L(1)},\beta_{L(2)},\ldots,\beta_{L(e)}\right)$.

(iv) Boundedness: Set the minimum and maximum LNNs as $\beta_{L\min} = \left\langle \min_j (\beta_{Lt(j)}),\ \max_j (\beta_{Lu(j)}),\ \max_j (\beta_{Lf(j)}) \right\rangle$ and $\beta_{L\max} = \left\langle \max_j (\beta_{Lt(j)}),\ \min_j (\beta_{Lu(j)}),\ \min_j (\beta_{Lf(j)}) \right\rangle$. Then $\beta_{L\min} \le LNNTDWG(\beta_{L(1)}, \beta_{L(2)}, \ldots, \beta_{L(e)}) \le \beta_{L\max}$.

A similar proof of Theorem 2 can be also applied to Theorem 4 (omitted).

**Example 2.** Set two LNNs on the SLT $\Psi_\beta = \{\beta_0, \beta_1, \ldots, \beta_8,\}$ as $\beta_{L(1)} = <\beta_7, \beta_2, \beta_3>$ and $\beta_{L(2)} = <\beta_6, \beta_3, \beta_4>$ with $\eta_1 = 0.3$, $\eta_2 = 0.7$, and $\varepsilon = 2$. Then, using Eqs. (9) and (12), their aggregation results are given by the following calculation:

$$LNNTDWA(\beta_{L1},\beta_{L2}) = \mathop{\oplus_{TD}}_{j=1}^{2} \eta_j \beta_{L(j)}$$

$$= \left\langle \begin{array}{l} \beta_{\frac{32}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{0.3\left(\frac{\tan(7\pi/32)}{1-\tan(7\pi/32)}\right)^2+0.7\left(\frac{\tan(6\pi/32)}{1-\tan(6\pi/32)}\right)^2\right\}^{1/2}}\right\}}, \\ \beta_{\frac{32}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{0.3\left(\frac{1-\tan(2\pi/32)}{\tan(2\pi/32)}\right)^2+0.7\left(\frac{1-\tan(3\pi/32)}{\tan(3\pi/32)}\right)^2\right\}^{1/2}}\right\}}, \\ \beta_{\frac{32}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{0.3\left(\frac{1-\tan(3\pi/32)}{\tan(3\pi/32)}\right)^2+0.7\left(\frac{1-\tan(4\pi/32)}{\tan(4\pi/32)}\right)^2\right\}^{1/2}}\right\}} \end{array} \right\rangle = \left\langle \beta_{6.9969}, \beta_{1.9070}, \beta_{3.8883} \right\rangle.$$

$$LNNTDWG(\beta_{L(1)},\beta_{L(2)}) = \mathop{\otimes_{TD}}_{j=1}^{2} \beta_{L(j)}^{\eta_j}$$

$$= \left\langle \begin{array}{l} \beta_{\frac{32}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{0.3\left(\frac{1-\tan(7\pi/28)}{\tan(7\pi/28)}\right)^2+0.7\left(\frac{1-\tan(6\pi/28)}{\tan(6\pi/28)}\right)^2\right\}^{1/2}}\right\}}, \\ \beta_{\frac{32}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{0.3\left(\frac{\tan(2\pi/28)}{1-\tan(2\pi/28)}\right)^2+0.7\left(\frac{\tan(3\pi/28)}{1-\tan(3\pi/28)}\right)^2\right\}^{1/2}}\right\}}, \\ \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{0.3\left(\frac{\tan(3\pi/28)}{1-\tan(3\pi/28)}\right)^2+0.7\left(\frac{\tan(4\pi/28)}{1-\tan(4\pi/28)}\right)^2\right\}^{1/2}}\right\}} \end{array} \right\rangle = \left\langle \beta_{7.5455}, \beta_{0.7148}, \beta_{1.7058} \right\rangle.$$

## 5. MAGDM Technique Using the LNNTDWA and LNNTDWG Operators

In terms of the LNNTDWA and LNNTDWG operators, this part establishes a MAGDM technique to perform MAGDM issues in the LNN scenario.

Regarding a MAGDM issue, there usually are a set of alternatives $\xi_A = \{\xi_{A1}, \xi_{A2}, ..., \xi_{Aq}\}$ and a set of attributes $\zeta_B = \{\zeta_{B1}, \zeta_{B2}, ..., \zeta_{Be}\}$, subject to the weight $\eta_j \in [0, 1]$ of each attribute $\zeta_{Bj}$ with $\sum_{j=1}^{e} \eta_j = 1$. In the MAGDM process, a team of decision makers, denoted as $\zeta_D = \{\zeta_{D1}, \zeta_{D2}, ..., \zeta_{Ds}\}$ with the weight $\delta_j \in [0, 1]$ of each $\zeta_{Dk}$ with $\sum_{k=1}^{s} \delta_k = 1$, will provide their suitability evaluation of the alternatives $\xi_{Ai}$ on the attributes $\zeta_{Bj}$ by the LNNs $\beta_{L(ij)}^{k} =< \beta_{Lt^k(ij)}, \beta_{Lu^k(ij)}, \beta_{Lf^k(ij)} >$ ($j$ = 1, 2, ..., $e$; $i$ = 1, 2, ..., q; $k$ = 1, 2,

…, $s$) yielded from the SLT $\Psi_\beta = \{\beta_0, \beta_1, \ldots, \beta_g\}$. Thus, the $k$-th LNN decision matrix is represented as $M_\beta^k = \left(\beta_{L(ij)}^k\right)_{q\times e}$.

Based on the LNNTDWA and LNNTDWG operators, a MAGDM technique is built to tackle a MAGDM issue in the LNN scenario by the following algorithm.

**Step 1:** Obtain the aggregated matrix $M_\beta = (\beta_{L(ij)})_{q\times e}$ by one of the LNNTDWA and LNNTDWA operators:

$$\beta_{L(ij)} = LNNTDWA(\beta_{L(ij)}^1, \beta_{L(ij)}^2, \ldots, \beta_{L(ij)}^s) = \mathop{\oplus_{TD}}_{k=1}^{s} \delta_k \beta_{L(ij)}^k$$
$$= \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{k=1}^{s}\delta_k\left(\frac{\tan(Lt^k(ij)\pi/(4g))}{1-\tan(Lt^k(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{k=1}^{s}\delta_k\left(\frac{1-\tan(Lu^k(ij)\pi/(4g))}{\tan(Lu^k(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{k=1}^{s}\delta_k\left(\frac{1-\tan(Lf^k(ij)\pi/(4g))}{\tan(Lf^k(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle. \quad (13)$$

$$\beta_{L(ij)} = LNNTDWG(\beta_{L(ij)}^1, \beta_{L(ij)}^2, \ldots, \beta_{L(ij)}^s) = \mathop{\otimes_{TD}}_{k=1}^{s} \left(\beta_{L(ij)}^k\right)^{\delta_k}$$
$$= \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{k=1}^{s}\delta_k\left(\frac{1-\tan(Lt^k(ij)\pi/(4g))}{\tan(Lt^k(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{k=1}^{s}\delta_k\left(\frac{\tan(Lu^k(ij)\pi/(4g))}{1-\tan(Lu^k(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{k=1}^{s}\delta_k\left(\frac{\tan(Lf^k(j)\pi/(4g))}{1-\tan(Lf^k(j)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle. \quad (14)$$

**Step 2:** Yield each aggregated LNN $\beta_{L(i)}$ for each alternative $\xi_{Ai}$ ($i$ = 1, 2, …, $q$) by one of the following LNNTDWA and LNNTDWG operators with $\varepsilon \ge 1$:

$$\beta_{L(i)} = LNNTDWA(\beta_{L(i1)}, \beta_{L(i2)}, \ldots, \beta_{L(ie)}) = \mathop{\oplus_{TD}}_{j=1}^{e} \eta_j \beta_{L(ij)}$$
$$= \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\tan(Lt(ij)\pi/(4g))}{1-\tan(Lt(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lu(ij)\pi/(4g))}{\tan(Lu(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lf(ij)\pi/(4g))}{\tan(Lf(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle, \quad (15)$$

$$\beta_{L(i)} = LNNTDWG(\beta_{L(i1)}, \beta_{L(i2)}, \ldots, \beta_{L(ie)}) = \mathop{\otimes_{TD}}_{j=1}^{e} \beta_{L(ij)}^{\eta_j}$$
$$= \left\langle \beta_{\frac{4g}{\pi}\tan^{-1}\left\{\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{1-\tan(Lt(ij)\pi/(4g))}{\tan(Lt(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\tan(Lu(ij)\pi/(4g))}{1-\tan(Lu(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}}, \beta_{\frac{4g}{\pi}\tan^{-1}\left\{1-\frac{1}{1+\left\{\sum_{j=1}^{e}\eta_j\left(\frac{\tan(Lf(ij)\pi/(4g))}{1-\tan(Lf(ij)\pi/(4g))}\right)^{\varepsilon}\right\}^{1/\varepsilon}}\right\}} \right\rangle. \quad (16)$$

**Step 2:** Get the score (accuracy if necessary) values of $D(\beta_{L(i)})$ ($E(\beta_{L(i)})$) ($i$ = 1, 2, ..., $q$) by Eq. (3) (Eq. (4)).

**Step 3**: Sort the alternatives and decide the optimal one.

**Step 4**: End.

## 6. MAGDM Application

*6.1 Selection of NEDSs*

This section applies the built MAGDM technique in the selection of NEDSs for an investment project in Ningbo City, China, in the LNN scenario.

New energy development is becoming increasingly important and necessary to diversify energy sources and reduce carbon emissions. To effectively develop new energy sources, an energy company wants to develop some new energy generation as an investment project in Ningbo City. Therefore, the energy engineers provide four potential design schemes: solar energy ($\xi_{A1}$), wind energy ($\xi_{A2}$), wave energy ($\xi_{A3}$), tidal energy ($\xi_{A4}$), which are denoted as their set $\xi_A$ = {$\xi_{A1}$, $\xi_{A2}$, $\xi_{A3}$, $\xi_{A4}$}. In the assessment process, they must meet the requirements of important factors/attributes: technical performance ($\zeta_{D1}$), cost-effectiveness in terms of benefits and return on investment ($\zeta_{D2}$), environmental condition ($\zeta_{D3}$), and operational adaptability ($\zeta_{D4}$), then their importance is specified by the known weight vector $\boldsymbol{\eta}$ = (0.3, 0.25, 0.23, 0.22). The energy company invites a panel of three decision makers/experts, referred to as their set $\zeta_D$ = {$\zeta_{D1}$, $\zeta_{D2}$, $\zeta_{D3}$}, subject to their specified weight vector $\boldsymbol{\delta}$= (0.37, 0.33, 0.3), to evaluate the suitability degrees of each scheme $\xi_{Ai}$ ($i$ = 1, 2, 3, 4) over the factors $\zeta_{Dj}$ ($j$ = 1, 2, 3, 4) in terms of the SLT $\Psi_\beta$ = {$\beta_0$ (Extremely unsuitable), $\beta_1$ (Very unsuitable), $\beta_2$ (Unsuitable), $\beta_3$ (Slightly unsuitable), $\beta_4$ (Medium), $\beta_5$ (Slightly suitable), $\beta_6$ (Suitable), $\beta_7$ (Very suitable), $\beta_8$ (Extremely suitable)}. Thus, their true, false and indeterminate linguistic assessment values can be constructed as the LNNs $\beta_{L(ij)}^k = <\beta_{Lt^k(ij)}, \beta_{Lu^k(ij)}, \beta_{Lf^k(ij)}>$ ($j$, $i$ = 1, 2, 3, 4; $k$ = 1, 2, 3). Subsequently, the LNNs provided by the three decision makers/experts can be constructed as the three LNN evaluation matrices:

$$M_\beta^1 = \left(\beta_{L(ij)}^1\right)_{4\times 4} = \begin{bmatrix} \langle\beta_6,\beta_1,\beta_1\rangle & \langle\beta_7,\beta_2,\beta_2\rangle & \langle\beta_5,\beta_1,\beta_2\rangle & \langle\beta_6,\beta_2,\beta_2\rangle \\ \langle\beta_7,\beta_1,\beta_1\rangle & \langle\beta_6,\beta_1,\beta_3\rangle & \langle\beta_7,\beta_2,\beta_2\rangle & \langle\beta_6,\beta_1,\beta_2\rangle \\ \langle\beta_6,\beta_2,\beta_3\rangle & \langle\beta_5,\beta_2,\beta_3\rangle & \langle\beta_5,\beta_3,\beta_1\rangle & \langle\beta_4,\beta_1,\beta_2\rangle \\ \langle\beta_5,\beta_1,\beta_2\rangle & \langle\beta_5,\beta_3,\beta_2\rangle & \langle\beta_5,\beta_2,\beta_2\rangle & \langle\beta_6,\beta_2,\beta_3\rangle \end{bmatrix},$$

$$M_\beta^2 = \left(\beta_{L(ij)}^2\right)_{4\times 4} = \begin{bmatrix} \langle\beta_6,\beta_2,\beta_3\rangle & \langle\beta_7,\beta_2,\beta_2\rangle & \langle\beta_5,\beta_3,\beta_2\rangle & \langle\beta_6,\beta_3,\beta_2\rangle \\ \langle\beta_7,\beta_2,\beta_1\rangle & \langle\beta_7,\beta_1,\beta_3\rangle & \langle\beta_7,\beta_1,\beta_1\rangle & \langle\beta_6,\beta_1,\beta_2\rangle \\ \langle\beta_6,\beta_2,\beta_3\rangle & \langle\beta_6,\beta_3,\beta_3\rangle & \langle\beta_5,\beta_2,\beta_2\rangle & \langle\beta_5,\beta_2,\beta_3\rangle \\ \langle\beta_5,\beta_2,\beta_2\rangle & \langle\beta_6,\beta_1,\beta_2\rangle & \langle\beta_5,\beta_3,\beta_2\rangle & \langle\beta_5,\beta_4,\beta_3\rangle \end{bmatrix},$$

$$M_\beta^3 = \left(\beta_{L(ij)}^3\right)_{4\times 4} = \begin{bmatrix} \langle\beta_5,\beta_1,\beta_2\rangle & \langle\beta_7,\beta_2,\beta_2\rangle & \langle\beta_6,\beta_1,\beta_2\rangle & \langle\beta_6,\beta_2,\beta_3\rangle \\ \langle\beta_6,\beta_1,\beta_2\rangle & \langle\beta_7,\beta_1,\beta_2\rangle & \langle\beta_6,\beta_2,\beta_3\rangle & \langle\beta_7,\beta_2,\beta_4\rangle \\ \langle\beta_5,\beta_2,\beta_1\rangle & \langle\beta_6,\beta_3,\beta_3\rangle & \langle\beta_5,\beta_3,\beta_3\rangle & \langle\beta_6,\beta_3,\beta_2\rangle \\ \langle\beta_6,\beta_3,\beta_4\rangle & \langle\beta_5,\beta_3,\beta_4\rangle & \langle\beta_6,\beta_3,\beta_2\rangle & \langle\beta_5,\beta_2,\beta_1\rangle \end{bmatrix}.$$

Thus, the proposed MAGDM technique is utilized to tackle the MAGDM issue in the LNN scenario by the following decision algorithm.

First, set $\varepsilon$ = 1 as a calculational example in this MAGDM application. By Eq. (13) or (14), the aggregated LNN matrix is obtained below:

$$M_\beta = \left(\beta_{L(ij)}\right)_{4\times 4} = \begin{bmatrix} \langle \beta_{5.7765}, \beta_{1.1982}, \beta_{1.5903} \rangle & \langle \beta_{7}, \beta_{2}, \beta_{2} \rangle & \langle \beta_{5.3893}, \beta_{1.2841}, \beta_{2} \rangle & \langle \beta_{6}, \beta_{2.2485}, \beta_{2.2235} \rangle \\ \langle \beta_{6.823}, \beta_{1.1982}, \beta_{1.177} \rangle & \langle \beta_{6.7723}, \beta_{1}, \beta_{2.6107} \rangle & \langle \beta_{6.823}, \beta_{1.505}, \beta_{1.629} \rangle & \langle \beta_{6.4603}, \beta_{1.177}, \beta_{2.3574} \rangle \\ \langle \beta_{5.7765}, \beta_{2}, \beta_{1.7032} \rangle & \langle \beta_{5.7171}, \beta_{2.5337}, \beta_{3} \rangle & \langle \beta_{5}, \beta_{2.5771}, \beta_{1.5777} \rangle & \langle \beta_{5.157}, \beta_{1.5777}, \beta_{2.2485} \rangle \\ \langle \beta_{5}, \beta_{1.461}, \beta_{1.8098} \rangle & \langle \beta_{5.4229}, \beta_{1.8129}, \beta_{2.3574} \rangle & \langle \beta_{5.3893}, \beta_{2.5337}, \beta_{2} \rangle & \langle \beta_{5.4663}, \beta_{2.4002}, \beta_{1.881} \rangle \end{bmatrix}$$

or

$$M_\beta = \left(\beta_{L(ij)}\right)_{4\times 4} = \begin{bmatrix} \langle \beta_{5.6645}, \beta_{1.3599}, \beta_{2.0642} \rangle & \langle \beta_{7}, \beta_{2}, \beta_{2} \rangle & \langle \beta_{5.2665}, \beta_{1.7966}, \beta_{2} \rangle & \langle \beta_{6}, \beta_{2.3672}, \beta_{2.3355} \rangle \\ \langle \beta_{6.6715}, \beta_{1.3599}, \beta_{1.3285} \rangle & \langle \beta_{6.5987}, \beta_{1}, \beta_{2.7335} \rangle & \langle \beta_{6.6715}, \beta_{1.6988}, \beta_{2.0639} \rangle & \langle \beta_{6.2728}, \beta_{1.3285}, \beta_{2.7664} \rangle \\ \langle \beta_{5.6645}, \beta_{2}, \beta_{2.1688} \rangle & \langle \beta_{5.591}, \beta_{2.6675}, \beta_{3} \rangle & \langle \beta_{5}, \beta_{2.7054}, \beta_{2.0294} \rangle & \langle \beta_{4.8063}, \beta_{2.0294}, \beta_{2.3672} \rangle \\ \langle \beta_{5}, \beta_{1.6605}, \beta_{2.8167} \rangle & \langle \beta_{5.2946}, \beta_{2.4653}, \beta_{2.7664} \rangle & \langle \beta_{5.2665}, \beta_{2.6675}, \beta_{2} \rangle & \langle \beta_{5.3325}, \beta_{2.8333}, \beta_{2.5181} \rangle \end{bmatrix}.$$

Second, the aggregated LNNs $\beta_{L(i)}$ for the schemes $\xi_{Ai}$ ($i$ = 1, 2, …, $q$) are yielded by Eqs. (14) or (15) as follows:

$\beta_{L(1)}$ = <$\beta_{6.2825}$, $\beta_{1.5338}$, $\beta_{1.8958}$>, $\beta_{L(2)}$ = <$\beta_{6.7449}$, $\beta_{1.1904}$, $\beta_{1.7107}$>, $\beta_{L(3)}$ = <$\beta_{5.4896}$, $\beta_{2.0959}$, $\beta_{1.9893}$>, and $\beta_{L(4)}$ = <$\beta_{5.3110}$, $\beta_{1.9035}$, $\beta_{1.9853}$>.

Or $\beta_{L(1)}$ = <$\beta_{5.9241}$, $\beta_{1.8615}$, $\beta_{2.0957}$>, $\beta_{L(2)}$ = <$\beta_{6.5622}$, $\beta_{1.3485}$, $\beta_{2.2201}$>, $\beta_{L(3)}$ = <$\beta_{5.2808}$, $\beta_{2.3536}$, $\beta_{2.4116}$>, and $\beta_{L(4)}$ = <$\beta_{5.2047}$, $\beta_{2.3846}$, $\beta_{2.5668}$>.

Lastly, using Eq. (3), the score values of D($\beta_L(i)$) ($i$ = 1, 2, ..., $q$) are in the following:

$D(\beta_{L(1)})$ = 0.7855, $D(\beta_{L(2)})$ = 0.8268, $D(\beta_{L(3)})$ = 0.7252, and $D(\beta_{L(4)})$ = 0.7259.

Or $D(\beta_{L(1)})$ = 0.7486, $D(\beta_{L(2)})$ = 0.7914, $D(\beta_{L(3)})$ = 0.6882, and $D(\beta_{L(4)})$ = 0.6772.

Subsequently, the sorting orders of the four NEDSs are $\xi_{A2} > \xi_{A1} > \xi_{A4} > \xi_{A3}$ or $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$, then the optimal one is $\xi_{A2}$.

Similarly to the above calculation process, all the evaluation results with $\varepsilon$ = {1, 3, 5, 7, 9, 11} are shown in Table 1 and Table 2.

**Table 1.** Evaluation results based on the proposed MAGDM technique using the LNNTDWA operator

| $\varepsilon$ | Score value | Sorting | Optimal one |
|---|---|---|---|
| $\varepsilon$ = 1 | 0.7855, 0.8268, 0.7252, 0.7259 | $\xi_{A2} > \xi_{A1} > \xi_{A4} > \xi_{A3}$ | $\xi_{A2}$ |
| $\varepsilon$ = 3 | 0.8174, 0.8482, 0.7627, 0.7653 | $\xi_{A2} > \xi_{A1} > \xi_{A4} > \xi_{A3}$ | $\xi_{A2}$ |
| $\varepsilon$ = 5 | 0.8372, 0.8581, 0.7868, 0.7876 | $\xi_{A2} > \xi_{A1} > \xi_{A4} > \xi_{A3}$ | $\xi_{A2}$ |
| $\varepsilon$ = 7 | 0.8481, 0.8629, 0.8001, 0.8000 | $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$ | $\xi_{A2}$ |
| $\varepsilon$ = 9 | 0.8544, 0.8657, 0.8078, 0.8075 | $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$ | $\xi_{A2}$ |
| $\varepsilon$ = 11 | 0.8584, 0.8674, 0.8127, 0.8123 | $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$ | $\xi_{A2}$ |

**Table 2.** Evaluation results based on the proposed MAGDM technique using the LNNTDWG operator

| $\varepsilon$ | Score value | Sorting | Optimal one |
|---|---|---|---|
| $\varepsilon$ = 1 | 0.7486, 0.7914, 0.6882, 0.6772 | $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$ | $\xi_{A2}$ |
| $\varepsilon$ = 3 | 0.7184, 0.7568, 0.6620, 0.6391 | $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$ | $\xi_{A2}$ |
| $\varepsilon$ = 5 | 0.6962, 0.7325, 0.6438, 0.6139 | $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$ | $\xi_{A2}$ |
| $\varepsilon$ = 7 | 0.6805, 0.7170, 0.6309, 0.5974 | $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$ | $\xi_{A2}$ |
| $\varepsilon$ = 9 | 0.6696, 0.7068, 0.6218, 0.5863 | $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$ | $\xi_{A2}$ |
| $\varepsilon$ = 11 | 0.6620, 0.6999, 0.6153, 0.5786 | $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$ | $\xi_{A2}$ |

From the results of Table 1, we see that the different values of ε can affect the ranking of the four schemes, but the optimal one is always $\xi_{A2}$.

From the results of Table 2, we see that the four schemes corresponding to the different values of ε show the ranking robustness, and then the optimal one is always $\xi_{A2}$.

According to the sorting results in Tables 1 and 2, the energy company should select $\xi_{A2}$ (wind energy) as the investment project in Ningbo City. The wind power generation is reasonable because Ningbo City is closer to the east see.

*6.2 Comparison with the Existing MAGDM Technique Using the LNNDWA and LNNDWG Operators*

In this part, the proposed MAGDM technique is compared with the existing MAGDM technique using the LNNDWA and LNNDWG operators [29] by the selection case of NEDSs to demonstrate the validity and appropriateness of the former.

Based on the existing MAGDM technique using the LNNDWA and LNNDWG operators [29], it can be applied to the selection case of NEDSs. Using Eqs. (1) – (3), all the evaluation results with ε = {1, 3, 5, 7, 9, 11} are shown in Tables 3 and 4.

**Table 3.** Evaluation results based on the existing MAGDM technique using the LNNDWA operator [29]

| $\varepsilon$ | Score value | Sorting | Optimal one |
|---|---|---|---|
| $\varepsilon = 1$ | 0.7858, 0.8270, 0.7256, 0.7264 | $\xi_{A2} > \xi_{A1} > \xi_{A4} > \xi_{A3}$ | $\xi_{A2}$ |
| $\varepsilon = 3$ | 0.8184, 0.8487, 0.7640, 0.7666 | $\xi_{A2} > \xi_{A1} > \xi_{A4} > \xi_{A3}$ | $\xi_{A2}$ |
| $\varepsilon = 5$ | 0.8381, 0.8585, 0.7881, 0.7888 | $\xi_{A2} > \xi_{A1} > \xi_{A4} > \xi_{A3}$ | $\xi_{A2}$ |
| $\varepsilon = 7$ | 0.8488, 0.8632, 0.8011, 0.8010 | $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$ | $\xi_{A2}$ |
| $\varepsilon = 9$ | 0.8549, 0.8659, 0.8086, 0.8083 | $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$ | $\xi_{A2}$ |
| $\varepsilon = 11$ | 0.8588, 0.8676, 0.8133, 0.8130 | $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$ | $\xi_{A2}$ |

**Table 4.** Evaluation results based on the existing MAGDM technique using the LNNDWG operator [29]

| $\varepsilon$ | Score value | Sorting | Optimal one |
|---|---|---|---|
| $\varepsilon = 1$ | 0.7476, 0.7903, 0.6872, 0.6758 | $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$ | $\xi_{A2}$ |
| $\varepsilon = 3$ | 0.7164, 0.7547, 0.6603, 0.6366 | $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$ | $\xi_{A2}$ |
| $\varepsilon = 5$ | 0.6939, 0.7303, 0.6418, 0.6112 | $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$ | $\xi_{A2}$ |
| $\varepsilon = 7$ | 0.6783, 0.7150, 0.6290, 0.5949 | $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$ | $\xi_{A2}$ |
| $\varepsilon = 9$ | 0.6676, 0.7051, 0.6200, 0.5842 | $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$ | $\xi_{A2}$ |
| $\varepsilon = 11$ | 0.6603, 0.6984, 0.6138, 0.5768 | $\xi_{A2} > \xi_{A1} > \xi_{A3} > \xi_{A4}$ | $\xi_{A2}$ |

According to the sorting results in Tables 1–4, it is obvious that the sorting orders between the proposed MAGDM technique and the existing MAGDM technique [29] are very consistent in this decision application, which investigates the validity and appropriateness of the proposed MAGDM technique in the LNN scenario. However, the proposed MAGDM technique contains the periotic/multitemporal MAGDM capability, while the existing MAGDM technique [29] cannot imply it and shows its research gap in the LNN scenario. Therefore, the former fill the research gap of the latter in the LNN scenario.

## 7. Conclusions

This article first presented the linguistic tangent Dombi t-norm and t-conorm and the LNNDOLs, which imply the trigonometric periodicity. To overcome the inadequacy of existing LNN aggregation operators without periodicity, the LNNTDWA and LNNTDWG operators were developed to provide periodic aggregation tools for building periodic MAGDM techniques in the LNN scenario. Next, the developed MAGDM technique using the LNNTDWA and LNNTDWG operators can reasonably tackle MAGDM problems with periodic/multitemporal requirements in LNN scenarios. Finally, the developed MAGDM technique was applied to the decision case of NEDSs and compared with the existing MAGDM technique using the LNNDWA and LNNDWG operators in the LNN scenario. Then the decision results demonstrated that the developed MAGDM technique was effective and applicable. Meanwhile, the developed MAGDM technique has filled the research gap of the existing MAGDM techniques, which lack periodic/multitemporal capability in LNN scenarios.

Generally, this paper has developed the LNNTDWA and LNNTDWG operators and their MAGDM technique and used them for the first time in the choice of NEDSs. To present much more research work in the future, we need to further propose LNN trigonometric Einstein, Heronian, and Bonferroni operators and to apply them in supply chain management, medical diagnosis, and risk investment in LNN scenarios.

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