Neutrosophic Sets and Systems, Vol. 70, 2024
University of New Mexico

On The Solutions of Some Different Types of Two-Fold Fuzzy and Neutrosophic Differential Equations

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Abstract:

This paper is dedicated to studying for the first time the concept of two-fold differential equations of different orders and different types, where we present the solutions for two-fold fuzzy differential equations, and for two-fold neutrosophic differential equations. Also, we illustrate many numerical examples to clarify and explain the novelty of this work.

Keywords: two-fold fuzzy algebra, two-fold neutrosophic set, two-fold differential equation, numerical method.

Introduction

Differential equations are one of the most important concepts in mathematics due to their wide applications in all scientific fields, such as computer science, economics, and even in physics. The use of fuzzy logic or neutrosophic logic to study differential equations and their applications in various fields is not new, as we find in [8-9], many results describing the solutions of various rows of neutrosophic differential equations, and also studying their applications in other scientific and vital fields.

The concept of two-fold algebras was introduced for the first time recently in 2023 [1], and these ideas have been used in their application to various algebraic structures such as rings, algebraic modules and spaces, and metric spaces [3,4,6]. In [7], two-fold algebras were used to study some special functions, specifically the gamma function.

Previous studies have motivated us to study the possibility of applying two-fold fuzzy algebras and two-fold neutrosophic algebras for the first time in the field of differential equations, by using two-fold real numbers in defining different classes of differential equations, and also in studying their solutions and the behavior of these solutions.

For details about two-fold neutrosophic/fuzzy numbers and functions, see [5].

Main Discussion

Definition:

Let $F(x,y,y')=0, G(x,\mu,\mu')=0$ be two different differential equations of first order. The two-fold fuzzy differential equation is defined as follows:

$\left(F(x,y,y')\right)_{G(x,\mu,\mu')}=0_0(1)$.

A function $y_\mu=f_\mu$ is called a solution at $x_0\in\mathbb{R}$ if and only if:

$$\begin{cases} F(x_0,f,f')=0 \\ G(x_0,\mu,\mu')=0 \\ \mu(x_0)\in[0,1] \end{cases}$$

The set of all $x_0\in\mathbb{R}$ for which $\mu(x_0)\in[0,1]$ is called the solving set of equation (1).

We denote it by

$SS=\{x_0\in\mathbb{R} \quad ; F(x_0,f,f')=G(x_0,\mu,\mu')=0, \mu(x_0)\in[0,1]\}$.

Example:

Consider the following two-fold fuzzy differential equation:

$(y'-y)_{\mu'(x)-\cos x}=0_0$

It is equivalent to:

$$\begin{cases} y'-y=0 & (1) \\ \mu'(x)-\cos x=0 & (2) \end{cases}$$

The general solution of (1) is:

$f(x)=k \;\; e^x \;\; ; k\in\mathbb{R}$,

The general solution of (2) is:

$\mu(x)=\sin x+l \qquad ; l\in\mathbb{R}$.

$SS=\{a\in\mathbb{R} \;\; ; 0\le \sin a+l \;\; \le 1\}=\{a\in\mathbb{R} \;\; ; -l\le \sin a\le 1-l\}$.

The general solution of $(y'-y)_{\mu'-\cos x}=0_0$ is:

$y_\mu=(k \;\; e^x)_{\sin x+l} \quad ; k\in\mathbb{R} \;\; , l\in\mathbb{R} \;\; , x\in SS$.

Example:

Consider the following two-fold fuzzy first order differential equation:

$(y'-3)_{\mu'+2}=0_0$, then: $\begin{cases} y'=3 & (1) \\ \mu'=-2 & (2) \end{cases}$

Thus: $\begin{cases} f(x) = y = 3x + c \\ \mu(x) = -2x + a \end{cases} \quad ; a, c \in \mathbb{R}$

Also, $0 \le \mu(x) \le 1 \quad \Leftrightarrow \frac{a}{2} \ge x \ge \frac{a-1}{2}$.

The general solution of $(y' - 3)_{\mu'+2} = 0_0$ is:

$$y_\mu = (3x + c)_{-2x+a} \quad ; \quad \frac{a-1}{2} \le x \le \frac{a}{2}$$

Definition:

1] The first-second order two-fold fuzzy differential equation is defined as follows:

$\big(F(x, y, y')\big)_{G(x,\mu,\mu',\mu'')} = 0_0$.

2] The second-first order two-fold fuzzy differential equation is defined as follows:

$\big(F(x, y, y', y'')\big)_{G(x,\mu,\mu')} = 0_0$.

3] The second order two-fold fuzzy differential equation is defined as:

$\big(F(x, y, y', y'')\big)_{G(x,\mu,\mu',\mu'')} = 0_0$.

Remark:

The solving set (SS) for each previous kind of equations is defined under the condition

$\mu(x_0) \in [0,1] \ \forall \ x_0 \in SS$.

Example:

Consider the following second-first order equation:

$(y'' - 2x)_{\mu'-\mu} = 0_0$ it is equivalent to: $\begin{cases} y'' = 2x & (1) \\ \mu' = \mu & (2) \end{cases}$

The general solution of (1) is:

$y = f(x) = \frac{1}{3}x^3 + ax + b \ ; a, b \in \mathbb{R}$.

The general solution of (2) is:

$\mu(x) = k \ e^x \ ; k \in \mathbb{R}$.

$0 \le \mu(x) \le 1 \Leftrightarrow 0 \le k \ e^x \le 1$:

For $k > 0$: $\quad 0 \le e^x \le \frac{1}{k} \Leftrightarrow -\infty < x \le \ln(\frac{1}{k})$

For $k < 0$: $\quad 0 \ge e^x \ge \frac{1}{k}$ which is a contradiction.

For $k = 0$: $\quad \mu(x) = 0$ for all $x \in \mathbb{R}$.

The general solution of $(y'' - 2x)_{\mu' - \mu} = 0_0$ is:

$$y_\mu = \begin{cases} (\frac{1}{3}x^3 + ax + b)_0 & ; x \in \mathbb{R} \quad , a, b \in \mathbb{R} \\ (\frac{1}{3}x^3 + ax + b)_{k\ e^x} & ; k > 0 \quad , -\infty < x \leq \ln\left(\frac{1}{k}\right) \quad , a, b \in \mathbb{R} \end{cases}.$$

Example:

Consider the following second order equation:

$(y'' - 6x)_{\mu'' - 2} = 0_0$, then: $\begin{cases} y'' = 6x & (1) \\ \mu'' = 2 & (2) \end{cases}$

The general solution of (1) is:

$y = f(x) = x^3 + ax + b \ ; a, b \in \mathbb{R}.$

The general solution of (2) is:

$\mu(x) = x^2 + cx + d \ ; c, d \in \mathbb{R}.$

To determine the solving set, we write:

$0 \leq \mu(x) \leq 1 \Leftrightarrow \mu(x) = l \quad ; l \in [0,1]$, there for:

$x^2 + cx + d - l = 0, \ \Delta = c^2 - 4(d - l) \geq 0.$

$SS = \{x \in \mathbb{R} \ ; c^2 - 4(d - l) \geq 0\}$ with $l \in [0,1]$.

The general solution of $(y'' - 6x)_{\mu' - 2} = 0_0$ is:

$y_\mu = (x^3 + ax + b)_{x^2 + cx + d} \quad ; x \in SS.$

For example: let $d = 0$, then: $SS = \mathbb{R}$.

For $c = d = 1$, $c^2 - 4(d - l) = 1 - 4 + 4l = 4l - 3$, then:

$\begin{cases} 4l - 3 \geq 0 \ \Rightarrow \ l \geq \frac{3}{4} \\ 4l - 3 \leq 1 \ \Rightarrow l \leq 1 \end{cases} \Leftrightarrow \frac{3}{4} \leq l \leq 1$ and:

$$x = \begin{cases} \dfrac{-c + \sqrt{4l - 3}}{2} \\ \dfrac{-c - \sqrt{4l - 3}}{2} \end{cases}$$

The general solution in this case is:

$y_\mu = (x^3 + ax + b)_{x^2 + cx + d} \quad ; x \in \left\{\frac{-c + \sqrt{4l - 3}}{2}, \frac{-c - \sqrt{4l - 3}}{2}\right\} \ , l \in [\frac{3}{4}, 1].$

Example:

Consider the following first-second order differential equation:

$(y' - \frac{1}{x})_{\mu''-2} = 0_0$, then: $\begin{cases} y' = \frac{1}{x} & (1) \\ \mu'' = 2 & (2) \end{cases}$

The solution of (1) is:

$y = f(x) = \ln x + b \ \ ; x > 0, b \in \mathbb{R}.$

The solution of (2) is:

$\mu(x) = x^2 + cx + d \ \ ; c, d \in \mathbb{R}.$

$SS = \{x \in \mathbb{R} \ \ ; c^2 - 4(d - l) \geq 0 \ \ , 0 \leq l \leq 1\}.$

The general solution is:

$y_\mu = (\ln x + b \ \ )_{x^2+cx+d}$ with: $x \in]0, \infty[\cap SS.$

Definition:

Let $\begin{cases} F_1(x, y, y', \dots, y^n) = 0 \\ F_2(x, t, t', \dots, t^m) = 0 \\ F_3(x, i, i', \dots, i^s) = 0 \\ F_4(x, f, f', \dots, f^j) = 0 \end{cases}$

Be 4 differential equations of orders n, m, s, j respectively.

We define the two-fold neutrosophic differential equation of order (n, m, s, j) as follows:

$\left(F_1(x, y, y', \dots, y^{(n)})\right)_{(F_2,F_3,F_4)} = 0_{(0,0,0)}$ (1)

A two-fold neutrosophic function $(h)_{(t,i,f)}$ is called a solution of (1) if and only if (at the point x_0):

$\begin{cases} F_1(x_0, h, h', \dots, h^{(n)}) = 0 \\ F_2(x_0, t, t', \dots, t^{(m)}) = 0 \\ F_3(x_0, i, i', \dots, i^{(s)}) = 0 \\ F_1(x_0, f, f', \dots, f^{(j)}) = 0 \end{cases}$ with $t(x_0), i(x_0), f(x_0) \in [0,1].$

The solving set is defined as $= S_1 \cap S_2 \cap S_3$;

$\begin{cases} S_1 = \{x \in \mathbb{R} \ \ ; t(x) \in [0,1]\} \\ S_2 = \{x \in \mathbb{R} \ \ ; i(x) \in [0,1]\} \\ S_3 = \{x \in \mathbb{R} \ \ ; f(x) \in [0,1]\} \end{cases}$

Example:

Consider the two-fold neutrosophic differential equation of order (1,1,1,1):

$(y'-2)_{(t'-1,i'-1,f'+3)}=0_{(0,0,0)}$, thus: $\begin{cases} y'=2 \\ t'=1 \\ i'=1 \\ f'=-3 \end{cases}$

Therefor: $\begin{cases} y=2x+a_1 \\ t=x+a_2 \\ i=x+a_3 \\ f=-3x+a_4 \end{cases}$

On the other hand, we have:

$$\begin{cases} 0\le t(x)\le 1 & \Leftrightarrow -a_2\le x\le 1-a_2 \\ 0\le i(x)\le 1 & \Leftrightarrow -a_3\le x\le 1-a_3 \\ 0\le f(x)\le 1 & \Leftrightarrow \dfrac{+a_4}{3}\ge x\ge \dfrac{a_4-1}{3} \end{cases}$$

Hence $SS=[-a_2,1-a_2]\cap[-a_3,1-a_3]\cap\left[\frac{a_4-1}{3},\frac{+a_4}{3}\right]$.

The general solution is:

$(2x+a_1)_{(x+a_2,x+a_3,-3x+a_4)}$; $x\in SS$.

For example, take: $a_2=0$, $a_3=\frac{1}{2}, a_4=+1$, then:

$SS=[0,1]\cap\left[\frac{-1}{2},\frac{1}{2}\right]\cap\left[0,\frac{1}{3}\right]=[0,\frac{1}{3}]$, and the general solution in this case is:

$(2x+a_1)_{(x,x+\frac{1}{2},-3x+1)}$; $x\in\left[0,\frac{1}{3}\right]$.

Example:

Consider the following two-fold neutrosophic differential equation of order (2,1,1,1): $(y''+\sin x)_{(t'-2,i'+1,f'+2)}=0_{(0,0,0)}$, hence:

$$\begin{cases} y''=-\sin x \\ t'=2 \\ i'=-1 \\ f'=-2 \end{cases} \text{ and } \begin{cases} y=\sin x+ax+b & ;a,b\in\mathbb{R} \\ t=2x+c_1 \\ i=-x+c_2 \\ f=-2x+c_3 \end{cases} \quad ;c_1,c_2,c_3\in\mathbb{R}$$

$$\begin{cases} 0\le t(x)\le 1 & \Leftrightarrow \dfrac{-c_1}{2}\le x\le \dfrac{1-c_1}{2} \\ 0\le i(x)\le 1 & \Leftrightarrow c_2-1\le x\le c_2 \\ 0\le f(x)\le 1 & \Leftrightarrow \dfrac{c_3-1}{2}\ge x\ge \dfrac{c_3}{2} \end{cases}$$

$SS=[\frac{-c_1}{2},\frac{1-c_1}{2}]\cap[c_2-1,c_2]\cap\left[\frac{c_3-1}{2},\frac{c_3}{2}\right]$.

The general solution is:

$y_{(t,i,f)} = (\sin x + ax + b)_{(2x+c_1,-x+c_2,-2x+c_3)}$; $x \in SS$, $a, b, c_1, c_2, c_3 \in \mathbb{R}$.

For example: take $c_1 = -3, c_2 = 2, c_3 = 4$, then:

$SS = \left[\frac{3}{2}, 2\right] \cap [1,2] \cap \left[\frac{3}{2}, 2\right] = \left[\frac{3}{2}, 2\right]$.

The general solution in this case is:

$y_{(t,i,f)} = (\sin x + ax + b)_{(2x-3,-x+2,-2x+4)}$; $a, b \in \mathbb{R}$, $x \in \left[\frac{3}{2}, 2\right]$.

Another case if $c_1 = c_2 = c_3 = 1$, then:

$SS = \left[\frac{-1}{2}, 0\right] \cap [0,1] \cap \left[0, \frac{1}{2}\right] = \{0\}$, so that the corresponding solution is: $y_{(t,i,f)} = (\sin x + ax + b)_{(0,0,0)}$; $a, b \in \mathbb{R}$

Example:

Take the following equation of order (1,2,1,2):

$(y')_{(t''-2,i',f''-f')} = 0_{(0,0,0)}$, then:

$$\begin{cases} y' = 0 \\ t'' = 2 \\ i' = 0 \\ f'' = f' \end{cases} \text{ and } \begin{cases} y = c \quad ; c \in \mathbb{R} \\ t = x^2 + ax + b \quad ; a, b \in \mathbb{R} \\ i = m \quad ; m \in \mathbb{R} \\ f = k \ e^x \quad ; k \in \mathbb{R} \end{cases}$$

For determining the solving set (SS), we write:

$$\begin{cases} 0 \le i(x) \le 1 \Leftrightarrow m \in [0,1] \\ 0 \le f(x) \le 1 \Leftrightarrow 0 \le e^x \le \dfrac{1}{k} \Leftrightarrow -\infty < x \le \ln\left(\dfrac{1}{k}\right) \quad ; k > 0 \end{cases}$$

And $0 \le t(x) \le 1 \Leftrightarrow x^2 + ax + b = l \in [0,1] \Leftrightarrow x^2 + ax + b - l = 0 \Leftrightarrow a^2 - 4(b - l) \ge 0$, with $x = \frac{-a \mp \sqrt{a^2 - 4(b-l)}}{2}$

$SS = \left]-\infty, \ln\left(\frac{1}{k}\right)\right] \cap I$; $I = \{x \in \mathbb{R} \ ; x = \frac{-a \mp \sqrt{a^2-4(b-l)}}{2}$, $l \in [0,1], a^2 - 4(b - l) \ge 0\}$.

For example if $k = \frac{1}{2}$, $a = b = 0$, then:

$x = \sqrt{l}$; $0 \le l \le 1$, and $SS =]-\infty, \ln 2] \cap \{\sqrt{l} \ ; \ 0 \le l \le 1\} = [0, \ln 2]$; $l \in [0, (\ln 2)^2]$.

The solution in this case is:

$y_{(t,i,f)} = (c)_{(x^2, m, \frac{1}{2}e^x)}$; $x \in [0, \ln 2], m \in [0,1]$.

Example:

Consider the following equation of order (2,1,1,1):

$(y''-1)_{(t'-t,i'-i,f'-f)} = 0_{(0,0,0)}$, hence:

$$\begin{cases} y''=1 \\ t'=t \\ i'=i \\ f'=f \end{cases} \text{ and } \begin{cases} y=\frac{x^2}{2}+ax+b \\ t=k_1e^x \\ i=k_2e^x \\ f=k_3e^x \end{cases} \quad ; a,b,k_i \in \mathbb{R}.$$

$$\begin{cases} 0 \le t(x) \le 1 \\ 0 \le i(x) \le 1 \\ 0 \le f(x) \le 1 \end{cases} \Leftrightarrow \begin{cases} -\infty < x \le \ln\left(\frac{1}{k_1}\right) \\ -\infty < x \le \ln\left(\frac{1}{k_2}\right) \\ -\infty < x \le \ln\left(\frac{1}{k_3}\right) \end{cases} \quad ; \quad k_1,k_2,k_3 > 0$$

$$SS = \left]-\infty, \ln\left(\frac{1}{k_1}\right)\right] \cap \left]-\infty, \ln\left(\frac{1}{k_2}\right)\right] \cap \left]-\infty, \ln\left(\frac{1}{k_3}\right)\right] = \left]-\infty, \ln\left(\frac{1}{\max(k_1,k_2,k_3)}\right)\right].$$

The general solution is:

$$y_{(t,i,f)} = (\frac{x^2}{2}+ax+b)_{(k_1e^x,k_2e^x,k_3e^x)} \quad ; x \in \left]-\infty, \ln\left(\frac{1}{\max(k_1,k_2,k_3)}\right)\right].$$

Example:

Consider the following equation of order (3.1,1,1):

$(y'''-G)_{(t',i'+2,f'-2)} = (0)_{(0,0,0)}$, then:

$$\begin{cases} y'''=6 \\ t'=0 \\ i'=-2 \\ f'=2 \end{cases} \text{ and } \begin{cases} y=x^3+ax^2+bx+c \\ t'=m_1 \\ i'=-2x+m_2 \\ f=2x+m_3 \end{cases} \quad \text{with } a,b,c,m_1,m_2,m_3 \in \mathbb{R}.$$

$$\begin{cases} 0 \le t(x) \le 1 \\ 0 \le i(x) \le 1 \\ 0 \le f(x) \le 1 \end{cases} \Leftrightarrow \begin{cases} m_1 \in [0,1] \quad , x \in \mathbb{R} \\ x_2 \in \left[\frac{m_2-1}{2}, \frac{m_2}{2}\right] \\ x_3 \in \left[\frac{-m_3}{2}, \frac{-m_3+1}{2}\right] \end{cases}$$

$$SS = \left[\frac{m_2-1}{2}, \frac{m_2}{2}\right] \cap \left[\frac{-m_3}{2}, \frac{-m_3+1}{2}\right].$$

For $m_2 = 1 \ , m_3 = \frac{-1}{2}$, then $SS = \left[0,\frac{1}{2}\right] \cap \left[\frac{1}{4},\frac{3}{4}\right] = \left[\frac{1}{4},\frac{1}{2}\right]$.

The general solution in this case is:

$$y_{(t,i,f)} = (x^3+ax^2+bx+c \qquad)_{(m_1,-2x+\frac{1}{2},2x-\frac{1}{2})} \quad ; x \in \left[\frac{1}{4},\frac{1}{2}\right], m_1 \in [0,1].$$

Conclusion

In this paper, we have studied for the first time the concept of two-fold differential equations of different orders and different types, where we have presented the solutions for two-fold fuzzy differential equations, and for two-fold neutrosophic differential equations. Also, we illustrated many numerical examples to clarify and explain the novelty of this work.

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Received: March 22, 2024. Accepted: July 29, 2024