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A New Development of Entropy and Similarity Measures in Temporal Complex Neutrosophic Environments for Tourist Destination Selection

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Abstract. In human existence, making decisions is a common event. Various techniques have been devised to tackle decision-making troubles in practical situations. Over the past ten years, a great deal of study has concentrated on quantifying the degree of ambiguity and unpredictability in knowledge using the concept of neutrosophic sets or extensions thereof. An efficient framework for handling information in decision-making problems involving uncertain, indeterminate, and time-related aspects is the Temporal Complex Neurosophic Set (TCNS). Measures of entropy and similarity can be helpful for evaluating data to solve multicriteria decision-making (MCDM) challenges in practical situations. However, TCNS information measurements were of no concern or relevance to the current technique. In the context of the TCNS, this work suggests multiple novel similarity and entropy measurements. The proposed metrics have been validated and shown to comply with the explicit definition of the entropy measure and similarity for the TCNS. The novel similarity and entropy measures on the TCNS environment are proposed in this research. The four similarity measures on the TCNS contain Dice, Jaccard, Cosine, and Cotangent. Also, a numerical example concerning selecting a Vietnam tourist destination is provided to validate the usefulness of the suggested measures. The practical application shows that proposed TCNS similarity and entropy metrics can produce accurate and significant outcomes for real-world decision-making problems.

Keywords: Similarity measure; Entropy measure; Neutrosophic set; Temporal Complex Neutrosophic Set; MCDM;

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1. Introduction

In human activities, decision-making (DM) is a highly typical routine. A multi-attribute decision-making (MADM) problem is determining the most suitable option from a group of options that all need to be evaluated simultaneously. Numerous DM problems have been solved using MADM models, which have attracted the attention of multiple researchers in diverse domains [1, 2] However, real-world issues are excessively complex because they frequently need clarification, clarity, or insufficient data. As a result, decision-makers tackle those issues through the fuzzification process, which is a vital method to address humanistic systems that exist in real-world problems.

The concept of classical fuzzy sets (FS) was introduced by Zadeh [3] in 1965 as a way to handle uncertain data. By enabling elements to be a part of sets with varying levels of membership, FS successfully captures the fuzziness and ambiguity that are inherent in many common problems. After that, it was used by numerous scientists and scholars to look into various real-world events in different fields and recommend the best courses of action. However, the nature of covert human assessments of discontent was not explained by the FS theory, Atanassov [4] was prompted by this to suggest an intuitionistic fuzzy set (IFS), wherein the levels of each element's membership and non-membership are determined separately. Then, an extension of IFS, namely neutrosophic set (NS), is introduced by Smarandache [5] by introducing an uncertain membership function with degrees of falsity, indeterminacy, and truth into IFS. It's also an essential and effective technique for handling conflicting, ambiguous, and incomplete information in certain real-life situations [6, 7, 8].

Although FS or its extension versions have shown remarkable effectiveness in addressing issues emerging from vagueness and uncertainty in many real-world scenarios, they cannot solve the periodicity of some ambiguous information. It is challenging to express periodic forms using crisp sets, FS, IFS, or NS. In order to address this challenge, Ramot et al. [9] introduced the notion of the complex fuzzy set (CFS), which adds information to a phase component in order to provide details about a specific higher dimensional periodic problem. Since its introduction, many researchers have researched and developed CFS because it is considered a valuable tool for representing information with ambiguous, uncertain, and periodic components [10, 11].

Furthermore, the complex neutrosophic set (CNS) was described by Ali and Smarandache [12]. With the intention of integrating the "complex" component to make NS more flexible and sensitive to occasionally ambiguous information, CNS is an integrated version of the CFS and NS. Uncertainty, inconsistency, and indeterminacy are the three membership degree functions that define a CNS. These functions have a complex-valued range in the complex space unit circle. The CNS is limited if the total of the negative, abstinence, and positive grades is three or less. Following its introduction, CNS demonstrated its value in characterizing data that included unpredictable, inconsistent, periodic, and indeterminate factors.

In real-world decision-making problems, we encounter complex natural phenomena that require adding a second dimension to represent the degree of membership. By establishing the second dimension, information is fully gathered, and the problem of information loss is minimized. Therefore, since the beginning of the CNS, many hybrid versions have been developed and used in a range of life domains, such as Interval CNS [13], Bipolar CNS [14], Complex neutrosophic soft set (CNSS) [15], Interval-valued complex neutrosophic soft set (IVCNSS) [16]... In light of the theories mentioned above, it is evident that many MCDM researchers have found the information's uncertainty and time-periodic element interesting. Many of the decision-making scenarios we encounter in real life involve periodic events, which call for the representation of various time intervals (e.g., seasonal variations in weather information). However, the present experiments have yet to take into account the effect of temporal factors by CNS in an MCDM model concurrently. Recently, Lan et al. [17] introduced an extension of CNS, namely Temporal Complex neutrosophic set (TCNS), to represent data with dynamic and temporal factors of several real-life decision-making contexts. It covers decision-making issues with uncertain, temporal, and periodic aspects. It is a novel opening with several practical applications that concern the time cycle and time element.

Making decisions involves systematically solving issues in the real world and selecting the best option after determining which feasible alternatives are available. All possible options are sorted, allowing users to choose the suitable solution. When faced with multi-criteria decision-making difficulties, decision-makers can select the best option primarily based on information measures, including cross-entropy, entropy, similarity, and distance measures. Because of this, entropy and similarity measurements are consistently two crucial subjects that have been investigated widely by many scholars from various backgrounds. Nevertheless, prior research has yet to focus on refining and strengthening the theory of the TCNS environments using similarity metrics, mathematical operations, and entropy measures. That is the primary motivation for carrying out this research.

(1) Define novel measures of entropy and similarity for the Temporal Complex Neutronosophic Set, which addresses the unpredictable, periodic, and temporal aspects of DM problems.

(2) Build the decision-making model using the proposed entropy and similarity measurements in the TCNS environment.

(3) Using a real-world case study of selecting a tourism location in Vietnam, illustrate the viability and logic of applying the suggested model's decision-making process.

(4) Prove our proposed model's effectiveness and potential for real-life case study through a comparative analysis with related MCDM techniques.

The remaining portions of the paper are organized in this manner. Section 2 outlines the research gap and a few connected works. Next, in Section 3, we go over a few of the significant concepts of the TCNS. Novel metrics of entropy and similarity depending on the TCNS framework are defined in Section 4. The multi-criteria technique for making decisions using proposed indicators is provided in Section 5. In Section 6, a comparative experiment is utilized to demonstrate how our theory might be used in practice in a particular tourist location. The conclusions are finally presented in Section 7.

2. **Literature review**

2.1. *Related works*

Similarity is essential to finding solutions in many intricate and unpredictable situations in human life. Both the entropy and similarity measurements were able to show how similar two objects were in an unexpected and uncertain environment. As a result, many scholars have concentrated on using NS or extensions to quantify fuzziness and ambiguity in data. Many different entropy and similarity metrics established and have effectively dealt with practical applications such as clustering analysis [18, 19], decision-making [20, 21], health diagnostics [22, 23] and pattern recognition [24, 25].

According to Hausdorff distance, Broumi and Smarandache [26] presented some similarities for NS in the neutrosophic environment. Several metrics of entropy and similarity of a single-valued NS were proposed by Majumdar et al. [27]. The approach based on score function and divergence measure is suggested to determine the weights of decision experts using a single-valued NS [28]. Bin Ji et al. [29] published the multi-parameter similarity measure for interval-valued neutrosophic sets based on the tangent function. The Q-Neutrosophic Soft set was enhanced with similarity and entropy tools by Abu Qamar and Hassan [30], who also evaluated how well they worked for issues involving medical diagnosis. Additionally, entropy and similarity are frequently employed as crucial metrics for resolving real-world multi-criteria decision-making issues [31, 32, 33]

Since entropy and similarity measurements are two significant topics in the CFS and CNS, they have been comprehensively investigated from a variety of perspectives in multiple studies. Utilizing mixed models to handle the ambiguities of periodical data—where time is a crucial component in its representation—researchers made essential contributions to the literature on similarity measures domains. Axiomatic formulations of the similarity measure and entropy for Complex Multi Fuzzy Soft Sets were presented by Al-Qudah and Hassan [34]. Using a complex fuzzy soft set, Ganeshsree Selvachandran et al. [35] introduced metrics of distance and similarity to address pattern recognition challenges involving digital photographs. The Cosine similarity measures were used by the authors in [36] to calculate the coefficients of similarity and dissimilarity between two Complex linear Diophantine fuzzy sets. The Complex Multi-Fuzzy Hypersoft Set's Entropy and Similarity Measure concept was introduced in [37]. To demonstrate the validity and significance of the developed measures, some associated theorems and an application for a person's selection decision issue who intends to buy a car are established.

Keeping up with this development and improving the modeling of some real-world issues using our approach (CNS theory and its hybrid structures), novel dice similarity measurements utilizing CNS were proposed by Ali et al. [38]. The authors then used the pattern recognition model and the criteria they provided to test the validity and superiority of the pre-existing methods. Jaccard, Dice, and cosine are three examples of complex neutrosophic similarity metrics proposed by Mondal et al. [39]. A numerical case study demonstrates the suggested methods for handling multi-attribute decision-making scenarios in which students select an appropriate course for post-secondary education after passing a secondary exam. Xu et al. [40] introduced some distance measures based on Hamming, Hausdorff, and Euclidean metrics to deal with the interval CNS information. In order to help with decision-making in medical diagnosis, Faisal et al. [41] developed several similarity metrics of interval complex neutrosophic soft sets (ICNSS) based on the distance measurements: Euclidean and Hamming. An axiomatic definition of single-valued CNS entropy and normalized distance formulas are provided by [42]. Also, to demonstrate the usefulness and feasibility of our suggested entropy metric, we offer a real-world example that involves the choice of green suppliers.

Because physical problems are inherently complicated and unpredictable, the difficulty of researching MCDM has concentrated on the issues with criteria values in the form of intervals, drawing attention to this attractive study area. The research indicates they are effective at encapsulating and illustrating vague and ambiguous information, enabling decision-makers to make intelligent decisions.

2.2. *Research gap analysis*

The evaluations above suggest that these several measures of similarity and entropy and their applications play a significant role in efficiently managing uncertainty and inconsistency in decision-making contexts. However, there are several extant research gaps, as indicated below:

- All entropy and similarity measures in the literature are established based on the theory of CFS, CNS, and these hybrid structures.

- Entropy and similarity measures in the literature only deal with uncertain, imprecise, and periodic information. None of the above theories have the capability of managing information involving dynamic, temporal, and time-cycle factors in real-life problems.

- Previous theories focused on the time-varying component but neglected the temporal, time-cycle factor, and the impact of temporal factors.

- Many entropy measures exist for CFS, CNS, or extension of CNS in the literature that fail to deal with the problem of providing reasonable or consistent results to the decision-makers in the case of a TCNS environment.

- There hasn't been any study on entropy and similarity metrics especially created for TCNS information theory, even though they are essential to fuzzy set theory.

When taking into account the time cycle, periodic, and time components, TCNS is a novel opening with several practical applications. Unfortunately, previous studies have not concentrated on improving and solidifying the theory of the TCNS settings through entropy and similarity metrics. The above facts and the research gap in CNS neutrosophic information theory prompted us to pursue this work. Thus, the paper aims to develop novel entropy and similarity metrics for the TCNS.

3. Preliminaries

This section summarizes the core concepts of TCNS and the three operations (intersection, union, and complement) that will be applied in our study.

Definition 3.1 ([17]). Let U be a universal set, and let $\tilde{\tau} = \{\tau_1, \tau_2, ..., \tau_{n_\tau}\}$ be consecutive time periods. On U, a temporal complex neutrosophic set (TCNS) A is represented by form below:

$$\mathrm{A}(\theta, \tilde{\tau}) = \{\theta, \langle p(\theta, \tau_l) .e^{j\mu(\theta,\tau_l)}, q(\theta, \tau_l) .e^{j\nu(\theta,\tau_l)}, r(\theta, \tau_l) .e^{j\eta(\theta,\tau_l)}\rangle \mid \theta \in U\} \tag{1}$$

Where $p(\theta, \tau_l)$, $q(\theta, \tau_l)$, $r(\theta, \tau_l) \in [0, 1]$ and $\mu(\theta, \tau_l)$, $\nu(\theta, \tau_l)$, $\eta(\theta, \tau_l) \in [0, 2\pi]$

Definition 3.2 ([17]). Suppose there are two TCNSs $\Omega_1(\theta, \tilde{\tau})$ and $\Omega_2(\theta, \tilde{\tau})$ respectively.

$$\Omega_1(\theta, \tilde{\tau}) = \left\{\theta, \left\langle p_{\Omega_1}(\theta, \tilde{\tau}) e^{j\mu_{\Omega_1}(\theta,\tilde{\tau})}, q_{\Omega_1}(\theta, \tilde{\tau}) e^{j\nu_{\Omega_1}(\theta,\tilde{\tau})}, r_{\Omega_1}(\theta, \tilde{\tau}) e^{j\eta_{\Omega_1}(\theta,\tilde{\tau})}\right\rangle\right\}$$
$$\Omega_2(\theta, \tilde{\tau}) = \left\{\theta, \left\langle p_{\Omega_2}(\theta, \tilde{\tau}) e^{j\mu_{\Omega_2}(\theta,\tilde{\tau})}, q_{\Omega_2}(\theta, \tilde{\tau}) e^{j\nu_{\Omega_2}(\theta,\tilde{\tau})}, r_{\Omega_2}(\theta, \tilde{\tau}) e^{j\eta_{\Omega_2}(\theta,\tilde{\tau})}\right\rangle\right\}$$

The following describes the fundamental functions of TCNS:

Complement:

$$C(\Omega_1(\theta, \tilde{\tau})) = \left\{\theta, \left\langle \begin{array}{l} r_{\Omega_1}(\theta, \tilde{\tau}) e^{j\left(2\pi - \eta_{\Omega_1}(\theta,\tilde{\tau})\right)}, \\ (1 - q_{\Omega_1}(\theta, \tilde{\tau})) e^{j\left(2\pi - v_{\Omega_1}(\theta,\tilde{\tau})\right)}, \\ p_{\Omega_1}(\theta, \tilde{\tau}) e^{j\left(2\pi - \mu_{\Omega_1}(\theta,\tilde{\tau})\right)} \end{array} \right\rangle \mid \theta \in U \right\} \tag{2}$$

Union:

$$\Omega_1(\theta, \tilde{\tau}) \cup \Omega_2(\theta, \tilde{\tau}) = \left\{\theta, \left\langle \begin{array}{l} (p_{\Omega_1}(\theta, \tilde{\tau}) \vee p_{\Omega_2}(\theta, \tilde{\tau})) e^{j\mu_{\Omega_1 \cup \Omega_2}(\theta,\tilde{\tau})}, \\ (q_{\Omega_1}(\theta, \tilde{\tau}) \wedge q_{\Omega_2}(\theta, \tilde{\tau})) e^{j\nu_{\Omega_1 \cup \Omega_2}(\theta,\tilde{\tau})}, \\ (r_{\Omega_1}(\theta, \tilde{\tau}) \wedge r_{\Omega_2}(\theta, \tilde{\tau})) e^{j\eta_{\Omega_1 \cup \Omega_2}(\theta,\tilde{\tau})} \end{array} \right\rangle \mid \theta \in U \right\} \tag{3}$$

Intersect:

$$\Omega_1(\theta,\tilde{\tau}) \cap \Omega_2(\theta,\tilde{\tau}) = \left\{ \theta, \left\langle \begin{array}{l} (p_{\Omega_1}(\theta,\tilde{\tau}) \wedge p_{\Omega_2}(\theta,\tilde{\tau}))\, e^{j\mu_{\Omega_1\cap\Omega_2}(\theta,\tilde{\tau})}, \\ (q_{\Omega_1}(\theta,\tilde{\tau}) \vee q_{\Omega_2}(\theta,\tilde{\tau}))\, e^{j\nu_{\Omega_1\cap\Omega_2}(\theta,\tilde{\tau})}, \\ (r_{\Omega_1}(\theta,\tilde{\tau}) \vee r_{\Omega_2}(\theta,\tilde{\tau}))\, e^{j\eta_{\Omega_1\cap\Omega_2}(\theta,\tilde{\tau})} \end{array} \right\rangle \mid \theta \in U \right\} \quad (4)$$

4. Propose new measures of Temporal complex neutrosophic set

The study presents a few TCNS similarity metrics in this part, including the Dice, Jaccard, Cosine, and Cotangent similarity measures. Additionally, it is suggested that decision-making models in temporally complex neutrosophic contexts start with an entropy assessment of TCNS.

4.1. *TCNS - Dice similarity measure*

Definition 4.1. Assume that $\Omega_1 = \left\langle p_{\Omega_1}(\theta_i,\tau_l)\, e^{j\mu_{\Omega_1}(\theta_i,\tau_l)}, q_{\Omega_1}(\theta_i,\tau_l)\, e^{j\nu_{\Omega_1}(\theta_i,\tau_l)}, r_{\Omega_1}(\theta_i,\tau_l)\, e^{j\eta_{\Omega_1}(\theta_i,\tau_l)} \right\rangle$ and $\Omega_2 = \left\langle p_{\Omega_2}(\theta_i,\tau_l)\, e^{j\mu_{\Omega_2}(\theta_i,\tau_l)}, q_{\Omega_2}(\theta_i,\tau_l)\, e^{j\nu_{\Omega_2}(\theta_i,\tau_l)}, r_{\Omega_2}(\theta_i,\tau_l)\, e^{j\eta_{\Omega_2}(\theta_i,\tau_l)} \right\rangle$ be are two TCNSs on universe of discourse U for $\theta_i\,(1,2,...,n_U)$ and $\tau_l\,(1,2,...,n_\tau)$.
Suppose $Sim_D(\Omega_1,\Omega_2)$ is a TCNS - Dice similarity measure between Ω_1 and Ω_2, $Sim_D(\Omega_1,\Omega_2)$ is defined as follows:

$$Sim_D(\Omega_1,\Omega_2) = \frac{1}{n_U * n_\tau}\sum_{i=1}^{n_U}\sum_{l=1}^{n_\tau} \frac{2*\left[\begin{array}{l} (p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l))(p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l)) \\ +(q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l))(q_{\Omega_2}(\tau_l)+\nu_{\Omega_2}(\theta_i,\tau_l)) \\ +(r_{\Omega_1}(\theta_i,\tau_l)+\eta_{\Omega_1}(\theta_i,\tau_l))(r_{\Omega_2}(\theta_i,\tau_l)+\eta_{\Omega_2}(\theta_i,\tau_l)) \end{array}\right]}{\left(\begin{array}{l} (p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l))^2+(q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l))^2 \\ +(r_{\Omega_1}(\theta_i,\tau_l)+\eta_{\Omega_1}(\theta_i,\tau_l))^2+(p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l))^2 \\ +(q_{\Omega_2}(\theta_i,\tau_l)+\nu_{\Omega_2}(\theta_i,\tau_l))^2+(r_{\Omega_2}(\theta_i,\tau_l)+\eta_{\Omega_2}(\theta_i,\tau_l))^2 \end{array}\right)} \quad (5)$$

Example 4.2. Let $U = \{\theta_1,\theta_2\}$ be a universe of discourse; $\tilde{\tau} = \{\tau_1,\tau_2,\tau_3\}$ be time period; Ω_1 and Ω_2 be two TCNS in U

$$\Omega_1(\theta_1) = \left\{ \begin{array}{l} \langle 0.1e^{j0.2}, 0.6e^{j0.45}, 0.3e^{j0.56}\rangle, \langle 0.8e^{j0.77}, 0.2e^{j0.86}, 0.4e^{j0.65}\rangle, \\ \langle 0.6e^{j0.52}, 0.2e^{j0.2}, 0.36e^{j0.2}\rangle \end{array} \right\}$$

$$\Omega_1(\theta_2) = \left\{ \begin{array}{l} \langle 0.5e^{j0.12}, 0.2e^{j0.65}, 0.5e^{j0.85}\rangle, \langle 0.3e^{j0.8}, 0.7e^{j0.23}, 0.1e^{j0.32}\rangle, \\ \langle 0.74e^{j0.6}, 0.5e^{j0.15}, 0.66e^{j0.41}\rangle \end{array} \right\}$$

$$\Omega_2(\theta_1) = \left\{ \begin{array}{l} \langle 0.6e^{j0.8}, 0.9e^{j0.25}, 0.4e^{j0.52}\rangle, \langle 0.6e^{j0.62}, 0.5e^{j0.24}, 0.3e^{j0.18}\rangle, \\ \langle 0.3e^{j0.82}, 0.3e^{j0.6}, 0.4e^{j0.71}\rangle \end{array} \right\}$$

$$\Omega_2(\theta_2)=\left\{\begin{array}{l}\left\langle 0.3e^{j0.22},0.4e^{j0.56},0.6e^{j0.26}\right\rangle,\left\langle 0.1e^{j0.38},0.5e^{j0.43},0.5e^{j0.24}\right\rangle,\\ \left\langle 0.2e^{j0.29},0.4e^{j0.72},0.5e^{j0.65}\right\rangle\end{array}\right\}$$

Then, the TCNS - Dice similarity measure between Ω_1 and Ω_2 can be: $Sim_D(\Omega_1,\Omega_2)=0.87942$

Theorem 4.3. *Let Ω_1 and Ω_2 be two TCNSs then,*

(1) $0\leq Sim_D(\Omega_1,\Omega_2)\leq 1$

(2) $Sim_D(\Omega_1,\Omega_2)=Sim_D(\Omega_2,\Omega_1)$

(3) $Sim_D(\Omega_1,\Omega_2)=1$ *, if and only if* $\Omega_1=\Omega_2$

(4) if Ω_3 is a TCNS in U and $\Omega_1\subseteq\Omega_2\subseteq\Omega_3$ then $Sim_D(\Omega_1,\Omega_3)\leq Sim_D(\Omega_1,\Omega_2)$ and $Sim_D(\Omega_1,\Omega_3)\leq Sim_D(\Omega_1,\Omega_2)$

Proof

(1) We can have,

$$2*\left[\begin{array}{l}(p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l))(p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l))\\ +(q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l))(q_{\Omega_2}(\theta_i,\tau_l)+\nu_{\Omega_2}(\theta_i,\tau_l))\\ +(r_{\Omega_1}(\theta_i,\tau_l)+\eta_{\Omega_1}(\theta_i,\tau_l))(r_{\Omega_2}(\theta_i,\tau_l)+\eta_{\Omega_2}(\theta_i,\tau_l))\end{array}\right]$$

$$\leq\left(\begin{array}{l}(p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l))^2+(q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l))^2\\ +(r_{\Omega_1}(\theta_i,\tau_l)+\eta_{\Omega_1}(\theta_i,\tau_l))^2+(p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l))^2\\ +(q_{\Omega_2}(\theta_i,\tau_l)+\nu_{\Omega_2}(\theta_i,\tau_l))^2+(r_{\Omega_2}(\theta_i,\tau_l)+\eta_{\Omega_2}(\theta_i,\tau_l))^2\end{array}\right)$$

Hence, $0\leq Sim_D(\Omega_1,\Omega_2)\leq 1$. Thus, the first inequality is shown to be true.

(2) It is easy to observe that this is true

(3) When $\Omega_1=\Omega_2$, then it implies that $Sim_D(\Omega_1,\Omega_2)=1$. On the other hand, if $Sim_D(\Omega_1,\Omega_2)=1$ then,

$p_{\Omega_1}(\theta_i,\tau_l)=p_{\Omega_2}(\theta_i,\tau_l)$; $\mu_{\Omega_1}(\theta_i,\tau_l)=\mu_{\Omega_2}(\theta_i,\tau_l)$; $q_{\Omega_1}(\theta_i,\tau_l)=q_{\Omega_2}(\theta_i,\tau_l)$; $\nu_{\Omega_1}(\theta_i,\tau_l)=\nu_{\Omega_2}(\theta_i,\tau_l)$; $r_{\Omega_1}(\theta_i,\tau_l)=r_{\Omega_2}(\theta_i,\tau_l)$; $\eta_{\Omega_1}(\theta_i,\tau_l)=\eta_{\Omega_2}(\theta_i,\tau_l)$;

This implies that $\Omega_1=\Omega_2$. Thus, the third inequality is proved.

(4) When $\Omega_1\subseteq\Omega_2\subseteq\Omega_3$, we can have

$p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l)\leq p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l)\leq p_{\Omega_3}(\theta_i,\tau_l)+\mu_{\Omega_3}(\theta_i,\tau_l)$;

$q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l)\geq q_{\Omega_2}(\theta_i,\tau_l)+\nu_{\Omega_2}(\theta_i,\tau_l)\geq q_{\Omega_3}(\theta_i,\tau_l)+\nu_{\Omega_3}(\theta_i,\tau_l)$;

$r_{\Omega_1}(\theta_i,\tau_l)+\eta_{\Omega_1}(\theta_i,\tau_l)\geq r_{\Omega_2}(\theta_i,\tau_l)+\eta_{\Omega_2}(\theta_i,\tau_l)\geq r_{\Omega_3}(\theta_i,\tau_l)+\eta_{\Omega_3}(\theta_i,\tau_l)$;

So, $Sim_D(\Omega_1,\Omega_3)\leq Sim_D(\Omega_1,\Omega_2)$ and $Sim_D(\Omega_1,\Omega_3)\leq Sim_D(\Omega_2,\Omega_3)$. So, the fourth inequality is proved

4.2. *TCNS - Jaccard similarity measure*

Definition 4.4. Let two TCNSs

$\Omega_1 = \left\langle p_{\Omega_1}(\theta_i,\tau_l)\, e^{j\mu_{\Omega_1}(\theta_i,\tau_l)}, q_{\Omega_1}(\theta_i,\tau_l)\, e^{j\nu_{\Omega_1}(\theta_i,\tau_l)}, r_{\Omega_1}(\theta_i,\tau_l)\, e^{j\eta_{\Omega_1}(\theta_i,\tau_l)} \right\rangle$ and $\Omega_2 = \left\langle p_{\Omega_2}(\theta_i,\tau_l)\, e^{j\mu_{\Omega_2}(\theta_i,\tau_l)}, q_{\Omega_2}(\theta_i,\tau_l)\, e^{j\nu_{\Omega_2}(\theta_i,\tau_l)}, r_{\Omega_2}(\theta_i,\tau_l)\, e^{j\eta_{\Omega_2}(\theta_i,\tau_l)} \right\rangle$ and $\theta_i; i = 1, 2, ..., n_U$ and $\tau_l; l = 1, 2, ..., n_\tau$.

$Sim_J(\Omega_1,\Omega_2))$ is a TCNS - Jaccard similarity measure between Ω_1 and Ω_2 and $Sim_J(\Omega_1,\Omega_2))$ is defined as follows:

$$Sim_J(\Omega_1,\Omega_2) = \frac{1}{n_U * n_\tau}\sum_{i=1}^{n_U}\sum_{l=1}^{n_\tau}\frac{\begin{array}{c}(p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l))(p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l))\\ +(q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l))(q_{\Omega_2}(\theta_i,\tau_l)+q_{\Omega_2}(\theta_i,\tau_l))\\ +(r_{\Omega_1}(\theta_i,\tau_l)+r_{\Omega_1}(\theta_i,\tau_l))(r_{\Omega_2}(\theta_i,\tau_l)+r_{\Omega_2}(\theta_i,\tau_l))\end{array}}{\begin{array}{l}(p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l))^2+(q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l))^2\\ +(r_{\Omega_1}(\theta_i,\tau_l)+r_{\Omega_1}(\theta_i,\tau_l))^2+(p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l))^2\\ +(q_{\Omega_2}(\theta_i,\tau_l)+q_{\Omega_2}(\theta_i,\tau_l))^2+(r_{\Omega_2}(\theta_i,\tau_l)+r_{\Omega_2}(\theta_i,\tau_l))^2\\ -\left(\begin{array}{c}(p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l))(p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l))\\ +(q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l))(q_{\Omega_2}(\theta_i,\tau_l)+q_{\Omega_2}(\theta_i,\tau_l))\\ +(r_{\Omega_1}(\theta_i,\tau_l)+r_{\Omega_1}(\theta_i,\tau_l))(r_{\Omega_2}(\theta_i,\tau_l)+r_{\Omega_2}(\theta_i,\tau_l))\end{array}\right)\end{array}} \quad (6)$$

Example 4.5. Let $U = \{\theta_1,\theta_2\}; \tilde{\tau} = \{\tau_1,\tau_2,\tau_3\}$; and two TCNS Ω_1, Ω_2 in U.

$$\Omega_1(\theta_1) = \left\{\begin{array}{l}\langle 0.1e^{j0.2}, 0.6e^{j0.45}, 0.3e^{j0.56}\rangle, \langle 0.8e^{j0.77}, 0.2e^{j0.86}, 0.4e^{j0.65}\rangle,\\ \langle 0.6e^{j0.52}, 0.2e^{j0.2}, 0.36e^{j0.2}\rangle\end{array}\right\}$$

$$\Omega_1(\theta_2) = \left\{\begin{array}{l}\langle 0.5e^{j0.12}, 0.2e^{j0.65}, 0.5e^{j0.85}\rangle, \langle 0.3e^{j0.8}, 0.7e^{j0.23}, 0.1e^{j0.32}\rangle,\\ \langle 0.74e^{j0.6}, 0.5e^{j0.15}, 0.66e^{j0.41}\rangle\end{array}\right\}$$

$$\Omega_2(\theta_1) = \left\{\begin{array}{l}\langle 0.6e^{j0.8}, 0.9e^{j0.25}, 0.4e^{j0.52}\rangle, \langle 0.6e^{j0.62}, 0.5e^{j0.24}, 0.3e^{j0.18}\rangle,\\ \langle 0.3e^{j0.82}, 0.3e^{j0.6}, 0.4e^{j0.71}\rangle\end{array}\right\}$$

$$\Omega_2(\theta_2) = \left\{\begin{array}{l}\langle 0.3e^{j0.22}, 0.4e^{j0.56}, 0.6e^{j0.26}\rangle, \langle 0.1e^{j0.38}, 0.5e^{j0.43}, 0.5e^{j0.24}\rangle,\\ \langle 0.2e^{j0.29}, 0.4e^{j0.72}, 0.5e^{j0.65}\rangle\end{array}\right\}$$

$Sim_J(\Omega_1,\Omega_2)$ is The TTCNS - Jaccard similarity measure between Ω_1, Ω_2 can be: $Sim_J(\Omega_1,\Omega_2) = 0.78807$

Theorem 4.6. *Let Ω_1 and Ω_2 be two TCNSs then,*

(1) $0 \le Sim_J(\Omega_1,\Omega_2) \le 1$

(2) $Sim_J(\Omega_1,\Omega_2) = Sim_J(\Omega_2,\Omega_1)$

(3) $Sim_J(\Omega_1,\Omega_2) = 1$, *if and only if* $\Omega_1 = \Omega_2$

(4) if Ω_3 is a TCNS in U and $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3$ then $Sim_J(\Omega_1,\Omega_3) \le Sim_J(\Omega_1,\Omega_2)$ and $Sim_J(\Omega_1,\Omega_3) \le Sim_J(\Omega_2,\Omega_3)$

Proof

(1) Since:

$$\left(\begin{array}{l}\left(p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l)\right)\left(p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l)\right)\\+\left(q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l)\right)\left(q_{\Omega_2}(\theta_i,\tau_l)+\nu_{\Omega_2}(\theta_i,\tau_l)\right)\\+\left(r_{\Omega_1}(\theta_i,\tau_l)+\eta_{\Omega_1}(\theta_i,\tau_l)\right)\left(r_{\Omega_2}(\theta_i,\tau_l)+\eta_{\Omega_2}(\theta_i,\tau_l)\right)\end{array}\right)$$
$$\leq\left(\begin{array}{l}\left(p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l)\right)^2+\left(q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l)\right)^2\\+\left(r_{\Omega_1}(\theta_i,\tau_l)+\eta_{\Omega_1}(\theta_i,\tau_l)\right)^2+\left(p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l)\right)^2\\+\left(q_{\Omega_2}(\theta_i,\tau_l)+\nu_{\Omega_2}(\theta_i,\tau_l)\right)^2+\left(r_{\Omega_2}(\theta_i,\tau_l)+\eta_{\Omega_2}(\theta_i,\tau_l)\right)^2\\-\left[\begin{array}{l}\left(p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l)\right)\left(p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l)\right)\\+\left(q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l)\right)\left(q_{\Omega_2}(\theta_i,\tau_l)+\nu_{\Omega_2}(\theta_i,\tau_l)\right)\\+\left(r_{\Omega_1}(\theta_i,\tau_l)+\eta_{\Omega_1}(\theta_i,\tau_l)\right)\left(r_{\Omega_2}(\theta_i,\tau_l)+\eta_{\Omega_2}(\theta_i,\tau_l)\right)\end{array}\right]\end{array}\right)$$

Hence, we have, $0 \leq Sim_J(\Omega_1,\Omega_2) \leq 1$

(2) It is easily observed.

(3) When $\Omega_1=\Omega_2$, then, $Sim_J(\Omega_1,\Omega_2)=1$. On the other hand, if $Sim_J(\Omega_1,\Omega_2)=1$ then, $p_{\Omega_1}(\theta_i,\tau_l)=p_{\Omega_2}(\theta_i,\tau_l)$; $\mu_{\Omega_1}(\theta_i,\tau_l)=\mu_{\Omega_2}(\theta_i,\tau_l)$; $q_{\Omega_1}(\theta_i,\tau_l)=q_{\Omega_2}(\theta_i,\tau_l)$; $\nu_{\Omega_1}(\theta_i,\tau_l)=\nu_{\Omega_2}(\theta_i,\tau_l)$; $r_{\Omega_1}(\theta_i,\tau_l)=r_{\Omega_2}(\theta_i,\tau_l)$; $\eta_{\Omega_1}(\theta_i,\tau_l)=\eta_{\Omega_2}(\theta_i,\tau_l)$

This implies that $\Omega_1=\Omega_2$.

(4) When $\Omega_1\subseteq\Omega_2\subseteq\Omega_3$, we can write $p_{\Omega_1}(\theta_i,\tau_l)+\mu_{\Omega_1}(\theta_i,\tau_l)\leq p_{\Omega_2}(\theta_i,\tau_l)+\mu_{\Omega_2}(\theta_i,\tau_l)\leq p_{\Omega_3}(\theta_i,\tau_l)+\mu_{\Omega_3}(\theta_i,\tau_l)$;

$q_{\Omega_1}(\theta_i,\tau_l)+\nu_{\Omega_1}(\theta_i,\tau_l)\geq q_{\Omega_2}(\theta_i,\tau_l)+\nu_{\Omega_2}(\theta_i,\tau_l)\geq q_{\Omega_3}(\theta_i,\tau_l)+\nu_{\Omega_3}(\theta_i,\tau_l)$;

$r_{\Omega_1}(\theta_i,\tau_l)+\eta_{\Omega_1}(\theta_i,\tau_l)\geq r_{\Omega_2}(\theta_i,\tau_l)+\eta_{\Omega_2}(\theta_i,\tau_l)\geq r_{\Omega_3}(\theta_i,\tau_l)+\eta_{\Omega_3}(\theta_i,\tau_l)$;

So, $Sim_J(\Omega_1,\Omega_3)\leq S_J(\Omega_1,\Omega_2)$ and $Sim_J(\Omega_1,\Omega_3)\leq Sim_J(\Omega_2,\Omega_3)$.

4.3. *TCNS - Cosine Similarity*

The cosine similarity measure is one of the practical measures in the decision-making process. The similarity between two TCNS can be computed with the use of the temporal complex neutrosophic cosine measure. Time factors and time cycles are of interest to this metric, which employs complex neutrosophic numbers.

The cosine similarity measure for TCNSs is defined as follows:

Definition 4.7. Let $\Omega_1=\left\langle p_{\Omega_1}(\theta_i,\tau_l)e^{j\mu_{\Omega_1}(\theta_i,\tau_l)}, q_{\Omega_1}(\theta_i,\tau_l)e^{j\nu_{\Omega_1}(\theta_i,\tau_l)}, r_{\Omega_1}(\theta_i,\tau_l)e^{j\eta_{\Omega_1}(\theta_i,\tau_l)}\right\rangle$ and $\Omega_2=\left\langle p_{\Omega_2}(\theta_i,\tau_l)e^{j\mu_{\Omega_2}(\theta_i,\tau_l)}, q_{\Omega_2}(\theta_i,\tau_l)e^{j\nu_{\Omega_2}(\theta_i,\tau_l)}, r_{\Omega_2}(\theta_i,\tau_l)e^{j\eta_{\Omega_2}(\theta_i,\tau_l)}\right\rangle$ be two TCNSs and $\theta_i\,(1,2,3,...,n_U)$ belong to U and $\tau_l\,(1,2,3,...,n_\tau)$.

$Sim_{\text{Cos}}(\Omega_1, \Omega_2)$ is A TCNS - Cosine similarity measure between TCNSs Ω_1 and Ω_2 and $Sim_{\text{Cos}}(\Omega_1, \Omega_2)$ is proposed as follows:

$$Sim_{\text{Cos}}(\Omega_1, \Omega_2)$$
$$= \frac{1}{n_U * n_\tau} \sum_{i=1}^{n_U} \sum_{l=1}^{n_\tau} \frac{\begin{array}{l} (p_{\Omega_1}(\theta_i, \tau_l) + \mu_{\Omega_1}(\theta_i, \tau_l))(p_{\Omega_2}(\theta_i, \tau_l) + \mu_{\Omega_2}(\theta_i, \tau_l)) \\ + (q_{\Omega_1}(\theta_i, \tau_l) + \nu_{\Omega_1}(\theta_i, \tau_l))(q_{\Omega_2}(\theta_i, \tau_l) + q_{\Omega_2}(\theta_i, \tau_l)) \\ + (r_{\Omega_1}(\theta_i, \tau_l) + r_{\Omega_1}(\theta_i, \tau_l))(r_{\Omega_2}(\theta_i, \tau_l) + r_{\Omega_2}(\theta_i, \tau_l)) \end{array}}{\begin{array}{l} \sqrt{\begin{array}{l} (p_{\Omega_1}(\theta_i, \tau_l) + \mu_{\Omega_1}(\theta_i, \tau_l))^2 + (q_{\Omega_1}(\theta_i, \tau_l) + \nu_{\Omega_1}(\theta_i, \tau_l))^2 \\ + (r_{\Omega_1}(\theta_i, \tau_l) + r_{\Omega_1}(\theta_i, \tau_l))^2 \end{array}} \\ \times \sqrt{\begin{array}{l} (p_{\Omega_2}(\theta_i, \tau_l) + \mu_{\Omega_2}(\theta_i, \tau_l))^2 + (q_{\Omega_2}(\theta_i, \tau_l) + q_{\Omega_2}(\theta_i, \tau_l))^2 \\ + (r_{\Omega_2}(\theta_i, \tau_l) + r_{\Omega_2}(\theta_i, \tau_l))^2 \end{array}} \end{array}} \tag{7}$$

Example 4.8. Let $U = \{\theta_1, \theta_2\}$;$\tilde{\tau} = \{\tau_1, \tau_2, \tau_3\}$ and Ω_1 and Ω_2 are two TCNS in U

$$\Omega_1(\theta_1) = \left\{ \begin{array}{l} \langle 0.1e^{j0.2}, 0.6e^{j0.45}, 0.3e^{j0.56} \rangle, \langle 0.8e^{j0.77}, 0.2e^{j0.86}, 0.4e^{j0.65} \rangle, \\ \langle 0.6e^{j0.52}, 0.2e^{j0.2}, 0.36e^{j0.2} \rangle \end{array} \right\}$$

$$\Omega_1(\theta_2) = \left\{ \begin{array}{l} \langle 0.5e^{j0.12}, 0.2e^{j0.65}, 0.5e^{j0.85} \rangle, \langle 0.3e^{j0.8}, 0.7e^{j0.23}, 0.1e^{j0.32} \rangle, \\ \langle 0.74e^{j0.6}, 0.5e^{j0.15}, 0.66e^{j0.41} \rangle \end{array} \right\}$$

$$\Omega_2(\theta_1) = \left\{ \begin{array}{l} \langle 0.6e^{j0.8}, 0.9e^{j0.25}, 0.4e^{j0.52} \rangle, \langle 0.6e^{j0.62}, 0.5e^{j0.24}, 0.3e^{j0.18} \rangle, \\ \langle 0.3e^{j0.82}, 0.3e^{j0.6}, 0.4e^{j0.71} \rangle \end{array} \right\}$$

$$\Omega_2(\theta_2) = \left\{ \begin{array}{l} \langle 0.3e^{j0.22}, 0.4e^{j0.56}, 0.6e^{j0.26} \rangle, \langle 0.1e^{j0.38}, 0.5e^{j0.43}, 0.5e^{j0.24} \rangle, \\ \langle 0.2e^{j0.29}, 0.4e^{j0.72}, 0.5e^{j0.65} \rangle \end{array} \right\}$$

We have the following results:

$Sim_{\text{Cos}}(\Omega_1, \Omega_2) = 0.73203$

Theorem 4.9. *Let Ω_1 and Ω_2 be two TCNSs then,*

(1) $0 \leq Sim_{\text{Cos}}(\Omega_1, \Omega_2) \leq 1$

(2) $Sim_{\text{Cos}}(\Omega_1, \Omega_2) = Sim_{\text{Cos}}(\Omega_2, \Omega_1)$

(3) $Sim_{\text{Cos}}(\Omega_1, \Omega_2) = 1$, *if and only if* $\Omega_1 = \Omega_2$

(4) if Ω_3 *is a TCNS in* U *and* $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3$ *then* $Sim_{\text{Cos}}(\Omega_1, \Omega_3) \leq Sim_{\text{Cos}}(\Omega_1, \Omega_2)$ *and* $Sim_{\text{Cos}}(\Omega_1, \Omega_3) \leq Sim_{\text{Cos}}(\Omega_2, \Omega_3)$

Proof

(1) We can have,

$$\left(\begin{array}{l} \left(p_{\Omega_1}\left(\theta_i,\tau_l\right)+\mu_{\Omega_1}\left(\theta_i,\tau_l\right)\right)\left(p_{\Omega_2}\left(\theta_i,\tau_l\right)+\mu_{\Omega_2}\left(\theta_i,\tau_l\right)\right) \\ +\left(q_{\Omega_1}\left(\theta_i,\tau_l\right)+\nu_{\Omega_1}\left(\theta_i,\tau_l\right)\right)\left(q_{\Omega_2}\left(\theta_i,\tau_l\right)+\nu_{\Omega_2}\left(\theta_i,\tau_l\right)\right) \\ +\left(r_{\Omega_1}\left(\theta_i,\tau_l\right)+\eta_{\Omega_1}\left(\theta_i,\tau_l\right)\right)\left(r_{\Omega_2}\left(\theta_i,\tau_l\right)+\eta_{\Omega_2}\left(\theta_i,\tau_l\right)\right) \end{array}\right)^2$$
$$\leq\left(\begin{array}{l} \left(p_{\Omega_1}\left(\theta_i,\tau_l\right)+\mu_{\Omega_1}\left(\theta_i,\tau_l\right)\right)^2+\left(q_{\Omega_1}\left(\theta_i,\tau_l\right)+\nu_{\Omega_1}\left(\theta_i,\tau_l\right)\right)^2 \\ +\left(r_{\Omega_1}\left(\theta_i,\tau_l\right)+\eta_{\Omega_1}\left(\theta_i,\tau_l\right)\right)^2+\left(p_{\Omega_2}\left(\theta_i,\tau_l\right)+\mu_{\Omega_2}\left(\theta_i,\tau_l\right)\right)^2 \\ +\left(q_{\Omega_2}\left(\theta_i,\tau_l\right)+\nu_{\Omega_2}\left(\theta_i,\tau_l\right)\right)^2+\left(r_{\Omega_2}\left(\theta_i,\tau_l\right)+\eta_{\Omega_2}\left(\theta_i,\tau_l\right)\right)^2 \end{array}\right)$$

Hence, $0 \leq S_{\text{Cos}}\left(\Omega_1,\Omega_2\right) \leq 1$

(2) It is easily observed.

(3) Since $\Omega_1=\Omega_2$, then $Sim_{\text{Cos}}\left(\Omega_1,\Omega_2\right)=1$. On the other hand, if $Sim_{\text{Cos}}\left(\Omega_1,\Omega_2\right)=1$, we have,

$p_{\Omega_1}\left(\theta_i,\tau_l\right)=p_{\Omega_2}\left(\theta_i,\tau_l\right);\mu_{\Omega_1}\left(\theta_i,\tau_l\right)=\mu_{\Omega_2}\left(\theta_i,\tau_l\right);\ q_{\Omega_1}\left(\theta_i,\tau_l\right)=q_{\Omega_2}\left(\theta_i,\tau_l\right).;\ \nu_{\Omega_1}\left(\theta_i,\tau_l\right)=\nu_{\Omega_2}\left(\theta_i,\tau_l\right);\ r_{\Omega_1}\left(\theta_i,\tau_l\right)=r_{\Omega_2}\left(\theta_i,\tau_l\right);\ \eta_{\Omega_1}\left(\theta_i,\tau_l\right)=\eta_{\Omega_2}\left(\theta_i,\tau_l\right);$

This implies that $\Omega_1=\Omega_2$

(4) When $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3$, we can have

$p_{\Omega_1}\left(\theta_i,\tau_l\right)+\mu_{\Omega_1}\left(\theta_i,\tau_l\right) \leq p_{\Omega_2}\left(\theta_i,\tau_l\right)+\mu_{\Omega_2}\left(\theta_i,\tau_l\right) \leq p_{\Omega_3}\left(\theta_i,\tau_l\right)+\mu_{\Omega_3}\left(\theta_i,\tau_l\right);$

$q_{\Omega_1}\left(\theta_i,\tau_l\right)+\nu_{\Omega_1}\left(\theta_i,\tau_l\right) \geq q_{\Omega_2}\left(\theta_i,\tau_l\right)+\nu_{\Omega_2}\left(\theta_i,\tau_l\right) \geq q_{\Omega_3}\left(\theta_i,\tau_l\right)+\nu_{\Omega_3}\left(\theta_i,\tau_l\right);$

$r_{\Omega_1}\left(\theta_i,\tau_l\right)+\eta_{\Omega_1}\left(\theta_i,\tau_l\right) \geq r_{\Omega_2}\left(\theta_i,\tau_l\right)+\eta_{\Omega_2}\left(\theta_i,\tau_l\right) \geq r_{\Omega_3}\left(\theta_i,\tau_l\right)+\eta_{\Omega_3}\left(\theta_i,\tau_l\right);$

So, $Sim_J\left(\Omega_1,\Omega_3\right) \leq Sim_J\left(\Omega_1,\Omega_2\right)$ and $Sim_J\left(\Omega_1,\Omega_3\right) \leq Sim_J\left(\Omega_2,\Omega_3\right)$

Hence, Theorem 4.9 is proved.

4.4. *TCNS - Cotangent Similarity*

In this part, we utilize the special characteristics of the Cotangent function to present a novel method of similarity measurement in the context of TCNS.

Definition 4.10. Let two TCNSs

$\Omega_1=\left\langle p_{\Omega_1}\left(\theta_i,\tau_l\right)e^{j\mu_{\Omega_1}\left(\theta_i,\tau_l\right)}, q_{\Omega_1}\left(\theta_i,\tau_l\right)e^{j\nu_{\Omega_1}\left(\theta_i,\tau_l\right)}, r_{\Omega_1}\left(\theta_i,\tau_l\right)e^{j\eta_{\Omega_1}\left(\theta_i,\tau_l\right)}\right\rangle$ and $\Omega_2=\left\langle p_{\Omega_2}\left(\theta_i,\tau_l\right)e^{j\mu_{\Omega_2}\left(\theta_i,\tau_l\right)}, q_{\Omega_2}\left(\theta_i,\tau_l\right)e^{j\nu_{\Omega_2}\left(\theta_i,\tau_l\right)}, r_{\Omega_2}\left(\theta_i,\tau_l\right)e^{j\eta_{\Omega_2}\left(\theta_i,\tau_l\right)}\right\rangle$ for all $\theta_i\,(1,2,3,...,n_U)$ belong to U and $\tau_l\,(1,2,3,...,n_\tau)$.

$Sim_{\text{Cot\_1}}(\Omega_1, \Omega_2)$ is a TCNS - Cotangent similarity measure between TCNSs Ω_1 and Ω_2 and $Sim_{\text{Cot\_1}}(\Omega_1, \Omega_2)$ is defined as follows:

$$Sim_{\text{Cot\_1}}(\Omega_1, \Omega_2) = cot\left(\frac{\pi}{4} + \frac{\pi}{8} \times \frac{1}{n_U * n_\tau}\sum_{i=1}^{n_U}\sum_{l=1}^{n_\tau}\left[\begin{array}{l} max\begin{pmatrix} \mid p_{\Omega_1}(\theta_i, \tau_l) - p_{\Omega_2}(\theta_i, \tau_l) \mid, \\ \mid q_{\Omega_1}(\theta_i, \tau_l) - q_{\Omega_2}(\theta_i, \tau_l) \mid, \\ \mid r_{\Omega_1}(\theta_i, \tau_l) - r_{\Omega_2}(\theta_i, \tau_l) \mid \end{pmatrix} \\ + \frac{1}{2\pi} max\begin{pmatrix} \mid \mu_{\Omega_1}(\theta_i, \tau_l) - \mu_{\Omega_2}(\theta_i, \tau_l) \mid, \\ \mid \nu_{\Omega_1}(\theta_i, \tau_l) - \nu_{\Omega_2}(\theta_i, \tau_l) \mid, \\ \mid \eta_{\Omega_1}(\theta_i, \tau_l) - \eta_{\Omega_2}(\theta_i, \tau_l) \mid \end{pmatrix} \end{array}\right]\right) \tag{8}$$

$$Sim_{\text{Cot\_2}}(\Omega_1, \Omega_2) = cot\left(\frac{\pi}{4} + \frac{\pi}{24} \times \frac{1}{n_U * n_\tau}\sum_{i=1}^{n_U}\sum_{l=1}^{n_\tau}\left[\begin{array}{l} \begin{pmatrix} \mid p_{\Omega_1}(\theta_i, \tau_l) - p_{\Omega_2}(\theta_i, \tau_l) \mid \\ + \mid q_{\Omega_1}(\theta_i, \tau_l) - q_{\Omega_2}(\theta_i, \tau_l) \mid \\ + \mid r_{\Omega_1}(\theta_i, \tau_l) - r_{\Omega_2}(\theta_i, \tau_l) \mid \end{pmatrix} \\ + \frac{1}{2\pi} max\begin{pmatrix} \mid \mu_{\Omega_1}(\theta_i, \tau_l) - \mu_{\Omega_2}(\theta_i, \tau_l) \mid \\ + \mid \nu_{\Omega_1}(\theta_i, \tau_l) - \nu_{\Omega_2}(\theta_i, \tau_l) \mid \\ + \mid \eta_{\Omega_1}(\theta_i, \tau_l) - \eta_{\Omega_2}(\theta_i, \tau_l) \mid \end{pmatrix} \end{array}\right]\right) \tag{9}$$

Example 4.11. Let $U = \{\theta_1, \theta_2\}$; $\tilde{\tau} = \{\tau_1, \tau_2, \tau_3\}$; Ω_1 and Ω_2 are two TCNS in U

$$\Omega_1(\theta_1) = \left\{\begin{array}{l} \langle 0.1e^{j0.2}, 0.6e^{j0.45}, 0.3e^{j0.56}\rangle, \langle 0.8e^{j0.77}, 0.2e^{j0.86}, 0.4e^{j0.65}\rangle, \\ \langle 0.6e^{j0.52}, 0.2e^{j0.2}, 0.36e^{j0.2}\rangle \end{array}\right\}$$

$$\Omega_1(\theta_2) = \left\{\begin{array}{l} \langle 0.5e^{j0.12}, 0.2e^{j0.65}, 0.5e^{j0.85}\rangle, \langle 0.3e^{j0.8}, 0.7e^{j0.23}, 0.1e^{j0.32}\rangle, \\ \langle 0.74e^{j0.6}, 0.5e^{j0.15}, 0.66e^{j0.41}\rangle \end{array}\right\}$$

$$\Omega_2(\theta_1) = \left\{\begin{array}{l} \langle 0.6e^{j0.8}, 0.9e^{j0.25}, 0.4e^{j0.52}\rangle, \langle 0.6e^{j0.62}, 0.5e^{j0.24}, 0.3e^{j0.18}\rangle, \\ \langle 0.3e^{j0.82}, 0.3e^{j0.6}, 0.4e^{j0.71}\rangle \end{array}\right\}$$

$$\Omega_2(\theta_2) = \left\{\begin{array}{l} \langle 0.3e^{j0.22}, 0.4e^{j0.56}, 0.6e^{j0.26}\rangle, \langle 0.1e^{j0.38}, 0.5e^{j0.43}, 0.5e^{j0.24}\rangle, \\ \langle 0.2e^{j0.29}, 0.4e^{j0.72}, 0.5e^{j0.65}\rangle \end{array}\right\}$$

Since Definitions 4.10, we have: $Sim_{Cot\_1}(\Omega_1, \Omega_2) = 0.690485$; $Sim_{Cot\_2}(\Omega_1, \Omega_2) = 0.82243$

Theorem 4.12. *Let Ω_1 and Ω_2 be two TCNSs then,*

(1) $0 \le Sim_{Cot}(\Omega_1, \Omega_2) \le 1$

(2) $Sim_{Cot}(\Omega_1, \Omega_2) = Sim_{Cot}(\Omega_2, \Omega_1)$

(3) $Sim_{Cot}(\Omega_1, \Omega_2) = 1$, *if and only if* $\Omega_1 = \Omega_2$

(4) if Ω_3 is a TCNS in U and $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3$ then $Sim_{Cot}(\Omega_1, \Omega_3) \leq Sim_{Cot}(\Omega_1, \Omega_2)$ and $Sim_{Cot}(\Omega_1, \Omega_3) \leq Sim_{Cot}(\Omega_2, \Omega_3)$

Proof

The proof follows the same general approach as that used for Equations (5) and (6), with specific adaptations for this case.

(1) We have,

Hence, $0 \leq Sim_{Cot}(\Omega_1, \Omega_2) \leq 1$

(2) It is easily observed.

(3) When $\Omega_1 = \Omega_2$, then obviously$S_{Cot_2}(\Omega_1, \Omega_2) = 1$. On the other hand, if $S_{Cot\_2}(\Omega_1, \Omega_2) = 1$ then,

$p_{\Omega_1}(\theta_i, \tau_l) = p_{\Omega_2}(\theta_i, \tau_l);\mu_{\Omega_1}(\theta_i, \tau_l) = \mu_{\Omega_2}(\theta_i, \tau_l);q_{\Omega_1}(\theta_i, \tau_l) = q_{\Omega_2}(\theta_i, \tau_l);$ $\nu_{\Omega_1}(\theta_i, \tau_l) = \nu_{\Omega_2}(\theta_i, \tau_l);$ $r_{\Omega_1}(\theta_i, \tau_l) = r_{\Omega_2}(\theta_i, \tau_l);$ $\eta_{\Omega_1}(\theta_i, \tau_l) = \eta_{\Omega_2}(\theta_i, \tau_l);$

This implies that $\Omega_1 = \Omega_2$

(4) When $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3$, wen can have

$p_{\Omega_1}(\theta_i, \tau_l) + \mu_{\Omega_1}(\theta_i, \tau_l) \leq p_{\Omega_2}(\theta_i, \tau_l) + \mu_{\Omega_2}(\theta_i, \tau_l) \leq p_{\Omega_3}(\theta_i, \tau_l) + \mu_{\Omega_3}(\theta_i, \tau_l);$

$q_{\Omega_1}(\theta_i, \tau_l) + \nu_{\Omega_1}(\theta_i, \tau_l) \geq q_{\Omega_2}(\theta_i, \tau_l) + \nu_{\Omega_2}(\theta_i, \tau_l) \geq q_{\Omega_3}(\theta_i, \tau_l) + \nu_{\Omega_3}(\theta_i, \tau_l);$

$r_{\Omega_1}(\theta_i, \tau_l) + \eta_{\Omega_1}(\theta_i, \tau_l) \geq r_{\Omega_2}(\theta_i, \tau_l) + \eta_{\Omega_2}(\theta_i, \tau_l) \geq r_{\Omega_3}(\theta_i, \tau_l) + \eta_{\Omega_3}(\theta_i, \tau_l);$

So, $Sim_{Cot\_2}(\Omega_1, \Omega_3) \leq Sim_{Cot\_2}(\Omega_1, \Omega_2)$ and $Sim_{Cot\_2}(\Omega_1, \Omega_3) \leq Sim_{Cot\_2}(\Omega_2, \Omega_3)$. The Theorem 4.12 is proved.

4.5. *Entropy measures of Temporal complex neutrosophic set*

When Zadeh [3] first proposed the entropy metric, it was intended to quantify fuzziness in information. A measure of the uncertainty and fuzziness present in any set: fuzzy set, complex fuzzy set, neutrophilic, etc.—is the entropy measure. Since the TCNS can handle ambiguous and uncertain data in this work, finding a TCNS's entropy is particularly important.

Definition 4.13. A function E qualifies as a TCNS - Entropy measure on a finite set U if *E: TCNS(U)* $\rightarrow$ [0, 1] and it satisfies the following properties:

(a) $E(\Omega) = 0$ if Ω is a crisp set

(b) $E(\Omega) = 1$ if $p_\Omega(\theta, \tau) = 0.5;q_\Omega(\theta, \tau) = 0.5;r_\Omega(\theta, \tau) = 0.5$ and $\mu_\Omega(\theta, \tau) = \pi;\nu_\Omega(\theta, \tau) = \pi;\eta_\Omega(\theta, \tau) = \pi$ for $\forall \tau \in \tilde{\tau}; \forall \theta \in U$.

(c) $E(\Omega_1) \leq E(\Omega_2)$ if Ω_2 is more uncertain that Ω_1, that is $| p_{\Omega_1}(\theta, \tau) - 0.5 | \geq | p_{\Omega_2}(\theta, \tau) - 0.5 |$; $| q_{\Omega_1}(\theta, \tau) - 0.5 | \geq | q_{\Omega_2}(\theta, \tau) - 0.5 |$; $| r_{\Omega_1}(\theta, \tau) - 0.5 | \geq | r_{\Omega_2}(\theta, \tau) - 0.5 |$ and

$| \mu_{\Omega_1}(\theta, \tau) - \pi | \geq | \mu_{\Omega_2}(\theta, \tau) - \pi |$; $| \nu_{\Omega_1}(\theta, \tau) - \pi | \geq | \nu_{\Omega_2}(\theta, \tau) - \pi |$; $| \eta_{\Omega_1}(\theta, \tau) - \pi | \geq | \eta_{\Omega_2}(\theta, \tau) - \pi |$

(d) $E(\Omega) = E(\Omega^c)$, Ω^c is a TCNS complement of Ω

Theorem 4.14. *Let a Temporal Complex Neutrosophic Set Ω on U. A TCNS - The following equation suggests entropy metric of Ω:*

$$E(\Omega) = \tan\left(\frac{\pi}{6 * n_U * n_\tau} \sum_{i=1}^{n_U} \sum_{l=1}^{n_\tau} e_{il}(\Omega)\right) \tag{10}$$

Where

$$e_{il}(\Omega) = \begin{pmatrix} p_\Omega(\theta_i, \tau_l)(1 - p_\Omega(\theta_i, \tau_l)) + q_\Omega(\theta_i, \tau_l)(1 - q_\Omega(\theta_i, \tau_l)) \\ + r_\Omega(\theta_i, \tau_l)(1 - r_\Omega(\theta_i, \tau_l)) + \left(\frac{\mu_\Omega(\theta_i, \tau_l)}{2\pi}\right)\left(1 - \frac{\mu_\Omega(\theta_i, \tau_l)}{2\pi}\right) \\ + \left(\frac{\nu_\Omega(\theta_i, \tau_l)}{2\pi}\right)\left(1 - \frac{\nu_\Omega(\theta_i, \tau_l)}{2\pi}\right) + \left(\frac{\eta_\Omega(\theta_i, \tau_l)}{2\pi}\right)\left(1 - \frac{\eta_\Omega(\theta_i, \tau_l)}{2\pi}\right) \end{pmatrix}$$

Proof.

$E(\Omega)$ must strictly adhere to all conditions outlined in Definition 4.13.

(a) Assume that $E(\Omega) = 0$, it implies that $p(\theta_i, \tau_l) = 0$; $q(\theta_i, \tau_l) = 0$; $r(\theta_i, \tau_l) = 0$; $\mu(\theta_i, \tau_l) = 0$; $v(\theta_i, \tau_l) = 0$; $r(\theta_i, \tau_l) = 0$or $p(\theta_i, \tau_l) = 1$; $q(\theta_i, \tau_l) = 1$; $r(\theta_i, \tau_l) = 1$; $\mu(\theta_i, \tau_l) = 2\pi$; $v(\theta_i, \tau_l) = 2\pi$; $r(\theta_i, \tau_l) = 2\pi$;

(b) $E(\Omega) = 1$, if and only if $p(\theta_i, \tau_l) = 0.5$; $q(\theta_i, \tau_l) = 0.5$; $r(\theta_i, \tau_l) = 0.5$; $\mu(\theta_i, \tau_l) = \pi$; $v(\theta_i, \tau_l) = \pi$; $r(\theta_i, \tau_l) = \pi$

(c) Assum Ω_1, Ω_2 and Ω_2 more uncertain than that Ω_2. It implies that, if $p_{\Omega_1}(\theta, \tau) \leq p_{\Omega_2}(\theta, \tau) \leq 0.5$; $q_{\Omega_1}(\theta, \tau) \leq q_{\Omega_2}(\theta, \tau) \leq 0.5$; $r_{\Omega_1}(\theta, \tau) \leq r_{\Omega_2}(\theta, \tau) \leq 0.5$ or $p_{\Omega_1}(\theta, \tau) \geq p_{\Omega_2}(\theta, \tau) \geq 0.5$; $q_{\Omega_1}(\theta, \tau) \geq q_{\Omega_2}(\theta, \tau) \geq 0.5$; $r_{\Omega_1}(\theta, \tau) \geq r_{\Omega_2}(\theta, \tau) \geq 0.5$ for each $\theta_i \in X$.

Hence,

$p_{\Omega 1}(\theta_i, \tau_l)(1 - p_{\Omega_1}(\theta_i, \tau_l)) \leq p_{\Omega 2}(\theta_i, \tau_l)(1 - p_{\Omega_2}(\theta_i, \tau_l))$

$q_{\Omega 1}(\theta_i, \tau_l)(1 - q_{\Omega_1}(\theta_i, \tau_l)) \leq q_{\Omega 2}(\theta_i, \tau_l)(1 - q_{\Omega_2}(\theta_i, \tau_l))$;

$r_{\Omega 1}(\theta_i, \tau_l)(1 - r_{\Omega_1}(\theta_i, \tau_l)) \leq r_{\Omega 2}(\theta_i, \tau_l)(1 - r_{\Omega_2}(\theta_i, \tau_l))$;

and $\frac{\mu_{\Omega_1}(\theta_i,\tau_l)}{2\pi}\left(1 - \frac{\mu_{\Omega_1}(\theta_i,\tau_l)}{2\pi}\right) \leq \frac{\mu_{\Omega_2}(\theta_i,\tau_l)}{2\pi}\left(1 - \frac{\mu_{\Omega_2}(\theta_i,\tau_l)}{2\pi}\right)$;

$\frac{\nu_{\Omega 1}(\theta_i,\tau_l)}{2\pi}\left(1 - \frac{\nu_{\Omega_1}(\theta_i,\tau_l)}{2\pi}\right) \leq \frac{\nu_{\Omega 2}(\theta_i,\tau_l)}{2\pi}\left(1 - \frac{\nu_{\Omega_2}(\theta_i,\tau_l)}{2\pi}\right)$;

$\frac{\eta_{\Omega 1}(\theta_i,\tau_l)}{2\pi}\left(1 - \frac{\eta_{\Omega 1}(\theta_i,\tau_l)}{2\pi}\right) \leq \frac{\eta_{\Omega_2}(\theta_i,\tau_l)}{2\pi}\left(1 - \frac{\eta_{\Omega 2}(\theta_i,\tau_l)}{2\pi}\right)$;

which implies $E(\Omega_1) \leq E(\Omega_2)$

(d) $E(\Omega) = E(\Omega^c)$ is trivial

Example 4.15. Let $U = \{\theta_1, \theta_2\}$;$\tilde{\tau} = \{\tau_1, \tau_2, \tau_3\}$ and Ω_1 and Ω_2 are two TCNS in U

$$\Omega_1(\theta_1) = \left\{ \begin{array}{l} \left\langle 0.1e^{j0.2}, 0.6e^{j0.45}, 0.3e^{j0.56} \right\rangle, \left\langle 0.8e^{j0.77}, 0.2e^{j0.86}, 0.4e^{j0.65} \right\rangle, \\ \left\langle 0.6e^{j0.52}, 0.2e^{j0.2}, 0.36e^{j0.2} \right\rangle \end{array} \right\}$$

$$\Omega_1(\theta_2) = \left\{ \begin{array}{l} \left\langle 0.5e^{j0.12}, 0.2e^{j0.65}, 0.5e^{j0.85} \right\rangle, \left\langle 0.3e^{j0.8}, 0.7e^{j0.23}, 0.1e^{j0.32} \right\rangle, \\ \left\langle 0.74e^{j0.6}, 0.5e^{j0.15}, 0.66e^{j0.41} \right\rangle \end{array} \right\}$$

$$\Omega_2(\theta_1) = \left\{ \begin{array}{l} \left\langle 0.6e^{j0.8}, 0.9e^{j0.25}, 0.4e^{j0.52} \right\rangle, \left\langle 0.6e^{j0.62}, 0.5e^{j0.24}, 0.3e^{j0.18} \right\rangle, \\ \left\langle 0.3e^{j0.82}, 0.3e^{j0.6}, 0.4e^{j0.71} \right\rangle \end{array} \right\}$$

$$\Omega_2(\theta_2) = \left\{ \begin{array}{l} \left\langle 0.3e^{j0.22}, 0.4e^{j0.56}, 0.6e^{j0.26} \right\rangle, \left\langle 0.1e^{j0.38}, 0.5e^{j0.43}, 0.5e^{j0.24} \right\rangle, \\ \left\langle 0.2e^{j0.29}, 0.4e^{j0.72}, 0.5e^{j0.65} \right\rangle \end{array} \right\}$$

Entropy values can be estimated by applying Definition 7 as follows: $E_{\Omega_1} = 0.44476$; $E_{\Omega_2} = 0.47580$;

5. Multi-Criteria Decision Making

Consider that $\tilde{A}$, $\tilde{C}$, $\tilde{D}$ are sets of alternatives, requirements and decision makers; where $\tilde{A} = \{A_1, A_2, ..., A_{n_A}\}$ and $\tilde{C} = \{C_1, C_2, ..., C_{n_C}\}$ and $\tilde{D} = \{D_1, D_2, ..., D_{n_D}\}$. With regard to the decision maker D_{i_d}; $i_d = 1, 2, ..., n_D$, the evaluation description of an alternatives A_{i_a}; $i_a = 1, 2, ..., n_A$ on an attribute C_{i_c}; $i_c = 1, 2, ..., n_C$ in given time period τ_l; $l = 1, 2, ..., n_\tau$ is represented by matrix $U^{i_d}(\tau_l) = \left(\theta_{i_a i_c}^{i_d}(\tau_l)\right)_{m \times n}$. where $\theta_{i_a i_c}^{i_d}(\tau_l)$ chosen as the language identifier of CNS by given period τ_l.

Let
$\theta_{i_a i_c i_d}(\tau_l) = \left\langle p_{i_a i_c i_d}(\tau_l)\, e^{j\mu_{i_a i_c i_d}(\tau_l)}, q_{i_a i_c i_d}(\tau_l)\, e^{j\nu_{i_a i_c i_d}(\tau_l)}, r_{i_a i_c i_d}(\tau_l)\, e^{j\eta_{i_a i_c i_d}(\tau_l)} \right\rangle$

Step 1. In the absence of any pre-existing information regarding criterion weights, we proceed to calculate the weight w_{i_c} of criterion C_{i_c} as follows:

Equation (11) can be used to evaluate the decision makers' averaged rating.

$$\theta_{i_a i_c} = \frac{1}{n_D * n_\tau} \bigoplus_{l=1}^{n_\tau} \bigoplus_{i_a=1}^{n_D} x_{i_a i_c i_d}(\tau_l) = \left\langle \tilde{T}_{i_a i_c}, \tilde{I}_{i_a i_c}, \tilde{F}_{i_a i_c} \right\rangle \tag{11}$$

The weight w_{i_c} of criterion C_{i_c} is calculated by Equation (12)

$$w_{i_c} = \frac{\left(1 - E_{C_{i_c}}\right)}{n_c - \sum_{i=1}^{n_c} E_{C_{i_c}}} \tag{12}$$

Where $E_{C_{i_c}} = \frac{1}{n_A} \sum_{i_a=1}^{n_A} E\left(\theta_{i_a i_c}\right)$; each $E\left(\theta_{i_a i_c}\right)$ is calculated using Equation (10)

Step 2. The temporal complex neutrosophic positive ideal solution and temporal complex neutrosophic negative ideal solution are represented by two values, $TCN-PIS$ and $TCN-NIS$, respectively, and are determined in the following way:

$$TCN-PIS=\left\{\theta,\left\langle \max\left(p_{i_a i_c}\left(\theta,\tilde{\tau}\right)\right)e^{j\max\left(\mu_{i_a i_c}\left(\theta,\tilde{\tau}\right)\right)},0,0\right\rangle\right\} \tag{13}$$

$$TCN-NIS= \\ \left\{\theta,\left\langle 0,\min\left(q_{i_a i_c}\left(\theta,\tilde{\tau}\right)\right)e^{j\min\left(\nu_{i_a i_c}\left(\theta,\tilde{\tau}\right)\right)},\min\left(r_{i_a i_c}\left(\theta,\tilde{\tau}\right)\right)e^{j\min\left(\eta_{i_a i_c}\left(\theta,\tilde{\tau}\right)\right)}\right\rangle\right\} \tag{14}$$

Step 3 For alternatives to $TCN-PIS$ and $TCN-NIS$, we can analyze the weighted similarity index using Theorems 4.3-4.12. And equation (15) is used to calculate the weighted similarity measure $S_{i_a}^{+}$ and $S_{i_a}^{-}$ of A_{i_a} $(i_a=1,2,...,n_A)$-(16)

$$S_{i_a}^{+}=\overset{n_A}{\underset{i_a=1}{\oplus}}w_{i_c}Sim\left(\theta_{i_a i_c},TCN-PIS\right) \tag{15}$$

$$S_{i_a}^{-}=\overset{n_A}{\underset{i_a=1}{\oplus}}w_{i_c}S\left(\theta_{i_a i_c},TCN-NIS\right) \tag{16}$$

Step 4. We determine the relative closeness of alternative A_{i_a} to the ideal solution using the following calculation

$$RR_{i_a}=\frac{S_{i_a}^{-}}{S_{i_a}^{+}+S_{i_a}^{-}} \tag{17}$$

Step 5. Through a comparison based on relative closeness values, we have identified the most promising alternatives, listed in descending order of their suitability.

6. A case study for choosing a tourist destination

A practical use of recommended metrics to a decision problem analyzing a real-world scenario is shown in this section. The case study is a particular illustration of the challenge of selecting a travel destination in Vietnam. This model can be applied to travel companies supporting and advising tourists.

6.1. *Problem description*

The viability and efficacy of the suggested decision technique are illustrated by an example provided by Lan et al. [17] regarding the selection of a tourism location in Vietnam. Figure 1 presents an MCDM model for choosing a tourism destination in Vietnam. Assume for the moment that a business is looking for a tourism attraction in Vietnam. For example, L_i; $i=1,2,3,4,5$ is one of the five tourist locations that the company recommended. There are twenty sub-criteria that are specific to each place and three group criterion that determine which site is the best tourist destination. This enables a more precise evaluation of the different locations.

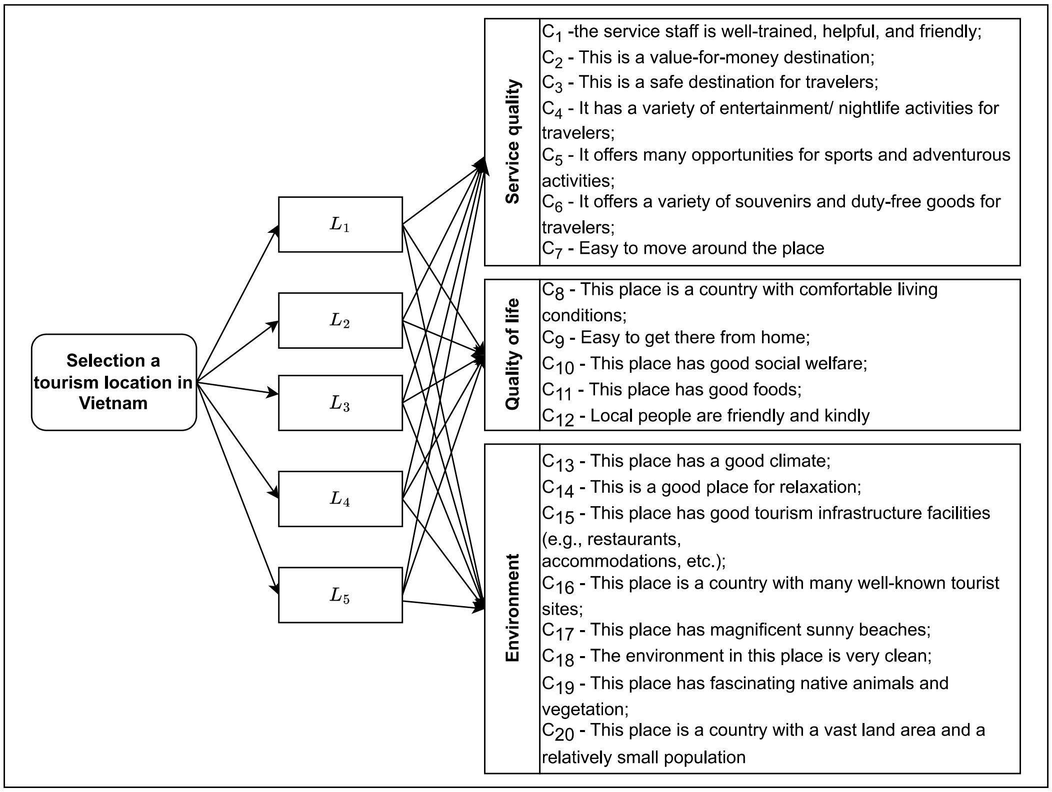

Figure 1. Selection a tourism location in Vietnam

Table 1. Language variables

Label	Number
Very low (VL)	$\langle 0.15e^{j0.55}, 0.65e^{j0.45}, 0.65e^{j0.35} \rangle$
Low (L)	$\langle 0.25e^{j0.65}, 0.55e^{j0.55}, 0.65e^{j0.45} \rangle$
Medium (M)	$\langle 0.40e^{j0.75}, 0.50e^{j0.65}, 0.45e^{j0.55} \rangle$
High (H)	$\langle 0.55e^{j0.85}, 0.45e^{j0.75}, 0.35e^{j0.65} \rangle$
Very high (VH)	$\langle 0.65e^{j0.95}, 0.25e^{j0.85}, 0.25e^{j0.75} \rangle$

Three experts with varying backgrounds and specialties have been invited to conduct a thorough study of the four locations. The estimates made by the expert are represented as TCN numbers.

6.2. *Solution approach using the proposed measures*

TCNS is a technology that helps characterize periodic, ambiguous information that we encounter in our daily lives. In order to show that the suggested similarity measures are useful in resolving real-world issues, such as selecting tourism sites in multi-criteria decision-making problems, we provide a real-world instance in the TCNS environment in this section.

TABLE 2. Similarity measure Values at step 3

Measures	Ideal solution	Alternatives				
		L_1	L_2	L_3	L_4	L_5
Sim_D	S^+	0.63893	0.63931	0.63727	0.63702	0.63459
	S^-	0.65671	0.65388	0.65677	0.66025	0.66090
Sim_j	S^+	0.46947	0.46989	0.46768	0.46740	0.46479
	S^-	0.48892	0.48580	0.48899	0.49283	0.49354
Sim_{Cos}	S^+	0.58630	0.58890	0.58585	0.58270	0.58158
	S^-	0.55933	0.55967	0.56067	0.56004	0.56200
Sim_{Cot_1}	S^+	0.64133	0.62623	0.63643	0.66239	0.65205
	S^-	0.59517	0.57743	0.59477	0.62114	0.61690
Sim_{Cot_2}	S^+	0.718926	0.71437	0.71740	0.72466	0.72094
	S^-	0.78670	0.76771	0.78476	0.81370	0.80613

Step 1. Equations 11 and 12 can be used for calculating the weight of each of the twenty criteria.

$w_1 = 0.04920; w_2 = 0.05787; w_3 = 0.05274; w_4 = 0.04981; w_5 = 0.03842;$

$w_6 = 0.04216; w_7 = 0.04805; w_8 = 0.05547; w_9 = 0.05323; w_{10} = 0.04190;$

$w_{11} = 0.04967; w_{12} = 0.0439; w_{13} = 0.04418; w_{14} = 0.04968; w_{15} = 0.05861;$

$w_{16} = 0.05000; w_{17} = 0.053227; w_{18} = 0.05968; w_{19} = 0.05397; w_{20} = 0.05745;$

Step 2. The positive ideal solution (TCN-PIS) and negative ideal solution (TCN-NIS) of the temporal complex neutrosophic should be estimated. Here's how the values are derived:

$$TCN - PIS = \left\{ \begin{array}{l} \left\langle 0.696238.\ \mathrm{e}^{\mathrm{j}0.916667}, 0, 0 \right\rangle, \left\langle 0.696238.\mathrm{e}^{\mathrm{j}0.916667}, 0, 0 \right\rangle, \\ \left\langle 0.696238.\mathrm{e}^{\mathrm{j}0.916667}, 0, 0 \right\rangle, \left\langle 0.696238.\mathrm{e}^{\mathrm{j}0.916667}, 0, 0 \right\rangle \end{array} \right\}$$

$$TCN - NIS = \left\{ \begin{array}{l} \left\langle 0, 0.30411.\mathrm{e}^{\mathrm{j}0.583333}, 0.279672.\mathrm{e}^{0.483333} \right\rangle, \\ \left\langle 0, 0.30411.\mathrm{e}^{\mathrm{j}0.583333}, 0.279672.\mathrm{e}^{0.483333} \right\rangle, \\ \left\langle 0, 0.30411.\mathrm{e}^{\mathrm{j}0.583333}, 0.279672.\mathrm{e}^{0.483333} \right\rangle, \\ \left\langle 0, 0.30411.\mathrm{e}^{\mathrm{j}0.583333}, 0.279672.\mathrm{e}^{0.483333} \right\rangle \end{array} \right\}$$

Step 3. Determine TCN-PIS and TCN-NIS's similarity metrics for five different places as shown in Table 2.

Step 4. Calculate the proximity of the optimal solution to five different places. They are described in Table 3.

Step 5. Table 3 displays the five places that are ordered according to their relative proximity.

TABLE 3. The relative proximity of places determined via similarity metrics

Mesuares		Alternatives				
		L_1	L_2	L_3	L_4	L_5
Sim_D	RR	0.50686	0.50563	0.50753	0.50895	**0.51015**
	In Order	4	5	3	2	1
Sim_J	RR	0.51015	0.50832	0.51114	0.51324	**0.51500**
	In Order	4	5	3	2	1
Sim_{Cos}	RR	0.48823	0.48728	0.48902	0.49009	**0.49144**
	In Order	4	5	3	2	1
Sim_{Cot_1}	RR	0.48134	0.47973	0.48308	0.48393	**0.48615**
	In Order	4	5	3	2	1
Sim_{Cot_2}	RR	0.52251	0.51800	0.52242	**0.52894**	0.52789
	In Order	3	5	4	1	2

6.3. *Result and Discussion*

In this section, We validate the proposed model's efficacy through comparison with Ali et al.'s models (2020) [38] in temporal complex neutrosophic environments. In Table 3, the ranking results of 5 locations at four different periods, $\tau_1, \tau_4, \tau_3, \tau_4$, are suitably explained for the problem of selecting a tourist destination. According to Table 4, the rankings of five locations at four times each of τ_1, τ_4, τ_3, and τ_4 are $L_5 \succ L_4 \succ L_1 \succ L_3 \succ L_2$; $L_5 \succ L_4 \succ L_3 \succ L_1 \succ L_2$; $L_5 \succ L_4 \succ L_3 \succ L_1 \succ L_2$ and $L_5 \succ L_4 \succ L_1 \succ L_3 \succ L_2$ and in all four time periods $L_5$is assigned as the optimal location. In the meantime, the suggested method's overall outcome at four different periods based on the metrics: Dice, Jaccard, Cosine, and Cotangent similarity, respectively, are $L_5 \succ L_4 \succ L_3 \succ L_1 \succ L_2$; $L_5 \succ L_4 \succ L_3 \succ L_1 \succ L_2$; $L_5 \succ L_4 \succ L_3 \succ L_1 \succ L_2$; $L_4 \succ L_5 \succ L_1 \succ L_3 \succ L_2$. The best location according to the combined results of the Cosine, Jaccard, and Dice similarity measures is L_5; the best location according to the Cotangent similarity measure is L_4.

This finding demonstrates how the benefits and practical application of the suggested measures handle time-related decision-making challenges in a TCNS. Furthermore, in TCN contexts, it is more flexible and generalized than the technique of Ali et al. [38]. The ability to handle periodic and temporal factors in the data effectively is one of TCNS's advantages over standard CNS. The proposed approach has the considerable advantage of taking into account the impact of temporal aspects in CNS disorders while also reducing information loss. In other words, we argue that new similarity and distance measures will provide decision-makers with various options based on their optimistic and pessimistic behavior during the decision-making process.

Table 4. Results of Ali et al. method for locations at four periods

	Measures	Alternitives				
		L_1	L_2	L_3	L_4	L_5
τ_1	RR	0.81427	0.80866	0.81220	0.81627	**0.81948**
	In Order	3	5	4	2	1
τ_2	RR	0.81330	0.81059	0.81335	0.81762	**0.82221**
	In Order	4	5	3	2	1
τ_3	RR	0.81413	0.80887	0.81348	0.81707	**0.82197**
	In Order	4	5	3	2	1
τ_4	RR	0.81316	0.80860	0.81309	0.81665	**0.81934**
	In Order	3	5	4	2	1

7. Conclusion, Limitations, and Future Works

TCNS theory is a valuable tool for solving problems concerning the uncertain, temporal, and periodical factors of decision-making. In this paper, we have introduced four similarity measures: Dice, Jaccard, Cosine, and cotangent in the context of the TCNS. Furthermore, we have established an entropy measure of TCNS to ascertain the weights of unknown attributes in MCDM. Next, based on suggested entropy and similarity metrics in the TCNS environment, a new MCDM technique has also been devised. Lastly, the TCN environment provides a numerical example of decision-making issues when selecting a tourist location in Vietnam. It is provided to highlight the benefits and real-world suitability of the suggested actions. It is demonstrated through the case study that the suggested TCNS entropy and similarity measurements can yield credible outcomes for decision-making issues. However, the proposed measures only use the discrete temporal variables and apply to only a practical problem of tourist destinations chosen in Vietnam. Hence, in future follow-up research, we can use TCNS to account for continuous time variables and apply our model to other real-life problems, including selecting workers, medical treatment, logistics center choosing, etc.

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