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Some Operators For Interval Generalized Set Valued Neutrosophic Quintuple Numbers And Sets

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Abstract: In this study, some constructions related to interval generalized set-valued neutrosophic quintuple sets and score and certainty functions are given. Moreover, optimistic union, pessimistic union, mean union, optimistic intersection, pessimistic intersection and mean intersection operators are defined for interval generalized set-valued neutrosophic quintuple set. Thus, interval generalized set-valued neutrosophic quintuple set are effectively used in decision making applications. An example application was realised using these operators and functions. With this application, the use of operators and functions in decision making methods was demonstrated. In addition, it was concluded that similar or different results can be obtained by using different operators. In addition, the importance of choosing the appropriate operator for the problem in decision making applications was mentioned for the decision making process. Researchers can obtain more objective results in decision making processes by using the operators and functions in this study.

Keywords: Interval Generalized Set Valued Neutrosophic Quintuple set, Interval Generalized Set Valued Neutrosophic Quintuple set, Operators, Score and Accuracy Functions, Decision Making Applications.

1. Introduction

The theory of neutrosophic sets and neutrosophic logic, developed by Florentin Smarandache in 1998, provides an important mathematical framework for handling uncertain concepts more effectively [1]. Furthermore, Wang et al. in 2010 defined single-valued neutrosophic sets (SVNS) [2]. This theory determines the degrees of truth (T), uncertainty (I) and falsity (F) of each element in the interval [0,1]. Thus, it recognises that each element has a degree of uncertainty, without being bound to a precise degree of truth or falsity. Neutrosophic set and logic theory is an important tool for understanding and solving complex situations full of uncertainties and finds applications in many different disciplines. Kargın and Şahin defined SuperHyper Groups and Neutro-SuperHyper Groups in 2023 [3]. Sahin et al. 2023 defined metric spaces and normed spaces for neutrosophic numbers [4]. J. Jakhar et al. In 2024, he studied to find the stability of Cauchy additive functional equation in neutrosophic normed space [6]. S.N. Suber Bathusha and S. Angelin Kavitha Raj did a study on a new approach on the energy of λJ-dominated single-valued neutrosophic graph structure in 2024 [7]. B. Banik et al. conducted a study on the Analysis of Economic Decline of Different Countries due to COVID-19 Surge in 2024 with Advanced Multiple MOORA Strategy under

Pentagonal Neutrosophic Area [8]. P. Agarwal et al. 2024 on Ideals and Filters in Neutrosophic Topologies Generated by Neutrosophic Relations [9].

Wang et al. In 2005, Wang et al. defined interval neutrosophic sets (INS) as an important step in the development of neutrosophic theory [9]. In INS; accuracy, uncertainty and inaccuracy values are used as intervals instead of a single value. This approach provides a more useful structure in decision-making applications because it is often not possible to make a definite judgement in a decision-making situation. In offer a more flexible approach when dealing with decision-making situations fraught with uncertainty. In 2022, Kargın and Şahin defined interval generalized set-valued neutrosophic quadruple sets and numbers [10]. In 2023, Sahin and Kargin defined interval generalized set-valued Pythagorean neutrosophic quadruple set [11]. In 2024, Lo, H. W. conducted a study on a new interval neutrosophic-based group decision-making approach for sustainable development assessment in the computer manufacturing industry [12]. Rosli, S. N. I. I., and Zulkifly, M. I. E. Z. 2024 on Interval Neutrosophic Cubic Bézier Curve Approximation Model for Complex Data [13]. Hamad, F. A., and Algamudi, B. M. In 2024, he did a study on N-Cylindrical Fuzzy Interval Neutrosophic set [14]. Li, H. In 2024, he conducted a study on an improved group decision-making framework for financial performance evaluation of high-tech enterprises in an interval neutrosophic environment [15].

Smarandache defined neutrosophic quadruple sets and numbers in 2015 [16].Neutrosophic quadruple set have truth (T), uncertainty (I) and falsity (F) values as in neutrosophic sets, but unlike neutrosophic sets, they also have known and unknown parts such that (i)+(jT+kI+lF) (i,j,k,l$\in \mathbb{R}$ or $\mathbb{C}$) [17]. Thus, this set is named with neutrosophic quadruple set because of four components (i, jT, kI, lF). Therefore, neutrosophic quadruple sets are a generalization of neutrosophic sets. Sahin et al. in 2020 defined generalized set-valued neutrosophic quadruple sets and numbers [18].Thanks to this new structure, neutrosophic quadruple sets can be used in decision-making applications. Şahin et al. In 2021, Şahin et al. defined generalized Euclidean measures based on generalized set-valued neutrosophic quadruple numbers and multi-criteria decision-making applications [19]. Kargın et al. In 2021, he studied on Generalized Hamming Similarity Measure Based on Neutrosophic Quadruple Sets and Applications to Legal Sciences [20]. In 2022, Şahin and Kargın defined interval generalized set-valued neutrosophic quadruple sets [21]. A new structure was obtained using interval neutrosophic sets and generalized set-valued neutrosophic quadruple sets together. Sahin et al. In 2022, Pythagorean neutrosophic quadruple set and set-valued Pythagorean neutrosophic quadruple set were defined [22]. In 2023, Kargın and Şahin defined neutrosophic triple normed spaces based on set-valued neutrosophic quadruple numbers [23]. Sahin et al. in 2023 defined interval generalized set-valued neutrosophic quadruple graphs [24]. Şahin and Kargın defined interval generalized set-valued Pythagorean neutrosophic quadruple numbers in 2023 [25].

Sahin et al. In 2022, neutrosophic quintuple sets and numbers were defined [26]. Neutrosophic quintuple set have truth (T), uncertainty (U), contradiction (C) and falsity (F) values as in quadripartitioned neutrosophic sets, but unlike quadripartitioned neutrosophic sets, they also have known and unknown parts such that (i)+(jT+kU+lC+mF) (i,j,k,l,m$\in \mathbb{R}$ or $\mathbb{C}$) [26]. Thus, this set is named with neutrosophic quadruple set because of five components (i, jT, kU, lC, mF). Thus, this new structure, which is a generalization of neutrosophic quadruple sets and quadripartitioned neutrosophic sets, provides the properties of both neutrosophic quadruple sets and quadripartitioned neutrosophic sets. In 2022, Sahin and Kargin defined interval set-valued neutrosophic quintuple sets (ISVNQS) and numbers (ISVNQN) [27]. These new sets, which are a generalization of neutrosophic quintuple sets and interval quadruple neutrosophic sets, satisfy the properties of both neutrosophic quintuple sets and interval quadruple neutrosophic sets. In 2022, Şahin and Kargın defined bipolar neutrosophic quintuple set and bipolar set-valued neutrosophic quintuple set [28]. Sahin et al. defined generalized set-valued neutrosophic quintuple set in 2023 [29]. This new structure, which is a generalized form of set-valued neutrosophic quintuple set, provides the properties of set-valued neutrosophic quintuple set. Şahin et al. in 2024 defined single-valued neutrosophic quintuple graphs [30]. Thus, a new structure using generalized set-valued neutrosophic quintuple set and graph theory is obtained. Sahin et al. 2024 defined interval generalized set-valued neutrosophic quintuple sets (IGSVNQS) and numbers (IGSVNQN) and (IGSVNQS) graphs [31]. Thus, a new structure was added to both neutrosophic quintuple set theory and graph theory. Also, Chatterjee et al. defined quadripartitioned neutrosophic set [32] in 2016. Pramanik obtained interval quadripartitioned neutrosophic set (IQNS) [33] in 2022.

In this study, in the Preliminaries Section, the basic definitions and properties to be used in the study are given. In the Research and Fundigs Section, some constructions related to (IGSVNQS) and score and certainty functions are defined. Moreover, optimistic union, pessimistic union, mean union, optimistic intersection, pessimistic intersection and mean intersection operators are defined for IGSVNQS. Thus, IGSVNQS are made available for decision making applications. In the Application Section, a sample application was made using the functions and some operators defined for IGSVNQS. With this application, the use of operators and functions in decision making processes is concretely demonstrated. In addition, it was shown that similar or different results can be obtained by using different operators and the importance of choosing the appropriate operator for the problem in the decision making process was emphasised. In the Conclusion Section, the findings of the study are summarised and suggestions for future studies are presented.

2. Preliminaries

Definition 2.1: [2] Let $\mathcal{P}$ be a universal set. For $\forall \breve{p} \in \mathcal{P}$,

a SVNS $\tilde{A}$ over $\mathcal{P}$ with functions; is defined as

$$\tilde{A} = \{\langle \breve{p}, T_{\tilde{A}(\breve{p})}, I_{\tilde{A}(\breve{p})}, F_{\tilde{A}(\breve{p})} \rangle : \breve{p} \in \mathcal{P}\}$$

such that

$$0 \leq T_{\tilde{A}(\breve{p})} + I_{\tilde{A}(\breve{p})} + F_{\tilde{A}(\breve{p})} \leq 3$$

$$T_{\tilde{A}}: \mathcal{P} \rightarrow [0,1] \ , \ I_{\tilde{A}}: \mathcal{P} \rightarrow [0,1] \quad \text{and} \quad F_{\tilde{A}}: \mathcal{P} \rightarrow [0,1]$$

Where $T_{\tilde{A}(\breve{p})}, I_{\tilde{A}(\breve{p})}$ and $F_{\tilde{A}(\breve{p})}$ are the degree of truth, degree of indeterminancy and degree of falsity of $\breve{p} \in \mathcal{P}$ respectively.

Definition 2.2:[10] Let $\boldsymbol{\mathcal{P}} \neq \emptyset$. An INS is defined as

$$\widetilde{\boldsymbol{A}} = \{< \breve{\boldsymbol{p}}, [\boldsymbol{T^L}_{\tilde{A}(\breve{p})}, \boldsymbol{T^U}_{\tilde{A}(\breve{p})}], [\boldsymbol{I^L}_{\tilde{A}(\breve{p})}, \boldsymbol{I^U}_{\tilde{A}(\breve{p})}], [\boldsymbol{F^L}_{\tilde{A}(\breve{p})}, \boldsymbol{F^U}_{\tilde{A}(\breve{p})}] >: \breve{\boldsymbol{p}} \in \boldsymbol{\mathcal{P}}\}.$$

Here,

$[\boldsymbol{T^L}_{\tilde{A}(\breve{p})}, \boldsymbol{T^U}_{\tilde{A}(\breve{p})}], [\boldsymbol{I^L}_{\tilde{A}(\breve{p})}, \boldsymbol{I^U}_{\tilde{A}(\breve{p})}]$ and $[\boldsymbol{F^L}_{\tilde{A}(\breve{p})}, \boldsymbol{F^U}_{\tilde{A}(\breve{p})}]$ are intervals

$\boldsymbol{T^L}_{\tilde{A}} : \boldsymbol{\mathcal{P}} \rightarrow [0,1]$, $\boldsymbol{T^U}_{\tilde{A}} : \boldsymbol{\mathcal{P}} \rightarrow [0,1]$ truth functions,

$\boldsymbol{I^L}_{\tilde{A}} : \boldsymbol{\mathcal{P}} \rightarrow [0,1]$, $\boldsymbol{I^U}_{\tilde{A}} : \boldsymbol{\mathcal{P}} \rightarrow [0,1]$ indeterminancy functions,

$\boldsymbol{F^L}_{\tilde{A}} : \boldsymbol{\mathcal{P}} \rightarrow [0,1]$, $\boldsymbol{F^U}_{\tilde{A}} : \boldsymbol{\mathcal{P}} \rightarrow [0,1]$ are falsity functions.

Definition 2.3: [32] Let $\boldsymbol{\mathcal{P}} \neq \emptyset$. An IQNS is defined as

$\widetilde{\boldsymbol{A}} = \{< \breve{\boldsymbol{p}}: [\boldsymbol{T^L}_{\widetilde{A}(\breve{p})}, \boldsymbol{T^U}_{\widetilde{A}(\breve{p})}], [\boldsymbol{U^L}_{\widetilde{A}(\breve{p})}, \boldsymbol{U^U}_{\widetilde{A}(\breve{p})}], [\boldsymbol{C^L}_{\widetilde{A}(\breve{p})}, \boldsymbol{C^U}_{\widetilde{A}(\breve{p})}], [\boldsymbol{F^L}_{\widetilde{A}(\breve{p})}, \boldsymbol{F^U}_{\widetilde{A}(\breve{p})}] >; \breve{\boldsymbol{p}} \in \boldsymbol{\mathcal{P}}\}$

Here,

$[\boldsymbol{T^L}_{\widetilde{A}(\breve{p})}, \boldsymbol{T^U}_{\widetilde{A}(\breve{p})}], [\boldsymbol{U^L}_{\widetilde{A}(\breve{p})}, \boldsymbol{U^U}_{\widetilde{A}(\breve{p})}], [\boldsymbol{C^L}_{\widetilde{A}(\breve{p})}, \boldsymbol{C^U}_{\widetilde{A}(\breve{p})}]$ and $[\boldsymbol{F^L}_{\widetilde{A}(\breve{p})}, \boldsymbol{F^U}_{\widetilde{A}(\breve{p})}]$ are intervals

$\boldsymbol{T^L}_{\widetilde{A}} : \boldsymbol{\mathcal{P}} \rightarrow [0,1]$, $\boldsymbol{T^U}_{\widetilde{A}} : \boldsymbol{\mathcal{P}} \rightarrow [0,1]$ truth functions,

$\boldsymbol{U^L}_{\widetilde{A}} : \boldsymbol{\mathcal{P}} \rightarrow [0,1]$ $\boldsymbol{U^U}_{\widetilde{A}} : \boldsymbol{\mathcal{P}} \rightarrow [0,1]$ unknownability functions,

$\boldsymbol{C^L}_{\widetilde{A}} : \boldsymbol{\mathcal{P}} \rightarrow [0,1]$, $\boldsymbol{C^U}_{\widetilde{A}} : \boldsymbol{\mathcal{P}} \rightarrow [0,1]$ contradiction functions,

$\boldsymbol{F^L}_{\widetilde{A}} : \boldsymbol{\mathcal{P}} \rightarrow [0,1]$, $\boldsymbol{F^U}_{\widetilde{A}} : \boldsymbol{\mathcal{P}} \rightarrow [0,1]$ falsity functions.

Definition 2.4:[19] Let $\boldsymbol{\mathcal{N}}$ be a set and $\boldsymbol{P}(\boldsymbol{\mathcal{N}})$ be the power set of $\boldsymbol{\mathcal{N}}$. An ISVNQN is represented as

$$\widetilde{\boldsymbol{\mathcal{R}}}_N = (\breve{\boldsymbol{\mathcal{P}}},\ \breve{\boldsymbol{\mathcal{P}}}_T[\boldsymbol{T^L},\ \boldsymbol{T^U}],\ \breve{\boldsymbol{\mathcal{P}}}_U[\boldsymbol{U^L},\ \boldsymbol{U^U}],\ \breve{\boldsymbol{\mathcal{P}}}_C[\boldsymbol{C^L},\ \boldsymbol{C^U}],\ \breve{\boldsymbol{\mathcal{P}}}_F[\boldsymbol{F^L},\ \boldsymbol{F^U}])$$

Here, the components $\boldsymbol{T^L},\ \boldsymbol{T^U}, \boldsymbol{U^L},\ \boldsymbol{U^U}, \boldsymbol{C^L},\ \boldsymbol{C^U}, \boldsymbol{F^L},\ \boldsymbol{F^U}$ are the truth, unknownability, contradiction and falsity functions in the IQNS. In addition,

$$\breve{\boldsymbol{\mathcal{P}}},\ \breve{\boldsymbol{\mathcal{P}}}_T, \breve{\boldsymbol{\mathcal{P}}}_U,\ \breve{\boldsymbol{\mathcal{P}}}_C, \breve{\boldsymbol{\mathcal{P}}}_F \in \mathbf{P}(\boldsymbol{\mathcal{N}})$$

Also, an ISVNQS

$$\widetilde{\boldsymbol{\mathcal{R}}} = \{(\breve{\boldsymbol{\mathcal{P}}},\ \breve{\boldsymbol{\mathcal{P}}}_T[\boldsymbol{T^L},\ \boldsymbol{T^U}],\ \breve{\boldsymbol{\mathcal{P}}}_U[\boldsymbol{U^L},\ \boldsymbol{U^U}],\ \breve{\boldsymbol{\mathcal{P}}}_C[\boldsymbol{C^L},\ \boldsymbol{C^U}],\ \breve{\boldsymbol{\mathcal{P}}}_F[\boldsymbol{F^L},\ \boldsymbol{F^U}]) : \breve{\boldsymbol{\mathcal{P}}},\ \breve{\boldsymbol{\mathcal{P}}}_T, \breve{\boldsymbol{\mathcal{P}}}_U,\ \breve{\boldsymbol{\mathcal{P}}}_C, \breve{\boldsymbol{\mathcal{P}}}_F \in \mathbf{P}(\boldsymbol{\mathcal{N}})\}$$

is shown in the form. Here it is,

$$(\breve{\boldsymbol{\mathcal{P}}},\ \breve{\boldsymbol{\mathcal{P}}}_T[\boldsymbol{T^L},\ \boldsymbol{T^U}],\ \breve{\boldsymbol{\mathcal{P}}}_U[\boldsymbol{U^L},\ \boldsymbol{U^U}],\ \breve{\boldsymbol{\mathcal{P}}}_C[\boldsymbol{C^L},\ \boldsymbol{C^U}],\ \breve{\boldsymbol{\mathcal{P}}}_F[\boldsymbol{F^L},\ \boldsymbol{F^U}])$$

represents a number, idea, object, etc. Also

$$``\breve{\boldsymbol{\mathcal{P}}}"$$

is called known part and

$$``(\breve{\boldsymbol{\mathcal{P}}}_T[\boldsymbol{T^L},\ \boldsymbol{T^U}],\ \breve{\boldsymbol{\mathcal{P}}}_U[\boldsymbol{U^L},\ \boldsymbol{U^U}],\ \breve{\boldsymbol{\mathcal{P}}}_C[\boldsymbol{C^L},\ \boldsymbol{C^U}],\ \breve{\boldsymbol{\mathcal{P}}}_F[\boldsymbol{F^L},\ \boldsymbol{F^U}])"$$

is called the unknown part.

Definition 2.5:[19] Let

$$(\widetilde{\boldsymbol{\mathcal{R}}}_N)_1 = (\breve{\boldsymbol{\mathcal{P}}}_1,\ \breve{\boldsymbol{\mathcal{P}}}_{1_T}[\boldsymbol{T^l},\ \boldsymbol{T^u}],\ \breve{\boldsymbol{\mathcal{P}}}_{1_U}[\boldsymbol{U^l},\ \boldsymbol{U^u}],\ \breve{\boldsymbol{\mathcal{P}}}_{1_C}[\boldsymbol{C^l},\ \boldsymbol{C^u}],\ \breve{\boldsymbol{\mathcal{P}}}_{1_F}[\boldsymbol{F^l},\ \boldsymbol{F^u}])$$

and

$$(\widetilde{\boldsymbol{\mathcal{R}}}_N)_2 = (\breve{\boldsymbol{\mathcal{P}}}_2,\ \breve{\boldsymbol{\mathcal{P}}}_{2_T}[\boldsymbol{T^l},\ \boldsymbol{T^u}],\ \breve{\boldsymbol{\mathcal{P}}}_{2_U}[\boldsymbol{U^l},\ \boldsymbol{U^u}],\ \breve{\boldsymbol{\mathcal{P}}}_{2_C}[\boldsymbol{C^l},\ \boldsymbol{C^u}],\ \breve{\boldsymbol{\mathcal{P}}}_{2_F}[\boldsymbol{F^l},\ \boldsymbol{F^u}])$$

be two ISVNQSs.

The operations ∪ and ∩ are defined for ISVNQSs as following

i)$(\widetilde{\mathcal{R}}_N)_1 \cup (\widetilde{\mathcal{R}}_N)_2$= $(\breve{\mathcal{P}}_1 \cup \breve{\mathcal{P}}_2, (\breve{\mathcal{P}}_{1_T} \cup \breve{\mathcal{P}}_{2_T})[T^l,\ T^u], (\breve{\mathcal{P}}_{1_U} \cup \breve{\mathcal{P}}_{2_U})[U^l,\ U^u], (\breve{\mathcal{P}}_{1_C} \cup \breve{\mathcal{P}}_{2_C})[C^l,\ C^u],$

$(\breve{\mathcal{P}}_{1_F} \cup \breve{\mathcal{P}}_{2_F})[F^l,\ F^u])$

ii)$(\widetilde{\mathcal{R}}_N)_1 \cap (\widetilde{\mathcal{R}}_N)_2$= $(\breve{\mathcal{P}}_1 \cap \breve{\mathcal{P}}_2, (\breve{\mathcal{P}}_{1_T} \cap \breve{\mathcal{P}}_{2_T})[T^l,\ T^u], (\breve{\mathcal{P}}_{1_U} \cap \breve{\mathcal{P}}_{2_U})[U^l,\ U^u], (\breve{\mathcal{P}}_{1_C} \cap \breve{\mathcal{P}}_{2_C})[C^l,\ C^u],$

$(\breve{\mathcal{P}}_{1_F} \cap \breve{\mathcal{P}}_{2_F})[F^l,\ F^u])$

Definition 2.6: [31] Let $\mathcal{N} \neq \emptyset$ and $P(\mathcal{N})$ be the power set of $\mathcal{N}$. An IGSVNQS

$\widetilde{\mathcal{R}} = \{< \breve{\mathcal{P}}_{1_{\widetilde{\mathcal{R}}_1}},\ \breve{\mathcal{P}}_{1_{T_{\widetilde{\mathcal{R}}_1}}}[\ T^L{}_{\widetilde{\mathcal{R}}_1}, T^U{}_{\widetilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{U_{\widetilde{\mathcal{R}}_1}}}[U^L{}_{\widetilde{\mathcal{R}}_1},\ U^U{}_{\widetilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{C_{\widetilde{\mathcal{R}}_1}}}[C^L{}_{\widetilde{\mathcal{R}}_1},\ C^U{}_{\widetilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1_{F_{\widetilde{\mathcal{R}}_1}}}[F^L{}_{\widetilde{\mathcal{R}}_1}, F^U{}_{\widetilde{\mathcal{R}}_1}];$

$\breve{\mathcal{P}}_{2_{\widetilde{\mathcal{R}}_2}},\ \breve{\mathcal{P}}_{2_{T_{\widetilde{\mathcal{R}}_2}}}[\ T^L{}_{\widetilde{\mathcal{R}}_2}, T^U{}_{\widetilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2_{U_{\widetilde{\mathcal{R}}_2}}}[U^L{}_{\widetilde{\mathcal{R}}_2},\ U^U{}_{\widetilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2_{C_{\widetilde{\mathcal{R}}_2}}}[C^L{}_{\widetilde{\mathcal{R}}_2},\ C^U{}_{\widetilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2_{F_{\widetilde{\mathcal{R}}_2}}}[F^L{}_{\widetilde{\mathcal{R}}_2}, F^U{}_{\widetilde{\mathcal{R}}_2}];$

...

$\breve{\mathcal{P}}_{s_{\widetilde{\mathcal{R}}_s}},\ \breve{\mathcal{P}}_{s_{T_{\widetilde{\mathcal{R}}_s}}}[\ T^L{}_{\widetilde{\mathcal{R}}_s}, T^U{}_{\widetilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{s_{U_{\widetilde{\mathcal{R}}_s}}}[U^L{}_{\widetilde{\mathcal{R}}_s},\ U^U{}_{\widetilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{s_{C_{\widetilde{\mathcal{R}}_s}}}[C^L{}_{\widetilde{\mathcal{R}}_s},\ C^U{}_{\widetilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{s_{F_{\widetilde{\mathcal{R}}_s}}}[F^L{}_{\widetilde{\mathcal{R}}_s}, F^U{}_{\widetilde{\mathcal{R}}_s}] >,$

$\breve{\mathcal{P}}_{l_{\widetilde{\mathcal{R}}_l}}, \breve{\mathcal{P}}_{l_{T_{\widetilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\widetilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\widetilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\widetilde{\mathcal{R}}_l}}} \in P(\mathcal{N}); l = 1, 2, 3, \dots, s\}$

is shown in the form. Where $T^L{}_{\widetilde{\mathcal{R}}_l}, T^U{}_{\widetilde{\mathcal{R}}_l}, U^L{}_{\widetilde{\mathcal{R}}_l}, U^U{}_{\widetilde{\mathcal{R}}_l}, C^L{}_{\widetilde{\mathcal{R}}_l}, C^U{}_{\widetilde{\mathcal{R}}_l}, F^L{}_{\widetilde{\mathcal{R}}_l}, F^U{}_{\widetilde{\mathcal{R}}_l}$ the components are the degree of truth, degree of uncertainty, degree of contradiction and degree of falsity.

Also, an IGSVNQN

$$\left(\widetilde{\mathcal{R}}_N\right)_1 = \{< \breve{\mathcal{P}}_{1_{\widetilde{\mathcal{R}}_1}},\ \breve{\mathcal{P}}_{1_{T_{\widetilde{\mathcal{R}}_1}}}[\ T^L{}_{\widetilde{\mathcal{R}}_1}, T^U{}_{\widetilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{U_{\widetilde{\mathcal{R}}_1}}}[U^L{}_{\widetilde{\mathcal{R}}_1},\ U^U{}_{\widetilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{C_{\widetilde{\mathcal{R}}_1}}}[C^L{}_{\widetilde{\mathcal{R}}_1},\ C^U{}_{\widetilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1_{F_{\widetilde{\mathcal{R}}_1}}}[F^L{}_{\widetilde{\mathcal{R}}_1}, F^U{}_{\widetilde{\mathcal{R}}_1}] >\}$$

is shown in the form. Here, it is shown as

$$\text{``}\breve{\mathcal{P}}_{1_{\widetilde{\mathcal{R}}_1}}\text{''}$$

known part and

$$\text{``}\breve{\mathcal{P}}_{1_{T_{\widetilde{\mathcal{R}}_1}}}[T^L{}_{\widetilde{\mathcal{R}}_1}, T^U{}_{\widetilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{U_{\widetilde{\mathcal{R}}_1}}}[U^L{}_{\widetilde{\mathcal{R}}_1},\ U^U{}_{\widetilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{C_{\widetilde{\mathcal{R}}_1}}}[C^L{}_{\widetilde{\mathcal{R}}_1},\ C^U{}_{\widetilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{F_{\widetilde{\mathcal{R}}_1}}}[F^L{}_{\widetilde{\mathcal{R}}_1}, F^U{}_{\widetilde{\mathcal{R}}_1}]\text{''}$$

is called the unknown part.

It can also be represented as

$$\widetilde{\mathcal{R}} = \{\left(\widetilde{\mathcal{R}}_N\right)_l : l = 1, 2, \dots, s\}.$$

3. Research and Findings

3.1. Some Structures Related to Interval Generalized Set Valued Neutrosophic Quintuple Sets

Definition 3.1.1: Let

$\tilde{\mathcal{R}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}}, \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}[T^L{}_{\tilde{\mathcal{R}}_1}, T^U{}_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1}}}[U^L{}_{\tilde{\mathcal{R}}_1}, U^U{}_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1}}}[C^L{}_{\tilde{\mathcal{R}}_1}, C^U{}_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1}}}[F^L{}_{\tilde{\mathcal{R}}_1}, F^U{}_{\tilde{\mathcal{R}}_1}];$

$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2}}, \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{R}}_2}}}[T^L{}_{\tilde{\mathcal{R}}_2}, T^U{}_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{R}}_2}}}[U^L{}_{\tilde{\mathcal{R}}_2}, U^U{}_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{R}}_2}}}[C^L{}_{\tilde{\mathcal{R}}_2}, C^U{}_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{R}}_2}}}[F^L{}_{\tilde{\mathcal{R}}_2}, F^U{}_{\tilde{\mathcal{R}}_2}];$

…

$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s}}, \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{R}}_s}}}[T^L{}_{\tilde{\mathcal{R}}_s}, T^U{}_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{R}}_s}}}[U^L{}_{\tilde{\mathcal{R}}_s}, U^U{}_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{R}}_s}}}[C^L{}_{\tilde{\mathcal{R}}_s}, C^U{}_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{R}}_s}}}[F^L{}_{\tilde{\mathcal{R}}_s}, F^U{}_{\tilde{\mathcal{R}}_s}] >,$

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$

and

$\tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{H}}_1}}, \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{H}}_1}}}[T^L{}_{\tilde{\mathcal{H}}_1}, T^U{}_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{H}}_1}}}[U^L{}_{\tilde{\mathcal{H}}_1}, U^U{}_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{H}}_1}}}[C^L{}_{\tilde{\mathcal{H}}_1}, C^U{}_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{H}}_1}}}[F^L{}_{\tilde{\mathcal{H}}_1}, F^U{}_{\tilde{\mathcal{H}}_1}];$

$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{H}}_2}}, \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{H}}_2}}}[T^L{}_{\tilde{\mathcal{H}}_2}, T^U{}_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{H}}_2}}}[U^L{}_{\tilde{\mathcal{H}}_2}, U^U{}_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{H}}_2}}}[C^L{}_{\tilde{\mathcal{H}}_2}, C^U{}_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{H}}_2}}}[F^L{}_{\tilde{\mathcal{H}}_2}, F^U{}_{\tilde{\mathcal{H}}_2}];$

…

$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{H}}_s}}, \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{H}}_s}}}[T^L{}_{\tilde{\mathcal{H}}_s}, T^U{}_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{H}}_s}}}[U^L{}_{\tilde{\mathcal{H}}_s}, U^U{}_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{H}}_s}}}[C^L{}_{\tilde{\mathcal{H}}_s}, C^U{}_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{H}}_s}}}[F^L{}_{\tilde{\mathcal{H}}_s}, F^U{}_{\tilde{\mathcal{H}}_s}] >,$

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$

be IGSVNQSs.

i) $\tilde{\mathcal{H}}$ is a subset of $\tilde{\mathcal{R}}$ $(\tilde{\mathcal{R}} \subset' \tilde{\mathcal{H}})$ if and only if

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \subset \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}$, $\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \subset \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}$, $\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \subset \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}$, $\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \subset \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}$, $\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \subset \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}$

$T^L{}_{\tilde{\mathcal{R}}_l} \leq T^L{}_{\tilde{\mathcal{H}}_l}$, $T^U{}_{\tilde{\mathcal{R}}_l} \leq T^U{}_{\tilde{\mathcal{H}}_l}$, $U^L{}_{\tilde{\mathcal{R}}_l} \geq U^L{}_{\tilde{\mathcal{H}}_l}$, $U^U{}_{\tilde{\mathcal{R}}_l} \geq U^U{}_{\tilde{\mathcal{H}}_l}$

$C^L{}_{\tilde{\mathcal{R}}_l} \geq C^L{}_{\tilde{\mathcal{H}}_l}$, $C^U{}_{\tilde{\mathcal{R}}_l} \geq C^U{}_{\tilde{\mathcal{H}}_l}$, $F^L{}_{\tilde{\mathcal{R}}_l} \geq F^L{}_{\tilde{\mathcal{H}}_l}$, $F^U{}_{\tilde{\mathcal{R}}_l} \geq F^U{}_{\tilde{\mathcal{H}}_l}$

ii) $\tilde{\mathcal{R}}$ is equal to $\tilde{\mathcal{H}}$ $(\tilde{\mathcal{R}} =' \tilde{\mathcal{H}})$ if and only if

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} = \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}$, $\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} = \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}$, $\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} = \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}$, $\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} = \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}$, $\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} = \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}$

$T^L{}_{\tilde{\mathcal{R}}_l} = T^L{}_{\tilde{\mathcal{H}}_l}$, $T^U{}_{\tilde{\mathcal{R}}_l} = T^U{}_{\tilde{\mathcal{H}}_l}$, $U^L{}_{\tilde{\mathcal{R}}_l} = U^L{}_{\tilde{\mathcal{H}}_l}$, $U^U{}_{\tilde{\mathcal{R}}_l} = U^U{}_{\tilde{\mathcal{H}}_l}$

$C^L{}_{\tilde{\mathcal{R}}_l} = C^L{}_{\tilde{\mathcal{H}}_l}$, $C^U{}_{\tilde{\mathcal{R}}_l} = C^U{}_{\tilde{\mathcal{H}}_l}$, $F^L{}_{\tilde{\mathcal{R}}_l} = F^L{}_{\tilde{\mathcal{H}}_l}$, $F^U{}_{\tilde{\mathcal{R}}_l} = F^U{}_{\tilde{\mathcal{H}}_l}$

Example 3.1.2: Let $\acute{A} = \{\acute{n}, \acute{t}, \acute{r}, \acute{s}, \acute{f}, \acute{k}, \acute{b}, \acute{l}\}$ be a set and

$\widehat{\mathcal{X}} = \{< \{\acute{n}, \acute{t}, \acute{l}\}, \{\acute{r}, \acute{f}, \acute{b}\}[0,0.6], \{\acute{n}\}[0.3,0,6], \{\acute{s}, \acute{k}\}[0.6,0.7], \{\acute{f}\}[0.6,0.9];$

$\{\acute{k}\}, \{\acute{t}, \acute{f}\}[0,0.5], \{\acute{b}\}[0.4,0.7], \{\acute{r}, \acute{s}\}[0.5,0.9], \{\acute{l}\}[0.2,0.6] >\},$

$\hat{\mathcal{Z}} = \{< \{\acute{n}, \acute{t}, \acute{s}, \acute{k}, \acute{l}\}, \{\acute{r}, \acute{f}, \acute{b}, \acute{l}\}[0,0.8], \{\acute{n}, \acute{l}\}[0.2,0.5], \{\acute{r}, \acute{s}, \acute{k}, \acute{l}\}[0.5,0.7], \{\acute{n}, \acute{f}\}[0.5,0.8];$

$\{\acute{k}, \acute{l}\}, \{\acute{t}, \acute{f}, \acute{l}\}[0.1,0.6], \{\acute{b}, \acute{l}\}[0.3,0.4], \{\acute{n}, \acute{r}, \acute{s}\}[0.5,0.8], \{\acute{r}, \acute{l}\}[0.2,0.3] >\}$

be two IGSVNQSs.

$(\widehat{\mathcal{X}}_N)_1$ and $(\hat{\mathcal{Z}}_N)_1$ satisfy the conditions

$$\left((\widehat{\mathcal{X}}_N)_1 \subset' (\hat{\mathcal{Z}}_N)_1\right)$$

such that

$$\{\acute{n}, \acute{t}, \acute{l}\} \subset \{\acute{n}, \acute{t}, \acute{s}, \acute{k}, \acute{l}\},\ \{\acute{r}, \acute{f}, \acute{b}\} \subset \{\acute{r}, \acute{f}, \acute{b}, \acute{l}\},\ \{\acute{n}\} \subset \{\acute{n}, \acute{l}\},\ \{\acute{s}, \acute{k}\} \subset \{\acute{r}, \acute{s}, \acute{k}, \acute{l}\},\ \{\acute{f}\} \subset \{\acute{n}, \acute{f}\}$$

$$0 \le 0,\ \ 0.6 \le 0.8,\ 0.3 \ge 0{,}2,\ 0.6 \ge 0.5,\ \ 0{,}6 \ge 0{,}5,\ 0.7 \ge 0.7,\ 0.6 \ge 0.5,\ 0.9 \ge 0.8.$$

Similarly, the $(\widehat{\mathcal{X}}_N)_2$and $(\hat{\mathcal{Z}}_N)_2$ satisfy the conditions

$$\left((\widehat{\mathcal{X}}_N)_2 \subset' (\hat{\mathcal{Z}}_N)_2\right)$$

such that

$$\{\acute{k}\} \subset \{\acute{k}, \acute{l}\},\ \{\acute{t}, \acute{f}\} \subset \{\acute{t}, \acute{f}, \acute{l}\},\ \{\acute{b}\} \subset \{\acute{b}, \acute{l}\},\ \{\acute{r}, \acute{s}\} \subset \{\acute{n}, \acute{r}, \acute{s}\},\ \{\acute{r}\} \subset \{\acute{r}, \acute{l}\}$$

$$0 \le 0.1,\ 0.5 \le 0.6,\ 0.4 \ge 0.3,\ 0.7 \ge 0.4,\ 0.5 \ge 0.5,\ 0.9 \ge 0.8,\ 0.2 \ge 0.2,\ 0.6 \ge 0.3.$$

Thus, we obtain that

$$\widehat{\mathcal{X}} \subset' \hat{\mathcal{Z}}$$

Definition 3.1.3: Let

$(\tilde{\mathcal{R}}_N)_1$={< $\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}}$, $\breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}$[$T^L{}_{\tilde{\mathcal{R}}_1}$,$T^U{}_{\tilde{\mathcal{R}}_1}$], $\breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1}}}$[$U^L{}_{\tilde{\mathcal{R}}_1}$, $U^U{}_{\tilde{\mathcal{R}}_1}$], $\breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1}}}$[$C^L{}_{\tilde{\mathcal{R}}_1}$, $C^U{}_{\tilde{\mathcal{R}}_1}$], $\breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1}}}$[$F^L{}_{\tilde{\mathcal{R}}_1}$,$F^U{}_{\tilde{\mathcal{R}}_1}$] >}

be an IGSVNQN. Score and accuracy functions for IGSVNQN is defined as

$$S_D((\mathcal{R}_N)_1) = \frac{\left|\ s\left(\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}} \cap \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}\right) - \ s\left(\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}} \cap \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1}}}\right) - \ s\left(\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}} \cap \breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1}}}\right) - \ s\left(\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}} \cap \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1}}}\right)\right|}{s\left(\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}}\right)} +$$

$$\frac{\left|T^L{}_{\tilde{\mathcal{R}}_1} - U^L{}_{\tilde{\mathcal{R}}_1} - C^L{}_{\tilde{\mathcal{R}}_1} - F^L{}_{\tilde{\mathcal{R}}_1}\right|}{3} - \frac{\left|T^U{}_{\tilde{\mathcal{R}}_1} - U^U{}_{\tilde{\mathcal{R}}_1} - C^U{}_{\tilde{\mathcal{R}}_1} - F^U{}_{\tilde{\mathcal{R}}_1}\right|}{3} + 1$$

$$K_D((\mathcal{R}_N)_1) = \frac{s\left(\breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}/\breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1}}}\right) + s\left(\breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}/\breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1}}}\right) + s\left(\breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}/\breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1}}}\right)}{2.s(\breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}})} + T^L{}_{\tilde{\mathcal{R}}_1} + T^U{}_{\tilde{\mathcal{R}}_1}$$

Where $s\left(\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}}\right)$ is the number of elements of the set $\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}}$,

$s\left(\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}} \cap \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}\right)$ is the number of elements of the intersection of the sets $\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}}$ and $\breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}$,

$s\left(\breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}} / \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1}}}\right)$ is the number of elements of the difference of the sets $\breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}$ and $\breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1}}}$.

3.2. Operators for Interval Generalized Set Values Neutrosophic Quintuple Sets

Definition 3.2.1: Let

$$\tilde{\mathcal{R}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}},\ \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}[\ T^L{}_{\tilde{\mathcal{R}}_1}, T^U{}_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1}}}[U^L{}_{\tilde{\mathcal{R}}_1},\ U^U{}_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1}}}[C^L{}_{\tilde{\mathcal{R}}_1},\ C^U{}_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1}}}[F^L{}_{\tilde{\mathcal{R}}_1}, F^U{}_{\tilde{\mathcal{R}}_1}];$$

$$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2}},\ \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{R}}_2}}}[\ T^L{}_{\tilde{\mathcal{R}}_2}, T^U{}_{\tilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{R}}_2}}}[U^L{}_{\tilde{\mathcal{R}}_2},\ U^U{}_{\tilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{R}}_2}}}[C^L{}_{\tilde{\mathcal{R}}_2},\ C^U{}_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{R}}_2}}}[F^L{}_{\tilde{\mathcal{R}}_2}, F^U{}_{\tilde{\mathcal{R}}_2}];$$

...

$$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s}},\ \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{R}}_s}}}[\ T^L{}_{\tilde{\mathcal{R}}_s}, T^U{}_{\tilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{R}}_s}}}[U^L{}_{\tilde{\mathcal{R}}_s},\ U^U{}_{\tilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{R}}_s}}}[C^L{}_{\tilde{\mathcal{R}}_s},\ C^U{}_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{R}}_s}}}[F^L{}_{\tilde{\mathcal{R}}_s}, F^U{}_{\tilde{\mathcal{R}}_s}] >,$$

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$$

and

$$\tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{H}}_1}},\ \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{H}}_1}}}[\ T^L{}_{\tilde{\mathcal{H}}_1}, T^U{}_{\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{H}}_1}}}[U^L{}_{\tilde{\mathcal{H}}_1},\ U^U{}_{\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{H}}_1}}}[C^L{}_{\tilde{\mathcal{H}}_1},\ C^U{}_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{H}}_1}}}[F^L{}_{\tilde{\mathcal{H}}_1}, F^U{}_{\tilde{\mathcal{H}}_1}];$$

$$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{H}}_2}},\ \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{H}}_2}}}[\ T^L{}_{\tilde{\mathcal{H}}_2}, T^U{}_{\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{H}}_2}}}[U^L{}_{\tilde{\mathcal{H}}_2},\ U^U{}_{\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{H}}_2}}}[C^L{}_{\tilde{\mathcal{H}}_2},\ C^U{}_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{H}}_2}}}[F^L{}_{\tilde{\mathcal{H}}_2}, F^U{}_{\tilde{\mathcal{H}}_2}];$$

...

$$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{H}}_s}},\ \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{H}}_s}}}[\ T^L{}_{\tilde{\mathcal{H}}_s}, T^U{}_{\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{H}}_s}}}[U^L{}_{\tilde{\mathcal{H}}_s},\ U^U{}_{\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{H}}_s}}}[C^L{}_{\tilde{\mathcal{H}}_s},\ C^U{}_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{H}}_s}}}[F^L{}_{\tilde{\mathcal{H}}_s}, F^U{}_{\tilde{\mathcal{H}}_s}] >,$$

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$$

be two IGSVNQSs.

i. The "mean ∪" operation for $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{H}}$ is defined as

$$\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}},\ \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[\ T^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, T^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[U^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1},\ U^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}],$$

$$\breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[C^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, C^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[F^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, F^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}];$$

$$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}},\ \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[\ T^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, T^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[U^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2},\ U^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}],$$

$$\breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[C^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, C^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[F^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, F^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}];$$

...

$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}, \ \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[\ T^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, T^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}], \ \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[U^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, \ U^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}],$

$\breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[C^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, C^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[F^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, F^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}] >,$

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} \in P(\mathcal{N}); l = 1,2,3, \ldots, s\}.$

Here, for $l = 1,2,3, \ldots, s$;

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right), \ \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}\right), \ \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}\right)$

$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}\right), \ \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}\right)$

$T^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{T^L{}_{\tilde{\mathcal{R}}_l} + T^L{}_{\tilde{\mathcal{H}}_l}}{2}, \ T^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{T^U{}_{\tilde{\mathcal{R}}_l} + T^U{}_{\tilde{\mathcal{H}}_l}}{2}, \ U^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{U^L{}_{\tilde{\mathcal{R}}_l} + U^L{}_{\tilde{\mathcal{H}}_l}}{2}, \ U^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{U^U{}_{\tilde{\mathcal{R}}_l} + U^U{}_{\tilde{\mathcal{H}}_l}}{2}$

$C^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{C^L{}_{\tilde{\mathcal{R}}_l} + C^L{}_{\tilde{\mathcal{H}}_l}}{2}, \ C^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{C^U{}_{\tilde{\mathcal{R}}_l} + C^U{}_{\tilde{\mathcal{H}}_l}}{2}, \ F^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{F^L{}_{\tilde{\mathcal{R}}_l} + F^L{}_{\tilde{\mathcal{H}}_l}}{2}, \ F^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{F^U{}_{\tilde{\mathcal{R}}_l} + F^U{}_{\tilde{\mathcal{H}}_l}}{2}$

ii. For $\mathcal{R}$ and $\mathcal{H}$, the operation "mean $\cap$ '' is defined as

$\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}, \ \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[\ T^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, T^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}], \ \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[U^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, \ U^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}],$

$\breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[C^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, C^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[F^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, F^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}];$

$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}, \ \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[\ T^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, T^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}], \ \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[U^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, \ U^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}],$

$\breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[C^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, C^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[F^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, F^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}];$

…

$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}, \ \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[\ T^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, T^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}], \ \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[U^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, \ U^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}],$

$\breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[C^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, C^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[F^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, F^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}] >,$

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} \in P(\mathcal{N}); l = 1,2,3, \ldots, s\}$

Here, for $l = 1,2,3, \ldots, s$;

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right), \ \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}\right), \ \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}\right)$

$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}\right), \ \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}\right)$

$T^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{T^L{}_{\tilde{\mathcal{R}}_l} + T^L{}_{\tilde{\mathcal{H}}_l}}{2}, \ T^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{T^U{}_{\tilde{\mathcal{R}}_l} + T^U{}_{\tilde{\mathcal{H}}_l}}{2}, \ U^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{U^L{}_{\tilde{\mathcal{R}}_l} + U^L{}_{\tilde{\mathcal{H}}_l}}{2}, \ U^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{U^U{}_{\tilde{\mathcal{R}}_l} + U^U{}_{\tilde{\mathcal{H}}_l}}{2}$

$C^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{C^L{}_{\tilde{\mathcal{R}}_l} + C^L{}_{\tilde{\mathcal{H}}_l}}{2}, \ C^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{C^U{}_{\tilde{\mathcal{R}}_l} + C^U{}_{\tilde{\mathcal{H}}_l}}{2}, \ F^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{F^L{}_{\tilde{\mathcal{R}}_l} + F^L{}_{\tilde{\mathcal{H}}_l}}{2}, \ F^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{F^U{}_{\tilde{\mathcal{R}}_l} + F^U{}_{\tilde{\mathcal{H}}_l}}{2}$

Example 3.2.2: From Example 3.1.2, for $\hat{\mathcal{X}}$ and $\hat{\mathcal{Z}}$, we obtain that

i. $\widehat{\mathcal{X}} \cup'_M \hat{\mathcal{Z}} = \{< \{\acute{n},\acute{t},\acute{r},\acute{s},\acute{l},\acute{k}\},\{\acute{n},\acute{r},\acute{f},\acute{b},\acute{l}\}[0.05,0.7],\{\acute{n},\acute{s},\acute{l}\}[0.25,0.65],\{\acute{n},\acute{r},\acute{s},\acute{l},\acute{f},\acute{k}\}[0.4,0.65],$

$\{\acute{t},\acute{f}\}[0.45,0.8];\{\acute{f},\acute{k},\acute{l}\},\{\acute{t},\acute{r},\acute{s},\acute{f},\acute{l}\}[0.3,0.6],\{\acute{t},\acute{r},\acute{b}\}[0.4,0.65],\{\acute{n},\acute{t},\acute{r},\acute{s},\acute{l}\}[0.25,0.7],\{\acute{l},\acute{k}\}[0.2,0.55] >\}$

ii. $\widehat{\mathcal{X}} \cap'_M \hat{\mathcal{Z}} = \{< \{\acute{n},\acute{t},\acute{l}\},\{\acute{f},\acute{l}\}[0.05,0.7],\{\acute{l}\}[0.25,0.65],\{\acute{r}\}[0.4,0.65],\{\acute{f}\}[0.45,0.8];$

$\{\acute{k}\},\{\acute{f},\acute{l}\}[0.3,0.6],\{\acute{b}\}[0.4,0.65],\{\acute{n},\acute{s}\}[0.25,0.7],\{\acute{l}\}[0.2,0.55] >\}$

Definition 3.2.3: Let

$\tilde{\mathcal{R}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}}, \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}[T^L{}_{\tilde{\mathcal{R}}_1}, T^U{}_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1}}}[U^L{}_{\tilde{\mathcal{R}}_1}, U^U{}_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1}}}[C^L{}_{\tilde{\mathcal{R}}_1}, C^U{}_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1}}}[F^L{}_{\tilde{\mathcal{R}}_1}, F^U{}_{\tilde{\mathcal{R}}_1}];$

$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2}}, \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{R}}_2}}}[T^L{}_{\tilde{\mathcal{R}}_2}, T^U{}_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{R}}_2}}}[U^L{}_{\tilde{\mathcal{R}}_2}, U^U{}_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{R}}_2}}}[C^L{}_{\tilde{\mathcal{R}}_2}, C^U{}_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{R}}_2}}}[F^L{}_{\tilde{\mathcal{R}}_2}, F^U{}_{\tilde{\mathcal{R}}_2}];$

...

$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s}}, \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{R}}_s}}}[T^L{}_{\tilde{\mathcal{R}}_s}, T^U{}_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{R}}_s}}}[U^L{}_{\tilde{\mathcal{R}}_s}, U^U{}_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{R}}_s}}}[C^L{}_{\tilde{\mathcal{R}}_s}, C^U{}_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{R}}_s}}}[F^L{}_{\tilde{\mathcal{R}}_s}, F^U{}_{\tilde{\mathcal{R}}_s}] >,$

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$

and

$\widetilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1_{\widetilde{\mathcal{H}}_1}}, \breve{\mathcal{P}}_{1_{T_{\widetilde{\mathcal{H}}_1}}}[T^L{}_{\widetilde{\mathcal{H}}_1}, T^U{}_{\widetilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{U_{\widetilde{\mathcal{H}}_1}}}[U^L{}_{\widetilde{\mathcal{H}}_1}, U^U{}_{\widetilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{C_{\widetilde{\mathcal{H}}_1}}}[C^L{}_{\widetilde{\mathcal{H}}_1}, C^U{}_{\widetilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F_{\widetilde{\mathcal{H}}_1}}}[F^L{}_{\widetilde{\mathcal{H}}_1}, F^U{}_{\widetilde{\mathcal{H}}_1}];$

$\breve{\mathcal{P}}_{2_{\widetilde{\mathcal{H}}_2}}, \breve{\mathcal{P}}_{2_{T_{\widetilde{\mathcal{H}}_2}}}[T^L{}_{\widetilde{\mathcal{H}}_2}, T^U{}_{\widetilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{U_{\widetilde{\mathcal{H}}_2}}}[U^L{}_{\widetilde{\mathcal{H}}_2}, U^U{}_{\widetilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{C_{\widetilde{\mathcal{H}}_2}}}[C^L{}_{\widetilde{\mathcal{H}}_2}, C^U{}_{\widetilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F_{\widetilde{\mathcal{H}}_2}}}[F^L{}_{\widetilde{\mathcal{H}}_2}, F^U{}_{\widetilde{\mathcal{H}}_2}];$

...

$\breve{\mathcal{P}}_{s_{\widetilde{\mathcal{H}}_s}}, \breve{\mathcal{P}}_{s_{T_{\widetilde{\mathcal{H}}_s}}}[T^L{}_{\widetilde{\mathcal{H}}_s}, T^U{}_{\widetilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{U_{\widetilde{\mathcal{H}}_s}}}[U^L{}_{\widetilde{\mathcal{H}}_s}, U^U{}_{\widetilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{C_{\widetilde{\mathcal{H}}_s}}}[C^L{}_{\widetilde{\mathcal{H}}_s}, C^U{}_{\widetilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F_{\widetilde{\mathcal{H}}_s}}}[F^L{}_{\widetilde{\mathcal{H}}_s}, F^U{}_{\widetilde{\mathcal{H}}_s}] >,$

$\breve{\mathcal{P}}_{l_{\widetilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T_{\widetilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\widetilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\widetilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\widetilde{\mathcal{H}}_l}}} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$

be two IGSVNQSs.

i. For $\tilde{\mathcal{R}}$ and $\widetilde{\mathcal{H}}$, the operation " optimistic ∪'' is defined as

$\tilde{\mathcal{R}} \cup'_O \widetilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1\widetilde{\mathcal{H}}_1}}, \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1\widetilde{\mathcal{H}}_1}}}[T^L{}_{\tilde{\mathcal{R}}_1\widetilde{\mathcal{H}}_1}, T^U{}_{\tilde{\mathcal{R}}_1\widetilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1\widetilde{\mathcal{H}}_1}}}[U^L{}_{\tilde{\mathcal{R}}_1\widetilde{\mathcal{H}}_1}, U^U{}_{\tilde{\mathcal{R}}_1\widetilde{\mathcal{H}}_1}],$

$\breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1\widetilde{\mathcal{H}}_1}}}[C^L{}_{\tilde{\mathcal{R}}_1\widetilde{\mathcal{H}}_1}, C^U{}_{\tilde{\mathcal{R}}_1\widetilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1\widetilde{\mathcal{H}}_1}}}[F^L{}_{\tilde{\mathcal{R}}_1\widetilde{\mathcal{H}}_1}, F^U{}_{\tilde{\mathcal{R}}_1\widetilde{\mathcal{H}}_1}];$

$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2\widetilde{\mathcal{H}}_2}}, \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{R}}_2\widetilde{\mathcal{H}}_2}}}[T^L{}_{\tilde{\mathcal{R}}_2\widetilde{\mathcal{H}}_2}, T^U{}_{\tilde{\mathcal{R}}_2\widetilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{R}}_2\widetilde{\mathcal{H}}_2}}}[U^L{}_{\tilde{\mathcal{R}}_2\widetilde{\mathcal{H}}_2}, U^U{}_{\tilde{\mathcal{R}}_2\widetilde{\mathcal{H}}_2}],$

$\breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{R}}_2\widetilde{\mathcal{H}}_2}}}[C^L{}_{\tilde{\mathcal{R}}_2\widetilde{\mathcal{H}}_2}, C^U{}_{\tilde{\mathcal{R}}_2\widetilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{R}}_2\widetilde{\mathcal{H}}_2}}}[F^L{}_{\tilde{\mathcal{R}}_2\widetilde{\mathcal{H}}_2}, F^U{}_{\tilde{\mathcal{R}}_2\widetilde{\mathcal{H}}_2}];$

...

$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s\widetilde{\mathcal{H}}_s}}, \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{R}}_s\widetilde{\mathcal{H}}_s}}}[T^L{}_{\tilde{\mathcal{R}}_s\widetilde{\mathcal{H}}_s}, T^U{}_{\tilde{\mathcal{R}}_s\widetilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{R}}_s\widetilde{\mathcal{H}}_s}}}[U^L{}_{\tilde{\mathcal{R}}_s\widetilde{\mathcal{H}}_s}, U^U{}_{\tilde{\mathcal{R}}_s\widetilde{\mathcal{H}}_s}],$

$\breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{R}}_s\widetilde{\mathcal{H}}_s}}}[C^L{}_{\tilde{\mathcal{R}}_s\widetilde{\mathcal{H}}_s}, C^U{}_{\tilde{\mathcal{R}}_s\widetilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{R}}_s\widetilde{\mathcal{H}}_s}}}[F^L{}_{\tilde{\mathcal{R}}_s\widetilde{\mathcal{H}}_s}, F^U{}_{\tilde{\mathcal{R}}_s\widetilde{\mathcal{H}}_s}] >,$

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$

Here, for $l = 1,2,3, \dots, s$;

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right)$, $\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}\right)$, $\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}\right)$

$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}\right)$, $\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}\right)$

$T^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{T^L{}_{\tilde{\mathcal{R}}_l}, T^L{}_{\tilde{\mathcal{H}}_l}\}$, $T^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{T^U{}_{\tilde{\mathcal{R}}_l}, T^U{}_{\tilde{\mathcal{H}}_l}\}$

$U^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{U^L{}_{\tilde{\mathcal{R}}_l}, U^L{}_{\tilde{\mathcal{H}}_l}\}$, $U^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{U^U{}_{\tilde{\mathcal{R}}_l}, U^U{}_{\tilde{\mathcal{H}}_l}\}$

$C^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{C^L{}_{\tilde{\mathcal{R}}_l}, C^L{}_{\tilde{\mathcal{H}}_l}\}$, $C^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{C^U{}_{\tilde{\mathcal{R}}_l}, C^U{}_{\tilde{\mathcal{H}}_l}\}$

$F^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{F^L{}_{\tilde{\mathcal{R}}_l}, F^L{}_{\mathcal{H}_l}\}$, $F^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{F^U{}_{\mathcal{R}_l}, F^U{}_{\tilde{\mathcal{H}}_l}\}$

ii. For $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{H}}$, the " optimistic $\cap$''operation is defined as

$\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}, \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[T^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, T^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[U^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, U^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}],$

$\breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[C^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, C^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[F^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, F^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}];$

$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}, \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[T^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, T^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[U^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, U^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}],$

$\breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[C^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, C^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[F^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, F^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}];$

…

$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}, \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[T^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, T^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[U^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, U^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}],$

$\breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[C^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, C^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[F^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, F^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}] >,$

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$

Here, for $l = 1,2,3, \dots, s$;

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right)$, $\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}\right)$, $\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}\right)$

$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}\right)$, $\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}\right)$

$T^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{T^L{}_{\tilde{\mathcal{R}}_l}, T^L{}_{\tilde{\mathcal{H}}_l}\}$, $T^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{T^U{}_{\tilde{\mathcal{R}}_l}, T^U{}_{\tilde{\mathcal{H}}_l}\}$

$U^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{U^L{}_{\tilde{\mathcal{R}}_l}, U^L{}_{\tilde{\mathcal{H}}_l}\}$, $U^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{U^U{}_{\tilde{\mathcal{R}}_l}, U^U{}_{\tilde{\mathcal{H}}_l}\}$

$C^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{C^L{}_{\tilde{\mathcal{R}}_l}, C^L{}_{\tilde{\mathcal{H}}_l}\}$, $C^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{C^U{}_{\tilde{\mathcal{R}}_l}, C^U{}_{\tilde{\mathcal{H}}_l}\}$

$F^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{F^L{}_{\tilde{\mathcal{R}}_l}, F^L{}_{\mathcal{H}_l}\}$, $F^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{F^U{}_{\mathcal{R}_l}, F^U{}_{\tilde{\mathcal{H}}_l}\}$

Example 3.2.4: From Example 3.1.2, for $\hat{\mathcal{X}}$ and $\hat{\mathcal{Z}}$, we obtain that

i. $\hat{\mathcal{X}} \cup'_O \hat{\mathcal{Z}} = \{<\{\acute{n},\acute{t},\acute{r},\acute{s},\acute{l},\acute{k}\},\{\acute{n},\acute{r},\acute{f},\acute{b},\acute{l}\}[0.1,0.8],\{\acute{n},\acute{s},\acute{l}\}[0.2,0.5],\{\acute{n},\acute{r},\acute{s},\acute{l},\acute{f},\acute{k}\}[0.3,0.6],$

$\{\acute{t},\acute{f}\}[0.3,0.7];\{\acute{f},\acute{k},\acute{l}\},\{\acute{t},\acute{r},\acute{s},\acute{f},\acute{l}\}[0.5,0.6],\{\acute{t},\acute{r},\acute{b}\}[0.3,0.4],\{\acute{n},\acute{t},\acute{r},\acute{s},\acute{l}\}[0.1,0.6],\{\acute{l},\acute{k}\}[0.1,0.3]>\}$

ii. $\hat{\mathcal{X}} \cap'_O \hat{\mathcal{Z}} = \{<\{\acute{n},\acute{t},\acute{l}\},\{\acute{f},\acute{l}\}[0.1,0.8],\{\acute{l}\}[0.2,0.5],\{\acute{r}\}[0.3,0.6],\{\acute{f}\}[0.3,0.7];$

$\{\acute{k}\},\{\acute{f},\acute{l}\}[0.5,0.6],\{\acute{b}\}[0.3,0.4],\{\acute{n},\acute{s}\}[0.1,0.6],\{\acute{l}\}[0.1,0.3]>\}$

Definition 3.2.5: Let

$\tilde{\mathcal{R}} = \{<\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}}, \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}[T^L{}_{\tilde{\mathcal{R}}_1}, T^U{}_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1}}}[U^L{}_{\tilde{\mathcal{R}}_1}, U^U{}_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1}}}[C^L{}_{\tilde{\mathcal{R}}_1}, C^U{}_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1}}}[F^L{}_{\tilde{\mathcal{R}}_1}, F^U{}_{\tilde{\mathcal{R}}_1}];$

$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2}}, \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{R}}_2}}}[T^L{}_{\tilde{\mathcal{R}}_2}, T^U{}_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{R}}_2}}}[U^L{}_{\tilde{\mathcal{R}}_2}, U^U{}_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{R}}_2}}}[C^L{}_{\tilde{\mathcal{R}}_2}, C^U{}_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{R}}_2}}}[F^L{}_{\tilde{\mathcal{R}}_2}, F^U{}_{\tilde{\mathcal{R}}_2}];$

...

$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s}}, \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{R}}_s}}}[T^L{}_{\tilde{\mathcal{R}}_s}, T^U{}_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{R}}_s}}}[U^L{}_{\tilde{\mathcal{R}}_s}, U^U{}_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{R}}_s}}}[C^L{}_{\tilde{\mathcal{R}}_s}, C^U{}_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{R}}_s}}}[F^L{}_{\tilde{\mathcal{R}}_s}, F^U{}_{\tilde{\mathcal{R}}_s}]>,$

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \in P(\mathcal{N}); l = 1,2,3,\dots,s\}$

and

$\tilde{\mathcal{H}} = \{<\breve{\mathcal{P}}_{1_{\tilde{\mathcal{H}}_1}}, \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{H}}_1}}}[T^L{}_{\tilde{\mathcal{H}}_1}, T^U{}_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{H}}_1}}}[U^L{}_{\tilde{\mathcal{H}}_1}, U^U{}_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{H}}_1}}}[C^L{}_{\tilde{\mathcal{H}}_1}, C^U{}_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{H}}_1}}}[F^L{}_{\tilde{\mathcal{H}}_1}, F^U{}_{\tilde{\mathcal{H}}_1}];$

$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{H}}_2}}, \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{H}}_2}}}[T^L{}_{\tilde{\mathcal{H}}_2}, T^U{}_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{H}}_2}}}[U^L{}_{\tilde{\mathcal{H}}_2}, U^U{}_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{H}}_2}}}[C^L{}_{\tilde{\mathcal{H}}_2}, C^U{}_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{H}}_2}}}[F^L{}_{\tilde{\mathcal{H}}_2}, F^U{}_{\tilde{\mathcal{H}}_2}];$

...

$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{H}}_s}}, \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{H}}_s}}}[T^L{}_{\tilde{\mathcal{H}}_s}, T^U{}_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{H}}_s}}}[U^L{}_{\tilde{\mathcal{H}}_s}, U^U{}_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{H}}_s}}}[C^L{}_{\tilde{\mathcal{H}}_s}, C^U{}_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{H}}_s}}}[F^L{}_{\tilde{\mathcal{H}}_s}, F^U{}_{\tilde{\mathcal{H}}_s}]>,$

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}} \in P(\mathcal{N}); l = 1,2,3,\dots,s\}$

be two IGSVNQSs.

i. The "pessimistic $\cup$" operation for $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{H}}$ is defined as

$\tilde{\mathcal{R}} \cup'_P \tilde{\mathcal{H}} = \{<\breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}, \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[T^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, T^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[U^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, U^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}],$

$\breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[C^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, C^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[F^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, F^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}];$

$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}, \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[T^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, T^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[U^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, U^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}],$

$\breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[C^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, C^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[F^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, F^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}];$

...

$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}, \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[T^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, T^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[U^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, U^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}],$

$\breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[C^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, C^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[F^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, F^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}] >,$

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$

Here, for $l = 1,2,3, \dots, s$;

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right)$, $\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}\right)$, $\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}\right)$

$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}\right)$, $\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}\right)$

$T^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{T^L{}_{\tilde{\mathcal{R}}_l}, T^L{}_{\tilde{\mathcal{H}}_l}\}$, $T^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{T^U{}_{\tilde{\mathcal{R}}_l}, T^U{}_{\tilde{\mathcal{H}}_l}\}$

$U^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{U^L{}_{\tilde{\mathcal{R}}_l}, U^L{}_{\tilde{\mathcal{H}}_l}\}$, $U^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{U^U{}_{\tilde{\mathcal{R}}_l}, U^U{}_{\tilde{\mathcal{H}}_l}\}$

$C^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\ \{C^L{}_{\tilde{\mathcal{R}}_l}, C^L{}_{\tilde{\mathcal{H}}_l}\}$, $C^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{C^U{}_{\tilde{\mathcal{R}}_l}, C^U{}_{\tilde{\mathcal{H}}_l}\}$

$F^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{F^L{}_{\tilde{\mathcal{R}}_l}, F^L{}_{\tilde{\mathcal{H}}_l}\}$, $F^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{F^U{}_{\tilde{\mathcal{R}}_l}, F^U{}_{\tilde{\mathcal{H}}_l}\}$

ii. For $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{H}}$, the " pessimistic $\cap$''operation is defined as

$\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}, \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[T^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, T^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[U^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, U^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}],$

$\breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[C^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, C^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}}}[F^L{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}, F^U{}_{\tilde{\mathcal{R}}_1\tilde{\mathcal{H}}_1}];$

$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}, \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[T^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, T^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[U^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, U^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}],$

$\breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[C^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, C^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}}}[F^L{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}, F^U{}_{\tilde{\mathcal{R}}_2\tilde{\mathcal{H}}_2}];$

...

$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}, \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[T^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, T^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[U^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, U^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}],$

$\breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[C^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, C^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}}}[F^L{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}, F^U{}_{\tilde{\mathcal{R}}_s\tilde{\mathcal{H}}_s}] >,$

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$

Here, for $l = 1,2,3, \dots, s$;

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right)$, $\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}\right)$, $\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}\right)$

$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}\right)$, $\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}\right)$

$T^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{T^L{}_{\tilde{\mathcal{R}}_l}, T^L{}_{\tilde{\mathcal{H}}_l}\}$, $T^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{T^U{}_{\tilde{\mathcal{R}}_l}, T^U{}_{\tilde{\mathcal{H}}_l}\}$

$U^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{U^L{}_{\tilde{\mathcal{R}}_l}, U^L{}_{\tilde{\mathcal{H}}_l}\}$, $U^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{U^U{}_{\tilde{\mathcal{R}}_l}, U^U{}_{\tilde{\mathcal{H}}_l}\}$

$C^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\ \{C^L{}_{\tilde{\mathcal{R}}_l}, C^L{}_{\tilde{\mathcal{H}}_l}\}$, $C^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{C^U{}_{\tilde{\mathcal{R}}_l}, C^U{}_{\tilde{\mathcal{H}}_l}\}$

$F^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{F^L{}_{\tilde{\mathcal{R}}_l}, F^L{}_{\tilde{\mathcal{H}}_l}\}$, $F^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{F^U{}_{\tilde{\mathcal{R}}_l}, F^U{}_{\tilde{\mathcal{H}}_l}\}$

Example 3.2.6: From Example 3.1.2, for $\hat{\mathcal{X}}$ and $\hat{\mathcal{Z}}$, we obtain that

i. $\hat{\mathcal{X}} \cup'_P \hat{\mathcal{Z}} = \{< \{\acute{n}, \acute{t}, \acute{r}, \acute{s}, \acute{l}, \acute{k}\}, \{\acute{n}, \acute{r}, \acute{f}, \acute{b}, \acute{l}\}[0,0.6], \{\acute{n}, \acute{s}, \acute{l}\}[0.3,0.8], \{\acute{n}, \acute{r}, \acute{s}, \acute{l}, \acute{f}, \acute{k}\}[0.5,0.7], \{\acute{t}, \acute{f}\}[0.6,0.9];$

$\{\acute{f}, \acute{k}, \acute{l}\}, \{\acute{t}, \acute{r}, \acute{s}, \acute{f}, \acute{l}\}[0.1,0.6], \{\acute{t}, \acute{r}, \acute{b}\}[0.5,0.9], \{\acute{n}, \acute{t}, \acute{r}, \acute{s}, \acute{l}\}[0.4,0.8], \{\acute{l}, \acute{k}\}[0.3,0.8] >\}$

ii. $\hat{\mathcal{X}} \cap'_P \hat{\mathcal{Z}} = \{< \{\acute{n}, \acute{t}, \acute{l}\}, \{\acute{f}, \acute{l}\}[0,0.6], \{\acute{l}\}[0.3,0.8], \{\acute{r}\}[0.5,0.7], \{\acute{f}\}[0.6,0.9];$

$\{\acute{k}\}, \{\acute{f}, \acute{l}\}[0.1,0.6], \{\acute{b}\}[0.5,0.9], \{\acute{n}, \acute{s}\}[0.4,0.8], \{\acute{l}\}[0.3,0.8] >\}$

Properties 3.2.9: Let

$\tilde{\mathcal{R}} = \{< \breve{\mathcal{P}}_{1\tilde{\mathcal{R}}_1}, \breve{\mathcal{P}}_{1T\tilde{\mathcal{R}}_1}[T^L{}_{\tilde{\mathcal{R}}_1}, T^U{}_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1U\tilde{\mathcal{R}}_1}[U^L{}_{\tilde{\mathcal{R}}_1}, U^U{}_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1C\tilde{\mathcal{R}}_1}[C^L{}_{\tilde{\mathcal{R}}_1}, C^U{}_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1F\tilde{\mathcal{R}}_1}[F^L{}_{\tilde{\mathcal{R}}_1}, F^U{}_{\tilde{\mathcal{R}}_1}];$

$\breve{\mathcal{P}}_{2\tilde{\mathcal{R}}_2}, \breve{\mathcal{P}}_{2T\tilde{\mathcal{R}}_2}[T^L{}_{\tilde{\mathcal{R}}_2}, T^U{}_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2U\tilde{\mathcal{R}}_2}[U^L{}_{\tilde{\mathcal{R}}_2}, U^U{}_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2C\tilde{\mathcal{R}}_2}[C^L{}_{\tilde{\mathcal{R}}_2}, C^U{}_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2F\tilde{\mathcal{R}}_2}[F^L{}_{\tilde{\mathcal{R}}_2}, F^U{}_{\tilde{\mathcal{R}}_2}];$

...

$\breve{\mathcal{P}}_{s\tilde{\mathcal{R}}_s}, \breve{\mathcal{P}}_{sT\tilde{\mathcal{R}}_s}[T^L{}_{\tilde{\mathcal{R}}_s}, T^U{}_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{sU\tilde{\mathcal{R}}_s}[U^L{}_{\tilde{\mathcal{R}}_s}, U^U{}_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{sC\tilde{\mathcal{R}}_s}[C^L{}_{\tilde{\mathcal{R}}_s}, C^U{}_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{sF\tilde{\mathcal{R}}_s}[F^L{}_{\tilde{\mathcal{R}}_s}, F^U{}_{\tilde{\mathcal{R}}_s}] >,$

$\breve{\mathcal{P}}_{l\tilde{\mathcal{R}}_l}, \breve{\mathcal{P}}_{lT\tilde{\mathcal{R}}_l}, \breve{\mathcal{P}}_{lU\tilde{\mathcal{R}}_l}, \breve{\mathcal{P}}_{lC\tilde{\mathcal{R}}_l}, \breve{\mathcal{P}}_{lF\tilde{\mathcal{R}}_l} \in P(\mathcal{N}); l = 1,2,3, \ldots, s\}$

$\tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1\tilde{\mathcal{H}}_1}, \breve{\mathcal{P}}_{1T\tilde{\mathcal{H}}_1}[T^L{}_{\tilde{\mathcal{H}}_1}, T^U{}_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1U\tilde{\mathcal{H}}_1}[U^L{}_{\tilde{\mathcal{H}}_1}, U^U{}_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1C\tilde{\mathcal{H}}_1}[C^L{}_{\tilde{\mathcal{H}}_1}, C^U{}_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1F\tilde{\mathcal{H}}_1}[F^L{}_{\tilde{\mathcal{H}}_1}, F^U{}_{\tilde{\mathcal{H}}_1}];$

$\breve{\mathcal{P}}_{2\tilde{\mathcal{H}}_2}, \breve{\mathcal{P}}_{2T\tilde{\mathcal{H}}_2}[T^L{}_{\tilde{\mathcal{H}}_2}, T^U{}_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2U\tilde{\mathcal{H}}_2}[U^L{}_{\tilde{\mathcal{H}}_2}, U^U{}_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2C\tilde{\mathcal{H}}_2}[C^L{}_{\tilde{\mathcal{H}}_2}, C^U{}_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2F\tilde{\mathcal{H}}_2}[F^L{}_{\tilde{\mathcal{H}}_2}, F^U{}_{\tilde{\mathcal{H}}_2}];$

...

$\breve{\mathcal{P}}_{s\tilde{\mathcal{H}}_s}, \breve{\mathcal{P}}_{sT\tilde{\mathcal{H}}_s}[T^L{}_{\tilde{\mathcal{H}}_s}, T^U{}_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{sU\tilde{\mathcal{H}}_s}[U^L{}_{\tilde{\mathcal{H}}_s}, U^U{}_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{sC\tilde{\mathcal{H}}_s}[C^L{}_{\tilde{\mathcal{H}}_s}, C^U{}_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{sF\tilde{\mathcal{H}}_s}[F^L{}_{\tilde{\mathcal{H}}_s}, F^U{}_{\tilde{\mathcal{H}}_s}] >,$

$\breve{\mathcal{P}}_{l\tilde{\mathcal{H}}_l}, \breve{\mathcal{P}}_{lT\tilde{\mathcal{H}}_l}, \breve{\mathcal{P}}_{lU\tilde{\mathcal{H}}_l}, \breve{\mathcal{P}}_{lC\tilde{\mathcal{H}}_l}, \breve{\mathcal{P}}_{lF\tilde{\mathcal{H}}_l} \in P(\mathcal{N}); l = 1,2,3, \ldots, s\}$

and

$\tilde{\mathcal{G}} = \{< \breve{\mathcal{P}}_{1\tilde{\mathcal{G}}_1}, \breve{\mathcal{P}}_{1T\tilde{\mathcal{G}}_1}[T^L{}_{\tilde{\mathcal{G}}_1}, T^U{}_{\tilde{\mathcal{G}}_1}], \breve{\mathcal{P}}_{1U\tilde{\mathcal{G}}_1}[U^L{}_{\tilde{\mathcal{G}}_1}, U^U{}_{\tilde{\mathcal{G}}_1}], \breve{\mathcal{P}}_{1C\tilde{\mathcal{G}}_1}[C^L{}_{\tilde{\mathcal{G}}_1}, C^U{}_{\tilde{\mathcal{G}}_1}], \breve{\mathcal{P}}_{1F\tilde{\mathcal{G}}_1}[F^L{}_{\tilde{\mathcal{G}}_1}, F^U{}_{\tilde{\mathcal{G}}_1}];$

$\breve{\mathcal{P}}_{2\tilde{\mathcal{G}}_2}, \breve{\mathcal{P}}_{2T\tilde{\mathcal{G}}_2}[T^L{}_{\tilde{\mathcal{G}}_2}, T^U{}_{\tilde{\mathcal{G}}_2}], \breve{\mathcal{P}}_{2U\tilde{\mathcal{G}}_2}[U^L{}_{\tilde{\mathcal{G}}_2}, U^U{}_{\tilde{\mathcal{G}}_2}], \breve{\mathcal{P}}_{2C\tilde{\mathcal{G}}_2}[C^L{}_{\tilde{\mathcal{G}}_2}, C^U{}_{\tilde{\mathcal{G}}_2}], \breve{\mathcal{P}}_{2F\tilde{\mathcal{G}}_2}[F^L{}_{\tilde{\mathcal{G}}_2}, F^U{}_{\tilde{\mathcal{G}}_2}];$

...

$\breve{\mathcal{P}}_{s\tilde{\mathcal{G}}_s}, \breve{\mathcal{P}}_{sT\tilde{\mathcal{G}}_s}[T^L{}_{\tilde{\mathcal{G}}_s}, T^U{}_{\tilde{\mathcal{G}}_s}], \breve{\mathcal{P}}_{sU\tilde{\mathcal{G}}_s}[U^L{}_{\tilde{\mathcal{G}}_s}, U^U{}_{\tilde{\mathcal{G}}_s}], \breve{\mathcal{P}}_{sC\tilde{\mathcal{G}}_s}[C^L{}_{\tilde{\mathcal{G}}_s}, C^U{}_{\tilde{\mathcal{G}}_s}], \breve{\mathcal{P}}_{sF\tilde{\mathcal{G}}_s}[F^L{}_{\tilde{\mathcal{G}}_s}, F^U{}_{\tilde{\mathcal{G}}_s}] >,$

$\breve{\mathcal{P}}_{l\tilde{\mathcal{G}}_l}, \breve{\mathcal{P}}_{lT\tilde{\mathcal{G}}_l}, \breve{\mathcal{P}}_{lU\tilde{\mathcal{G}}_l}, \breve{\mathcal{P}}_{lC\tilde{\mathcal{G}}_l}, \breve{\mathcal{P}}_{lF\tilde{\mathcal{G}}_l} \in P(\mathcal{N}); l = 1,2,3, \ldots, s\}$

be three IGSVNQS. Then, following conditions are satisfied.

i. $\left(\tilde{\mathcal{R}} \cup'_O \tilde{\mathcal{H}}\right) =' \left(\tilde{\mathcal{H}} \cup'_O \tilde{\mathcal{R}}\right)$

ii. $\left(\tilde{\mathcal{R}} \cup'_P \tilde{\mathcal{H}}\right) =' \left(\mathcal{H} \cup'_P \tilde{\mathcal{R}}\right)$

iii. $(\mathcal{R} \cap'_M \mathcal{H}) =' (\mathcal{H} \cap'_M \mathcal{R})$

iv. $(\mathcal{R} \cap'_O \mathcal{H}) =' (\mathcal{H} \cap'_O \mathcal{R})$

v. $(\mathcal{R} \cap'_P \mathcal{H}) =' (\mathcal{H} \cap'_P \mathcal{R})$

vi. If

$$T^L{}_{\tilde{\mathcal{R}}_l} = T^L{}_{\tilde{\mathcal{G}}_l} \;,\; T^U{}_{\tilde{\mathcal{R}}_l} = T^U{}_{\tilde{\mathcal{G}}_l} \;,\; U^L{}_{\tilde{\mathcal{R}}_l} = U^L{}_{\tilde{\mathcal{G}}_l} \;,\; U^U{}_{\tilde{\mathcal{R}}_l} = U^U{}_{\tilde{\mathcal{G}}_l}$$

$$C^L{}_{\tilde{\mathcal{R}}_l} = C^L{}_{\tilde{\mathcal{G}}_l} \;,\; C^U{}_{\tilde{\mathcal{R}}_l} = C^U{}_{\tilde{\mathcal{G}}_l} \;,\; F^L{}_{\tilde{\mathcal{R}}_l} = F^L{}_{\tilde{\mathcal{G}}_l} \;,\; F^U{}_{\tilde{\mathcal{R}}_l} = F^U{}_{\tilde{\mathcal{G}}_l}$$

then

$$\tilde{\mathcal{R}} \cup'_M \left(\tilde{\mathcal{H}} \cup'_M \tilde{\mathcal{G}}\right) =' \left(\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}}\right) \cup'_M \tilde{\mathcal{G}}$$

vii. $\tilde{\mathcal{R}} \cup'_O (\tilde{\mathcal{H}} \cup'_O \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cup'_O \tilde{\mathcal{H}}) \cup'_O \tilde{\mathcal{G}}$

viii. $\tilde{\mathcal{R}} \cup'_P (\tilde{\mathcal{H}} \cup'_P \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cup'_P \tilde{\mathcal{H}}) \cup'_P \tilde{\mathcal{G}}$

ix. If

$$T^L{}_{\tilde{\mathcal{R}}_l} = T^L{}_{\tilde{\mathcal{G}}_l} \;,\; T^U{}_{\tilde{\mathcal{R}}_l} = T^U{}_{\tilde{\mathcal{G}}_l} \;,\; U^L{}_{\tilde{\mathcal{R}}_l} = U^L{}_{\tilde{\mathcal{G}}_l} \;,\; U^U{}_{\tilde{\mathcal{R}}_l} = U^U{}_{\tilde{\mathcal{G}}_l}$$

$$C^L{}_{\tilde{\mathcal{R}}_l} = C^L{}_{\tilde{\mathcal{G}}_l} \;,\; C^U{}_{\tilde{\mathcal{R}}_l} = C^U{}_{\tilde{\mathcal{G}}_l} \;,\; F^L{}_{\tilde{\mathcal{R}}_l} = F^L{}_{\tilde{\mathcal{G}}_l} \;,\; F^U{}_{\tilde{\mathcal{R}}_l} = F^U{}_{\tilde{\mathcal{G}}_l}$$

then

$$\tilde{\mathcal{R}} \cap'_M (\tilde{\mathcal{H}} \cap'_M \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}) \cap'_M \tilde{\mathcal{G}}.$$

x. $\tilde{\mathcal{R}} \cap'_O (\tilde{\mathcal{H}} \cap'_O \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{H}}) \cap'_O \tilde{\mathcal{G}}$

xi. $\tilde{\mathcal{R}} \cap'_P (\tilde{\mathcal{H}} \cap'_P \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{H}}) \cap'_P \tilde{\mathcal{G}}$

xii. $\tilde{\mathcal{R}} \cap'_M (\tilde{\mathcal{H}} \cup'_M \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}) \cup'_M (\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{G}})$

xiii. $\tilde{\mathcal{R}} \cap'_O (\tilde{\mathcal{H}} \cup'_O \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{H}}) \cup'_O (\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{G}})$

xiv. $\tilde{\mathcal{R}} \cap'_P (\tilde{\mathcal{H}} \cup'_P \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{H}}) \cup'_P (\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{G}})$

xv. $\tilde{\mathcal{R}} \cup'_M (\tilde{\mathcal{H}} \cap'_M \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}}) \cap'_M (\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{G}})$

xvi. $\tilde{\mathcal{R}} \cup'_O (\tilde{\mathcal{H}} \cap'_O \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cup'_O \tilde{\mathcal{H}}) \cap'_O (\tilde{\mathcal{R}} \cup'_O \tilde{\mathcal{G}})$

xvii. $\tilde{\mathcal{R}} \cup'_P (\tilde{\mathcal{H}} \cap'_P \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cup'_P \tilde{\mathcal{H}}) \cap'_P (\tilde{\mathcal{R}} \cup'_P \tilde{\mathcal{G}})$

xviii. İf $\tilde{\mathcal{R}} =' \tilde{\mathcal{H}}$ then $\left(\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}}\right) =' \left(\tilde{\mathcal{R}} \cup'_O \tilde{\mathcal{H}}\right) =' \left(\tilde{\mathcal{R}} \cup'_P \tilde{\mathcal{H}}\right)$

xix. İf $\tilde{\mathcal{R}} =' \tilde{\mathcal{H}}$ then $\left(\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}\right) =' \left(\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{H}}\right) =' \left(\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{H}}\right)$

Proof:

i. From Definition 3.2.1 for $(\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}})$, we get

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = (\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}})$, $\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = (\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}})$, $\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = (\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}})$

$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = (\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}})$, $\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = (\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}})$

$T^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{T^L{}_{\tilde{\mathcal{R}}_l}+T^L{}_{\tilde{\mathcal{H}}_l}}{2}$, $T^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{T^U{}_{\tilde{\mathcal{R}}_l}+T^U{}_{\tilde{\mathcal{H}}_l}}{2}$, $U^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{U^L{}_{\tilde{\mathcal{R}}_l}+U^L{}_{\tilde{\mathcal{H}}_l}}{2}$, $U^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{U^U{}_{\tilde{\mathcal{R}}_l}+U^U{}_{\tilde{\mathcal{H}}_l}}{2}$

$C^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{C^L{}_{\tilde{\mathcal{R}}_l}+C^L{}_{\tilde{\mathcal{H}}_l}}{2}$, $C^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{C^U{}_{\tilde{\mathcal{R}}_l}+C^U{}_{\tilde{\mathcal{H}}_l}}{2}$, $F^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{F^L{}_{\tilde{\mathcal{R}}_l}+F^L{}_{\tilde{\mathcal{H}}_l}}{2}$, $F^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{F^U{}_{\tilde{\mathcal{R}}_l}+F^U{}_{\tilde{\mathcal{H}}_l}}{2}$ (1)

From Definition 3.2.1 for $(\tilde{\mathcal{H}} \cup'_M \tilde{\mathcal{R}})$, we get

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}} = (\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}})$, $\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}}} = (\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}})$, $\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}}} = (\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}})$

$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}}} = (\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}})$, $\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}}} = (\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}})$

$$T^L{}_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{T^L{}_{\tilde{\mathcal{H}}_l} + T^L{}_{\tilde{\mathcal{R}}_l}}{2}, T^U{}_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{T^U{}_{\tilde{\mathcal{H}}_l} + T^U{}_{\tilde{\mathcal{R}}_l}}{2}, U^L{}_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{U^L{}_{\tilde{\mathcal{H}}_l} + U^L{}_{\tilde{\mathcal{R}}_l}}{2}, U^U{}_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{U^U{}_{\tilde{\mathcal{H}}_l} + U^U{}_{\tilde{\mathcal{R}}_l}}{2}$$

$$C^L{}_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{C^L{}_{\tilde{\mathcal{H}}_l} + C^L{}_{\tilde{\mathcal{R}}_l}}{2}, C^U{}_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{C^U{}_{\tilde{\mathcal{H}}_l} + C^U{}_{\tilde{\mathcal{R}}_l}}{2}, F^L{}_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{F^L{}_{\tilde{\mathcal{H}}_l} + F^L{}_{\tilde{\mathcal{R}}_l}}{2}, F^U{}_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{F^U{}_{\tilde{\mathcal{H}}_l} + F^U{}_{\tilde{\mathcal{R}}_l}}{2} \quad (2)$$

and from Definition 3.1.1, (1) and (2), we get

$$(\mathcal{R} \cup'_M \mathcal{H}) =' (\mathcal{H} \cup'_M \mathcal{R}).$$

Proof of {ii, iii} are obtained similar to i.

iv. From Definition 3.2.1 for $(\mathcal{R} \cap'_M \mathcal{H})$, we get

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = (\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}})$, $\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = (\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}})$, $\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = (\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}})$

$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = (\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}})$, $\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = (\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}})$

$T^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{T^L{}_{\tilde{\mathcal{R}}_l}+T^L{}_{\tilde{\mathcal{H}}_l}}{2}$, $T^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{T^U{}_{\tilde{\mathcal{R}}_l}+T^U{}_{\tilde{\mathcal{H}}_l}}{2}$, $U^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{U^L{}_{\tilde{\mathcal{R}}_l}+U^L{}_{\tilde{\mathcal{H}}_l}}{2}$, $U^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{U^U{}_{\tilde{\mathcal{R}}_l}+U^U{}_{\tilde{\mathcal{H}}_l}}{2}$

$C^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{C^L{}_{\tilde{\mathcal{R}}_l}+C^L{}_{\tilde{\mathcal{H}}_l}}{2}$, $C^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{C^U{}_{\tilde{\mathcal{R}}_l}+C^U{}_{\tilde{\mathcal{H}}_l}}{2}$, $F^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{F^L{}_{\tilde{\mathcal{R}}_l}+F^L{}_{\tilde{\mathcal{H}}_l}}{2}$, $F^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \frac{F^U{}_{\tilde{\mathcal{R}}_l}+F^U{}_{\tilde{\mathcal{H}}_l}}{2}$

(3)

Definition 3.2.1 for $(\tilde{\mathcal{H}} \cap'_M \tilde{\mathcal{R}})$

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}}\right)$, $\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}}\right)$, $\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}}\right)$

$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}}\right)$, $\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}}\right)$

$T^L{}_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{T^L{}_{\tilde{\mathcal{H}}_l}+T^L{}_{\tilde{\mathcal{R}}_l}}{2}$, $T^U{}_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{T^U{}_{\tilde{\mathcal{H}}_l}+T^U{}_{\tilde{\mathcal{R}}_l}}{2}$, $U^L{}_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{U^L{}_{\tilde{\mathcal{H}}_l}+U^L{}_{\tilde{\mathcal{R}}_l}}{2}$, $U^U{}_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{U^U{}_{\tilde{\mathcal{H}}_l}+U^U{}_{\tilde{\mathcal{R}}_l}}{2}$

$C^L{}_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{C^L{}_{\tilde{\mathcal{H}}_l}+C^L{}_{\tilde{\mathcal{R}}_l}}{2}$, $C^U{}_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{C^U{}_{\tilde{\mathcal{H}}_l}+C^U{}_{\tilde{\mathcal{R}}_l}}{2}$, $F^L{}_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{F^L{}_{\tilde{\mathcal{H}}_l}+F^L{}_{\tilde{\mathcal{R}}_l}}{2}$, $F^U{}_{\tilde{\mathcal{H}}_l\tilde{\mathcal{R}}_l} = \frac{F^U{}_{\tilde{\mathcal{H}}_l}+F^U{}_{\tilde{\mathcal{R}}_l}}{2}$ (4)

From Definition 3.1.1, (3) and (4) we get

$$\left(\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}\right) =' \left(\tilde{\mathcal{H}} \cap'_M \tilde{\mathcal{R}}\right)$$

Proof of {v, vi} are obtained similar to iv.

vii. We assume that

$$T^L{}_{\tilde{\mathcal{R}}_l} = T^L{}_{\tilde{\mathcal{G}}_l}\ ,\ T^U{}_{\tilde{\mathcal{R}}_l} = T^U{}_{\tilde{\mathcal{G}}_l}\ ,\ U^L{}_{\tilde{\mathcal{R}}_l} = U^L{}_{\tilde{\mathcal{G}}_l}\ ,\ U^U{}_{\tilde{\mathcal{R}}_l} = U^U{}_{\tilde{\mathcal{G}}_l}$$

$$C^L{}_{\tilde{\mathcal{R}}_l} = C^L{}_{\tilde{\mathcal{G}}_l}\ ,\ C^U{}_{\tilde{\mathcal{R}}_l} = C^U{}_{\tilde{\mathcal{G}}_l}\ ,\ F^L{}_{\tilde{\mathcal{R}}_l} = F^L{}_{\tilde{\mathcal{G}}_l}\ ,\ F^U{}_{\tilde{\mathcal{R}}_l} = F^U{}_{\tilde{\mathcal{G}}_l}.$$

For $\tilde{\mathcal{R}} \cup'_M \left(\tilde{\mathcal{H}} \cup'_M \tilde{\mathcal{G}}\right)$ from Definition 3.2.1, we get

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup (\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{G}}_l}})\right)$, $\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cup (\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{G}}_l}}})\right)$,

$\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cup (\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{G}}_l}}})\right)$, $\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cup (\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{G}}_l}}})\right)$

$\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cup (\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{G}}_l}}})\right)$

$$T^L{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{T^L{}_{\tilde{\mathcal{R}}_l} + \frac{T^L{}_{\tilde{\mathcal{H}}_l} + T^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad T^U{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{T^U{}_{\tilde{\mathcal{R}}_l} + \frac{T^U{}_{\tilde{\mathcal{H}}_l} + T^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$U^L{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{U^L{}_{\tilde{\mathcal{R}}_l} + \frac{U^L{}_{\tilde{\mathcal{H}}_l} + U^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad U^U{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{U^U{}_{\tilde{\mathcal{R}}_l} + \frac{U^U{}_{\tilde{\mathcal{H}}_l} + U^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$C^L{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{C^L{}_{\tilde{\mathcal{R}}_l} + \frac{C^L{}_{\tilde{\mathcal{H}}_l} + C^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad C^U{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{C^U{}_{\tilde{\mathcal{R}}_l} + \frac{C^U{}_{\tilde{\mathcal{H}}_l} + C^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$F^L{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{F^L{}_{\tilde{\mathcal{R}}_l} + \frac{F^L{}_{\tilde{\mathcal{H}}_l} + F^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2},$$

$$F^U{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{F^U{}_{\tilde{\mathcal{R}}_l} + \frac{F^U{}_{\tilde{\mathcal{H}}_l} + F^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}. \quad (5)$$

From Definition 3.2.1 for $(\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}}) \cup'_M \tilde{\mathcal{G}}$,

$$\breve{\mathcal{P}}_{l_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}} = \left((\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}) \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{G}}_l}}\right),\ \breve{\mathcal{P}}_{l_{T_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}}} = \left((\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}) \cup \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{G}}_l}}}\right)$$

$$\breve{\mathcal{P}}_{l_{U_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}}} = \left((\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}) \cup \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{G}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{C_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}}} = \left((\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}) \cup \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{G}}_l}}}\right)$$

$$\breve{\mathcal{P}}_{l_{F_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}}} = \left((\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}) \cup \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{G}}_l}}}\right)$$

$$T^L{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{T^L{}_{\tilde{\mathcal{R}}_l} + T^L{}_{\tilde{\mathcal{H}}_l}}{2} + T^L{}_{\tilde{\mathcal{G}}_l}}{2}, \qquad T^U{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{T^U{}_{\tilde{\mathcal{R}}_l} + T^U{}_{\tilde{\mathcal{H}}_l}}{2} + T^U{}_{\tilde{\mathcal{G}}_l}}{2}$$

$$U^L{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{U^L{}_{\tilde{\mathcal{R}}_l} + U^L{}_{\tilde{\mathcal{H}}_l}}{2} + U^L{}_{\tilde{\mathcal{G}}_l}}{2}, \qquad U^U{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{U^U{}_{\tilde{\mathcal{R}}_l} + U^U{}_{\tilde{\mathcal{H}}_l}}{2} + U^U{}_{\tilde{\mathcal{G}}_l}}{2}$$

$$C^L{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{C^L{}_{\tilde{\mathcal{R}}_l} + C^L{}_{\tilde{\mathcal{H}}_l}}{2} + C^L{}_{\tilde{\mathcal{G}}_l}}{2}, \qquad C^U{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{C^U{}_{\tilde{\mathcal{R}}_l} + C^U{}_{\tilde{\mathcal{H}}_l}}{2} + C^U{}_{\tilde{\mathcal{G}}_l}}{2}$$

$$F^L{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{F^L{}_{\tilde{\mathcal{R}}_l} + F^L{}_{\tilde{\mathcal{H}}_l}}{2} + F^L{}_{\tilde{\mathcal{G}}_l}}{2},$$

$$F^U{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{F^U{}_{\tilde{\mathcal{R}}_l} + F^U{}_{\tilde{\mathcal{H}}_l}}{2} + F^U{}_{\tilde{\mathcal{G}}_l}}{2}. \tag{6}$$

From Definition 3.1.1, (5) and (6)

we get

$$\tilde{\mathcal{R}} \cup'_M (\tilde{\mathcal{H}} \cup'_M \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}}) \cup'_M \tilde{\mathcal{G}}$$

Proof of {viii, ix} are obtained similar to vii.

x. We assume that

$$T^L{}_{\tilde{\mathcal{R}}_l} = T^L{}_{\tilde{\mathcal{G}}_l}\ ,\ T^U{}_{\tilde{\mathcal{R}}_l} = T^U{}_{\tilde{\mathcal{G}}_l}\ ,\ U^L{}_{\tilde{\mathcal{R}}_l} = U^L{}_{\tilde{\mathcal{G}}_l}\ ,\ U^U{}_{\tilde{\mathcal{R}}_l} = U^U{}_{\tilde{\mathcal{G}}_l}$$

$$C^L{}_{\tilde{\mathcal{R}}_l} = C^L{}_{\tilde{\mathcal{G}}_l}\ ,\ C^U{}_{\tilde{\mathcal{R}}_l} = C^U{}_{\tilde{\mathcal{G}}_l}\ ,\ F^L{}_{\tilde{\mathcal{R}}_l} = F^L{}_{\tilde{\mathcal{G}}_l}\ ,\ F^U{}_{\tilde{\mathcal{R}}_l} = F^U{}_{\tilde{\mathcal{G}}_l}.$$

From Definition 3.2.1 for $\mathcal{R} \cap'_M (\mathcal{H} \cap'_M \mathcal{G})$, we get

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap (\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{G}}_l}})\right),\ \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cap (\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{G}}_l}}})\right)$$

$$\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cap (\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{G}}_l}}})\right),\ \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cap (\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{G}}_l}}})\right)$$

$$\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cap (\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{G}}_l}}})\right)$$

$$T^L{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{T^L{}_{\tilde{\mathcal{R}}_l} + \frac{T^L{}_{\tilde{\mathcal{H}}_l} + T^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \quad T^U{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{T^U{}_{\tilde{\mathcal{R}}_l} + \frac{T^U{}_{\tilde{\mathcal{H}}_l} + T^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$U^L{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{U^L{}_{\tilde{\mathcal{R}}_l} + \frac{U^L{}_{\tilde{\mathcal{H}}_l} + U^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \quad U^U{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{U^U{}_{\tilde{\mathcal{R}}_l} + \frac{U^U{}_{\tilde{\mathcal{H}}_l} + U^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$C^L{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{C^L{}_{\tilde{\mathcal{R}}_l} + \frac{C^L{}_{\tilde{\mathcal{H}}_l} + C^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \quad C^U{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{C^U{}_{\tilde{\mathcal{R}}_l} + \frac{C^U{}_{\tilde{\mathcal{H}}_l} + C^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$F^L{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{F^L{}_{\tilde{\mathcal{R}}_l} + \frac{F^L{}_{\tilde{\mathcal{H}}_l} + F^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \quad F^U{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{F^U{}_{\tilde{\mathcal{R}}_l} + \frac{F^U{}_{\tilde{\mathcal{H}}_l} + F^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}. \tag{7}$$

From definition 3.2.1 for $(\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}) \cap'_M \tilde{\mathcal{G}}$, we get

$$\breve{\mathcal{P}}_{l_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}} = \left((\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}) \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{G}}_l}}\right),\ \breve{\mathcal{P}}_{l_{T(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}} = \left((\breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{H}}_l}}) \cap \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{G}}_l}}\right)$$

$$\breve{\mathcal{P}}_{l_{U(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}} = \left((\breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{H}}_l}}) \cap \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{G}}_l}}\right), \breve{\mathcal{P}}_{l_{C(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}} = \left((\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{H}}_l}}) \cap \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{G}}_l}}\right)$$

$$\breve{\mathcal{P}}_{l_{F(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l}} = \left((\breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{H}}_l}}) \cap \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{G}}_l}}\right)$$

$$T^L{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{T^L{}_{\tilde{\mathcal{R}}_l} + T^L{}_{\tilde{\mathcal{H}}_l}}{2} + T^L{}_{\tilde{\mathcal{G}}_l}}{2}, \quad T^U{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{T^U{}_{\tilde{\mathcal{R}}_l} + T^U{}_{\tilde{\mathcal{H}}_l}}{2} + T^U{}_{\tilde{\mathcal{G}}_l}}{2}$$

$$U^L{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{U^L{}_{\tilde{\mathcal{R}}_l} + U^L{}_{\tilde{\mathcal{H}}_l}}{2} + U^L{}_{\tilde{\mathcal{G}}_l}}{2}, \quad U^U{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{U^U{}_{\tilde{\mathcal{R}}_l} + U^U{}_{\tilde{\mathcal{H}}_l}}{2} + U^U{}_{\tilde{\mathcal{G}}_l}}{2}$$

$$C^L{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{C^L{}_{\tilde{\mathcal{R}}_l} + C^L{}_{\tilde{\mathcal{H}}_l}}{2} + C^L{}_{\tilde{\mathcal{G}}_l}}{2}, \quad C^U{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{C^U{}_{\tilde{\mathcal{R}}_l} + C^U{}_{\tilde{\mathcal{H}}_l}}{2} + C^U{}_{\tilde{\mathcal{G}}_l}}{2}$$

$$F^L{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{F^L{}_{\tilde{\mathcal{R}}_l} + F^L{}_{\tilde{\mathcal{H}}_l}}{2} + F^L{}_{\tilde{\mathcal{G}}_l}}{2}, \quad F^U{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)\tilde{\mathcal{G}}_l} = \frac{\frac{F^U{}_{\tilde{\mathcal{R}}_l} + F^U{}_{\tilde{\mathcal{H}}_l}}{2} + F^U{}_{\tilde{\mathcal{G}}_l}}{2}. \tag{8}$$

Definition 3.1.1, from (7) and (8) we get

$$\tilde{\mathcal{R}} \cap'_M (\tilde{\mathcal{H}} \cap'_M \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}) \cap'_M \tilde{\mathcal{G}}$$

Proof of {xi, xii} are obtained similar to x.

xiii. From Definition 3.2.1 for $\tilde{\mathcal{R}} \cap'_M (\tilde{\mathcal{H}} \cup'_M \tilde{\mathcal{G}})$, we get

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{G}}_l}}\right)\right)$, $\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cap \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{G}}_l}}}\right)\right)$

$\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cap \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{G}}_l}}}\right)\right)$, $\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cap \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{G}}_l}}}\right)\right)$

$\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cap \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}} \cup \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{G}}_l}}}\right)\right)$

$$T^L{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{T^L{}_{\tilde{\mathcal{R}}_l} + \frac{T^L{}_{\tilde{\mathcal{H}}_l} + T^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad T^U{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{T^U{}_{\tilde{\mathcal{R}}_l} + \frac{T^U{}_{\tilde{\mathcal{H}}_l} + T^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$U^L{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{U^L{}_{\tilde{\mathcal{R}}_l} + \frac{U^L{}_{\tilde{\mathcal{H}}_l} + U^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad U^U{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{U^U{}_{\tilde{\mathcal{R}}_l} + \frac{U^U{}_{\tilde{\mathcal{H}}_l} + U^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$C^L{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{C^L{}_{\tilde{\mathcal{R}}_l} + \frac{C^L{}_{\tilde{\mathcal{H}}_l} + C^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad C^U{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{C^U{}_{\tilde{\mathcal{R}}_l} + \frac{C^U{}_{\tilde{\mathcal{H}}_l} + C^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$F^L{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{F^L{}_{\tilde{\mathcal{R}}_l} + \frac{F^L{}_{\tilde{\mathcal{H}}_l} + F^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \quad F^U{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{F^U{}_{\tilde{\mathcal{R}}_l} + \frac{F^U{}_{\tilde{\mathcal{H}}_l} + F^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}. \tag{9}$$

Definition 3.2.1 for $(\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}) \cup'_M (\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{G}})$, we get

$\breve{\mathcal{P}}_{l_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}} = \left(\left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right) \cup \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{G}}_l}}\right)\right)$,

$\breve{\mathcal{P}}_{l_{T_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}}} = \left(\left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}\right) \cup \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{G}}_l}}}\right)\right)$,

$\breve{\mathcal{P}}_{l_{U_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}}} = \left(\left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}\right) \cup \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{G}}_l}}}\right)\right)$,

$\breve{\mathcal{P}}_{l_{C_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}}} = \left(\left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}\right) \cup \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{G}}_l}}}\right)\right)$,

$\breve{\mathcal{P}}_{l_{F_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}}} = \left(\left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}\right) \cup \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{G}}_l}}}\right)\right)$,

$$T^L{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{T^L{}_{\tilde{\mathcal{R}}_l} + T^L{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{T^L{}_{\tilde{\mathcal{R}}_l} + T^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad T^U{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{T^U{}_{\tilde{\mathcal{R}}_l} + T^U{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{T^U{}_{\tilde{\mathcal{R}}_l} + T^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$U^L{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{U^L{}_{\tilde{\mathcal{R}}_l} + U^L{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{U^L{}_{\tilde{\mathcal{R}}_l} + U^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \qquad U^U{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{U^U{}_{\tilde{\mathcal{R}}_l} + U^U{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{U^U{}_{\tilde{\mathcal{R}}_l} + U^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$C^L{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{C^L{}_{\tilde{\mathcal{R}}_l} + C^L{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{C^L{}_{\tilde{\mathcal{R}}_l} + C^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \quad C^U{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{C^U{}_{\tilde{\mathcal{R}}_l} + C^U{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{C^U{}_{\tilde{\mathcal{R}}_l} + C^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$F^L{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{F^L{}_{\tilde{\mathcal{R}}_l} + F^L{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{F^L{}_{\tilde{\mathcal{R}}_l} + F^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2},$$

$$F^U{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{F^U{}_{\tilde{\mathcal{R}}_l} + F^U{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{F^U{}_{\tilde{\mathcal{R}}_l} + F^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2} \tag{10}$$

From Definition 3.1.1, (9) and (10) we get

$$\tilde{\mathcal{R}} \cap'_M \left(\tilde{\mathcal{H}} \cup'_M \tilde{\mathcal{G}}\right) =' \left(\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}\right) \cup'_M \left(\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{G}}\right).$$

Proof of {xiv, xv} are obtained similar to xiii.

xvi. From Definition 3.2.1 for $\tilde{\mathcal{R}} \cup'_M \left(\tilde{\mathcal{H}} \cap'_M \tilde{\mathcal{G}}\right)$, we get

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{G}}_l}}\right)\right), \quad \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cup \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{G}}_l}}}\right)\right)$$

$$\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cup \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{G}}_l}}}\right)\right), \quad \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cup \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{G}}_l}}}\right)\right)$$

$$\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cup \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}} \cap \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{G}}_l}}}\right)\right)$$

$$T^L{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{T^L{}_{\tilde{\mathcal{R}}_l} + \frac{T^L{}_{\tilde{\mathcal{H}}_l} + T^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \quad T^U{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{T^U{}_{\tilde{\mathcal{R}}_l} + \frac{T^U{}_{\tilde{\mathcal{H}}_l} + T^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$U^L{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{U^L{}_{\tilde{\mathcal{R}}_l} + \frac{U^L{}_{\tilde{\mathcal{H}}_l} + U^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \quad U^U{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{U^U{}_{\tilde{\mathcal{R}}_l} + \frac{U^U{}_{\tilde{\mathcal{H}}_l} + U^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$C^L{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{C^L{}_{\tilde{\mathcal{R}}_l} + \frac{C^L{}_{\tilde{\mathcal{H}}_l} + C^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \quad C^U{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{C^U{}_{\tilde{\mathcal{R}}_l} + \frac{C^U{}_{\tilde{\mathcal{H}}_l} + C^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$F^L{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{F^L{}_{\tilde{\mathcal{R}}_l} + \frac{F^L{}_{\tilde{\mathcal{H}}_l} + F^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2},$$

$$F^U{}_{\tilde{\mathcal{R}}_l(\tilde{\mathcal{H}}_l\tilde{\mathcal{G}}_l)} = \frac{F^U{}_{\tilde{\mathcal{R}}_l} + \frac{F^U{}_{\tilde{\mathcal{H}}_l} + F^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}. \tag{11}$$

Definition 3.2.1 for $(\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}}) \cap'_M (\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{G}})$ we get

$$\breve{\mathcal{P}}_{l_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}} = \left(\left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right) \cap \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{G}}_l}}\right)\right),$$

$$\breve{\mathcal{P}}_{l_{T(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}} = \left(\left(\breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{H}}_l}}\right) \cap \left(\breve{\mathcal{P}}_{l_{T\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{T\tilde{\mathcal{G}}_l}}\right)\right)$$

$$\breve{\mathcal{P}}_{l_{U(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}} = \left(\left(\breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{H}}_l}}\right) \cap \left(\breve{\mathcal{P}}_{l_{U\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{U\tilde{\mathcal{G}}_l}}\right)\right),$$

$$\breve{\mathcal{P}}_{l_{C(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}} = \left(\left(\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{H}}_l}}\right) \cap \left(\breve{\mathcal{P}}_{l_{C\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{C\tilde{\mathcal{G}}_l}}\right)\right)$$

$$\breve{\mathcal{P}}_{l_{F(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)}} = \left(\left(\breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{H}}_l}}\right) \cap \left(\breve{\mathcal{P}}_{l_{F\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{F\tilde{\mathcal{G}}_l}}\right)\right)$$

$$T^L{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{T^L{}_{\tilde{\mathcal{R}}_l} + T^L{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{T^L{}_{\tilde{\mathcal{R}}_l} + T^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \quad T^U{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{T^U{}_{\tilde{\mathcal{R}}_l} + T^U{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{T^U{}_{\tilde{\mathcal{R}}_l} + T^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$U^L{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{U^L{}_{\tilde{\mathcal{R}}_l} + U^L{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{U^L{}_{\tilde{\mathcal{R}}_l} + U^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \quad U^U{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{U^U{}_{\tilde{\mathcal{R}}_l} + U^U{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{U^U{}_{\tilde{\mathcal{R}}_l} + U^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$C^L{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{C^L{}_{\tilde{\mathcal{R}}_l} + C^L{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{C^L{}_{\tilde{\mathcal{R}}_l} + C^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2}, \quad C^U{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{C^U{}_{\tilde{\mathcal{R}}_l} + C^U{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{C^U{}_{\tilde{\mathcal{R}}_l} + C^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}$$

$$F^L{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{F^L{}_{\tilde{\mathcal{R}}_l} + F^L{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{F^L{}_{\tilde{\mathcal{R}}_l} + F^L{}_{\tilde{\mathcal{G}}_l}}{2}}{2},$$

$$F^U{}_{(\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l)(\tilde{\mathcal{R}}_l\tilde{\mathcal{G}}_l)} = \frac{\frac{F^U{}_{\tilde{\mathcal{R}}_l} + F^U{}_{\tilde{\mathcal{H}}_l}}{2} + \frac{F^U{}_{\tilde{\mathcal{R}}_l} + F^U{}_{\tilde{\mathcal{G}}_l}}{2}}{2}. \quad (12)$$

Definition 3.1.1, from (11) and (12), we get

$$\tilde{\mathcal{R}} \cup'_M (\tilde{\mathcal{H}} \cap'_M \tilde{\mathcal{G}}) =' (\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}}) \cap'_M (\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{G}}).$$

Proof of {xvii, xviii} are obtained similar to xvi.

xix. We assume that $\tilde{\mathcal{R}} =' \tilde{\mathcal{H}}$. From, Definition 3.1.1, Definition 3.2.1, Definition 3.2.3, Definition 3.2.5 we obtain

$$T^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{T^L{}_{\tilde{\mathcal{R}}_l}, T^L{}_{\tilde{\mathcal{H}}_l}\} = \frac{T^L{}_{\tilde{\mathcal{R}}_l} + T^L{}_{\tilde{\mathcal{H}}_l}}{2} = \max\{T^L{}_{\tilde{\mathcal{R}}_l}, T^L{}_{\tilde{\mathcal{H}}_l}\}$$

$$T^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \min\{T^U{}_{\tilde{\mathcal{R}}_l}, T^U{}_{\tilde{\mathcal{H}}_l}\} = \frac{T^U{}_{\tilde{\mathcal{R}}_l} + T^U{}_{\tilde{\mathcal{H}}_l}}{2} = \max\{T^U{}_{\tilde{\mathcal{R}}_l}, T^U{}_{\tilde{\mathcal{H}}_l}\}$$

$$U^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{U^L{}_{\tilde{\mathcal{R}}_l}, U^L{}_{\tilde{\mathcal{H}}_l}\} = \frac{U^L{}_{\tilde{\mathcal{R}}_l} + U^L{}_{\tilde{\mathcal{H}}_l}}{2} = \min\{U^L{}_{\tilde{\mathcal{R}}_l}, U^L{}_{\tilde{\mathcal{H}}_l}\}$$

$$U^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{U^U{}_{\tilde{\mathcal{R}}_l}, U^U{}_{\tilde{\mathcal{H}}_l}\} = \frac{U^U{}_{\tilde{\mathcal{R}}_l} + U^U{}_{\tilde{\mathcal{H}}_l}}{2} = \min\{U^U{}_{\tilde{\mathcal{R}}_l}, U^U{}_{\tilde{\mathcal{H}}_l}\}$$

$$C^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\ \{C^L{}_{\tilde{\mathcal{R}}_l}, C^L{}_{\tilde{\mathcal{H}}_l}\} = \frac{C^L{}_{\tilde{\mathcal{R}}_l} + C^L{}_{\tilde{\mathcal{H}}_l}}{2} = \min\{C^L{}_{\tilde{\mathcal{R}}_l}, C^L{}_{\tilde{\mathcal{H}}_l}\}$$

$$C^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{C^U{}_{\tilde{\mathcal{R}}_l}, C^U{}_{\tilde{\mathcal{H}}_l}\} = \frac{C^U{}_{\tilde{\mathcal{R}}_l} + C^U{}_{\tilde{\mathcal{H}}_l}}{2} = \min\{C^U{}_{\tilde{\mathcal{R}}_l}, C^U{}_{\tilde{\mathcal{H}}_l}\}$$

$$F^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{F^L{}_{\tilde{\mathcal{R}}_l}, F^L{}_{\tilde{\mathcal{H}}_l}\} = \frac{F^L{}_{\tilde{\mathcal{R}}_l} + F^L{}_{\tilde{\mathcal{H}}_l}}{2} = \min\{F^L{}_{\tilde{\mathcal{R}}_l}, F^L{}_{\mathcal{H}_l}\}$$

$$F^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} = \max\{F^U{}_{\tilde{\mathcal{R}}_l}, F^U{}_{\tilde{\mathcal{H}}_l}\} = \frac{F^U{}_{\tilde{\mathcal{R}}_l} + F^U{}_{\tilde{\mathcal{H}}_l}}{2} = \min\{F^U{}_{\mathcal{R}_l}, F^U{}_{\tilde{\mathcal{H}}_l}\}. \quad (13)$$

Also,

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cup \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}\right)$$

$$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cup \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}\right) \quad (14)$$

Thusi from (13) and(14) we get

$$(\mathcal{R} \cup'_M \mathcal{H}) =' (\mathcal{R} \cup'_O \mathcal{H}) =' (\mathcal{R} \cup'_P \mathcal{H}).$$

Proof of xx are obtained similar to xix.

Theorem 3.2.10: Let

$\tilde{\mathcal{R}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}},\ \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}[\ T^L{}_{\tilde{\mathcal{R}}_1}, T^U{}_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1}}}[U^L{}_{\tilde{\mathcal{R}}_1},\ U^U{}_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1}}}[C^L{}_{\tilde{\mathcal{R}}_1},\ C^U{}_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1}}}[F^L{}_{\tilde{\mathcal{R}}_1}, F^U{}_{\tilde{\mathcal{R}}_1}];$

$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2}},\ \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{R}}_2}}}[\ T^L{}_{\tilde{\mathcal{R}}_2}, T^U{}_{\tilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{R}}_2}}}[U^L{}_{\tilde{\mathcal{R}}_2},\ U^U{}_{\tilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{R}}_2}}}[C^L{}_{\tilde{\mathcal{R}}_2},\ C^U{}_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{R}}_2}}}[F^L{}_{\tilde{\mathcal{R}}_2}, F^U{}_{\tilde{\mathcal{R}}_2}];$

...

$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s}},\ \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{R}}_s}}}[\ T^L{}_{\tilde{\mathcal{R}}_s}, T^U{}_{\tilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{R}}_s}}}[U^L{}_{\tilde{\mathcal{R}}_s},\ U^U{}_{\tilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{R}}_s}}}[C^L{}_{\tilde{\mathcal{R}}_s},\ C^U{}_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{R}}_s}}}[F^L{}_{\tilde{\mathcal{R}}_s}, F^U{}_{\tilde{\mathcal{R}}_s}];>,$

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$

and

$\tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{H}}_1}},\ \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{H}}_1}}}[\ T^L{}_{\tilde{\mathcal{H}}_1}, T^U{}_{\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{H}}_1}}}[U^L{}_{\tilde{\mathcal{H}}_1},\ U^U{}_{\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{H}}_1}}}[C^L{}_{\tilde{\mathcal{H}}_1},\ C^U{}_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{H}}_1}}}[F^L{}_{\tilde{\mathcal{H}}_1}, F^U{}_{\tilde{\mathcal{H}}_1}];$

$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{H}}_2}},\ \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{H}}_2}}}[\ T^L{}_{\tilde{\mathcal{H}}_2}, T^U{}_{\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{H}}_2}}}[U^L{}_{\tilde{\mathcal{H}}_2},\ U^U{}_{\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{H}}_2}}}[C^L{}_{\tilde{\mathcal{H}}_2},\ C^U{}_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{H}}_2}}}[F^L{}_{\tilde{\mathcal{H}}_2}, F^U{}_{\tilde{\mathcal{H}}_2}];$

...

$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{H}}_s}},\ \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{H}}_s}}}[\ T^L{}_{\tilde{\mathcal{H}}_s}, T^U{}_{\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{H}}_s}}}[U^L{}_{\tilde{\mathcal{H}}_s},\ U^U{}_{\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{H}}_s}}}[C^L{}_{\tilde{\mathcal{H}}_s},\ C^U{}_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{H}}_s}}}[F^L{}_{\tilde{\mathcal{H}}_s}, F^U{}_{\tilde{\mathcal{H}}_s}];>,$

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$

be two IGSVNQSs. Then, following conditions are satisfied.

i. $\left(\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{H}}\right) \subset' \left(\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}\right) \subset' \left(\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{H}}\right)$

ii. $\left(\tilde{\mathcal{R}} \cup'_P \tilde{\mathcal{H}}\right) \subset' \left(\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}}\right) \subset' \left(\tilde{\mathcal{R}} \cup'_O \tilde{\mathcal{H}}\right)$

iii. $(\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}) \subset' (\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}})$

$(\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{H}}) \subset' (\tilde{\mathcal{R}} \cup'_O \tilde{\mathcal{H}})$

$(\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{H}}) \subset' (\tilde{\mathcal{R}} \cup'_P \tilde{\mathcal{H}})$

Proof:

i. From Definition 3.1.1, Definition 3.2.1, Definition 3.2.3 and Definition 3.2.5, we get

$$\min\{T^L{}_{\tilde{\mathcal{R}}_l}, T^L{}_{\tilde{\mathcal{H}}_l}\} \le \frac{T^L{}_{\tilde{\mathcal{R}}_l} + T^L{}_{\tilde{\mathcal{H}}_l}}{2} \le \max\{T^L{}_{\tilde{\mathcal{R}}_l}, T^L{}_{\tilde{\mathcal{H}}_l}\}$$

$$\min\{T^U{}_{\tilde{\mathcal{R}}_l}, T^U{}_{\tilde{\mathcal{H}}_l}\} \le \frac{T^U{}_{\tilde{\mathcal{R}}_l} + T^U{}_{\tilde{\mathcal{H}}_l}}{2} \le \max\{T^U{}_{\tilde{\mathcal{R}}_l}, T^U{}_{\tilde{\mathcal{H}}_l}\}$$

$$\max\{U^L{}_{\tilde{\mathcal{R}}_l}, U^L{}_{\tilde{\mathcal{H}}_l}\} \ge \frac{U^L{}_{\tilde{\mathcal{R}}_l} + U^L{}_{\tilde{\mathcal{H}}_l}}{2} \ge \min\{U^L{}_{\tilde{\mathcal{R}}_l}, U^L{}_{\tilde{\mathcal{H}}_l}\}$$

$$\max\{U^U{}_{\tilde{\mathcal{R}}_l}, U^U{}_{\tilde{\mathcal{H}}_l}\} \ge \frac{U^U{}_{\tilde{\mathcal{R}}_l} + U^U{}_{\tilde{\mathcal{H}}_l}}{2} \ge \min\{U^U{}_{\tilde{\mathcal{R}}_l}, U^U{}_{\tilde{\mathcal{H}}_l}\}$$

$$\max\ \{C^L{}_{\tilde{\mathcal{R}}_l}, C^L{}_{\tilde{\mathcal{H}}_l}\} \ge \frac{C^L{}_{\tilde{\mathcal{R}}_l} + C^L{}_{\tilde{\mathcal{H}}_l}}{2} \ge \min\{C^L{}_{\tilde{\mathcal{R}}_l}, C^L{}_{\tilde{\mathcal{H}}_l}\}$$

$$\max\{C^U{}_{\tilde{\mathcal{R}}_l}, C^U{}_{\tilde{\mathcal{H}}_l}\} \ge \frac{C^U{}_{\tilde{\mathcal{R}}_l} + C^U{}_{\tilde{\mathcal{H}}_l}}{2} \ge \min\{C^U{}_{\tilde{\mathcal{R}}_l}, C^U{}_{\tilde{\mathcal{H}}_l}\}$$

$$\max\{F^L{}_{\tilde{\mathcal{R}}_l}, F^L{}_{\tilde{\mathcal{H}}_l}\} \ge \frac{F^L{}_{\tilde{\mathcal{R}}_l} + F^L{}_{\tilde{\mathcal{H}}_l}}{2} \ge \min\{F^L{}_{\tilde{\mathcal{R}}_l}, F^L{}_{\mathcal{H}_l}\}$$

$$\max\left\{F^U{}_{\tilde{\mathcal{R}}_l}, F^U{}_{\tilde{\mathcal{H}}_l}\right\} \ge \frac{F^U{}_{\tilde{\mathcal{R}}_l} + F^U{}_{\tilde{\mathcal{H}}_l}}{2} \ge \min\left\{F^U{}_{\mathcal{R}_l}, F^U{}_{\tilde{\mathcal{H}}_l}\right\}. \quad (15)$$

Also,

$$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right),\ \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}\right)$$

$$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}\right),\ \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}\right) \quad (16)$$

Thus, from (15), (16) and Definition 3.1.1, we get

$$\left(\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{H}}\right) \subset' \left(\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}\right) \subset' \left(\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{H}}\right).$$

The proofs of {ii, iii} are similar to i.

Theorem 3.2.11: Let

$\tilde{\mathcal{R}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{R}}_1}},\ \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{R}}_1}}}[\ T^L{}_{\tilde{\mathcal{R}}_1}, T^U{}_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{R}}_1}}}[U^L{}_{\tilde{\mathcal{R}}_1},\ U^U{}_{\tilde{\mathcal{R}}_1}],\ \breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{R}}_1}}}[C^L{}_{\tilde{\mathcal{R}}_1},\ C^U{}_{\tilde{\mathcal{R}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{R}}_1}}}[F^L{}_{\tilde{\mathcal{R}}_1}, F^U{}_{\tilde{\mathcal{R}}_1}];$

$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{R}}_2}},\ \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{R}}_2}}}[\ T^L{}_{\tilde{\mathcal{R}}_2}, T^U{}_{\tilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{R}}_2}}}[U^L{}_{\tilde{\mathcal{R}}_2},\ U^U{}_{\tilde{\mathcal{R}}_2}],\ \breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{R}}_2}}}[C^L{}_{\tilde{\mathcal{R}}_2},\ C^U{}_{\tilde{\mathcal{R}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{R}}_2}}}[F^L{}_{\tilde{\mathcal{R}}_2}, F^U{}_{\tilde{\mathcal{R}}_2}];$

…

$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{R}}_s}},\ \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{R}}_s}}}[\ T^L{}_{\tilde{\mathcal{R}}_s}, T^U{}_{\tilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{R}}_s}}}[U^L{}_{\tilde{\mathcal{R}}_s},\ U^U{}_{\tilde{\mathcal{R}}_s}],\ \breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{R}}_s}}}[C^L{}_{\tilde{\mathcal{R}}_s},\ C^U{}_{\tilde{\mathcal{R}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{R}}_s}}}[F^L{}_{\tilde{\mathcal{R}}_s}, F^U{}_{\tilde{\mathcal{R}}_s}];>,$

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$

and

$\tilde{\mathcal{H}} = \{< \breve{\mathcal{P}}_{1_{\tilde{\mathcal{H}}_1}},\ \breve{\mathcal{P}}_{1_{T_{\tilde{\mathcal{H}}_1}}}[\ T^L{}_{\tilde{\mathcal{H}}_1}, T^U{}_{\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1_{U_{\tilde{\mathcal{H}}_1}}}[U^L{}_{\tilde{\mathcal{H}}_1},\ U^U{}_{\tilde{\mathcal{H}}_1}],\ \breve{\mathcal{P}}_{1_{C_{\tilde{\mathcal{H}}_1}}}[C^L{}_{\tilde{\mathcal{H}}_1},\ C^U{}_{\tilde{\mathcal{H}}_1}], \breve{\mathcal{P}}_{1_{F_{\tilde{\mathcal{H}}_1}}}[F^L{}_{\tilde{\mathcal{H}}_1}, F^U{}_{\tilde{\mathcal{H}}_1}];$

$\breve{\mathcal{P}}_{2_{\tilde{\mathcal{H}}_2}},\ \breve{\mathcal{P}}_{2_{T_{\tilde{\mathcal{H}}_2}}}[\ T^L{}_{\tilde{\mathcal{H}}_2}, T^U{}_{\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2_{U_{\tilde{\mathcal{H}}_2}}}[U^L{}_{\tilde{\mathcal{H}}_2},\ U^U{}_{\tilde{\mathcal{H}}_2}],\ \breve{\mathcal{P}}_{2_{C_{\tilde{\mathcal{H}}_2}}}[C^L{}_{\tilde{\mathcal{H}}_2},\ C^U{}_{\tilde{\mathcal{H}}_2}], \breve{\mathcal{P}}_{2_{F_{\tilde{\mathcal{H}}_2}}}[F^L{}_{\tilde{\mathcal{H}}_2}, F^U{}_{\tilde{\mathcal{H}}_2}];$

…

$\breve{\mathcal{P}}_{s_{\tilde{\mathcal{H}}_s}},\ \breve{\mathcal{P}}_{s_{T_{\tilde{\mathcal{H}}_s}}}[\ T^L{}_{\tilde{\mathcal{H}}_s}, T^U{}_{\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{s_{U_{\tilde{\mathcal{H}}_s}}}[U^L{}_{\tilde{\mathcal{H}}_s},\ U^U{}_{\tilde{\mathcal{H}}_s}],\ \breve{\mathcal{P}}_{s_{C_{\tilde{\mathcal{H}}_s}}}[C^L{}_{\tilde{\mathcal{H}}_s},\ C^U{}_{\tilde{\mathcal{H}}_s}], \breve{\mathcal{P}}_{s_{F_{\tilde{\mathcal{H}}_s}}}[F^L{}_{\tilde{\mathcal{H}}_s}, F^U{}_{\tilde{\mathcal{H}}_s}];>,$

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}, \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}, \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}} \in P(\mathcal{N}); l = 1,2,3, \dots, s\}$

be two IGSVNQSs and let $(\tilde{\mathcal{R}} \subset' \tilde{\mathcal{H}})$. Then, following conditions are satisfied.

i. $\tilde{\mathcal{R}} \subset' (\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}) \subset' \tilde{\mathcal{H}}$

$\tilde{\mathcal{R}} \subset' \left(\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{H}}\right) =' \tilde{\mathcal{H}}$

$\tilde{\mathcal{R}} =' (\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{H}}) \subset' \tilde{\mathcal{H}}$

ii. $\tilde{\mathcal{R}} \subset' (\tilde{\mathcal{R}} \cup'_M \tilde{\mathcal{H}}) \subset' \tilde{\mathcal{H}}$

$\tilde{\mathcal{R}} \subset' \left(\tilde{\mathcal{R}} \cup'_O \tilde{\mathcal{H}}\right) =' \tilde{\mathcal{H}}$

$\tilde{\mathcal{R}} =' (\tilde{\mathcal{R}} \cup'_P \tilde{\mathcal{H}}) \subset' \tilde{\mathcal{H}}$

Proof:

i. From Definition 3.1.1, we get

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \subset \breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} \subset \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}$, $\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \subset \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} \subset \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}$, $\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \subset \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} \subset \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}$

$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \subset \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} \subset \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}$, $\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \subset \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} \subset \breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}$

$T^L{}_{\tilde{\mathcal{R}}_l} \le T^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \le T^L{}_{\tilde{\mathcal{H}}_l}$, $T^U{}_{\tilde{\mathcal{R}}_l} \le T^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \le T^U{}_{\tilde{\mathcal{H}}_l}$

$U^L{}_{\tilde{\mathcal{R}}_l} \ge U^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \ge U^L{}_{\tilde{\mathcal{H}}_l}$, $U^U{}_{\tilde{\mathcal{R}}_l} \ge U^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \ge U^U{}_{\tilde{\mathcal{H}}_l}$

$C^L{}_{\tilde{\mathcal{R}}_l} \ge C^L{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \ge C^L{}_{\tilde{\mathcal{H}}_l}$, $C^U{}_{\tilde{\mathcal{R}}_l} \ge C^U{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \ge C^U{}_{\tilde{\mathcal{H}}_l}$

$F^{L}{}_{\tilde{\mathcal{R}}_l} \geq F^{L}{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \geq F^{L}{}_{\tilde{\mathcal{H}}_l}$,$F^{U}{}_{\tilde{\mathcal{R}}_l} \geq F^{U}{}_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l} \geq$
$F^{U}{}_{\tilde{\mathcal{H}}_l}$.

From Definition 3.2.1, we get

$\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}} = \left(\breve{\mathcal{P}}_{l_{\tilde{\mathcal{R}}_l}} \cap \breve{\mathcal{P}}_{l_{\tilde{\mathcal{H}}_l}}\right)$, $\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{T_{\tilde{\mathcal{H}}_l}}}\right)$, $\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{U_{\tilde{\mathcal{H}}_l}}}\right)$

$\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{R}}_l}}} \cap \breve{\mathcal{P}}_{l_{C_{\tilde{\mathcal{H}}_l}}}\right)$, $\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l\tilde{\mathcal{H}}_l}}} = \left(\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{R}}_l}}} \cap\right.$
$\left.\breve{\mathcal{P}}_{l_{F_{\tilde{\mathcal{H}}_l}}}\right)$.

Thus, from the proof of (i) in Theorem 3.2.10, conditions (15), Definition 3.1.1, Definition 3.2.1, Definition 3.2.3, Definition 3.2.5, (17) and (18), we get

$$\tilde{\mathcal{R}} \subset' (\tilde{\mathcal{R}} \cap'_M \tilde{\mathcal{H}}) \subset' \tilde{\mathcal{H}}$$

$$\tilde{\mathcal{R}} \subset' \left(\tilde{\mathcal{R}} \cap'_O \tilde{\mathcal{H}}\right) =' \tilde{\mathcal{H}}$$

$$\tilde{\mathcal{R}} =' \left(\tilde{\mathcal{R}} \cap'_P \tilde{\mathcal{H}}\right) \subset' \tilde{\mathcal{H}}.$$

The proof of {ii} is similar to i.

4. Application

4.1 Algorithm and Example for Operators Based on Interval Generalized Set-Valued Neutrosophic Quintuple Set

In this Section, an algorithm and its application are given simultaneously in an example to demonstrate the use of the operators obtained in the study in a decision-making application. Thanks to this application, researchers will be able to use operators to solve decision-making problems.

Example 4.2: A decision-making exercise will be carried out in which three experts evaluate three alternatives on the basis of seven different criteria.

Step 1: Experts, alternatives and criteria required for the application are determined.

Let $\boldsymbol{\mathcal{D}} = \{\breve{\boldsymbol{\mathcal{D}}}_1, \breve{\boldsymbol{\mathcal{D}}}_2, \breve{\boldsymbol{\mathcal{D}}}_3\}$ be the set of experts.

The criteria to be used in the application are determined. Set of criteria

Let $\boldsymbol{\mathcal{Y}} = \{\ddot{\boldsymbol{y}}_1, \ddot{\boldsymbol{y}}_2, \ddot{\boldsymbol{y}}_3, \ddot{\boldsymbol{y}}_4, \ddot{\boldsymbol{y}}_5, \ddot{\boldsymbol{y}}_6, \ddot{\boldsymbol{y}}_7\}$ be the set of criterias.

Alternatives are identified. Let $\boldsymbol{\mathcal{S}} = \{\dot{\boldsymbol{\mathcal{S}}}^1, \dot{\boldsymbol{\mathcal{S}}}^2, \dot{\boldsymbol{\mathcal{S}}}^3\}$ be the set of alternatives.

Step 2: Opinions of experts about the alternatives are obtained.

Experts' opinions about an alternative are expressed using IGSVNQS and denoted by $\dot{\mathcal{S}}^{i}{}_{\breve{\mathcal{D}}_j}$.Where $\boldsymbol{i},\boldsymbol{j} = \{\mathbf{1},\mathbf{2},\mathbf{3}\}$, $\boldsymbol{i}$ stands for alternatives and $\boldsymbol{j}$ stands for experts. Opinions of experts on alternatives are expressed using IGSVNQS and denoted by $\widetilde{\mathcal{G}}_{\breve{\mathcal{D}}_j}$.

The IGSVNQS expressing the opinion of expert $\breve{\mathcal{D}}_1$ is denoted by $\widetilde{\mathcal{G}}_{\breve{\mathcal{D}}_1}$.This set is

$$\widetilde{\mathcal{G}}_{\breve{\mathcal{D}}_1} = \left\{\begin{array}{c} \dot{\mathcal{S}}^1{}_{\breve{\mathcal{D}}_1} = \{<\{\ddot{y}_1,\ddot{y}_2,\ddot{y}_3,\ddot{y}_4,\ddot{y}_5,\ddot{y}_6,\ddot{y}_7\},\{\ddot{y}_1,\ddot{y}_3,\ddot{y}_4,\ddot{y}_7\}[0.7,0.8],\{\ddot{y}_3,\ddot{y}_4,\ddot{y}_7\}[0.4,0.5], \\ \{\ddot{y}_1,\ddot{y}_2\}[0.55,0.6],\{\ddot{y}_5,\ddot{y}_6\}[0.3,0.45]>\} \\ \dot{\mathcal{S}}^2{}_{\breve{\mathcal{D}}_1} = \{<\{\ddot{y}_1,\ddot{y}_2,\ddot{y}_3,\ddot{y}_4,\ddot{y}_5,\ddot{y}_6,\ddot{y}_7\},\{\ddot{y}_1,\ddot{y}_2,\ddot{y}_5\}[0.7,0.85],\{\ddot{y}_2,\ddot{y}_4,\ddot{y}_5\}[0.25,0.5], \\ \{\ddot{y}_1,\ddot{y}_2\}[0.45,0.7],\{\ddot{y}_3,\ddot{y}_6,\ddot{y}_7\}[0.1,0.2]>\} \\ \dot{\mathcal{S}}^3{}_{\breve{\mathcal{D}}_1} = \{<\{\ddot{y}_1,\ddot{y}_2,\ddot{y}_3,\ddot{y}_4,\ddot{y}_5,\ddot{y}_6,\ddot{y}_7\},\{\ddot{y}_1,\ddot{y}_6,\ddot{y}_7\}[0.5,0.65],\{\ddot{y}_1,\ddot{y}_2,\ddot{y}_6\}[0.35,0.5], \\ \{\ddot{y}_3,\ddot{y}_5,\ddot{y}_6\}[0.4,0.6],\{\ddot{y}_4,\ddot{y}_7\}[0.2,0.3]>\} \end{array}\right\}$$

The expressing the opinion of the expert $\breve{\mathcal{D}}_2$ is denoted by the IGSVNQS $\widetilde{\mathcal{G}}_{\breve{\mathcal{D}}_2}$. This set is

$$\widetilde{\mathcal{G}}_{\breve{\mathcal{D}}_2} = \left\{\begin{array}{c} \dot{\mathcal{S}}^1{}_{\breve{\mathcal{D}}_2} = \{<\{\ddot{y}_1,\ddot{y}_2,\ddot{y}_3,\ddot{y}_4,\ddot{y}_6,\ddot{y}_7\},\{\ddot{y}_2,\ddot{y}_3,\ddot{y}_4\}[0.5,0.75],\{\ddot{y}_1,\ddot{y}_4,\ddot{y}_7\}[0.35,0.6], \\ \{\ddot{y}_1,\ddot{y}_2,\ddot{y}_7\}[0.4,0.7],\{\ddot{y}_3,\ddot{y}_6\}[0.1,0.25]>\} \\ \dot{\mathcal{S}}^2{}_{\breve{\mathcal{D}}_2} = \{<\{\ddot{y}_1,\ddot{y}_2,\ddot{y}_3,\ddot{y}_4,\ddot{y}_5,\ddot{y}_7\},\{\ddot{y}_2,\ddot{y}_4,\ddot{y}_5,\ddot{y}_7\}[0.5,0.9],\{\ddot{y}_2,\ddot{y}_3,\ddot{y}_5\}[0.2,0.4], \\ \{\ddot{y}_2,\ddot{y}_4\}[0.3,0.5],\{\ddot{y}_1,\ddot{y}_3,\ddot{y}_7\}[0.15,0.3]>\} \\ \dot{\mathcal{S}}^3{}_{\breve{\mathcal{D}}_2} = \{<\{\ddot{y}_1,\ddot{y}_3,\ddot{y}_5,\ddot{y}_6,\ddot{y}_7\},\{\ddot{y}_5,\ddot{y}_6\}[0.7,0.95],\{\ddot{y}_1,\ddot{y}_6\}[0.3,0.45], \\ \{\ddot{y}_3,\ddot{y}_5,\ddot{y}_7\}[0.3,0.5],\{\ddot{y}_1,\ddot{y}_7\}[0.15,0.2]>\} \end{array}\right\}$$

The expressing the opinion of the expert $\breve{\mathcal{D}}_3$ is denoted by the The expressing the opinion of the expert $\breve{\mathcal{D}}_2$ is denoted by the IGSVNQS $\widetilde{\mathcal{G}}_{\breve{\mathcal{D}}_2}$. This set is

$$\widetilde{\mathcal{G}}_{\breve{\mathcal{D}}_3} = \left\{\begin{array}{c} \dot{\mathcal{S}}^1{}_{\breve{\mathcal{D}}_3} = \{<\{\ddot{y}_1,\ddot{y}_3,\ddot{y}_4,\ddot{y}_6,\ddot{y}_7\},\{\ddot{y}_1,\ddot{y}_3,\ddot{y}_4\}[0.6,0.9],\{\ddot{y}_1,\ddot{y}_3,\ddot{y}_7\}[0.3,0.55], \\ \{\ddot{y}_1,\ddot{y}_7\}[0.3,0.5],\{\ddot{y}_4,\ddot{y}_6\}[0.2,0.5]>\} \\ \dot{\mathcal{S}}^2{}_{\breve{\mathcal{D}}_3} = \{<\{\ddot{y}_1,\ddot{y}_2,\ddot{y}_3,\ddot{y}_4,\ddot{y}_5,\ddot{y}_6\},\{\ddot{y}_2,\ddot{y}_4,\ddot{y}_6\}[0.8,1.0],\{\ddot{y}_2,\ddot{y}_3,\ddot{y}_4,\ddot{y}_5\}[0.1,0.6], \\ \{\ddot{y}_2,\ddot{y}_4,\ddot{y}_6\}[0.2,0.6],\{\ddot{y}_1,\ddot{y}_3\}[0.2,0.35]>\} \\ \dot{\mathcal{S}}^3{}_{\breve{\mathcal{D}}_3} = \{<\{\ddot{y}_1,\ddot{y}_2,\ddot{y}_3,\ddot{y}_5,\ddot{y}_6,\ddot{y}_7\},\{\ddot{y}_1,\ddot{y}_4,\ddot{y}_7\}[0.6,0.8],\{\ddot{y},\ddot{y}_6\}[0.4,0.5], \\ \{\ddot{y}_5,\ddot{y}_7\}[0.35,0.55],\{\ddot{y}_4,\ddot{y}_7\}[0.3,0.45]>\} \end{array}\right\}$$

Step 3: The opinions of the experts are expressed as a common opinion using one operators which is obtained Section 3.

Using the mean intersection operator in Definition 3.2.1, the consensus opinion of the experts is expressed as an IGSVNQS. It is denoted by $\widetilde{\mathcal{G}}_{\cap'_M}$ and calculated such that

$$\widetilde{\mathcal{G}}_{\cap'_M} = \left(\widetilde{\mathcal{G}}_{\breve{\mathcal{D}}_1} \cap \left(\widetilde{\mathcal{G}}_{\breve{\mathcal{D}}_2} \cap \widetilde{\mathcal{G}}_{\breve{\mathcal{D}}_3}\right)\right) = (\dot{\mathcal{S}}^1{}_{\cap'_M},\ \dot{\mathcal{S}}^2{}_{\cap'_M},\ \dot{\mathcal{S}}^3{}_{\cap'_M})$$

Also, we calculate

$$\dot{\mathcal{S}}^1{}_{\cap'_M} = \left(\dot{\mathcal{S}}^1{}_{\breve{\mathcal{D}}_1} \cap (\dot{\mathcal{S}}^1{}_{\breve{\mathcal{D}}_2} \cap \dot{\mathcal{S}}^1{}_{\breve{\mathcal{D}}_3})\right)$$

$$= \{<\{\ddot{y}_1,\ddot{y}_3,\ddot{y}_4,\ddot{y}_6,\ddot{y}_7\},\{\ddot{y}_3,\ddot{y}_4,\ddot{y}_7\}[0.62,0.81],\{\ddot{y}_4,\ddot{y}_7\}[0.36,0.53], \{\ddot{y}_1\}[0.45,0.6],\{\ddot{y}_6\}[0.22,0.41]>\}$$

such that

$$\{\ddot{y}_1,\ddot{y}_3,\ddot{y}_4,\ddot{y}_6,\ddot{y}_7\} = \left(\{\ddot{y}_1,\ddot{y}_2,\ddot{y}_3,\ddot{y}_4,\ddot{y}_5,\ddot{y}_6,\ddot{y}_7\} \cap (\{\ddot{y}_1,\ddot{y}_2,\ddot{y}_3,\ddot{y}_4,\ddot{y}_6,\ddot{y}_7\} \cap \{\ddot{y}_1,\ddot{y}_3,\ddot{y}_4,\ddot{y}_6,\ddot{y}_7\})\right)$$

$$\{\ddot{y}_3, \ddot{y}_4, \ddot{y}_7\} = (\{\ddot{y}_1, \ddot{y}_3, \ddot{y}_4, \ddot{y}_7\} \cap (\{\ddot{y}_2, \ddot{y}_3, \ddot{y}_4\} \cap \{\ddot{y}_1, \ddot{y}_3, \ddot{y}_4\}))$$

$$\{\ddot{y}_4, \ddot{y}_7\} = (\{\ddot{y}_3, \ddot{y}_4, \ddot{y}_7\} \cap (\{\ddot{y}_1, \ddot{y}_4, \ddot{y}_7\} \cap \{\ddot{y}_1, \ddot{y}_3, \ddot{y}_7\}))$$

$$\{\ddot{y}_1\} = (\{\ddot{y}_1, \ddot{y}_2\} \cap (\{\ddot{y}_1, \ddot{y}_2, \ddot{y}_7\} \cap \{\ddot{y}_1, \ddot{y}_7\}))$$

$$\{\ddot{y}_6\} = (\{\ddot{y}_5, \ddot{y}_6\} \cap (\{\ddot{y}_3, \ddot{y}_6\} \cap \{\ddot{y}_4, \ddot{y}_6\}))$$

$$0.62 = \frac{0.7 + \frac{0.5 + 0.6}{2}}{2}, 0.81 = \frac{0.8 + \frac{0.75 + 0,9}{2}}{2},$$

$$0.36 = \frac{0.4 + \frac{0.35 + 0.3}{2}}{2}, 0.53 = \frac{0.5 + \frac{0.6 + 0.55}{2}}{2}$$

$$0.45 = \frac{0.55 + \frac{0.4 + 0.3}{2}}{2}, 0.6 = \frac{0.6 + \frac{0.7 + 0.5}{2}}{2},$$

$$0.22 = \frac{0.3 + \frac{0.1 + 0.2}{2}}{2}, 0.41 = \frac{0.45 + \frac{0.25 + 0.5}{2}}{2}$$

Similarly, $\dot{S}^2{}_{\cap'_M}$ and $\dot{S}^3{}_{\cap'_M}$ are obtained such that

$$\tilde{\mathcal{G}}_{\cap'_M} = \begin{Bmatrix} \dot{S}^1{}_{\cap'_M} = \{< \{\ddot{y}_1, \ddot{y}_3, \ddot{y}_4, \ddot{y}_6, \ddot{y}_7\}, \{\ddot{y}_3, \ddot{y}_4, \ddot{y}_7\}[0.62, 0.81], \{\ddot{y}_4, \ddot{y}_7\}[0.36, 0.53], \\ \{\ddot{y}_1\}[0.45, 0.6], \{\ddot{y}_6\}[0.22, 0.41] >\} \\ \dot{S}^2{}_{\cap'_M} = \{< \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_3, \ddot{y}_4, \ddot{y}_5\}, \{\ddot{y}_2\}[0.67, 0.9], \{\ddot{y}_2, \ddot{y}_5\}[0.2, 0.5], \{\ddot{y}_2\} \\ [0.35, 0.62], \{\ddot{y}_3\}[0.13, 0.26] >\} \\ \dot{S}^3{}_{\cap'_M} = \{< \{\ddot{y}_1, \ddot{y}_3, \ddot{y}_4, \ddot{y}_5\}, \emptyset[0.57, 0.76], \{\ddot{y}_6\}[0.35, 0.48], \{\ddot{y}_5\}[0.36, 0.56], \\ \{\ddot{y}_7\}[0.21, 0.31] >\} \end{Bmatrix}$$

Using the optimistic intersection operator in Definition 3.2.3, the consensus opinions of the experts are expressed as an IGSVNQS and denoted by $\tilde{\mathcal{G}}_{\cap'_O}$ and calculated by

$$\tilde{\mathcal{G}}_{\cap'_O} = (\tilde{\mathcal{G}}_{\mathcal{D}_1} \cap (\tilde{\mathcal{G}}_{\mathcal{D}_2} \cap \tilde{\mathcal{G}}_{\mathcal{D}_3}))$$

$$= \begin{Bmatrix} \dot{S}^1{}_{\cap'_O} = \{< \{\ddot{y}_1, \ddot{y}_3, \ddot{y}_4, \ddot{y}_6, \ddot{y}_7\}, \{\ddot{y}_3, \ddot{y}_4, \ddot{y}_7\}[0.7, 0.9], \{\ddot{y}_4, \ddot{y}_7\}[0.3, 0.5], \\ \{\ddot{y}_1\}[0.3, 0.5], \{\ddot{y}_6\}[0.1, 0.25] >\} \\ \dot{S}^2{}_{\cap'_O} = \{< \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_3, \ddot{y}_4, \ddot{y}_5\}, \{\ddot{y}_2\}[0.8, 1.0], \{\ddot{y}_2, \ddot{y}_5\}[0.1, 0.4], \\ \{\ddot{y}_2\}[0.2, 0.5], \{\ddot{y}_3\}[0.1, 0.2] >\} \\ \dot{S}^3{}_{\cap'_O} = \{< \{\ddot{y}_1, \ddot{y}_3, \ddot{y}_4, \ddot{y}_5\}, \emptyset[0.7, 0.95], \{\ddot{y}_6\}[0.1, 0.45], \{\ddot{y}_5\}[0.3, 0.5], \\ \{\ddot{y}_7\}[0.15, 0.2] >\} \end{Bmatrix}$$

Using the pessimistic union operatör in Definition 3.2.5, the consensus opinions of experts are expressed as an IGSVNQS is denoted by $\tilde{\mathcal{G}}_{\cup'_K}$ and is calculated by

$$\tilde{\mathcal{G}}_{\cup'_P} = (\tilde{\mathcal{G}}_{\mathcal{D}_1} \cup (\tilde{\mathcal{G}}_{\mathcal{D}_2} \cup \tilde{\mathcal{G}}_{\mathcal{D}_3}))$$

$$= \left\{ \begin{array}{c} \dot{S}^1{}_{\cup'_P} = \{< \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_3, \ddot{y}_4, \ddot{y}_5, \ddot{y}_6, \ddot{y}_7\}, \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_3, \ddot{y}_4, \ddot{y}_7\}[0.5, 0.75], \\ \{\ddot{y}_1, \ddot{y}_3, \ddot{y}_4, \ddot{y}_7\}[0.4, 0.6], \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_7\}[0.55, 0.7], \{\ddot{y}_4, \ddot{y}_5, \ddot{y}_6\}[0.3, 0.5] >\} \\ \dot{S}^2{}_{\cup'_P} = \{< \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_3, \ddot{y}_4, \ddot{y}_5, \ddot{y}_6, \ddot{y}_7\}, \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_4, \ddot{y}_5, \ddot{y}_6, \ddot{y}_7\}[0.5, 0.85], \\ \{\ddot{y}_2, \ddot{y}_3, \ddot{y}_4, \ddot{y}_5\}[0.25, 0.6], \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_4, \ddot{y}_6\}[0.45, 0.7], \{\ddot{y}_1, \ddot{y}_3, \ddot{y}_6, \ddot{y}_7\}[0.2, 0.35] >\} \\ \dot{S}^3{}_{\cup'_P} = \{< \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_3, \ddot{y}_4, \ddot{y}_5, \ddot{y}_6, \ddot{y}_7\}, \{\ddot{y}_1, \ddot{y}_3, \ddot{y}_4, \ddot{y}_5, \ddot{y}_6, \ddot{y}_7\}[0.5, 0.65], \\ \{\ddot{y}_1, \ddot{y}_2, \ddot{y}_6\}[0.4, 0.5], \{\ddot{y}_3, \ddot{y}_5, \ddot{y}_6, \ddot{y}_7\}[0.4, 0.6], \{\ddot{y}_1, \ddot{y}_4, \ddot{y}_7\}[0.3, 0.45] >\} \end{array} \right\}$$

Step 4: The obtained consensus is expressed as net value by using score and accuracy functions.

The score values of the common opinion obtained by the $\tilde{\mathcal{G}}_{\cap'_M}$ are calculated and the net values obtained by the mean intersection operator for the alternatives are given in Table 1.

Table 1: Net Values of The Views Obtained with The Mean Intercept Operator

Alternative	Net Value
$\dot{S}^1{}_{\cap'_M}$	1.09
$\dot{S}^2{}_{\cap'_M}$	1.44
$\dot{S}^3{}_{\cap'_M}$	1.42

The score values of the common opinion obtained by the $\tilde{\mathcal{G}}_{\cap'_O}$ are calculated and the net values obtained by the optimistic intersection operator for the alternatives are given in Table 2.

Table 2: Net Values of The Opinions Obtained by Optimistic Intersection Operation

Alternative	Net Value
$\dot{S}^1{}_{\cap'_O}$	1.08
$\dot{S}^2{}_{\cap'_O}$	1.7
$\dot{S}^3{}_{\cap'_O}$	1.48

The score values of the common opinion obtained by the $\tilde{\mathcal{G}}_{\cup'_P}$ are calculated as follows

$$S_D(\dot{S}^1{}_{\cup'_P}) = \frac{|5-4-3-3|}{7} + \frac{|0.5-0.4-0.55-0.3|}{3} - \frac{|0.75-0.6-0.7-0.5|}{3} + 1 = 1,61$$

$$S_D(\dot{S}^2{}_{\cup'_P}) = \frac{|6-4-4-4|}{7} + \frac{|0.5-0.25-0.45-0.2|}{3} - \frac{|0.85-0.6-0.7-0.35|}{3} + 1 = 1,72$$

$$S_D(\dot{S}^3{}_{\cup'_P}) = \frac{|5-3-4-3|}{7} + \frac{|0.5-0.4-0.4-0.3|}{3} - \frac{|0.65-0.5-0.6-0.45|}{3} + 1 = 1.61$$

Where $S_D(\dot{S}^1{}_{\cup'_K})$ and $S_D(\dot{S}^3{}_{\cup'_K})$ score values are equal. Thus, the accuracy values are calculated as follows

$$K_D(\dot{S}^1{}_{\cup'_P}) = \frac{|1+2+4|}{2*5} + 0.5 + 0.75 = \frac{7}{10} + 0.5 + 0.75 = 1.95$$

$$K_D(\dot{S}^3{}_{\cup'_P}) = \frac{|3+2+2|}{2*5} + 0.5 + 0.65 = \frac{7}{10} + 0.5 + 0.65 = 1.85$$

the net values are obtained with the pessimistic union operator for the alternatives in Table 3.

Table 3: Net Values of The Opinions Obtained with The Pessimistic Union Operator

Alternative	Net Value
$\dot{S}^1{}_{\cup'_P}$	1.95
$\dot{S}^2{}_{\cup'_P}$	1.72
$\dot{S}^3{}_{\cup'_P}$	1.85

Step 5: Rankings are made among the alternatives with the net values obtained.

i) The ranking of the alternatives with the common opinion obtained by using the mean intersection operator from Table 1,

$$\dot{S}^2 > \dot{S}^3 > \dot{S}^1.$$

According to this ranking, alternative $\dot{S}^2$ is the best alternative while alternative $\dot{S}^1$ is the least good alternative.

ii) The ranking of the alternatives with the common opinion obtained by using the optimistic intersection operatör from Table 2,

$$\dot{S}^2 > \dot{S}^3 > \dot{S}^1.$$

According to this ranking, alternative $\dot{S}^2$ is the best alternative while alternative $\dot{S}^1$ is the least good alternative.

iii) The ranking of the alternatives with the common opinion obtained by using the pessimistic union operator from Table 3,

$$\dot{S}^1 > \dot{S}^3 > \dot{S}^2.$$

According to this ranking, alternative $\dot{S}^1$ is the best alternative while alternative $\dot{S}^2$ is the least good alternative.

4.2 Comparision Method

From Step 5 in Section 4.1, we obtain Table 4.

Table 4: Rankings Obtained from The Operators

Operator	Obtained Sequence
Mean Intersection	$\dot{S}^2 > \dot{S}^3 > \dot{S}^1$
Optimistic Intersection	$\dot{S}^2 > \dot{S}^3 > \dot{S}^1$
Pessimistic Union	$\dot{S}^1 > \dot{S}^3 > \dot{S}^2$

It is clearly seen in Table 4 that similar rankings are obtained with the mean intersection and optimistic intersection operators, but the ranking obtained with the pessimistic intersection operator is different from these rankings. These results show that different operators can give similar or different results on the same data sets. For example, the alternative $\dot{S}^2$, which was determined as the best alternative when the mean intersection and optimistic intersection operators were used, was determined as the worst alternative when the pessimistic union operator was used. Similarly, the alternative $\dot{S}^1$ was determined as the best alternative when the pessimistic union operator was used and as the worst alternative when the mean intersection and optimistic intersection operators were used. Therefore, the best alternative according to one operator may be the worst alternative according to another operator. This shows that similar or different results can be obtained in the ranking of alternatives by using different operators. In this context, it should be kept in mind that each operator represents different operations and these operations may give different results according to the problem domain. Therefore, it is necessary to make an important decision about which operator to use in the decision-making process. In the process of determining the operator to be used, the characteristics of the problem domain and the objectives should be taken into consideration. Because the selected operator can significantly affect the results. In particular, the selection of the most appropriate operator is of great importance for the accuracy of the decision-making process and the reliability of the results. Therefore, careful consideration should be given to operator selection.

5. Conclusion

In this study, some structures related to IGSVNQS and score and certainty functions are defined. Moreover, optimistic union, pessimistic union, mean union, optimistic intersection, pessimistic intersection and mean intersection operators are defined for IGSVNQS. Thus, **IGSVNQS** are made available for decision making applications. An example application was made with these operators and functions and how to use operators and functions in decision making applications was shown through the example application. When different operators are used in decision making applications, similar or different results can be obtained. Therefore, the operator to be used in decision making applications should be selected by considering the characteristics and objectives of the problem domain. Because the selected operator can greatly affect the results. Researchers can obtain more objective results by using the operators and functions in this study in decision-making applications. Also, by using IGSVNQS and operators, distance and similarity measures based on IGSVNQS can be defined for decision making problems. They can also define arithmetic operations for some decision making applications (VIKOR, ELECTRE, PROMETHEE, ...).

Abbreviations

SVNS: Single Valued Neutrosophic Set

IQNS: Interval Quadripartitioned Neutrosophic Set

INS: Interval Neutrosophic Set

ISVNQN: Interval Set Valued Neutrosophic Quadruple Number

ISVNQS: Interval Set Valued Neutrosophic Quadruple Set

IGSVNQN: Generalized Set Valued Neutrosophic Quadruple Number

IGSVNQS: Generalized Set Valued Neutrosophic Quadruple Set

IGSVNQN: Interval Generalized Set Valued Neutrosophic Quadruple Number

IGSVNQS: Interval Generalized Set Valued Neutrosophic Quadruple Set

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