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Neutrosophic Regular Rings and Properties of Their Ideals

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Abstract: The goal of the research is to discover the required and sufficient conditions for $R(I)$ to be a neutrosophic regular as well as the properties of the neutrosophic ideals in the neutrosophic regular ring. Despite differences in the form of the elements and ideals in neutrosophic and classical rings, we discovered that the necessary and sufficient requirements for the classical ring to be regular also apply to neutrosophic rings.

Keywords: Neutrosophic ring, Neutrosophic regular, center, Neutrosophic ideal.

1. Introduction

The notion of a regular element within rings was introduced by J. Von Neumann [1]. Since then, regular rings and their properties have been comprehensively studied by numerous scholars and researchers [2,3,4].

Neutosophy signifies an advanced interpretation of intuitionistic fuzzy logic. This concept significantly influences decision making [5], and medical studies [6]. Further applications of neutrosophy are detailed in [7,8,9,10,11].

Neutosophy, as a novel area of philosophy, may be used in algebraic structures, leading to a greater understanding and evolution of these structures. Smarandache initially proposed the neutrosophic idea in 1980. Neutrosophic structures, a new concept in algebraic structures, have many applications and discoveries such as neutrosophic topologies [12, 13], and rings [14,15].

Kandasamy and Smarandache introduced neutrosophic groups, rings, and fields [16], which has been widely investigated [17,18,19,20], and is still being studied.

Numerous intriguing discoveries about neutrosophic rings have recently been discussed [21,22,23,24,25]. This study is still ongoing among researchers. Alabdullah discussed the basic features of regular neutrosophic elements and their relationship with the elements of the neutrosophic ring [21].

In our research, we present the required and sufficient conditions for neutrosophic ideals for $R(I)$ to be neutrosophic regular.

Our motivation is to the display of properties determining properties of neutrosophic ideals in neutrosophic regular rings.

2. Definitions and notations

Since academics conversant in classical regular rings possess comprehensive insight into their properties, this section encompasses definitions and essential findings concerning neutrosophic rings. Those interested in discovering more about neutrosophic rings can investigate the reference list.

Definition 2.1 [16] Supposing $\boldsymbol{R}$ is a ring. The set $\boldsymbol{R(I)} = \{\boldsymbol{a} + \boldsymbol{bI}\,; \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}$ **and** $\boldsymbol{I^2} = \boldsymbol{I}\}$, is known as a neutrosophic ring. When $\boldsymbol{R}$ is a field, $\boldsymbol{R(I)}$ is referred to as a neutrosophic field.

Properties 2.2 [16,17]

1) $\boldsymbol{R}$ is a unity commutative ring iff $\boldsymbol{R(I)}$ is a unity commutative neutrosophic ring with neutrosophic unity I.

2) $\boldsymbol{I^m} = \boldsymbol{I}$ for every $\boldsymbol{m} \in \mathbb{Z}^+$

3) $\boldsymbol{aI} = \boldsymbol{Ia}\ \forall \boldsymbol{a} \in \boldsymbol{R}$.

4) $\boldsymbol{0I} = \boldsymbol{0}$, $\underbrace{\boldsymbol{I} + \boldsymbol{I} + \boldsymbol{I} + \ldots\ . + \boldsymbol{I}}_{m\ time} = \boldsymbol{mI}$

Definition 2.3 [16,17] In any neutrosophic ring $\boldsymbol{R(I)}$, an element $\boldsymbol{e}$ that satisfies $\boldsymbol{e^2} = \boldsymbol{e}$ is considered a neutrosophic idempotent.

Theorem 2.4 [25] An element $\boldsymbol{a} + \boldsymbol{bI} \in \boldsymbol{R(I)}$ is idempotent iff $\boldsymbol{a}$ is idempotent and $\boldsymbol{b} = \boldsymbol{c} - \boldsymbol{a}$, where $\boldsymbol{c} \in \boldsymbol{R}$ is an idempotent.

Definition 2.5 [16] The neutrosophic ring's center is defined as $\mathrm{C}(R(I)) = \{x \in R(I) \mid yx = xy$ for all $y \in R(I)\}$.

Theorem 2.6 [23]

If $J + KI \subseteq R(\mathrm{I})$, then $J + KI$ is a neutrosophic ideal iff J *and* K are ideals in R, where $J \subseteq K$.

Definition 2.7 [21] An element $x \in R(\mathrm{I})$ is called neutrosophic regular if there is an element y where $x = xyx$. $R(\mathrm{I})$ is neutrosophic regular when all its elements are regular.

3. Results

In this section, all neutrosophic rings are unity. In the $R(I)$, we denote the collection of neutrosophic regular elements as $NReg_{R(I)}$, $NId_{R(I)}$ is the set of neutrosophic idempotent elements, $NC_{R(I)}$ is the center of $R(I)$, and $N\mathfrak{I}_{R(I)}$ the set of neutrosophic ideals. In classical ring R, let Reg_R denote the regular elements, Id_R the idempotents, C_R the center, and $\mathfrak{I}_R$ the ideals.

Theorem 3.1 In R(I), $\forall r_1 + r_2 I \in NReg_{R(I)} \Leftrightarrow r_1 \; and \; r_1 + r_2 \in Reg_R$

Proof.

(⇒): If $r_1 + r_2 I \in NReg_{R(I)}$, then $\exists x + yI \in R$(I), where, $r_1 + r_2 I = (r_1 + r_2 I)(x + yI)(r_1 + r_2 I) = r_1 x r_1 + [r_1 x r_2 + r_1 y r_1 + r_1 y r_2 + r_2 x r_1 + r_2 x r_2 + r_2 y r_1 + r_2 y r_2]I$

So $r_1 = r_1 x r_1 \Rightarrow r_1 \in Reg_R$

And $r_2 = r_1 x r_2 + r_1 y r_1 + r_1 y r_2 + r_2 x r_1 + r_2 x r_2 + r_2 y r_1 + r_2 y r_2$

$\Rightarrow r_1 + r_2 = r_1 x r_1 + r_1 x r_2 + r_1 y r_1 + r_1 y r_2 + r_2 x r_1 + r_2 x r_2 + r_2 y r_1 + r_2 y r_2 = (r_1 + r_2)(x + y)(r_1 + r_2)$.

Thus $r_1 + r_2 \in Reg_R$

(⇐): If $r_1 \; and \; r_1 + r_2 \in Reg_R$, then $\exists x, r \in R$, where $r_1 = r_1 x r_1 \; and \; r_1 + r_2 = (r_1 + r_2) r (r_1 + r_2)$

Firstly, Let's put $y = r - x \in R \Rightarrow r = x + y$. Subsequently, $r_1 + r_2 = (r_1 + r_2)(x + y)(r_1 + r_2)$

$$\Rightarrow r_1 + r_2 = r_1 x r_1 + r_1 x r_2 + r_1 y r_1 + r_1 y r_2 + r_2 x r_1 + r_2 x r_2 + r_2 y r_1 + r_2 y r_2 \Rightarrow r_2 = r_1 x r_2 + r_1 y r_1 + r_1 y r_2 + r_2 x r_1 + r_2 x r_2 + r_2 y r_1 + r_2 y r_2$$

At other hand, we have $r_1 + r_2 I = r_1 x r_1 + [r_1 x r_2 + r_1 y r_1 + r_1 y r_2 + r_2 x r_1 + r_2 x r_2 + r_2 y r_1 + r_2 y r_2]I = (r_1 + r_2 I)(x + yI)(r_1 + r_2 I)$, wherefore $r_1 + r_2 I \in NReg_{R(I)}$

The following corollary was mentioned in reference [21], and we have presented another proof for it:

Corollary 3.2 Every neutrosophic field R(I) is a regular.

Proof. $\forall r_1 + r_2 I \in R(I)$, where $r_1 and \; r_2 \in R$.

We know that every classical field is regular, so $r_1 \; and \; r_1 + r_2 \in Reg_R$. Subsequently, $r_1 + r_2 I \in NReg_{R(I)}$, according to theorem.3.1.

Example 3.3 $\mathbb{R}(I)$, $\mathbb{Q}(I), \mathbb{C}(I)$, $and \; \mathbb{Z}_p(I)$ are neutrosophic regular, where p is the prime.

Corollary 3.4 $R(I)$ is regular iff R is regular. (The proof results directly from the theorem.3.1)

Theorem 3.5 In R(I), $\forall r_1 + r_2 I \in NC_{R(I)} \Leftrightarrow r_1 \; and \; r_1 + r_2 \in C_R$

Proof.

(⇒): If $r_1 + r_2 I \in NC_{R(I)}$, then $\forall x + yI \in R$(I); $(r_1 + r_2 I)(x + yI) = (x + yI)(r_1 + r_2 I)$

$\Rightarrow r_1 x + [r_1 y + r_2 x + r_2 y]I = x r_1 + [y r_1 + x r_2 + y r_2]I$. So $r_1 x = x r_1 \forall x \in R \Rightarrow r_1 \in C_R$

And $r_1 y + r_2 x + r_2 y = y r_1 + x r_2 + y r_2$

$$since \; r_1 x = x r_1 \Rightarrow r_1 x + r_1 y + r_2 x + r_2 y = x r_1 + y r_1 + x r_2 + y r_2$$
$$\Rightarrow (r_1 + r_2)(x + y) = (x + y)(r_1 + r_2), \forall x + y = r \in R$$
$$\Rightarrow r_1 + r_2 \in C_R$$

(⇐): If $r_1 \; and \; r_1 + r_2 \in C_R$, then $\forall x, r \in R$; $r_1 x = x r_1 \; and \; (r_1 + r_2) r = r (r_1 + r_2)$

First, let us put $y = r - x \in R$. Subsequently, $r = x + y$

Subsequently, $(r_1 + r_2)(x + y) = (x + y)(r_1 + r_2) \Rightarrow r_1 x + r_1 y + r_2 x + r_2 y = x r_1 + y r_1 + x r_2 + y r_2$

$$since \; r_1 x = x r_1 \Rightarrow r_1 y + r_2 x + r_2 y = y r_1 + x r_2 + y r_2$$

At other hand, we have $(r_1 + r_2 I)(x + yI) = r_1 x + [r_1 y + r_2 x + r_2 y]I = xr_1 + [yr_1 + xr_2 + yr_2]I$ $= (x + yI)(r_1 + r_2 I)$ $\forall x + yI \in R(\mathrm{I})$, therefore $r_1 + r_2 I \in NC_{R(I)}$

Theorem 3.6 The center of the neutrosophic regular ring is also a regular.

Proof.

Suppose $R(I)$ is a neutrosophic regular. We have $NC_{R(I)}$ is a neutrosophic subring. Using theorem 3.5, if $r_1 + r_2 I \in NC_{R(I)}$, then $r_1 and\ r_1 + r_2 \in C_R$.

Since $R(I)$ is regular, so R also is regular. Therefore C_R is regular.

Thus $r_1 and\ r_1 + r_2 \in Reg_R$. Subsequently $r_1 + r_2 I \in NReg_{R(I)}$, according to theorem.3.1. Thus $NC_{R(I)}$ is regular.

Theorem 3.7 Assuming that $\boldsymbol{R(I)}$ is a neutrosophic ring, then $NC_{R(I)}$ is regular ⇔ C_R is regular.

Proof.

(⇒): $\forall r \in C_R \Rightarrow r\ and\ r + 0 \in C_R \Rightarrow \mathrm{r} + 0\mathrm{I} \in NC_{R(I)}$, according to theorem.3.5. Since $NC_{R(I)}$ is regular, so r is regular according to theorem.3.1. Thus C_R is regular.

(⇐): $\forall r_1 + r_2 I \in NC_{R(I)} \Rightarrow r_1\ and\ r_1 + r_2 \in C_R$, according to theorem.3.5. Since C_R is regular, so $r_1\ and\ r_1 + r_2$ is regular. Therefore $r_1 + r_2 I$ is regular according to theorem.3.1. Thus $NC_{R(I)}$ is regular

Remark.3.8 Figure 1 shows the resulting relationship between the neutrosophic and classical regular ring, as follows:

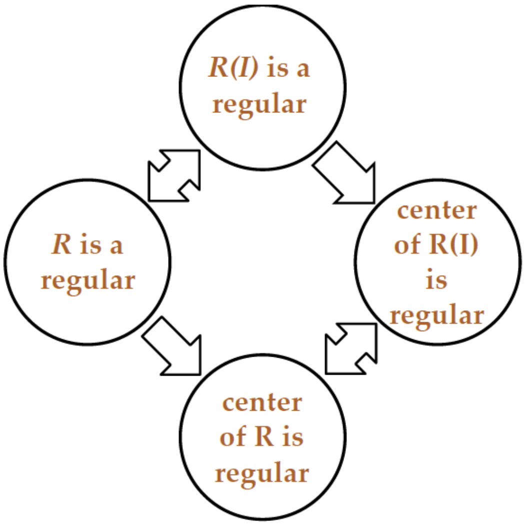

Figure 1. Relationship between the neutrosophic and classical regular rings.

Theorem 3.9 $R(I)$ is a regular ⇔ every neutrosophic principal ideal is generated by a neutrosophic idempotent element.

Proof.

(⇒): If $R(I) = NReg_{R(I)}$ and $J + KI$ is a neutrosophic ideal generated by $r_1 + r_2 I$. So $J + KI =$ $(r_1 + r_2 I)R(I) = (r_1 + r_2 I)(R + RI) = r_1 R + [(r_1 + r_2)R]I$. Therefore, $J = r_1 R\ and\ K = (r_1 + r_2)R$ are principal ideals of R. Because $R(I) = NReg_{R(I)}$, R is regular according to corollary.3.4. Therefore,

$J = r_1R = e_1R \; and \; K = (r_1 + r_2)R = e_2R, where, e_1, e_2 \in Id_R$. Thus $J + KI = e_1R + e_2RI = e_1R + e_2RI - e_1RI + e_1RI = e_1R + [e_1R + (e_2 - e_1)R]I = e_1R + [e_1R + (e_2 - e_1)R + (e_2 - e_1)R]I = [e_1 + (e_2 - e_1)I][R + RI]$

It can easily be verified that $e_1 + (e_2 - e_1)I \in NId_{R(I)}$. Therefore, the desired result is achieved.

(⇐): If $r_1 + r_2I \in R(I)$, then $(r_1 + r_2I)\mathrm{R(I)} = (e_1 + e_2I)R(I)$ is the principal ideal of $R(I)$ generated by $e_1 + e_2I \in NId_{R(I)}$. First, $(r_1 + r_2I) \in (e_1 + e_2I)R(I) \Rightarrow \exists s_1 + s_2I \in R(I), where \; r_1 + r_2I = (e_1 + e_2I)(s_1 + s_2I) \Rightarrow (e_1 + e_2I)(r_1 + r_2I) = (e_1 + e_2I)(s_1 + s_2I) = r_1 + r_2I$

At other hand, $e_1 + e_2I \in (e_1 + e_2I)R(I) = (r_1 + r_2I)\mathrm{R(I)} \Rightarrow \exists s_3 + s_4I \in R(I), where, e_1 + e_2I = (r_1 + r_2I)(s_3 + s_4I) \Rightarrow (e_1 + e_2I)(r_1 + r_2I) = (r_1 + r_2I)(s_3 + s_4I)(r_1 + r_2I)$

$\Rightarrow r_1 + r_2I = (r_1 + r_2I)(s_3 + s_4I)(r_1 + r_2I) \Rightarrow r_1 + r_2I \in NReg_{R(I)} \Rightarrow R(I) = NReg_{R(I)}$

Example 3.10 In any neutrosophic field $R(I)$, we have only three ideals. $\{0\}, RI, and \; R(I)$.

We note that each of the previous ideals is generated by the idempotent elements $0, I \; and \; 1$, respectively.

Theorem 3.11 $R(I)$ is regular iff every neutrosophic finitely generated right (left) ideal in $R(I)$ can be generated by an idempotent element.

Proof.

(⇒): Suppose $J + KI$ is a neutrosophic right ideal generated by n elements. We will use the principle of mathematical induction.

If $n = 1$, then using theorem 3.9, the desired result is achieved.

If $n = 2$, then $J + K = (r_1 + r_2I)(R + RI) + (r_3 + r_4I)(R + RI) = r_1R + [(r_1 + r_2)R]I + r_3R + [(r_3 + r_4)R]I = (r_1 + r_3)R + [(r_1 + r_2) + (r_3 + r_4)]RI \Rightarrow \; J = (r_1 + r_3)R \quad and \quad K = [(r_1 + r_2) + (r_3 + r_4)]R \; \Rightarrow J \; and \; K$ are finitely generated. Because R is regular, $J \; and \; K$ are generated by an idempotent element.

$$\Rightarrow \; J = (r_1 + r_3)R = e_1R \quad and \quad K = [(r_1 + r_2) + (r_3 + r_4)]R = e_2R, where, e_1, e_2 \in Id_R$$

Finally, $J + KI = e_1R + e_2RI = e_1R + e_2RI - e_1RI + e_1RI = e_1R + [e_1R + (e_2 - e_1)R]I$

$$= e_1R + [e_1R + (e_2 - e_1)R + (e_2 - e_1)R]I = [e_1 + (e_2 - e_1)I][R + RI]$$

It can easily be verified that $e_1 + (e_2 - e_1)I \in NId_{R(I)}$. Therefore, the desired result is achieved.

(⇐): This results directly from theorem 3.9.

In the same way, we prove it in the case of the left ideal.

Theorem 3.12 $R(I)$ is a regular iff every neutrosophic principal right (left) ideal is a direct actor in $R(I)$.

Proof.

(⇒): Suppose $K_1 + K_2I$ is a neutrosophic principal right ideal. Therefore, $K_1 + K_2I = (e_1 + e_2I)R(I)$, where $e_1 + e_2I \in NId_{R(I)}$, according to theorem 3.9.

We have $K_3 + K_4I = [(1 - e_1) - e_2I)]R(I)$ is a right ideal is generated by $(1 - e_1) - e_2I$, where $(1 - e_1) - e_2I \in NId_{R(I)}$. Now let us prove that $R(I) = (K_1 + K_2I) \oplus (K_3 + K_4I)$.

First, we have $(K_1 + K_2I) + (K_3 + K_4I) \subseteq R(I)$. At other hand, if $r_1 + r_2I \in \mathrm{R(I)}$, then

$$r_1 + r_2I = (e_1 + e_2I)(r_1 + r_2I) + [(r_1 + r_2I) - (e_1 + e_2I)(r_1 + r_2I)]$$

$= (e_1 + e_2I)(r_1 + r_2I) + [1 - (e_1 + e_2I)](r_1 + r_2I) \in (e_1 + e_2I)R(I) + [(1 - e_1) - e_2I]R(I)$.

Thus $R(I) = (K_1 + K_2I) + (K_3 + K_4I)$.

Finally, $\forall s+tI \in (K_1+K_2I)\cap(K_3+K_4I) \Rightarrow s+tI \in (K_1+K_2I)$ and $s+tI \in (K_3+K_4I)$
Therefore, $s+tI=(e_1+e_2I)(r_1+r_2I)$ and $s+tI=[(1-e_1)-e_2I](r_3+r_4I)$, where r_1+r_2I and $r_3+r_4I \in R(I)$.
At other hand, we note $s+tI=(e_1+e_2I)(r_1+r_2I) \Rightarrow (e_1+e_2I)(s+tI)=s+tI$
Subsequently, $s+tI=(e_1+e_2I)(s+tI)=(e_1+e_2I)[(1-e_1)-e_2I](r_3+r_4I)$

$$= [e_1(1-e_1)+(-e_1e_2+e_2(1-e_1)-{e_2}^2)I](r_3+r_4I)$$
$$= [(e_1-{e_1}^2)+(-e_1e_2+e_2-e_2e_1-{e_2}^2+{e_1}^2-{e_1}^2)I](r_3+r_4I)$$
$$= [(e_1-{e_1}^2)+(-({e_1}^2+e_1e_2+e_2e_1+{e_2}^2)+e_2+{e_1}^2)I](r_3+r_4I)$$
$$= [(e_1-{e_1}^2)+(-(e_1+e_2)^2+e_2+{e_1}^2)I](r_3+r_4I)$$

Since $e_1+e_2I \in NId_{R(I)}$, so e_1 and $e_1+e_2 \in Id_R$, According to theorem 2.4.
Therefore, $s+tI=[(e_1-e_1)+(-(e_1+e_2)+e_2+e_1)I](r_3+r_4I)=0$
So,$(K_1+K_2I)\cap(K_3+K_4I)=\{0\}$. Therefore, $R(I)=(K_1+K_2I)\oplus(K_3+K_4I)$.
Thus K_1+K_2I is a direct actor in $R(I)$.
(⇐): $\forall r_1+r_2I \in R(I)$, then $(r_1+r_2I)R(I)$ is a principal right ideal in $R(I)$.
Therefore, according to this assumption, there is a neutrosophic right ideal K_1+K_2I, where, $R(I)=(r_1+r_2I)R(I)\oplus(K_1+K_2I)$. Thus $1=(r_1+r_2I)(x+yI)+(j+kI)$, where $x+yI \in R(I)$ and $j+kI \in K_1+K_2I$.
And from it we find $r_1+r_2I=(r_1+r_2I)(x+yI)(r_1+r_2I)+(j+kI)(r_1+r_2I)\ldots.(*)$
Since K_1+K_2I is a neutrosophic right ideal, so $(j+kI)(r_1+r_2I)\in K_1+K_2I$.
At other hand, we note $(j+kI)(r_1+r_2I)=r_1+r_2I-(r_1+r_2I)(x+yI)(r_1+r_2I)=(r_1+r_2I)[1-(x+yI)(r_1+r_2I)] \in (r_1+r_2I)R(I)$. Thus $(j+kI)(r_1+r_2I)\in(r_1+r_2I)R(I)\cap(K_1+K_2I)=\{0\}$.
Subsequently, $(j+kI)(r_1+r_2I)=0$. Returning to $(*)$, we find that $r_1+r_2I=(r_1+r_2I)(x+yI)(r_1+r_2I)$. Therefore, $r_1+r_2I \in NReg_{R(I)} \Rightarrow R(I)=NReg_{R(I)}$.
In the same way, we prove it in the case of the left ideal.

Example 3.13 In any neutrosophic field $R(I)$, we have RI a neutrosophic principal ideal is generated by a neutrosophic idempotent I. At other hand, we have $(1-I)R(I)$ is a neutrosophic principal ideal is generated by a neutrosophic idempotent $1-I$.
Now , we note $(1-I)R(I)+RI=R(I)$. Also $\forall x \in (1-I)R(I)\cap RI \Rightarrow x \in (1-I)R(I)$ and $x \in RI$.
So $x=(1-I)(r_1+r_2I)=r_1+r_2I-r_1I-r_2I=r_1-r_1I$. Since $x \in RI$, so $r_1=0$. Therefore, $x=(1-I)r_2I=0$. Subsequently, $(1-I)R(I)\oplus RI=R(I)$. Therefore, RI is a direct actor in $R(I)$.

Theorem 3.14 $R(I)$ is a regular ⇔ each neutrosophic finitely generated right (left) ideal in $R(I)$ is a direct actor in $R(I)$.

Proof.
(⇒): If K_1+K_2I is a finitely generated right ideal, then $K_1+K_2I=(e_1+e_2I)R(I)$, where, $(e_1+e_2I)^2=e_1+e_2I$ according to theorem.3.11, and from it we find that

$$R(I)=(e_1+e_2I)R(I)\oplus[(1-e_1)-e_2I)]R(I)$$

(⇐): It results from theorem.3.11 and theorem.3.12.
In the same way, we prove it in the case of the left ideal.

Remark 3.15 Figure 2 shows the resulting relationship between the neutrosophic regular ring and their ideals, as follows:

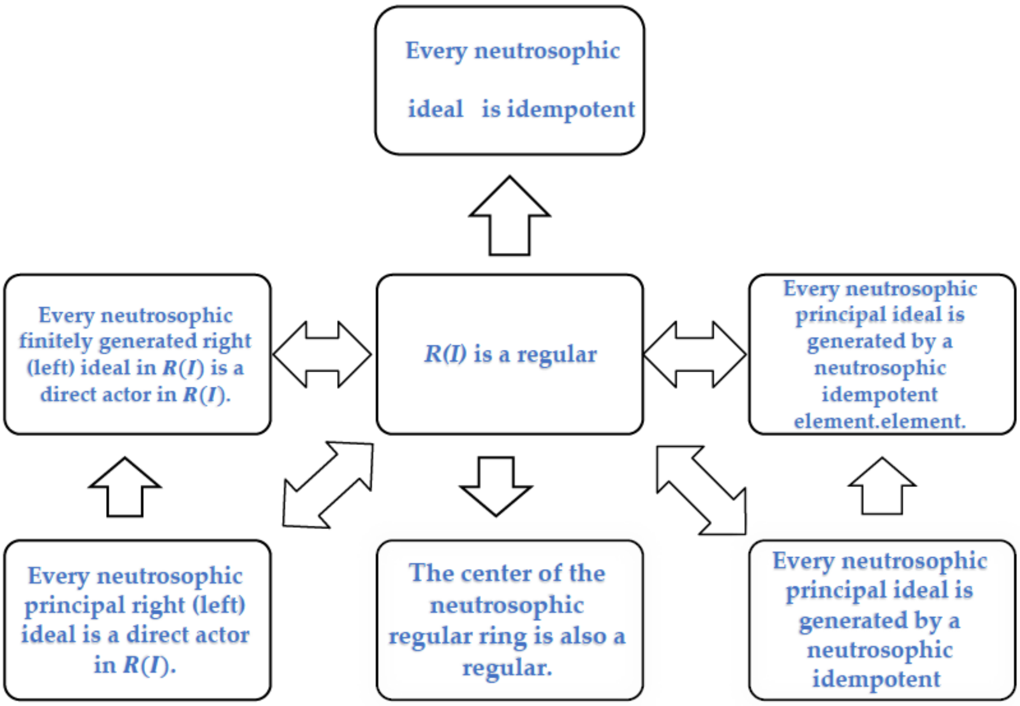

Figure 2. Relationship between the neutrosophic regular ring and their ideals.

Theorem 3.16 $R(I) = NReg_{R(I)}$ iff $(K_1 + K_2I) \cap (K_3 + K_4I) = (K_1 + K_2I)(K_3 + K_4I) = K_1K_3 + K_2K_4I$ for every neutrosophic right ideal $K_1 + K_2I$ and every neutrosophic left ideal $K_3 + K_4I$.

Proof.

$(\Rightarrow)$: We have $(K_1 + K_2I) \cap (K_3 + K_4I) = (K_1 \cap K_3) + (K_2 \cap K_4)I$ and also K_1 *and* K_2 $(K_3$ *and* $K_4)$ are right (left) ideals in R. Since $R(I)$ is regular, so R is regular according to theorem.3.4. Therefore, $K_1 \cap K_3 = K_1K_3$ *and* $K_2 \cap K_4 = K_2K_4$. Thus, $(K_1 + K_2I) \cap (K_3 + K_4I) = (K_1 \cap K_3) + (K_2 \cap K_4)I = K_1K_3 + K_2K_4I$.

At other hand, we have $(K_1 + K_2I)(K_3 + K_4I) \subseteq (K_1 + K_2I) \cap (K_3 + K_4I) = K_1K_3 + K_2K_4I \subseteq K_1K_3 + K_2K_4I + [K_1K_4 + K_2K_3]I = (K_1 + K_2I)(K_3 + K_4I)$.

Thus $(K_1 + K_2I) \cap (K_3 + K_4I) = (K_1 + K_2I)(K_3 + K_4I) = K_1K_3 + K_2K_4I$

$(\Leftarrow)$: $\forall\, r_1 + r_2I \in R(I)$, then $(r_1 + r_2I)R(I)$ is a right ideal and $R(I)(r_1 + r_2I)$ is a left ideal. At other hand, $r_1 + r_2I \in (r_1 + r_2I)R(I) \cap R(I)(r_1 + r_2I)$

$$= (r_1 + r_2I)R(I)R(I)(r_1 + r_2I)$$

Since $R(I)$ is unity, so $R(I) = R(I)^2$. Subsequently, $r_1 + r_2I = (r_1 + r_2I)R(I)(r_1 + r_2I)$

$\Rightarrow \exists\, r_3 + r_4I \in R(I)$ such that $r_1 + r_2I = (r_1 + r_2I)(r_3 + r_4I)(r_1 + r_2I) \Rightarrow r_1 + r_2I \in NReg_{R(I)}$

$\Rightarrow R(I) = NReg_{R(I)}$.

Example 3.17 In $\mathbb{Z}_5(I)$, we have $\{0\}$, *and* $\mathbb{Z}_5I$ are neutrosophic ideals, and it comes true $\{0\} \cap \mathbb{Z}_5I = \{0\} = \{0\}.\mathbb{Z}_5$

Theorem 3.18 If $R(I) = NReg_{R(I)}$, then $\forall\, K_1 + K_2I \in N\mathfrak{I}_{R(\mathrm{I})}$ is idempotent.

Proof. $\forall\, K_1 + K_2I \in N\mathfrak{I}_{R(\mathrm{I})}$. First, we have $(K_1 + K_2I)^2 \subseteq K_1 + K_2I$.

At other hand, $\forall\, r_1 + r_2I \in K_1 + K_2I \Rightarrow r_1 + r_2I \in R(I) \xRightarrow[R(I)=NReg_{R(I)}]{} \exists r_3 + r_4I \in R(I)$;

$$r_1 + r_2 I = \underbrace{(r_1 + r_2 I)}_{\in K_1 + K_2 I} \underbrace{(r_3 + r_4 I)(r_1 + r_2 I)}_{\in K_1 + K_2 I} \in (K_1 + K_2 I)^2$$

Thus $K_1 + K_2 I$ is idempotent.

Example 3.19 In a neutrosophic field $\mathbb{Z}_5$, we have only $\{0\}, \mathbb{Z}_5 I, and\ \mathbb{Z}_5(I)$ are neutrosophic ideals, and it comes true $\{0\}^2 = \{0\}$, $(\mathbb{Z}_5 I)^2 = \mathbb{Z}_5 I, and\ [\mathbb{Z}_5(I)]^2 = \mathbb{Z}_5(I)$.

4. Conclusion and Future Works

During our search, we created the essential and sufficient conditions for the neutrosophic ring to become a regular. Furthermore, several instances were created to demonstrate the search's reliability. We intend to generalize the regular refined neutrosophic rings and their attributes in future studies.

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Conflicts of Interest

The authors declare that the research has no conflicts of interest.

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