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Statistical Convergence Sequences in Neutrosophic Metric Spaces

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Abstract. In this study, motivated by the notion of statistical convergence in intuitionistic fuzzy metric space, we present the ideas of statistical convergence and cauchy sequences with regard to the neutrosophic metric spaces. This study also discusses the statistical completeness on neutrosophic metric space with an example and results.

Keywords: Neutrosophic metric; Statistically convergent sequence; Statistically Cauchy sequence

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1. **Introduction**

In 2004, utilising the intuitionistic fuzzy set [1], Fuzzy Metric Spaces ($\mathfrak{FMS}$) has been widened to Intuitionistic Fuzzy Metric Space ($\mathfrak{IFMS}$) by Park [13]. Park used continuous triangular norms as well as continuous triangular conorms to describe this idea. For $\mathfrak{FMS}$, Changqing et al. [4] developed Statistical Convergence ($\mathfrak{SC}$) sequences in 2020. On $\mathfrak{IFMS}$, similar ideas were used by several auhtors and their work, who achieved considerable outcomes. Fast [5] established the concept of $\mathfrak{SC}$ in 1951, garnering interest from researchers across both purely mathematical and practical disciplines. $\mathfrak{FMS}$ and $\mathfrak{IFS}$ were used for investigating several new breakthroughs, including fixed point theories and convergence by Granados et. al [7]. In $\mathfrak{NMS}$, similar outcomes can be derived by Jeyaraman et al [9]. In [10], Kramosil and Michalek presented the idea of $\mathfrak{FMS}$. Fuzzy sets were first presented by Zadeh [21], and numerous writers have since addressed their notions under numerous contexts, including $\mathfrak{FMS}$.

The concept of generalization of convergence, denoted by $\mathfrak{SC}$, is defined as follows: Let $\mathfrak{M} \subseteq \mathfrak{I}N$, where $\mathfrak{I}N$ represents the set of all positive integers. For every $\mathfrak{n} \in \mathfrak{I}N$, $\mathfrak{M}(\mathfrak{n})$ is defined as $\{\lambda \leq \mathfrak{n} : \lambda \in \mathfrak{M}\}$. The natural (or asymptotic) density of $\mathfrak{M}$, denoted by $\Delta(\mathfrak{M})$, is defined by $\Delta(\mathfrak{M}) = \lim_{n\to\infty} \frac{|\mathfrak{M}(\mathfrak{n})|}{\mathfrak{n}}$ if the limit exists, where $|\mathfrak{M}(\mathfrak{n})|$ represents the cardinality of the set $\mathfrak{M}(\mathfrak{n})$. The value of $\Delta(\mathfrak{M})$ lies in the interval $[0, 1]$ and satisfies $\Delta(\mathfrak{I}N\backslash\mathfrak{M}) = 1 - \Delta(\mathfrak{M})$ if $\Delta(\mathfrak{M})$ exists.

For instance, $\Delta(\mathfrak{I}\varrho) = 1$, $\Delta(\mathfrak{P}) = \frac{1}{2}$ where $\mathfrak{P}$ denotes the set of even positive integers, and $\Delta(\mathfrak{B}) = 0$ where $\mathfrak{B}$ is a finite subset of $\mathfrak{I}N$. A set $\mathfrak{M}$ is considered statistically dense if $\Delta(\mathfrak{M}) = 1$. A sequence $(\zeta_\mathfrak{n}) \subset \mathfrak{I}R$ is said to be $\mathfrak{SC}$ to $\zeta_0 \in \mathfrak{I}R$ if for every $\mathfrak{e} > 0$, the condition $\Delta(\{\mathfrak{n} \in \mathfrak{I}N : |\zeta_\mathfrak{n} - \zeta_0| < \mathfrak{e}\}) = 1$ is satisfied, where $\mathfrak{I}R$ denotes the set of all real numbers. Numerous significant results on $\mathfrak{SC}$ have been presented by various authors ([?]- [19]).

in this work, convergent and $\mathfrak{SC}$ relations in $\mathfrak{NMS}$ are then investigated. On $\mathfrak{NMS}$, we also investigate $\mathfrak{SCa}$ sequences and statistical completeness.

2. Preliminaries

Definition 2.1. [7] Let κ, ϱ, and φ be fuzzy sets on $\Omega^2 \times (0, \infty)$, where $\odot$ is a continuous triangular norm and $\oplus$ is a continuous triangular conorm. We say that $(\kappa, \varrho, \varphi)$ forms a Neutrosophic Metric ($\mathfrak{NM}$) on Ω if κ and ϱ satisfy the following conditions:

(1) $\kappa(\zeta, \xi, \varpi) + \varrho(\zeta, \xi, \varpi) + \varphi(\zeta, \xi, \varpi) \leq 3$;
(2) $0 < \kappa(\zeta, \xi, \varpi) < 1$, $0 < \varrho(\zeta, \xi, \varpi) < 1$ and $0 < \varphi(\zeta, \xi, \varpi) < 1$;
(3) $\kappa(\zeta, \xi, \varpi) > 0$;
(4) $\kappa(\zeta, \xi, \varpi) = 1 \iff \zeta = \xi$;
(5) $\kappa(\zeta, \xi, \varpi) = \kappa(\xi, \zeta, \varpi)$;
(6) $\kappa(\zeta, \xi, \varpi) \odot \kappa(\xi, \tilde{v}, \mathfrak{u}) \leq \kappa(\zeta, \tilde{v}, \varpi + \mathfrak{u})$;
(7) $\kappa(\zeta, \xi, .) : (0, \infty) \to (0, 1]$ is continuous;
(8) $\varrho(\zeta, \xi, \varpi) > 0$;
(9) $\varrho(\zeta, \xi, \varpi) = 0 \iff \zeta = \xi$;
(10) $\varrho(\zeta, \xi, \varpi) = \varrho(\xi, \zeta, \varpi)$;
(11) $\varrho(\zeta, \xi, \varpi) \oplus \varrho(\xi, \tilde{v}, \mathfrak{u}) \geq \varrho(\zeta, \tilde{v}, \varpi + \mathfrak{u})$;
(12) $\varrho(\zeta, \xi, .) : (0, \infty) \to (0, 1]$ is continuous.
(13) $\varphi(\zeta, \xi, \varpi) > 0$;
(14) $\varphi(\zeta, \xi, \varpi) = 0 \iff \zeta = \xi$;
(15) $\varphi(\zeta, \xi, \varpi) = \varphi(\xi, \zeta, \varpi)$;
(16) $\varphi(\zeta, \xi, \varpi) \oplus \varphi(\xi, \tilde{v}, \mathfrak{u}) \geq \varphi(\zeta, \tilde{v}, \varpi + \mathfrak{u})$;
(17) $\varphi(\zeta, \xi, .) : (0, \infty) \to (0, 1]$ is continuous.

A 6-tuple $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ is called $\mathfrak{NMS}$.

The functions $\kappa(\zeta, \xi, \varpi)$, $\varrho(\zeta, \xi, \varpi)$ and $\varphi(\zeta, \xi, \varpi)$ denote nearness degree, non-nearness degree and indeterminacy degree of ζ to ξ at ϖ, respectively.

Example 2.2. Let $\Omega = \Im R, \vartheta \odot \omega = \vartheta\omega$ and $\vartheta \oplus \omega = \min\{\vartheta + \omega, 1\}$ for all $\vartheta, \omega \in [0, 1]$. Define κ, ϱ and φ by $\kappa(\zeta, \xi, \varpi) = \frac{\varpi}{\varpi + |\zeta - \xi|}$, $\varrho(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi + |\zeta - \xi|}$, $\varphi(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi}$ and for all $\zeta, \xi \in \Omega$ and $\varpi > 0$. Then $(\Im R, \kappa, \varrho, \odot, \oplus)$ is a $\mathfrak{NMS}$.

Definition 2.3. Let $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ be a $\mathfrak{NMS}$ and $\varpi > 0, \mathfrak{v} \in (0, 1)$ and $\zeta \in \Omega$. The set $\mathfrak{B}_\zeta(\mathfrak{v}, \varpi) = \{\xi \in \Omega : \kappa(\zeta, \xi, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta, \xi, \varpi) < \mathfrak{v}, \varphi(\zeta, \xi, \varpi) < \mathfrak{v}\}$ is described as an open sphere with center ζ and radius $\mathfrak{v}$ with regard to ϖ.

Example 2.4. Let $\Omega = \Im R, \vartheta \odot \omega = \vartheta\omega$ and $\vartheta \oplus \omega = \min\{\vartheta + \omega, 1\}$ for all $\vartheta, \omega \in [0, 1]$. Define κ, ϱ and φ by $\kappa(\zeta, \xi, \varpi) = \frac{\varpi}{\varpi + |\zeta - \xi|}$, $\varrho(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi + |\zeta - \xi|}$, $\varphi(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi}$ and for all $\zeta, \xi \in \Omega$ and $\varpi > 0$. Then $(\Im R, \kappa, \varrho, \odot, \oplus)$ is an $\mathfrak{NMS}$.

Now define a sequence $(\zeta_\mathfrak{n})$ by $\zeta_\mathfrak{n} = \begin{cases} \frac{1}{\sqrt{\mathfrak{n}}}, & \mathfrak{n} = \lambda^3, \lambda \in \Im N; \\ 0, & \text{otherwise} \end{cases}$

Then, for every $\mathfrak{v} \in (0, 1)$ and for any $\varpi > 0$. we have $(\zeta_\mathfrak{n})$. Here $(\zeta_\mathfrak{n}) \in \mathfrak{B}_\zeta(0, \varpi)$.

Definition 2.5. Let $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ be a $\mathfrak{NMS}$.

(i) A sequence $(\zeta_\mathfrak{n})$ is known to be convergent to ζ if for all $\varpi > 0$ and $\mathfrak{v} \in (0, 1)$ there exists $\mathfrak{n}_0 \in \Im N$ so that $\kappa(\zeta_\mathfrak{n}, \zeta, \varpi) > 1 - \mathfrak{v}$, $\varrho(\zeta_\mathfrak{n}, \zeta, \varpi) < \mathfrak{v}$ and $\varphi(\zeta_\mathfrak{n}, \zeta, \varpi) < \mathfrak{v}$ for every $\mathfrak{n} \geq \mathfrak{n}_0$. It is symbolised as $\zeta_\mathfrak{n} \to \zeta$ as $\mathfrak{n} \to \infty$.

(ii) $\odot\kappa(\zeta_\mathfrak{n}, \zeta, \varpi) \to 1$, $\varrho(\zeta_\mathfrak{n}, \zeta, \varpi) \to 0$ and $\varphi(\zeta_\mathfrak{n}, \zeta, \varpi) \to 0$ as $\mathfrak{n} \to \infty$ for each $\varpi > 0$.

(iii) $(\zeta_\mathfrak{n})$ is known to be a Cauchy sequence if, for $\varpi > 0$ and $\mathfrak{v} \in (0, 1)$, there is $\mathfrak{n}_0 \in \Im N$ such that $\kappa(\zeta_\mathfrak{n}, \zeta_\mathfrak{m}, \varpi) > 1 - \mathfrak{v}$, $\varrho(\zeta_\mathfrak{n}, \zeta_\mathfrak{m}, \varpi) < \mathfrak{v}$ and $\varphi(\zeta_\mathfrak{n}, \zeta_\mathfrak{m}, \varpi) < \mathfrak{v}$ for all $\mathfrak{n}, \mathfrak{m} \geq \mathfrak{n}_0$.

(iv) $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ is called $(\kappa, \varrho, \varphi)$-complete if every Cauchy sequence is convergent.

Example 2.6. Let $\Omega = \Im R, \vartheta \odot \omega = \vartheta\omega$ and $\vartheta \oplus \omega = \min\{\vartheta + \omega, 1\}$ for all $\vartheta, \omega \in [0, 1]$. Define κ, ϱ and φ by $\kappa(\zeta, \xi, \varpi) = \frac{\varpi}{\varpi + |\zeta - \xi|}$, $\varrho(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi + |\zeta - \xi|}$, $\varphi(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi}$ and for all $\zeta, \xi \in \Omega$ and $\varpi > 0$. Then $(\Im R, \kappa, \varrho, \odot, \oplus)$ is an $\Im\mathfrak{FMS}$.

Now define a sequence $(\zeta_\mathfrak{n})$ by $\zeta_\mathfrak{n} = \begin{cases} \frac{1}{\sqrt[3]{\mathfrak{n}}}, & \mathfrak{n} = \lambda^2, \lambda \in \Im N; \\ 0, & \text{otherwise} \end{cases}$

Then, for every $\mathfrak{v} \in (0, 1)$ and for any $\varpi > 0$,

$\kappa(\zeta_\mathfrak{n}, 0, \varpi) \leq 1 - \mathfrak{v}, \varrho(\zeta_\mathfrak{n}, 0, \varpi) \geq \mathfrak{v}, \varphi(\zeta_\mathfrak{n}, 0, \varpi) \geq \mathfrak{v}$.

$\zeta_\mathfrak{n}$ is both convergent and Cauchy sequence in Ω.

Definition 2.7. Let $(\Omega, \kappa, \odot)$ be a $\mathfrak{FMS}$.

(1) A sequence $(\zeta_{\mathfrak{n}}) \subset \Omega$ is termed $\mathfrak{SC}$ to $\zeta_0 \in \Omega$ if, for every $\mathfrak{v} \in (0,1)$ and $\varpi > 0$, the condition $\Delta(\{\mathfrak{n} \in \Im N : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, 0, \varpi) < \mathfrak{v}\}) = 1$ holds.

(2) A sequence $(\zeta_{\mathfrak{n}}) \subset \Omega$ is referred to as $\mathfrak{SCa}$ if, for every $\mathfrak{v} \in (0,1)$ and $\varpi > 0$, there exists an $\mathfrak{m} \in \Im N$ such that $\Delta(\{\mathfrak{n} \in \Im N : \kappa(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{m}}, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{m}}, \varpi) < \mathfrak{v}\}) = 0$.

Example 2.8. Let $\Omega = \Im R, \vartheta \odot \omega = \vartheta\omega$ and $\vartheta \oplus \omega = \min\{\vartheta + \omega, 1\}$ for all $\vartheta, \omega \in [0,1]$. Define κ, ϱ and φ by $\kappa(\zeta, \xi, \varpi) = \frac{\varpi}{\varpi + |\zeta - \xi|}$, $\varrho(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi + |\zeta - \xi|}$, $\varphi(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi}$ and for all $\zeta, \xi \in \Omega$ and $\varpi > 0$. Then $(\Im R, \kappa, \varrho, \odot, \oplus)$ is an $\Im\mathfrak{FMS}$.

Now define a sequence $(\zeta_{\mathfrak{n}})$ by $\zeta_{\mathfrak{n}} = \begin{cases} \frac{\sqrt{n}}{2}, & \mathfrak{n} = \lambda^3, \lambda \in \Im N; \\ 0, & \text{otherwise} \end{cases}$

Then, for every $\mathfrak{v} \in (0,1)$ and for any $\varpi > 0$, let $\mathfrak{M} = \{\mathfrak{n} \leq \mathfrak{m} : \kappa(\zeta_{\mathfrak{n}}, 0, \varpi) \leq 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, 0, \varpi) \geq \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, 0, \varpi) \geq \mathfrak{v}\} = \left\{\mathfrak{n} \leq \mathfrak{m} : \frac{\varpi}{\varpi + |\zeta_{\mathfrak{n}}|} \leq 1 - \mathfrak{v}, \frac{|\zeta_{\mathfrak{n}}|}{\varpi + |\zeta_{\mathfrak{n}}|} \geq \mathfrak{v}, \frac{|\zeta_{\mathfrak{n}}|}{\varpi} \geq \mathfrak{v}\right\} = \{\mathfrak{n} \leq \mathfrak{m} : |\zeta_{\mathfrak{n}}| \geq \frac{\mathfrak{v}\varpi}{1 - \mathfrak{v}} > 0\} = \{\mathfrak{n} \leq \mathfrak{m} : \zeta_{\mathfrak{n}} = \frac{\sqrt{n}}{2}\} = \{\mathfrak{n} \leq \mathfrak{m} : \mathfrak{n} = \lambda^2, \lambda \in \Im N\}$, and we obtain $\frac{1}{\mathfrak{m}}|\mathfrak{M}| \leq \frac{1}{\mathfrak{m}}|\{\mathfrak{n} \leq \mathfrak{m} : \mathfrak{n} = \lambda^3, \mathfrak{n} \in \Im N\}| \leq \frac{\sqrt[3]{\mathfrak{m}}}{\mathfrak{m}} \to 0, \mathfrak{m} \to \infty$. As a result, we have $(\zeta_{\mathfrak{n}})$ is $\mathfrak{SC}$ to 0 with regard to the $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$.

3. Statical Convergence in Neutrosophic Metric Spaces

Definition 3.1. Let $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ be a $\mathfrak{NMS}$. A sequence $(\zeta_{\mathfrak{n}}) \subset \Omega$ is called $\mathfrak{SC}$ to $\zeta_0 \in \Omega$ with respect to $\mathfrak{NM}$ obtained that, for every $\mathfrak{v} \in (0,1)$ and $\varpi > 0$, $\Delta(\{\mathfrak{n} \in \Im N : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}\}) = 1$.

We assert that $(\zeta_{\mathfrak{n}})$ is $\mathfrak{SC}$ to ζ_0. We can observe as $\Delta(\{\mathfrak{n} \in \Im N : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta, \varpi) < \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta, \varpi) < \mathfrak{v}\}) = 1$

$\Leftrightarrow \lim_{n \to \infty} \frac{|\{k \leq n : \kappa(\zeta_\lambda, \zeta_0, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_\lambda, \zeta_0, \varpi) < \mathfrak{v}, \varphi(\zeta_\lambda, \zeta_0, \varpi) < \mathfrak{v}\}|}{\mathfrak{n}} = 1$

Example 3.2. Let $\Omega = \Im R, \vartheta \odot \omega = \vartheta\omega$ and $\vartheta \oplus \omega = \min\{\vartheta + \omega, 1\}$ for all $\vartheta, \omega \in [0,1]$. Define κ, ϱ and φ by $\kappa(\zeta, \xi, \varpi) = \frac{\varpi}{\varpi + |\zeta - \xi|}$, $\varrho(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi + |\zeta - \xi|}$, $\varphi(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi}$ and for all $\zeta, \xi \in \Omega$ and $\varpi > 0$. Then $(\Im R, \kappa, \varrho, \odot, \oplus)$ is an $\Im\mathfrak{FMS}$.

Now define a sequence $(\zeta_{\mathfrak{n}})$ by $\zeta_{\mathfrak{n}} = \begin{cases} 1, & \mathfrak{n} = \lambda^2, \lambda \in \Im N; \\ 0, & \text{otherwise} \end{cases}$

Then, for every $\mathfrak{v} \in (0,1)$ and for any $\varpi > 0$, let $\mathfrak{M} = \{\mathfrak{n} \leq \mathfrak{m} : \kappa(\zeta_{\mathfrak{n}}, 0, \varpi) \leq 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, 0, \varpi) \geq \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, 0, \varpi) \geq \mathfrak{v}\} = \left\{\mathfrak{n} \leq \mathfrak{m} : \frac{\varpi}{\varpi + |\zeta_{\mathfrak{n}}|} \leq 1 - \mathfrak{v}, \frac{|\zeta_{\mathfrak{n}}|}{\varpi + |\zeta_{\mathfrak{n}}|} \geq \mathfrak{v}, \frac{|\zeta_{\mathfrak{n}}|}{\varpi} \geq \mathfrak{v}\right\} = \{\mathfrak{n} \leq \mathfrak{m} : |\zeta_{\mathfrak{n}}| \geq \frac{\mathfrak{v}\varpi}{1 - \mathfrak{v}} > 0\} = \{\mathfrak{n} \leq \mathfrak{m} : \zeta_{\mathfrak{n}} = 1\} = \{\mathfrak{n} \leq \mathfrak{m} : \mathfrak{n} = \lambda^2, \lambda \in \Im N\}$, and we obtain $\frac{1}{\mathfrak{m}}|\mathfrak{M}| \leq \frac{1}{\mathfrak{m}}|\{\mathfrak{n} \leq \mathfrak{m} : \mathfrak{n} = \lambda^2, \mathfrak{n} \in \Im N\}| \leq \frac{\sqrt{\mathfrak{m}}}{\mathfrak{m}} \to 0, \mathfrak{m} \to \infty$. As a result, we have $(\zeta_{\mathfrak{n}})$ is $\mathfrak{SC}$ to 0 with regard to the $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$.

Lemma 3.3. *Let $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ be an $\mathfrak{NMS}$. Then, for every $\mathfrak{v} \in (0,1)$ and $\varpi > 0$. These are comparable to each other:*

(i) $(\zeta_{\mathfrak{n}})$ is $\mathfrak{SC}$ to ζ_0;

(ii) $\Delta(\{\mathfrak{n} \in \Im N : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) \leq 1 - \mathfrak{v}\}) = \Delta(\{\varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) \geq \mathfrak{v}, \Delta(\{\varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) \geq \mathfrak{v}\}) = 0$;

(iii) $\Delta(\{\mathfrak{n} \in \Im N : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v}\}) = \Delta(\{\varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}, \Delta(\{\varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}\}) = 1.$

Proof: Utilizing Definition (3.1) and we possess a lemma regarding density characteristics.

Theorem 3.4. *Let $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ be $\mathfrak{NMS}$. When a sequence $(\zeta_{\mathfrak{n}})$ is $\mathfrak{SC}$ with regards to the $\mathfrak{NM}$, then the $\mathfrak{SC}$ limit is unique.*

Proof: Let us consider $(\zeta_{\mathfrak{n}})$ be $\mathfrak{SC}$ to ζ_1 and ζ_2. Given $\mathfrak{v} \in (0, 1)$, take $\varpi > 0$ such that $(1\varpi) \odot (1\varpi) > 1\mathfrak{v}$ and $\varpi \oplus \varpi < \mathfrak{v}$.

Then classify the following sets, for any $\epsilon > 0$:

$\mathfrak{M}_{\kappa 1}(\varpi, \epsilon) := \{\mathfrak{n} \in \Im N : \kappa(\zeta_{\mathfrak{n}}, \zeta_1, \epsilon) > 1 - \varpi\}$

$\mathfrak{M}_{\kappa 2}(\varpi, \epsilon) := \{\mathfrak{n} \in \Im N : \kappa(\zeta_{\mathfrak{n}}, \zeta_2, \epsilon) > 1 - \varpi\}$

$\mathfrak{M}_{\varrho 1}(\varpi, \epsilon) := \{\mathfrak{n} \in \Im N : \varrho(\zeta_{\mathfrak{n}}, \zeta_1, \epsilon) < \varpi\}$

$\mathfrak{M}_{\varrho 2}(\varpi, \epsilon) := \{\mathfrak{n} \in \Im N : \varrho(\zeta_{\mathfrak{n}}, \zeta_2, \epsilon) < \varpi\}$

$\mathfrak{M}_{\varphi 1}(\varpi, \epsilon) := \{\mathfrak{n} \in \Im N : \varphi(\zeta_{\mathfrak{n}}, \zeta_1, \epsilon) < \varpi\}$

$\mathfrak{M}_{\varphi 2}(\varpi, \epsilon) := \{\mathfrak{n} \in \Im N : \varphi(\zeta_{\mathfrak{n}}, \zeta_2, \epsilon) < \varpi\}$

Since $(\zeta_{\mathfrak{n}})$ is $\mathfrak{SC}$ with respect to ζ_1 and ζ_2, we get

$\Delta\{\mathfrak{M}_{\kappa 1}(\varpi, \epsilon)\} = \Delta\{\mathfrak{M}_{\varrho 1}(\varpi, \epsilon)\} = \Delta\{\mathfrak{M}_{\varphi 1}(\varpi, \epsilon)\} = 1$ and $\Delta\{\mathfrak{M}_{\kappa 2}(\varpi, \epsilon)\} = \Delta\{\mathfrak{M}_{\varrho 2}(\varpi, \epsilon)\} = 1$, for every $\epsilon > 0$.

Let $K_{\kappa\varrho\varphi}(\varpi, \epsilon) := \{\mathfrak{M}_{\kappa 1}(\varpi, \epsilon) \cup \mathfrak{M}_{\kappa 2}(\varpi, \epsilon)\} \cap \{\mathfrak{M}_{\varrho 1}(\varpi, \epsilon) \cup \mathfrak{M}_{\varrho 2}(\varpi, \epsilon)\} \cap \{\mathfrak{M}_{\varphi 1}(\varpi, \epsilon) \cup \mathfrak{M}_{\varphi 2}(\varpi, \epsilon)\}$.

Therefore, $\Delta\{\mathfrak{M}_{\kappa\varrho\varphi}(\varpi, \epsilon)\} = 1$ which implies that $\Delta\{\Im N \backslash \mathfrak{M}_{\kappa\varrho\varphi}(\varpi, \epsilon)\} = 0$.

When $\mathfrak{n} \in \Im N \backslash \mathfrak{M}_{\kappa\varrho\varphi}(t, \epsilon)$, then there are two potential outcomes:

$\mathfrak{n} \in \Im N \backslash \{\mathfrak{M}_{\kappa 1}(\varpi, \epsilon) \cup \mathfrak{M}_{\kappa 2}(\varpi, \epsilon)\}$ or $\mathfrak{n} \in \Im N \backslash \{\mathfrak{M}_{\varrho 1}(\varpi, \epsilon) \cup \mathfrak{M}_{\varrho 2}(\varpi, \epsilon)\}$ or $\mathfrak{n} \in \Im N \backslash \{\mathfrak{M}_{\varphi 1}(\varpi, \epsilon) \cup \mathfrak{M}_{\varphi 2}(\varpi, \epsilon)\}$.

Let us consider $\mathfrak{n} \in \Im N \backslash \{\mathfrak{M}_{\kappa 1}(\varpi, \epsilon) \cup \mathfrak{M}_{\kappa 2}(\varpi, \epsilon)\}$.

Then we achieve $\kappa(\zeta_1, \zeta_2, \epsilon) \geq \kappa\left(\zeta_1, \zeta_{\mathfrak{n}}, \frac{\epsilon}{2}\right) \odot \kappa\left(\zeta_{\mathfrak{n}}, \zeta_2, \frac{\epsilon}{2}\right) > (1 - \varpi) \odot (1 - \varpi) > 1 - \mathfrak{v}$.

Therefore, $\kappa(\zeta_1, \zeta_2, \epsilon) > 1\mathfrak{v}$ and since $\mathfrak{v} > 0$ is arbitrary, we achieve $\kappa(\zeta_1, \zeta_2, \epsilon) = 1$ for all $\epsilon > 0$, which infers $\zeta_1 = \zeta_2$.

Let us consider $\mathfrak{n} \in \Im N \backslash \{\mathfrak{M}_{\varrho 1}(\varpi, \epsilon) \cup \mathfrak{M}_{\varrho 2}(\varpi, \epsilon)\}$.

Then, $\varrho(\zeta_1, \zeta_2, \epsilon) \leq \varrho(\zeta_1, \zeta_{\mathfrak{n}}, \epsilon) \oplus \varrho(\zeta_{\mathfrak{n}}, \zeta_2, \epsilon) < \varpi \oplus \varpi < \mathfrak{v}$.

Since $\mathfrak{v} > 0$ is arbitrary, we obtain $\varrho(\zeta_1, \zeta_2, \epsilon) = 0$ for all $\epsilon > 0$, which suggests $\zeta_1 = \zeta_2$.

Now let us consider $\mathfrak{n} \in \Im N \backslash \{\mathfrak{M}_{\varphi 1}(\varpi, \epsilon) \cup \mathfrak{M}_{\varphi 2}(\varpi, \epsilon)\}$.

Then, $\varphi(\zeta_1, \zeta_2, \epsilon) \leq \varphi(\zeta_1, \zeta_{\mathfrak{n}}, \epsilon) \oplus \varphi(\zeta_{\mathfrak{n}}, \zeta_2, \epsilon) < \varpi \oplus \varpi < \mathfrak{v}$.

Since $\mathfrak{v} > 0$ is arbitrary, we obtain $\varphi(\zeta_1, \zeta_2, \epsilon) = 0$ for all $\epsilon > 0$, which refers $\zeta_1 = \zeta_2$.

Theorem 3.5. *Consider the sequence $(\zeta_{\mathfrak{n}})$ in the Neutrosophic Metric Space $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$. If $(\zeta_{\mathfrak{n}})$ converges to ζ_0 with respect to the Neutrosophic Metric $\mathfrak{NM}$, then $(\zeta_{\mathfrak{n}})$ is $\mathfrak{SC}$ to ζ_0 in the context of the $\mathfrak{NM}$.*

Proof: Let $(\zeta_{\mathfrak{n}})$ be a sequence convergent to ζ_0. Then, for every $\mathfrak{v} \in (0,1)$ and $\varpi > 0$, there exists an $\mathfrak{n}_0 \in \Im N$ such that $\kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v}$, $\varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}$, and $\varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}$. We obtain $|\{\lambda \leq \mathfrak{n} : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v},\ \varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v} \text{ and } \varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}\}| \geq \mathfrak{n}\mathfrak{n}_0$.
Hence, the set $\{\lambda \leq \mathfrak{n} : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v}\ ,\ \varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}$ and $\varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}\}$ has a finite number of terms.
Then, $\lim_{\mathfrak{n}\to\infty} \dfrac{|\{\lambda \leq \mathfrak{n} : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}\}|}{\mathfrak{n}} \geq \lim_{\mathfrak{n}\to\infty} \dfrac{\mathfrak{n}\mathfrak{n}_0}{\mathfrak{n}} = 1.$
Consequently, $\Delta(\{\mathfrak{n} \in \Im N : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}\}) = 1.$
The converse of the theorem need not true.

Example 3.6. Let $\Omega = [1,3]$, and define the operations $\vartheta \odot \omega = \vartheta\omega$ and $\vartheta \oplus \omega = \min\{\vartheta + \omega, 1\}$ for all $\vartheta, \omega \in [0,1]$. Define κ, ϱ, and φ by the following formulas: $\kappa(\zeta, \xi, \varpi) = \frac{\varpi}{\varpi + |\zeta - \xi|}$, $\varrho(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi + |\zeta - \xi|}$, and $\varphi(\zeta, \xi, \varpi) = \frac{|\zeta - \xi|}{\varpi}$ for all $\zeta, \xi \in \Omega$ and $\varpi > 0$. Then $(\Im R, \kappa, \varrho, \varphi, \odot, \oplus)$ forms a Neutrosophic Metric Space ($\mathfrak{NMS}$).
Now define a sequence $(\zeta_{\mathfrak{n}})$ by $\zeta_{\mathfrak{n}} = \begin{cases} 2, & \mathfrak{n} = \lambda^2, \lambda \in \Im N; \\ 1, & \text{otherwise} \end{cases}$.
We can see that $(\zeta_{\mathfrak{n}})$ is not convergent to 1.
We need to show that $(\zeta_{\mathfrak{n}})$ is $\mathfrak{SC}$ to 1.
Let $\mathfrak{v} \in (0,1)$ and $\varpi > 0$. $\mathfrak{M} = \{\mathfrak{n} \in \Im N : \kappa(\zeta_{\mathfrak{n}}, 1, \varpi) > 1\mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, 1, \varpi) < \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, 1, \varpi) < \mathfrak{v}\}$.
Case 1. $\mathfrak{v} \in \left(0, \frac{1}{\varpi+1}\right]$. If $n \neq \lambda^2$ for all $\lambda \in \Im N$, then $\kappa(\zeta_{\mathfrak{n}}, 1, \varpi) = 1 > 1 - \mathfrak{v}$, $\varrho(\zeta_{\mathfrak{n}}, 1, \varpi) = 0 < \mathfrak{v}$ and $\varphi(\zeta_{\mathfrak{n}}, 1, \varpi) = 0 < \mathfrak{v}$. If $\mathfrak{n} = \lambda^2$ for some $\lambda \in \Im N$, then $\kappa(\zeta_{\mathfrak{n}}, 1, \varpi) = \frac{\varpi}{\varpi+1} = 1\frac{1}{\varpi+1} \leq 1\mathfrak{v}$, $\varrho(\zeta_{\mathfrak{n}}, 1, \varpi) = \frac{1}{\varpi+1} \geq \mathfrak{v}$ and $\varphi(\zeta_{\mathfrak{n}}, 1, \varpi) = \frac{1}{\varpi+1} \geq \mathfrak{v}$.
Now, let $\mathfrak{n} \in \Im N$. If $\mathfrak{n} = \lambda_0^2$ for an $\lambda_0 \in \Im N$, then $\lim_{\mathfrak{n}\to\infty} \dfrac{|\mathfrak{M}(\mathfrak{n})|}{\mathfrak{n}} = \lim_{\lambda_0\to\infty} \dfrac{\lambda_0^2\lambda_0}{\lambda_0^2} = 1$. If $\mathfrak{n} \neq \lambda^2$ for all $\lambda \in \Im N$, Therefore, it follows that $\lambda_1 \in \Im N$ such that $\mathfrak{n} = \lambda_1^2 l$ with $l \in \Im N$ and $1 \leq l \leq \lambda_1$. $\lim_{\mathfrak{n}\to\infty} |\mathfrak{M}(\mathfrak{n})|\mathfrak{n} = \lim_{\lambda_1\to\infty} \frac{\lambda_1^2 l(\lambda_1 l)}{\lambda_1^2 l} = \lim_{\lambda_1\to\infty} \frac{\lambda_1^2\lambda_1 l+1}{\lambda_1^2 l} = 1$.
Case 2. $\mathfrak{v} \in \left(\frac{1}{\varpi+1}, 1\right)$. If $\mathfrak{n} \neq \lambda^2$ for all $\lambda \in \Im N$, then $\kappa(\zeta_{\mathfrak{n}}, 1, \varpi) = 1 > 1 - \mathfrak{v}$, $\varrho(\zeta_{\mathfrak{n}}, 1, \varpi) = 0 < \mathfrak{v}$ and $\varphi(\zeta_{\mathfrak{n}}, 1, \varpi) = 0 < \mathfrak{v}$. If $\mathfrak{n} = \lambda^2$ for some $\lambda \in \Im N$, then $\kappa(\zeta_{\mathfrak{n}}, 1, \varpi) = \frac{\varpi}{\varpi+1} = 1\frac{1}{\varpi+1} > 1\mathfrak{v}$, $\varrho(\zeta_{\mathfrak{n}}, 1, \varpi) = \frac{1}{\varpi+1} < \mathfrak{v}$ and $\varphi(\zeta_{\mathfrak{n}}, 1, \varpi) = \frac{1}{\varpi+1} < \mathfrak{v}$.
Hence, $\kappa(\zeta_{\mathfrak{n}}, 1, \varpi) > 1 - \mathfrak{v}$, $\varrho(\zeta_{\mathfrak{n}}, 1, \varpi) < \mathfrak{v}$ and $\varphi(\zeta_{\mathfrak{n}}, 1, \varpi) < \mathfrak{v}$ for all $\mathfrak{n} \in \Im N$. Therefore, $\lim_{\mathfrak{n}\to\infty} \dfrac{|\mathfrak{M}(\mathfrak{n})|}{\mathfrak{n}} = \lim_{\mathfrak{n}\to\infty} \dfrac{\mathfrak{n}}{\mathfrak{n}} = 1$.
Therefore, $\Delta(\{\mathfrak{n} \in \Im N : \kappa(\zeta_{\mathfrak{n}}, 1, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, 1, \varpi) < \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, 1, \varpi) < \mathfrak{v}\}) = 1$ for all $\mathfrak{v} \in (0,1)$ and $\varpi > 0$.

Theorem 3.7. *Consider the sequence $(\zeta_{\mathfrak{n}})$ in the Neutrosophic Metric Space $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$. The sequence $(\zeta_{\mathfrak{n}})$ is $\mathfrak{SC}$ to ζ_0 if and only if there exists an increasing index sequence $\mathfrak{P} = \{\mathfrak{n}_i\}_{i \in \Im N}$ of natural numbers such that $(\zeta_{\mathfrak{n}_i})$ converges to ζ_0 and $\Delta(\mathfrak{P}) = 1$.*

Proof: Given that the sequence $(\zeta_{\mathfrak{n}})$ statistically converges to ζ_0, define $\mathfrak{M}_{\kappa\varrho\varphi}(j, \varpi) := \{\mathfrak{n} \in \Im N : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \frac{1}{j}, \varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \frac{1}{j}$, and $\varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \frac{1}{j}\}$ for any $\varpi > 0$ and $j \in \Im N$.

We establish that $\mathfrak{M}_{\kappa\varrho\varphi}(j+1,\varpi) \subset \mathfrak{M}_{\kappa\varrho\varphi}(j,\varpi)$ for all $\varpi > 0$ and $j \in \mathfrak{I}N$. Since $(\zeta_{\mathfrak{n}})$ is $\mathfrak{SC}$ to ζ_0,

$$\Delta(\mathfrak{M}_{\kappa\varrho\varphi}(j,\varpi)) = 1 \tag{1}$$

Take $\mathfrak{u}_1 \in \mathfrak{M}_{\kappa\varrho\varphi}(1,\varpi)$. Since $\Delta(\mathfrak{M}_{\kappa\varrho\varphi}(2,\varpi)) = 1$ (by Equation (1)) we have a number $\mathfrak{u}_2 \in (\mathfrak{M}_{\kappa\varrho\varphi}(2,\varpi)(\mathfrak{u}_2 > \mathfrak{u}_1)$ such that $\frac{|\{\lambda \le \mathfrak{n}: \kappa(\zeta_\lambda,\zeta_0,\varpi)>1-\frac{1}{2}, \varrho(\zeta_\lambda,\zeta_0,\varpi)<\frac{1}{2}, \varphi(\zeta_\lambda,\zeta_0,\varpi)<\frac{1}{2}\}|}{\mathfrak{n}} > \frac{1}{2}$, for every $\mathfrak{n} \ge \mathfrak{u}_2$.

According to Equation (1), $\Delta(\mathfrak{M}_{\kappa\varrho\varphi}(3,\varpi)) = 1$. We can select $\mathfrak{u}_3 \in \mathfrak{M}_{\kappa\varrho\varphi}(3,\varpi)$ (where $\mathfrak{u}_3 > \mathfrak{u}_2$) such that for all $\mathfrak{n} \ge \mathfrak{u}_3$, the fraction $\frac{|\{\lambda \le \mathfrak{n}: \kappa(\zeta_\lambda,\zeta_0,\varpi)>1-\frac{1}{3}, \varrho(\zeta_\lambda,\zeta_0,\varpi)<\frac{1}{3}, \varphi(\zeta_\lambda,\zeta_0,\varpi)<\frac{1}{3}\}|}{\mathfrak{n}} > \frac{2}{3}$.

We proceed in this manner.

Subsequently, we can construct an increasing sequence of indices $\{\mathfrak{u}_j\}_{j\in\mathfrak{I}N}$ from natural numbers such that each $\mathfrak{u}_j$ belongs to $\mathfrak{M}_{\kappa\varrho\varphi}(j,\varpi)$.

$$\frac{|\{\lambda \le \mathfrak{n} : \kappa(\zeta_\lambda,\zeta_0,\varpi) > 1-\frac{1}{j}, \varrho(\zeta_\lambda,\zeta_0,\varpi) < \frac{1}{j}, \varphi(\zeta_\lambda,\zeta_0,\varpi) < \frac{1}{j}\}|}{\mathfrak{n}} > \frac{j-1}{j}, \tag{2}$$

for all $\mathfrak{n} \ge \mathfrak{u}_j$, where j ranges over $\mathfrak{I}N$. We define the increasing index sequence $\mathfrak{P}$ as:

$$\mathfrak{P} := \{\mathfrak{n} \in \mathfrak{I}N : 1 < \mathfrak{n} < \mathfrak{u}_1\} \cup \left(\bigcup_{j\in\mathfrak{I}N} \{\mathfrak{n} \in \mathfrak{I}N : \mathfrak{u}_j \le \mathfrak{n} < \mathfrak{u}_{j+1}\} \cap \mathfrak{M}_{\kappa\varrho\varphi}(j,\varpi) \right).$$

By Equation (2) and $\mathfrak{M}_{\kappa\varrho\varphi}(j+1,\varpi) \subset \mathfrak{M}_{\kappa\varrho\varphi}(j,\varpi)$, we write

$$\frac{|\{\lambda \le \mathfrak{n} : \lambda \in \mathfrak{P}\}|}{\mathfrak{n}} \ge \frac{|\{\lambda \le \mathfrak{n} : \kappa(\zeta_\lambda,\zeta_0,\varpi) > 1-\frac{1}{j}, \varrho(\zeta_\lambda,\zeta_0,\varpi) < \frac{1}{j}, \varphi(\zeta_\lambda,\zeta_0,\varpi) < \frac{1}{j}\}|}{\mathfrak{n}} > \frac{j-1}{j}$$

for all $\mathfrak{n}, (\mathfrak{u}_j \le \mathfrak{n} < \mathfrak{u}_{j+1})$.

Given that $j \to \infty$, at which $\mathfrak{n} \to \infty$, we have $\lim_{\mathfrak{n}\to\infty} \frac{|\{\lambda \le \mathfrak{n}: \lambda\in\mathfrak{P}\}|}{\mathfrak{n}} = 1$, i.e., $\Delta(\mathfrak{P}) = 1$.

Now, we demonstrate the convergence of $(\zeta_{\mathfrak{n}_i})$ to ζ_0. Let $\mathfrak{v} \in (0,1)$ and $\varpi > 0$. Choose $N_0 > \mathfrak{u}_2$ sufficiently large such that there exists an index $s_0 \in \mathfrak{I}N$ satisfying $\mathfrak{u}_{s_0} \le N_0 < \mathfrak{u}_{s_0+1}$ and $\frac{1}{s_0} < \mathfrak{v}$. Consider $\mathfrak{n}_\mathfrak{m} \ge N_0$ where $\mathfrak{n}_\mathfrak{m} \in \mathfrak{P}$. By the definition of $\mathfrak{P}$, there exists $s \in \mathfrak{I}N$ such that $\mathfrak{u}_s \le \mathfrak{n}_\mathfrak{m} < \mathfrak{u}_{s+1}$ and $\mathfrak{n}_\mathfrak{m} \in \mathfrak{M}_{\kappa\varrho\varphi}(s,\varpi)$ (with $s \ge s_0$). Thus,
$\kappa(\zeta_{\mathfrak{n}_\mathfrak{m}},\zeta_0,\varpi) \ge \kappa\left(\zeta_{\mathfrak{n}_\mathfrak{m}},\zeta_0,\frac{1}{s_0}\right) \ge \kappa\left(\zeta_{\mathfrak{n}_\mathfrak{m}},\zeta_0,\frac{1}{s}\right) > 1-\frac{1}{s} \ge 1-\frac{1}{s_0} > 1-\mathfrak{v}$, $\varrho(\zeta_{\mathfrak{n}_\mathfrak{m}},\zeta_0,\varpi) > \frac{1}{s_0} < \mathfrak{v}$ and $\varphi(\zeta_{\mathfrak{n}_\mathfrak{m}},\zeta_0,\varpi) > \frac{1}{s_0} < \mathfrak{v}$. Therefore, $(\zeta_{\mathfrak{n}i})$ converges to ζ_0.

On the otherhand, Let's suppose there exists a sequence of increasing indices $\mathfrak{P} = \{\mathfrak{n}_i\}_{i\in\mathfrak{I}N}$ of natural numbers such that $\Delta(\mathfrak{P}) = 1$ and $(\zeta_{\mathfrak{n}_i})$ converges to ζ_0. Take $\mathfrak{v} \in (0,1)$ and $\varpi > 0$. Then, there exists a number $\mathfrak{n}_0 \in \mathfrak{I}N$ such that for every $\mathfrak{n} \ge \mathfrak{n}_0$, the conditions $\kappa(\zeta_{\mathfrak{n}_i},\zeta_0,\varpi) > 1-\mathfrak{v}$, $\varrho(\zeta_{\mathfrak{n}_i},\zeta_0,\varpi) < \mathfrak{v}$, and $\varphi(\zeta_{\mathfrak{n}_i},\zeta_0,\varpi) < \mathfrak{v}$ are satisfied.

Let us define $\mathfrak{M}_{\kappa\varrho\varphi}(\mathfrak{v},\varpi) := \{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_{\mathfrak{n}i},\zeta_0,\varpi) \le 1-\mathfrak{v} \,\text{or}\, \varrho(\zeta_{\mathfrak{n}i},\zeta_0,\varpi) \ge \mathfrak{v} \,\text{and}\, \varphi(\zeta_{\mathfrak{n}i},\zeta_0,\varpi) \ge \mathfrak{v}\}$. We have $\mathfrak{M}_{\kappa\varrho\varphi}(\mathfrak{v},\varpi) \subset \mathfrak{I}N \backslash \{\mathfrak{n}_{\mathfrak{n}_0}, \mathfrak{n}_{\mathfrak{n}_0+1}, \mathfrak{n}_{\mathfrak{n}_0+2}, \dots\}$. Since $\Delta(\mathfrak{P}) = 1$, we have $\Delta(\mathfrak{I}N \backslash \{\mathfrak{n}_{\mathfrak{n}_0}, \mathfrak{n}_{\mathfrak{n}_0+1}, \mathfrak{n}_{\mathfrak{n}_0+2}, \dots\}) = 0$, so we deduce $\Delta(\mathfrak{M}_{\kappa\varrho\varphi}(\mathfrak{v},\varpi)) = 0$. Hence, $\Delta(\{\mathfrak{n} \in \mathfrak{I}N : \kappa(\zeta_\mathfrak{n},\zeta_0,\varpi) < 1-\mathfrak{v}, \varrho(\zeta_\mathfrak{n},\zeta_0,\varpi) < \mathfrak{v} \,\text{and}\, \varphi(\zeta_\mathfrak{n},\zeta_0,\varpi) < \mathfrak{v}\}) = 1$.

Therefore, $(\zeta_\mathfrak{n})$ $\mathfrak{SC}$ to ζ_0.

Corollary 3.8. *If* $(\zeta_{\mathfrak{n}})$ *is a sequence within an* $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ *that is both* $\mathfrak{SC}$ *to* ζ_0 *and convergent, then* $(\zeta_{\mathfrak{n}})$ *converges to* ζ_0.

Definition 3.9. Let $(\Omega_1, \kappa_1, \varrho_1, \varphi_1, \odot_1, \oplus_1)$ and $(\Omega_2, \kappa_2, \varrho_2, \varphi_2, \odot_2, \oplus_2)$ be two $\mathfrak{NM}$ spaces.

(i) A map $\chi : \Omega_1 \to \Omega_2$ is termed an isometry if for every $\zeta, \xi \in \Omega_1$ and $\varpi > 0, \kappa_1(\zeta, \xi, \varpi) = \kappa_2(\chi(\zeta), \chi(\xi), \varpi)$, $\varrho_1(\zeta, \xi, \varpi) = \varrho_2(\chi(\zeta), \chi(\xi), \varpi)$ and $\varphi_1(\zeta, \xi, \varpi) = \varphi_2(\chi(\zeta), \chi(\xi), \varpi)$.

(ii) $(\Omega_1, \kappa_1, \varrho_1, \varphi_1, \odot_1, \oplus_1)$ and $(\Omega_2, \kappa_2, \varrho_2, \varphi_2, \odot_2, \oplus_2)$ are termed isometric if there exists a bijective mapping (an isometry) from Ω_1 to Ω_2.

(iii) A neutrosophic completion of $(\Omega_1, \kappa_1, \varrho_1, \odot_1, \oplus_1)$ is defined as a complete $\mathfrak{NMS}$ $(\Omega_2, \kappa_2, \varrho_2, \odot_2, \oplus_2)$ such that $(\Omega_1, \kappa_1, \varrho_1, \odot_1, \oplus_1)$ is isometrically embedded as a dense subspace within Ω_2.

(iv) $(\Omega_1, \kappa_1, \varrho_1, \odot_1, \oplus_1)$ is termed completable if it can be extended to form a complete $\mathfrak{NMS}$.

Proposition 3.10. *Suppose* $(\zeta_{\mathfrak{n}})$ *is a sequence in a completable* $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$. *If* $(\zeta_{\mathfrak{n}})$ *is a Cauchy sequence in* Ω *and it is statistically dense around* ζ_0, *then* $(\zeta_{\mathfrak{n}})$ *converges to* ζ_0.

Proof: Let $(\Omega_1, \kappa_1, \varrho_1, \varphi_1, \odot_1, \oplus_1)$ be the completion of $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$. Consequently, there exists $\zeta_1 \in \Omega_1$ such that the sequence $(\zeta_{\mathfrak{n}})$ converges to ζ_1. We have $\kappa_1(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) = \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi)$, $\varrho_1(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) = \varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi)$ and $\varphi_1(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) = \varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi)$ for all $\varpi > 0$ and $\mathfrak{n} \in \Im N$.
Let $\mathfrak{v} \in (0, 1)$ and $\varpi > 0$. Since $\Delta(\{\mathfrak{n} \in \Im N : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v} \text{ and } \varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}\}) = 1$, we obtain $\Delta(\{\mathfrak{n} \in \Im N : \kappa_1(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1 - \mathfrak{v}, \varrho_1(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v} \text{and } \varphi_1(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}\}) = 1$. Hence, we see that $(\zeta_{\mathfrak{n}})$ statistically converges to $\zeta_0 \in \Omega_1$ with respect to $(\kappa_1, \varrho_1, \varphi_1)$. By Corollary (3.8), we have $\zeta_1 = \zeta_0$.

4. Statically Complete $\mathfrak{NMS}$

Definition 4.1. Consider a sequence $(\zeta_{\mathfrak{n}}) \subset \Omega$, where $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ is a $\mathfrak{NMS}$. The sequence is termed a $\mathfrak{SCa}$ if, for every $\mathfrak{v} \in (0, 1)$ and $\varpi > 0$, there exists $\mathfrak{m} \in \Im N$ such that $\Delta(\{\mathfrak{n} \in \Im N : \kappa(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{m}}, \varpi) > 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{m}}, \varpi) < \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{m}}, \varpi) < \mathfrak{v}\}) = 1$.

Example 4.2. In Example(2.4), $\mathfrak{M} = \{\mathfrak{n} \leq \mathfrak{m} : \kappa(\zeta_{\mathfrak{n}}, 0, \varpi) \leq 1 - \mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, 0, \varpi) \geq \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, 0, \varpi) \geq \mathfrak{v}\} = \left\{\mathfrak{n} \leq \mathfrak{m} : \frac{\varpi}{\varpi + |\zeta_{\mathfrak{n}}|} \leq 1 - \mathfrak{v}, \frac{|\zeta_{\mathfrak{n}}|}{\varpi + |\zeta_{\mathfrak{n}}|} \geq \mathfrak{v}, \frac{|\zeta_{\mathfrak{n}}|}{\varpi} \geq \mathfrak{v}\right\} = \{\mathfrak{n} \leq \mathfrak{m} : |\zeta_{\mathfrak{n}}| \geq \frac{\mathfrak{v}\varpi}{1 - \mathfrak{v}} > 0\} = \{\mathfrak{n} \leq \mathfrak{m} : \zeta_{\mathfrak{n}} = 1\} = \{\mathfrak{n} \leq \mathfrak{m} : \mathfrak{n} = \lambda^3, \lambda \in \Im N\}$, and we obtain $\frac{1}{\mathfrak{m}}|\mathfrak{M}| \leq \frac{1}{\mathfrak{m}}|\{\mathfrak{n} \leq \mathfrak{m} : \mathfrak{n} = \lambda^3, \mathfrak{n} \in \Im N\}| \leq \frac{\sqrt[3]{\mathfrak{m}}}{\mathfrak{m}} \to 0, \mathfrak{m} \to \infty$. As a result, we have $(\zeta_{\mathfrak{n}})$ is $\mathfrak{SC}$ to 0 with regard to the $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$.

Theorem 4.3. *Let* $(\zeta_{\mathfrak{n}})$ *denote a sequence within the framework of* $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$. *The following statements are equivalent:*

(i) $(\zeta_{\mathfrak{n}})$ *is classified as a* $\mathfrak{SCa}$ *sequence.*

(ii) *There exists a progressively increasing index sequence* $\mathfrak{M} = \{\mathfrak{n}_i\}_{i \in \Im N}$ *of natural numbers such that* $(\zeta_{\mathfrak{n}_i})$ *is characterized by being Cauchy and achieving* $\Delta(\mathfrak{M}) = 1$.

Proof: Direct

Theorem 4.4. *Consider a sequence* $(\zeta_{\mathfrak{n}})$ *within the framework of* $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$. *If* $(\zeta_{\mathfrak{n}})$ *is* $\mathfrak{SC}$ *with respect to the* $\mathfrak{NM}$*, then* $(\zeta_{\mathfrak{n}})$ *is* $\mathfrak{SCa}$ *with respect to the* $\mathfrak{NM}$ *.*

Proof: If $(\zeta_{\mathfrak{n}})$ is statistically convergent to ζ_0 and for given $\mathfrak{v} \in (0,1)$ and $\varpi > 0$, there exists $\mathfrak{v}_1 \in (0,1)$ such that $(1-\mathfrak{v}_1) \odot (1-\mathfrak{v}_1) > 1-\mathfrak{v}$ and $\mathfrak{v}_1 \oplus \mathfrak{v}_1 < \mathfrak{v}$. We have $\Delta(\{\mathfrak{n} \in \Im N : \kappa(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) > 1-\mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, \zeta_0, \varpi) < \mathfrak{v}\}) = 1$. Refer Theorem (3.4), there exists an increasing index sequence $\{\mathfrak{n}_i\}_{i \in \Im N}$ so that $(\zeta_{\mathfrak{n}_i})$ is convergent to ζ_0. Hence, there exists $\mathfrak{n}_{i_0} \in \{\mathfrak{n}_i\}_{i \in \Im N} : \kappa\left(\zeta_{\mathfrak{n}_i}, \zeta_0, \frac{\varpi}{2}\right) > 1-\mathfrak{v}_1$, $\varrho\left(\zeta_{\mathfrak{n}_i}, \zeta_0, \frac{\varpi}{2}\right) < \mathfrak{v}_1$ and $\varphi\left(\zeta_{\mathfrak{n}_i}, \zeta_0, \frac{\varpi}{2}\right) < \mathfrak{v}_1$ for all $\mathfrak{n}_i \geq \mathfrak{n}_{i_0}$.
Since $\kappa\left(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{n}_{i_0}}, \varpi\right) \geq \kappa\left(\zeta_{\mathfrak{n}}, \zeta_0, \frac{\varpi}{2}\right) \odot \kappa\left(\zeta_0, \zeta_{\mathfrak{n}_{i_0}}, \frac{\varpi}{2}\right) \geq (1-\mathfrak{v}_1) \odot (1-\mathfrak{v}_1) > 1-\mathfrak{v}$,
$\varrho\left(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{n}_{i_0}}, \varpi\right) \leq \varrho\left(\zeta_{\mathfrak{n}}, \zeta_0, \frac{\varpi}{2}\right) \oplus \varrho\left(\zeta_0, \zeta_{\mathfrak{n}_{i_0}}, \frac{\varpi}{2}\right) < \mathfrak{v}_1 \oplus \mathfrak{v}_1 < \mathfrak{v}$ and
$\varphi\left(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{n}_{i_0}}, \varpi\right) \leq \varphi\left(\zeta_{\mathfrak{n}}, \zeta_0, \frac{\varpi}{2}\right) \oplus \varphi\left(\zeta_0, \zeta_{\mathfrak{n}_{i_0}}, \frac{\varpi}{2}\right) < \mathfrak{v}_1 \oplus \mathfrak{v}_1 < \mathfrak{v}$, we have $\Delta(\{\mathfrak{n} \in \Im N : \kappa(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{n}_{i_0}}, \varpi) > 1-\mathfrak{v}, \varrho(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{n}_{i_0}}, \varpi) < \mathfrak{v}, \varphi(\zeta_{\mathfrak{n}}, \zeta_{\mathfrak{n}_{i_0}}, \varpi) < \mathfrak{v}\}) = 1$. Therefore, $(\zeta_{\mathfrak{n}})$ is statistically Cauchy with respect to the $\mathfrak{NM}$.

Remark 4.5. Given that a sequence in a $\mathfrak{NMS}$ is Cauchy, it consequently meets the criteria to be classified as $\mathfrak{SCa}$.

Definition 4.6. The $\mathfrak{NMS}$ $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ is termed statistically complete if every $\mathfrak{SCa}$ sequence in Ω is also $\mathfrak{SC}$.

Theorem 4.7. *If* $(\Omega, \kappa, \varrho, \varphi, \odot, \oplus)$ *is a* $\mathfrak{NMS}$ *where* Ω *is statistically complete, then it is also complete according to the* $\mathfrak{NM}$*.*

Proof: The proof follows a similar approach to Theorem (4.4).

5. Conclusion

This paper has discussed and proved some results of $\mathfrak{SC}$ and $\mathfrak{SCa}$ on $\mathfrak{NMS}$. Additionally looked at the attributes of statistical completeness on $\mathfrak{NMS}$. Our results can be extended to other spaces and be used to arrive at more results in fixed point theory.

References

1. Atanassov ,K.T.,, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986), 87-96.

2. Abdel Monem , A., A.Nabeeh , N., & Abouhawwash, M. An Integrated Neutrosophic Regional Management Ranking Method for Agricultural Water Management. Neutrosophic Systems with Applications, vol.1, (2023): pp. 22 28. (Doi: https://doi.org/10.5281/zenodo.81711 94)
3. Ahmed Abdelhafeez , Hoda K.Mohamed, Nariman A.Khalil, Rank and Analysis Several Solutions of Healthcare Waste to Achieve Cost Effectiveness and Sustainability Using Neutrosophic MCDM Model, Neutrosophic Systems with Applications, vo l.2, (2023): pp. 25 37. (Doi: https://doi.org/10.5281/zenodo.8185213)
4. Changqing, L.; Zhangb, Y.; Zhanga, J. On statistical convergence in fuzzy metric spaces. J. Intell. Fuzzy Syst. 2020, 39, 3987–3993.
5. Fast, H. Sur la convergence statistique. Colloq. Math. 1951, 2, 241–244.
6. Granados, C., Dhital, A. Statistical convergence of double sequences in neutrosophic normed spaces, Neutrosophic Sets Syst. 42 (2021) 333-344.
7. Gregori, V.; Miñana, J.; Miravet, D. Extended Fuzzy Metrics and Fixed Point Theorems. Mathematics 2019, 7, 303.
8. Jeyaraman, M., Mangayarkkarasi, A.N., Jeyanthi, V., and Pandiselvi, R., Hyers- Ulam-Rassias stability for functional equation in neutrosophic normed spaces,International Journal of Neutrosophic Science, Vol.18, No. 1, (2022), 127-143.
9. Jeyaraman, M, Sowndrarajan, S, Common Fixed Point Results in Neutrosophic Metric Spaces, Neutrosophic Sets and Systems, 42, 208 – 220.
10. Kramosil, O.; Michalek, J. Fuzzy metric and statistical metric spaces. Kybernetika 1975, 11, 326-334.
11. Kirisci M, Simsek N. (2019) Neutrosophic metric spaces, arXiv;1907.00798.
12. O. Kisi, "I-Lacunary statistical convergence in intuitionistic fuzzy normed spaces," Afyon Kocatepe University Journal of Science and Engineering, vol. 20, pp. 207-212, 2020.
13. Maio, G.D.; Kocinac, L. Statistical convergence in topology. Topol. Appl. 2008, 156, 28-45.
14. Manas Karak, Animesh Mahata, Mahendra Rong, Supriya Mukherjee, Sankar Prasad Mondal, Said Broumi, Banamali Roy , A Solution Technique of Transportation Problem in Neutrosophic Environment , Neutrosophic Systems with Applications, vo l. 3 , (2023): pp. 1 7 34 . (Doi:https://doi.org/10.5281/zenodo.8197046.
15. Park, J.H. Intuitionistic fuzzy metric spaces. Chaos Solitons Fractals 2004, 22, 1039-1046.
16. Reich, S.; Zaslavski, A.J. Fixed points and convergence results for a class of contractive mappings. J. Nonlinear Var. Anal. 2021, 5, 665–671.
17. Samarandache, F. Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, Int J. Pure Appl. Math., 24, 287-297,(2005).
18. Sowndrarajan, S., Jeyaraman, M., Smarandache, F. Fixed Point Results for Contraction Theorems in Neutrosophic Metric Spaces, Neutrosophic Sets and Systems, 36(2020),308-318.
19. Schweizer, B.; Sklar, A. Statistical metric spaces. Pacific J. Math. 1960, 10, 313–334.
20. Vasuki, R.; Veeramani, P. Fixed point theorems and Cauchy sequences in fuzzy metric spaces. Fuzzy Sets Syst. 2003, 135, 415-417.
21. Zadeh, L.A. Fuzzy sets. Inf. Control. 1965, 8, 338-53.

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