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The 2-refined neutrosophic fraction function and its integrals

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Abstract: The goal of this article is to study the 2-refined neutrosophic fraction function and its integrals. We were defining the 2-refined neutrosophic fraction function and discussing Algorithm for finding the integral of the 2-refined neutrosophic proper rational function and introduce the integral of the 2-refined neutrosophic improper rational functions. Where each case was explained by given specific examples.

Keywords: 2-refined neutrosophic; partial fraction; proper rational function; integrals.

1. Introduction and Preliminaries

As an alternative to the existing logics, Smarandache proposed the neutrosophic Logic to characterize a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, and contradiction. Refined neutrosophic numbers were made available by Smarandache in the following format: $(a, b_1I_1, b_2I_2, \dots, b_nI_n)$ where $a, b_1, b_2, \dots, b_n \in R$ or C [1].

The notion of refined neutrosophic algebraic structures was first proposed by Agboola[2]. Furthermore, paper [3] examined the refined neutrosophic rings I, assuming that they divide into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. It then follows logically that: [3]

$$I_1I_1 = {I_1}^2 = I_1 \quad (1)$$

$$I_2I_2 = {I_2}^2 = I_2 \quad (2)$$

$$I_1I_2 = I_2I_1 = I_1 \quad (3)$$

Furthermore, a large number of papers [4-5-6-7-8-12-13-14-15] present research on revised neutrosophic numbers. A study on the refined ah-isometry and its applications in refined neutrosophic surfaces was given by Mehmet Celik and Ahmed Hatip. Smarandache talked about neutrosophic indefinite integral [11]. Additionally, Alhasan gave multiple calculus presentations in

which he covered neutrosophic definite and indefinite integrals. Also, he demonstrated the most significant uses of definite integrals in neutrosophic logic [9–10].
This work included several of subjects: the first part provided an introduction and background material, while the main discussion section covered the 2-refined neutrosophic integrals by partial fraction. The last portion contains the conclusion.

Main Discussion

2. The 2-refined neutrosophic fraction function

Definition3.1
A polynomial whose coefficients are 2-refined neutrosophic numbers call 2-refined neutrosophic real polynomials:

$$Q(x, I_1, I_2) = (a_0 + b_0 I_1 + c_0 I_2) + (a_1 + b_1 I_1 + c_1 I_2)x + (a_2 + b_2 I_1 + c_2 I_2)x^2 + \cdots + (a_n + b_n I_1 + c_n I_2)x^n$$

Where $a_0, b_0, c_0, a_1, b_1, c_1, a_2, b_2, c_2, \dots, a_n, b_n, c_n$ are real numbers, I_1, I_2 represent indeterminacy and n is positive integer.

Definition3.2
2-Refined neutrosophic fraction function takes the formula:

$$f(x, I_1, I_2) = \frac{Q_1(x, I_1, I_2)}{Q_2(x, I_1, I_2)}$$

Where $Q_1(x, I_1, I_2)$, $Q_2(x, I_1, I_2)$ are 2-refined neutrosophic real polynomials and $Q_2(x, I_1, I_2) \neq 0$, at least the numerator or denominator is a 2-refined neutrosophic real polynomials.

Example 1

1) $f(x, I_1, I_2) = \dfrac{(-4 + 5I_1 + I_2)x^2 + (7I_1 + 8I_2)x - 2I_2}{8I_2 x - 7 + 4I_1 + 2I_2}$

2) $f(x, I_1, I_2) = \dfrac{(6 - 7I_1 + 14I_2)x}{(I_1 + I_2)x^2 - 11I_2 x - 5 - I_1}$

3) $f(x, I_1, I_2) = \dfrac{1}{(2I_1 + 3I_2)x^2 - 1 + 2I_1}$

Remark 1

- If degree of $Q_1(x, I_1, I_2)$ is less than degree of $Q_2(x, I_1, I_2)$, then:

  $$f(x, I_1, I_2) = \frac{Q_1(x, I_1, I_2)}{Q_2(x, I_1, I_2)}$$

  Is a 2-refined neutrosophic proper rational function.

- If degree of $Q_1(x, I_1, I_2)$ is greater than degree of $Q_2(x, I_1, I_2)$, then:

  $$f(x, I_1, I_2) = \frac{Q_1(x, I_1, I_2)}{Q_2(x, I_1, I_2)}$$

  Is a 2-refined neutrosophic improper rational function.

2.1. Integral of the 2-refined neutrosophic proper rational functions

2.1.1 Cases of the 2-refined neutrosophic proper rational function

- State 1: when it is possible to express the denominator as the product of linear factors that non-repeated.

Let $Q_2(x,I_1,I_2)=\big((a_1+b_1I_1+c_1I_2)x+d_1+s_1I_1+t_1I_2\big)\big((a_2+b_2I_1+c_2I_2)x+d_2+s_2I_1+t_2I_2\big)\dots\big((a_n+b_nI_1+c_nI_2)x+d_n+s_nI_1+t_nI_2\big)$

Where $a_1,b_1,c_1,d_1,s_1,t_1,a_2,b_2,c_2,d_2,s_2,t_2,\dots,a_n,b_n,c_n,d_ns_n,t_n$ are real numbers, then:

$$\frac{Q_1(x,I_1,I_2)}{Q_2(x,I_1,I_2)}=\frac{A_1}{(a_1+b_1I_1+c_1I_2)x+d_1+s_1I_1+t_1I_2}+\frac{A_2}{(a_2+b_2I_1+c_2I_2)x+d_2+s_2I_1+t_2I_2}+\cdots+\frac{A_n}{(a_n+b_nI_1+c_nI_2)x+d_n+s_nI_1+t_nI_2}$$

The values of the constants $A_1,A_2,\dots,A_n$ are to be determined.

- State 2: when it is possible to express the denominator as the product of linear factors that repeated.

Let $Q_2(x,I_1,I_2)=\big((a_1+b_1I_1+c_1I_2)x+d_1+s_1I_1+t_1I_2\big)\big((a_1+b_1I_1+c_1I_2)x+d_1+s_1I_1+t_1I_2\big)\dots\big((a_1+b_1I_1+c_1I_2)x+d_1+s_1I_1+t_1I_2\big)$
$=\big((a_1+b_1I_1+c_1I_2)x+d_1+s_1I_1+t_1I_2\big)^n$

Where a_1,b_1,c_1,d_1,s_1,t_1 are real numbers and n is positive integer, then:

$$\frac{Q_1(x,I_1,I_2)}{Q_2(x,I_1,I_2)}=\frac{A_1}{(a+bI_1+cI_2)x+d+sI_1+tI_2}+\frac{A_2}{\big((a+bI_1+cI_2)x+d+sI_1+tI_2\big)^2}+\cdots+\frac{A_n}{\big((a+bI_1+cI_2)x+d+sI_1+tI_2\big)^n}$$

The values of the constants $A_1,A_2,\dots,A_n$ are to be determined.

- State 3: When the denominator can be express as the product of repeated and non-repeated linear factors.

Let $Q_2(x,I_1,I_2)=\big((a_1+b_1I_1+c_1I_2)x+d_1+s_1I_1+t_1I_2\big)\big((a_2+b_2I_1+c_2I_2)x+d_2+s_2I_1+t_2I_2\big)\dots\big((a_n+b_nI_1+c_nI_2)x+d_n+s_nI_1+t_nI_2\big)\big((a+bI_1+cI_2)x+d+sI_1+tI_2\big)^m$

Where $a_1,b_1,c_1,d_1,s_1,t_1,a_2,b_2,c_2,d_2,s_2,t_2,\dots,a_n,b_n,c_n,d_n,s_n,t_n,a,b,c,d,s,t$ are real numbers and n,m is positive integer, then we can write:

$$\frac{Q_1(x,I_1,I_2)}{Q_2(x,I_1,I_2)}=\frac{A_1}{(a_1+b_1I_1+c_1I_2)x+d_1+s_1I_1+t_1I_2}+\frac{A_2}{(a_2+b_2I_1+c_2I_2)x+d_2+s_2I_1+t_2I_2}+\cdots+\frac{A_n}{(a_n+b_nI_1+c_nI_2)x+d_n+s_nI_1+t_nI_2}+\frac{B_1}{(a+bI_1+cI_2)x+d+sI_1+tI_2}+\frac{B_2}{\big((a+bI_1+cI_2)x+d+sI_1+tI_2\big)^2}+\cdots+\frac{B_m}{\big((a+bI_1+cI_2)x+d+sI_1+tI_2\big)^m}$$

The values of the constants $A_1,A_2,\dots,A_n$, $B_1,B_2,\dots,B_m$ are to be determined.

- State 4: When the denominator can be express as the product of non-repeated quadratic factors, which cannot be further factorize to linear factors.

Let: $Q_2(x,I_1,I_2)=\big((a_1+b_1I_1+c_1I_2)x^2+d_1+s_1I_1+t_1I_2)x+e_1+k_1I_1+r_1I_2\big)\big((a_2+b_2I_1+c_2I_2)x^2+d_2+s_2I_1+t_2I_2)x+e_2+k_2I_1+r_2I_2\big)\dots\big((a_n+b_nI_1+c_nI_2)x^2+d_n+s_nI_1+t_nI_2)x+e_n+k_nI_1+r_nI_2\big)$

Where $a_1,b_1,c_1,d_1,s_1,t_1,e_1,k_1,r_1,a_2,b_2,c_2,d_2,s_2,t_2,e_2,k_2,r_2\dots,a_n,b_n,c_n,d_n,s_n,t_n,e_n,k_n,r_n$ are real numbers and n,m is positive integer, then we can write:

Then we can write:

$$\frac{Q_1(x,I_1,I_2)}{Q_2(x,I_1,I_2)}=\frac{A_1x+B_1}{(a_1+b_1I_1+c_1I_2)x^2+d_1+s_1I_1+t_1I_2)x+e_1+k_1I_1+r_1I_2}+\frac{A_2x+B_2}{(a_2+b_2I_1+c_2I_2)x^2+d_2+s_2I_1+t_2I_2)x+e_2+k_2I_1+r_2I_2}+\cdots+\frac{A_nx+B_n}{(a_n+b_nI_1+c_nI_2)x^2+d_n+s_nI_1+t_nI_2)x+e_n+k_nI_1+r_nI_2}$$

Where $A_1,A_2,\dots,A_n,\ B_1,B_2,\dots,B_n$ are constants whose values are to be determined.

2.1.2 Algorithm for finding Integral of the 2-refined neutrosophic proper rational functions

To evaluate $\int\frac{Q_1(x,I_1,I_2)}{Q_2(x,I_1,I_2)}$ we follow the following steps:

1) Reformulate the form of the function in one of the previous four cases according to the form of the denominator as sum of 2-refined neutrosophic partial fractions.
2) Integrate both sides.

Example 2

Evaluate:

$$\int\frac{dx}{(x+3-I_1+2I_2)(x+1+I_1+I_2)}$$

Solution:

$$\frac{1}{(x+3-I_1+2I_2)(x+1+I_1+I_2)}=\frac{A}{x+3-I_1+2I_2}+\frac{B}{x+1+I_1+I_2}\qquad(*)$$

let's multiply both sides $(*)$ by $(x+3-I_1+2I_2)$ to find value of A:

$$\frac{1}{x+1+I_1+I_2}=A+\frac{(x+3-I_1+2I_2)B}{x+1+I_1+I_2}\qquad(1)$$

By substituting $x=-3+I_1-2I_2$ in (1), we get:

$$A=\frac{1}{-3+I_1-2I_2+1+I_1+I_2}=\frac{1}{-2+2I_1-I_2}$$

let's multiply both sides $(*)$ by $(x+1+I_1+I_2)$ to find value of B:

$$\frac{1}{x+3-I_1+2I_2}=\frac{(x+1+I_1+I_2)A}{x+3-I_1+2I_2}+B\qquad(2)$$

By substituting $x = -1 - I_1 - I_2$ in (2), we get:

$$B = \frac{1}{-1 - I_1 - I_2 + 3 - I_1 + 2I_2} = \frac{1}{2 - 2I_1 + I_2}$$

By substituting in (*) we get:

$$\frac{1}{(x + 3 - I_1 + 2I_2)(x + 1 + I_1 + I_2)} = \frac{\frac{1}{-2 + 2I_1 - I_2}}{x + 3 - I_1 + 2I_2} + \frac{\frac{1}{2 - 2I_1 + I_2}}{x + 1 + I_1 + I_2}$$

$$\Rightarrow \int \frac{dx}{(x + 3 - I_1 + 2I_2)(x + 1 + I_1 + I_2)} dx = \int \left(\frac{\frac{1}{-2 + 2I_1 - I_2}}{x + 3 - I_1 + 2I_2} + \frac{\frac{1}{2 - 2I_1 + I_2}}{x + 1 + I_1 + I_2} \right) dx$$

$$= \frac{1}{-2 + 2I_1 - I_2} ln|x + 3 - I_1 + 2I_2| + \frac{1}{2 - 2I_1 + I_2} ln|x + 1 + I_1 + I_2|$$

$$= \frac{1}{-2 + 2I_1 - I_2} (ln|x + 3 - I_1 + 2I_2| - ln|x + 1 + I_1 + I_2|)$$

$$= \frac{1}{-2 + 2I_1 - I_2} ln\left|\frac{x + 3 - I_1 + 2I_2}{x + 1 + I_1 + I_2}\right|$$

$$= \left(-\frac{1}{2} - \frac{2}{3}I_1 + \frac{1}{6}I_2\right) ln\left|\frac{x + 3 - I_1 + 2I_2}{x + 1 + I_1 + I_2}\right| + C$$

Whereas $C = a + bI_1 + cI_2$ and a, b, c are real numbers.

Let's check the answer:

$$\frac{d}{dx}\left[\left(-\frac{1}{2} - \frac{2}{3}I_1 + \frac{1}{6}I_2\right) ln\left|\frac{x + 3 - I_1 + 2I_2}{x + 1 + I_1 + I_2}\right| + C\right]$$

$$= \frac{d}{dx}\left[\left(-\frac{1}{2} - \frac{2}{3}I_1 + \frac{1}{6}I_2\right)(ln|x + 3 - I_1 + 2I_2| - ln|x + 1 + I_1 + I_2|) + C\right]$$

$$= \left(-\frac{1}{2} - \frac{2}{3}I_1 + \frac{1}{6}I_2\right)\left(\frac{1}{x + 3 - I_1 + 2I_2} - \frac{1}{x + 1 + I_1 + I_2}\right)$$

$$= \left(-\frac{1}{2} - \frac{2}{3}I_1 + \frac{1}{6}I_2\right)\left(\frac{x + 1 + I_1 + I_2 - x - 3 + I_1 - 2I_2}{(x + 3 - I_1 + 2I_2)(x + 1 + I_1 + I_2)}\right)$$

$$= \left(-\frac{1}{2} - \frac{2}{3}I_1 + \frac{1}{6}I_2\right)\left(\frac{-2 + 2I_1 - I_2}{(x + 3 - I_1 + 2I_2)(x + 1 + I_1 + I_2)}\right) = \frac{\left(-\frac{1}{2} - \frac{2}{3}I_1 + \frac{1}{6}I_2\right)(-2 + 2I_1 - I_2)}{(x + 3 - I_1 + 2I_2)(x + 1 + I_1 + I_2)}$$

$$= \frac{1}{(x + 3 - I_1 + 2I_2)(x + 1 + I_1 + I_2)}$$

The same integral function.

Example 3

Evaluate:

$$\int \frac{2+I_1+I_2}{((4+I_1+I_2)x+4-I_1-I_2)((1+2I_1+I_2)x+4+2I_1+I_2)}dx$$

Solution:

$$\frac{2+I_1+I_2}{((4+I_1+I_2)x+4-I_1-I_2)((1+2I_1+I_2)x+4+2I_1+I_2)} = \frac{A}{(4+I_1+I_2)x+4-I_1-I_2} + \frac{B}{(1+2I_1+I_2)x+4+2I_1+I_2} \quad (*)$$

To find value of A, we multiply both sides of $(*)$ by $((4+I_1+I_2)x+4-I_1-I_2)$:

$$\frac{2+I_1+I_2}{(1+2I_1+I_2)x+4+2I_1+I_2} = A + \frac{((4+I_1+I_2)x+4-I_1-I_2)B}{(1+2I_1+I_2)x+4+2I_1+I_2} \quad (1)$$

By substituting

$$x = \frac{-4+I_1+I_2}{4+I_1+I_2} = -1+\frac{4}{15}I_1+\frac{2}{5}I_2$$

in (1), we get:

$$A = \frac{2+I_1+I_2}{(1+2I_1+I_2)\left(-1+\frac{4}{15}I_1+\frac{2}{5}I_2\right)+4+2I_1+I_2} = \frac{2+I_1+I_2}{3+\frac{28}{15}I_1+\frac{4}{5}I_2} = \frac{2}{3}-\frac{27}{323}I_1+\frac{7}{57}I_2$$

To find value of B, we multiply both sides $(*)$ by $((1+2I_1+I_2)x+4+2I_1+I_2)$:

$$\frac{2+I_1+I_2}{(4+I_1+I_2)x+4-I_1-I_2} = \frac{((1-2I_1+I_2)x+4+2I_1+I_2)A}{(4+I_1+I_2)x+4-I_1-I_2} + B \quad (2)$$

By substituting

$$x = \frac{-4-2I_1-I_2}{1+2I_1+I_2} = -4+\frac{3}{4}I_1+\frac{3}{2}I_2$$

in (2), we get:

$$B = \frac{2+I_1+I_2}{(4+I_1+I_2)\left(-4+\frac{3}{4}I_1+\frac{3}{2}I_2\right)+4-I_1-I_2} = \frac{2+I_1+I_2}{-12+I_1+\frac{5}{2}I_2} = -\frac{1}{6}-\frac{50}{323}I_1-\frac{17}{114}I_2$$

By substituting in (*) we get:

$$\frac{2+I_1+I_2}{\left((4+I_1+I_2)x+4-I_1-I_2\right)\left((1+2I_1+I_2)x+4+2I_1+I_2\right)} = \frac{\frac{2}{3}-\frac{27}{323}I_1+\frac{7}{57}I_2}{(4+I_1+I_2)x+4-I_1-I_2} + \frac{-\frac{1}{6}-\frac{50}{323}I_1-\frac{17}{114}I_2}{(1+2I_1+I_2)x+4+2I_1+I_2}$$

$$\Rightarrow \quad \int \frac{2+I_1+I_2}{\left((4+I_1+I_2)x+4-I_1-I_2\right)\left((1+2I_1+I_2)x+4+2I_1+I_2\right)} = \int\left(\frac{\frac{2}{3}-\frac{27}{323}I_1+\frac{7}{57}I_2}{(4+I_1+I_2)x+4-I_1-I_2} + \frac{-\frac{1}{6}-\frac{50}{323}I_1-\frac{17}{114}I_2}{(1+2I_1+I_2)x+4+2I_1+I_2}\right)dx$$

$$=\frac{\frac{2}{3}-\frac{27}{323}I_1+\frac{7}{57}I_2}{4+I_1+I_2}ln|(4+I_1+I_2)x+4-I_1-I_2| + \frac{-\frac{1}{6}-\frac{50}{323}I_1-\frac{17}{114}I_2}{1+2I_1+I_2}ln|(1+2I_1+I_2)x+4+2I_1+I_2|$$

$$=\left(\frac{1}{6}-\frac{13}{323}I_1-\frac{1}{114}I_2\right)ln|(4+I_1+I_2)x+4-I_1-I_2| + \left(-\frac{1}{6}+\frac{13}{323}I_1+\frac{1}{114}I_2\right)ln|(1+2I_1+I_2)x+4+2I_1+I_2|$$

$$=\left(\frac{1}{6}-\frac{13}{323}I_1-\frac{1}{114}I_2\right)(ln|(4+I_1+I_2)x+4-I_1-I_2|-ln|(1+2I_1+I_2)x+4+2I_1+I_2|)$$

$$=\left(\frac{1}{6}-\frac{13}{323}I_1-\frac{1}{114}I_2\right)ln\left|\frac{(4+I_1+I_2)x+4-I_1-I_2}{(1+2I_1+I_2)x+4+2I_1+I_2}\right|+C$$

Whereas $C=a+bI_1+cI_2$ and a,b,c are real numbers.

Example 4

Evaluate:

$$\int\frac{2x-3I_1+I_2}{(x+2+2I_1)^2(x-1)}dx$$

Solution:

$$\frac{2x-3I_1+I_2}{(x+2+2I_1)^2(x-1)}=\frac{A}{x+2+2I_1}+\frac{B}{(x+2+2I_1)^2}+\frac{D}{x-1} \qquad (*)$$

To find value of D, We multiply both sides $(*)$ by $(x-1)$:

$$\frac{2x-3I_1+I_2}{(x+2+2I_1)^2}=\frac{(x-1)A}{x+2+2I_1}+\frac{(x-1)B}{(x+2+2I_1)^2}+D \qquad (1)$$

By substituting $x=1$ in (1), we get:

$$D=\frac{2(1)-3I_1+I_2}{(3+2I_1)^2}=\frac{2-3I_1+I_2}{9+16I_1}$$

$$\Rightarrow \qquad D=\frac{2}{9}-\frac{1}{3}I_1+\frac{1}{9}I_2$$

To find value of B, We multiply both sides $(*)$ by $(x+2+2I_1)^2$:

$$\frac{2x-3I_1+I_2}{x-1}=\frac{(x+2+2I_1)^2A}{x+2+2I_1}+B+\frac{(x+2+2I_1)^2D}{x-1} \qquad (2)$$

By substituting $x=-2-2I_1$ in (2), we get:

$$B=\frac{2(-2-2I_1)-3I_1+I_2}{-2-2I_1-1}=\frac{-4-7I_1+I_2}{-3-2I_1}$$

$$\Rightarrow \qquad B=\frac{4}{3}+I_1-\frac{1}{3}I_2$$

To find value of A, we substitute value of B, D and $x=-1$ in (*) , we get:

$$\frac{-2-3I_1+I_2}{(-1+2+2I_1)^2(-1-1)}=\frac{A}{-1+2+2I_1}+\frac{\frac{4}{3}+I_1-\frac{1}{3}I_2}{(-1+2+2I_1)^2}+\frac{\frac{2}{9}-\frac{1}{3}I_1+\frac{1}{9}I_2}{-1-1}$$

$$\frac{-2-3I_1+I_2}{-2-16I_1}=\frac{A}{1+2I_1}+\frac{\frac{4}{3}+I_1-\frac{1}{3}I_2}{1+8I_1}-\frac{1}{9}+\frac{1}{6}I_1-\frac{1}{18}I_2$$

$$1-\frac{5}{18}I_1-\frac{1}{2}I_2=\frac{A}{1+2I_1}+\frac{4}{3}-\frac{7}{9}I_1-\frac{1}{3}I_2-\frac{1}{9}+\frac{1}{6}I_1-\frac{1}{18}I_2$$

$$\frac{A}{1+2I_1}=1-\frac{5}{18}I_1-\frac{1}{2}I_2-\frac{4}{3}+\frac{7}{9}I_1+\frac{1}{3}I_2+\frac{1}{9}-\frac{1}{6}I_1+\frac{1}{18}I_2$$

$$A=(1+2I_1)\left(-\frac{2}{9}+\frac{1}{3}I_1-\frac{1}{9}I_2\right)=-\frac{2}{9}+\frac{1}{3}I_1-\frac{1}{9}I_2-\frac{4}{9}I_1+\frac{2}{3}I_1-\frac{2}{9}I_1$$

$$\Rightarrow \qquad A=-\frac{2}{9}+\frac{1}{3}I_1-\frac{1}{9}I_2$$

By substituting in (*) we get:

$$\frac{2x-3I_1+I_2}{(x+2+2I_1)^2(x-1)}=\frac{-\frac{2}{9}+\frac{1}{3}I_1-\frac{1}{9}I_2}{x+2+2I_1}+\frac{\frac{4}{3}+I_1-\frac{1}{3}I_2}{(x+2+2I_1)^2}+\frac{\frac{2}{9}-\frac{1}{3}I_1+\frac{1}{9}I_2}{x-1}$$

$$\Rightarrow \int\frac{2x-3I_1+I_2}{(x+2+2I_1)^2(x-1)}dx=\int\left(\frac{-\frac{2}{9}+\frac{1}{3}I_1-\frac{1}{9}I_2}{x+2+2I_1}+\frac{\frac{4}{3}+I_1-\frac{1}{3}I_2}{(x+2+2I_1)^2}+\frac{\frac{2}{9}-\frac{1}{3}I_1+\frac{1}{9}I_2}{x-1}\right)dx$$

$$=\left(-\frac{2}{9}+\frac{1}{3}I_1-\frac{1}{9}I_2\right)ln|x+2+2I_1|-\left(\frac{4}{3}+I_1-\frac{1}{3}I_2\right)\frac{1}{x+2+2I_1}+\left(\frac{2}{9}-\frac{1}{3}I_1+\frac{1}{9}I_2\right)ln|x-1|+C$$

$$=\left(-\frac{2}{9}+\frac{1}{3}I_1-\frac{1}{9}I_2\right)(ln|x+2+2I_1|-ln|x-1|)-\left(\frac{4}{3}+I_1-\frac{1}{3}I_2\right)\frac{1}{x+2+2I_1}+C$$

$$=\left(-\frac{2}{9}+\frac{1}{3}I_1-\frac{1}{9}I_2\right)ln\left|\frac{x+2+2I_1}{x-1}\right|-\left(\frac{4}{3}+I_1-\frac{1}{3}I_2\right)\frac{1}{x+2+2I_1}+C$$

Whereas $C=a+bI_1+cI_2$ *and* a,b,c are real numbers.

Example 5

Evaluate:

$$\int\frac{I_1+I_2}{x^2-16-11I_1-9I_2}dx$$

Solution:
to find the denominator factors

$$x^2-16-11I_1-9I_2=x^2-\left(\sqrt{16+11I_1+9I_2}\right)^2$$

Let's find: $\sqrt{16+11I_1+9I_2}$

$$\sqrt{16+11I_1+9I_2}=x+yI_1+zI_2$$
$$16+11I_1+9I_2=(x+yI_1+zI_2)^2$$

$$16+11I_1+9I_2=(x+yI_1)^2+2(x+yI_1)(zI_2)+(zI_2)^2$$

$$16+11I_1+9I_2=x^2+2xyI_1+(yI_1)^2+2(x+yI_1)(zI_2)+(zI_2)^2$$

$$16+11I_1+9I_2=x^2+2xyI_1+y^2I_1+2xzI_2+2yzI_1+z^2I_2$$

$$16+11I_1+9I_2=x^2+(y^2+2xy+2yz)I_1+(z^2+2xz)I_2$$

Whence:

$$\begin{cases} x^2=16 \\ y^2+2xy+2yz=11 \\ z^2+2xz=9 \end{cases}$$

$$\Rightarrow \begin{cases} x=\pm 4 \\ y^2+2xy+2yz=11 \quad (1) \\ z^2+2xz=9 \quad (2) \end{cases}$$

Case1: $x=4$ by substituting in (2)

$$z^2+8z-9=0$$
$$(z-1)(z+9)=0$$

Then:

$$\begin{cases} z=1 \\ z=-9 \end{cases}$$

For $z=1$ we substitute in (1)

$$y^2+10y-11=0$$
$$(y+11)(y-1)=0$$

Then:

$$\begin{cases} y=1 \\ y=-11 \end{cases}$$

Hence:

$\sqrt{16+11I_1+9I_2}=4+I_1+I_2$

Or:

$=4-11I_1+I_2$

For $z=-9$ we substitute in (1)

$$y^2-10y-11=0$$
$$(y-11)(y+1)=0$$

Then:

$$\begin{cases} y=11 \\ y=-1 \end{cases}$$

Hence:

$\sqrt{16+11I_1+9I_2}=4+11I_1-9I_2$

Or:

$=4-I_1-9I_2$

Case2: $x=-4$ by substituting in (2)

$$z^2 - 8z - 9 = 0$$
$$(z-9)(z+1) = 0$$

Then:

$$\begin{cases} z = 9 \\ z = -1 \end{cases}$$

For $z = 9$ we substitute in (1)

$$y^2 + 10y - 11 = 0$$
$$(y-1)(y+11) = 0$$

Then:

$$\begin{cases} y = 1 \\ y = -11 \end{cases}$$

Hence:

$\sqrt{16 + 11I_1 + 9I_2} = -4 + I_1 + 9I_2$

Or:

$= -4 - 11I_1 + 9I_2$

For $z = -1$ we substitute in (1)

$$y^2 - 10y - 11 = 0$$
$$(y-11)(y+1) = 0$$
$$\begin{cases} y = 11 \\ y = -1 \end{cases}$$

Hence:

$\sqrt{16 + 11I_1 + 9I_2} = -4 + 11I_1 - I_2$

Or:

$= -4 - I_1 - I_2$

Hence, we got eight 2-refined neutrosophic solutions:

$\sqrt{16 + 11I_1 + 9I_2} = 4 + I_1 + I_2$

Or:

$= 4 - 11I_1 + I_2$

Or:

$= 4 + 11I_1 - 9I_2$

Or:

$= 4 - I_1 - 9I_2$

Or:

$= -4 + I_1 + 9I_2$

Or:

$= -4 - 11I_1 + 9I_2$

Or:

$= -4 + 11I_1 - I_2$

Or:

$= -4 - I_1 - I_2$

First 2-refined neutrosophic factoring:

$$x^2 - 16 - 11I_1 - 9I_2 = (x - 4 - I_1 - I_2)(x + 4 + I_1 + I_2)$$

Second 2-refined neutrosophic factoring:

$$x^2 - 16 - 11I_1 - 9I_2 = (x - 4 + 11I_1 - I_2)(x + 4 - 11I_1 + I_2)$$

Third 2-refined neutrosophic factoring:

$$x^2 - 16 - 11I_1 - 9I_2 = (x - 4 - 11I_1 + 9I_2)(x + 4 + 11I_1 - 9I_2)$$

Fourth 2-refined neutrosophic factoring:

$$x^2-16-11I_1-9I_2=(x-4+I_1+9I_2)(x+4-I_1-9I_2)$$

Thus, the denominator factors can be write in four cases:
Case1:

$$x^2-16-11I_1-9I_2=(x-4-I_1-I_2)(x+4+I_1+I_2)$$

$$\frac{I_1+I_2}{x^2-16-11I_1-9I_2}=\frac{I_1+I_2}{(x-4-I_1-I_2)(x+4+I_1+I_2)}$$

$$\frac{I_1+I_2}{(x-4-I_1-I_2)(x+4+I_1+I_2)}=\frac{A}{x-4-I_1-I_2}+\frac{B}{x+4+I_1+I_2} \qquad (*)$$

To find value of A, We multiply both sides $(*)$ by $(x-4-I_1-I_2)$:

$$\frac{I_1+I_2}{x+4+I_1+I_2}=A+\frac{(x-4-I_1-I_2)B}{x+4+I_1+I_2} \qquad (1)$$

By substituting $x=4+I_1+I_2$ in (1), we get:

$$A=\frac{I_1+I_2}{8+2I_1+2I_2}=\frac{1}{15}I_1+\frac{1}{10}I_2$$

To find value of A, We multiply both sides $(*)$ by $(x+4+I_1+I_2)$:

$$\frac{I_1+I_2}{x-4-I_1-I_2}=\frac{(x+4+I_1+I_2)A}{x-4-I_1-I_2}+B \qquad (2)$$

By substituting $x=-4-I_1-I_2$ in (2), we get:

$$B=\frac{I_1+I_2}{-8-2I_1-2I_2}=-\frac{1}{15}I_1-\frac{1}{10}I_2$$

By substituting in (*) we get:

$$\frac{I_1+I_2}{(x-4-I_1-I_2)(x+4+I_1+I_2)}=\frac{\frac{1}{15}I_1+\frac{1}{10}I_2}{x-4-I_1-I_2}+\frac{-\frac{1}{15}I_1-\frac{1}{10}I_2}{x+4+I_1+I_2}$$

$$\Rightarrow \int \frac{I_1+I_2}{(x-4-I_1-I_2)(x+4+I_1+I_2)}dx=\int\left(\frac{\frac{1}{15}I_1+\frac{1}{10}I_2}{x-4-I_1-I_2}+\frac{-\frac{1}{15}I_1-\frac{1}{10}I_2}{x+4+I_1+I_2}\right)dx$$

$$=\left(\frac{1}{15}I_1+\frac{1}{10}I_2\right)ln|x-4-I_1-I_2|+\left(-\frac{1}{15}I_1-\frac{1}{10}I_2\right)ln|x+4+I_1+I_2|$$

$$=\left(\frac{1}{15}I_1+\frac{1}{10}I_2\right)(ln|x-4-I_1-I_2|-ln|x+4+I_1+I_2|)$$

$$=\left(\frac{1}{15}I_1+\frac{1}{10}I_2\right)ln\left|\frac{x-4-I_1-I_2}{x+4+I_1+I_2}\right|+C$$

Case2:

$$x^2-16-11I_1-9I_2=(x-4+11I_1-I_2)(x+4-11I_1+I_2)$$

$$\frac{I_1+I_2}{x^2-16-11I_1-9I_2}=\frac{I_1+I_2}{(x-4+11I_1-I_2)(x+4-11I_1+I_2)}$$

$$\frac{I_1+I_2}{(x-4+11I_1-I_2)(x+4-11I_1+I_2)}=\frac{A}{x-4+11I_1-I_2}+\frac{B}{x+4-11I_1+I_2} \qquad (*)'$$

To find value of A, We multiply both sides $(*)'$ by $(x-4+11I_1-I_2)$:

$$\frac{I_1+I_2}{x+4-11I_1+I_2}=A+\frac{(x-4+11I_1-I_2)B}{x+4-11I_1+I_2} \qquad (1)'$$

By substituting $x=4-11I_1+I_2$ in $(1)'$, we get:

$$A=\frac{I_1+I_2}{8-22I_1+2I_2}=-\frac{4}{15}I_1+\frac{1}{10}I_2$$

To find value of A, We multiply both sides $(*)'$ by $(x+4-11I_1+I_2)$:

$$\frac{I_1+I_2}{x-4+11I_1-I_2}=\frac{(x+4-11I_1+I_2)A}{x-4+11I_1-I_2}+B \qquad (2)'$$

By substituting $x=-4+11I_1-I_2$ in $(2)'$, we get:

$$B=\frac{I_1+I_2}{-8+22I_1-2I_2}=\frac{4}{15}I_1-\frac{1}{10}I_2$$

By substituting in $(*)'$ we get:

$$\frac{I_1+I_2}{(x-4+11I_1-I_2)(x+4-11I_1+I_2)}=\frac{-\frac{4}{15}I_1+\frac{1}{10}I_2}{x-4+11I_1-I_2}+\frac{\frac{4}{15}I_1-\frac{1}{10}I_2}{x+4-11I_1+I_2}$$

$$\Rightarrow \int \frac{I_1+I_2}{(x-4+11I_1-I_2)(x+4-11I_1+I_2)}dx=\int\left(\frac{-\frac{4}{15}I_1+\frac{1}{10}I_2}{x-4+11I_1-I_2}+\frac{\frac{4}{15}I_1-\frac{1}{10}I_2}{x+4-11I_1+I_2}\right)dx$$

$$=\left(-\frac{4}{15}I_1+\frac{1}{10}I_2\right)ln|x-4+11I_1-I_2|+\left(\frac{4}{15}I_1-\frac{1}{10}I_2\right)ln|x+4-11I_1+I_2|$$

$$=\left(-\frac{4}{15}I_1+\frac{1}{10}I_2\right)(ln|x-4+11I_1-I_2|-ln|x+4-11I_1+I_2|)$$

$$=\left(-\frac{4}{15}I_1+\frac{1}{10}I_2\right)ln\left|\frac{x-4+11I_1-I_2}{x+4-11I_1+I_2}\right|+C$$

Case3:

$$x^2-16-11I_1-9I_2=(x-4-11I_1+9I_2)(x+4+11I_1-9I_2)$$

$$\frac{I_1+I_2}{x^2-16-11I_1-9I_2}=\frac{I_1+I_2}{(x-4-11I_1+9I_2)(x+4+11I_1-9I_2)}$$

$$\frac{I_1+I_2}{(x-4-11I_1+9I_2)(x+4+11I_1-9I_2)}=\frac{A}{x-4-11I_1+9I_2}+\frac{B}{x+4+11I_1-9I_2} \qquad (**)$$

To find value of A, We multiply both sides $(**)$ by $(x-4-11I_1+9I_2)$:

$$\frac{I_1+I_2}{x+4+11I_1-9I_2}=A+\frac{(x-4-11I_1+9I_2)B}{x+4+11I_1-9I_2} \qquad (1)''$$

By substituting $x=4+11I_1-9I_2$ in $(1)''$, we get:

$$A=\frac{I_1+I_2}{8+22I_1-18I_2}=\frac{4}{15}I_1-\frac{1}{10}I_2$$

To find value of A, We multiply both sides $(**)$ by $(x+4+11I_1-9I_2)$:

$$\frac{I_1+I_2}{x-4-11I_1+9I_2}=\frac{(x+4+11I_1-9I_2)A}{x-4-11I_1+9I_2}+B \qquad (2)''$$

By substituting $x=-4-11I_1+9I_2$ in $(2)''$, we get:

$$B=\frac{I_1+I_2}{-8-22I_1+18I_2}=-\frac{4}{15}I_1+\frac{1}{10}I_2$$

By substituting in $(**)$ we get:

$$\frac{I_1+I_2}{(x-4-11I_1+9I_2)(x+4+11I_1-9I_2)}=\frac{A}{x-4-11I_1+9I_2}+\frac{B}{x+4+11I_1-9I_2}$$

$$\Rightarrow \int \frac{I_1+I_2}{(x-4-11I_1+9I_2)(x+4+11I_1-9I_2)}dx=\int\left(\frac{\frac{4}{15}I_1-\frac{1}{10}I_2}{x-4-11I_1+9I_2}+\frac{-\frac{4}{15}I_1+\frac{1}{10}I_2}{x+4+11I_1-9I_2}\right)dx$$

$$=\left(\frac{4}{15}I_1-\frac{1}{10}I_2\right)ln|x-4-11I_1+9I_2|+\left(-\frac{4}{15}I_1+\frac{1}{10}I_2\right)ln|x+4+11I_1-9I_2|$$

$$=\left(\frac{4}{15}I_1-\frac{1}{10}I_2\right)(ln|x-4-11I_1+9I_2|-ln|x+4+11I_1-9I_2|)$$

$$=\left(\frac{4}{15}I_1-\frac{1}{10}I_2\right)ln\left|\frac{x-4-11I_1+9I_2}{x+4+11I_1-9I_2}\right|+C$$

Case4:

$$x^2-16-11I_1-9I_2=(x-4+I_1+9I_2)(x+4-I_1-9I_2)$$

$$\frac{I_1+I_2}{x^2-16-11I_1-9I_2}=\frac{I_1+I_2}{(x-4+I_1+9I_2)(x+4-I_1-9I_2)}$$

$$\frac{I_1+I_2}{(x-4+I_1+9I_2)(x+4-I_1-9I_2)}=\frac{A}{x-4+I_1+9I_2}+\frac{B}{x+4-I_1-9I_2} \qquad (**)'$$

To find value of A, We multiply both sides $(**)'$ by $(x-4+I_1+9I_2)$:

$$\frac{I_1+I_2}{x+4-I_1-9I_2}=A+\frac{(x-4+I_1+9I_2)B}{x+4-I_1-9I_2} \qquad (1)'''$$

By substituting $x=-4+I_1+9I_2$ in $(1)'''$, we get:

$$A = \frac{I_1 + I_2}{8 - 2I_1 - 18I_2} = -\frac{1}{15}I_1 - \frac{1}{10}I_2$$

To find value of A, We multiply both sides $(**)'$ by $(x + 4 - I_1 - 9I_2)$:

$$\frac{I_1 + I_2}{x - 4 + I_1 + 9I_2} = \frac{(x + 4 - I_1 - 9I_2)A}{x - 4 - 11I_1 + 9I_2} + B \qquad (2)'''$$

By substituting $x = -4 + I_1 + 9I_2$ in $(2)'''$, we get:

$$B = \frac{I_1 + I_2}{-8 + 2I_1 + 18I_2} = \frac{1}{15}I_1 + \frac{1}{10}I_2$$

By substituting in $(**)'$ we get:

$$\frac{I_1 + I_2}{(x - 4 + I_1 + 9I_2)(x + 4 - I_1 - 9I_2)} = \frac{A}{x - 4 + I_1 + 9I_2} + \frac{B}{x + 4 - I_1 - 9I_2}$$

$$\Rightarrow \int \frac{I_1 + I_2}{(x - 4 + I_1 + 9I_2)(x + 4 - I_1 - 9I_2)} dx = \int \left(\frac{-\frac{1}{15}I_1 - \frac{1}{10}I_2}{x - 4 + I_1 + 9I_2} + \frac{\frac{1}{15}I_1 + \frac{1}{10}I_2}{x + 4 - I_1 - 9I_2} \right) dx$$

$$= \left(-\frac{1}{15}I_1 - \frac{1}{10}I_2\right) ln|x - 4 + I_1 + 9I_2| + \left(\frac{1}{15}I_1 + \frac{1}{10}I_2\right) ln|x + 4 - I_1 - 9I_2|$$

$$= \left(-\frac{1}{15}I_1 - \frac{1}{10}I_2\right)(ln|x - 4 + I_1 + 9I_2| - ln|x + 4 - I_1 - 9I_2|)$$

$$= \left(-\frac{1}{15}I_1 - \frac{1}{10}I_2\right) ln\left|\frac{x - 4 + I_1 + 9I_2}{x + 4 - I_1 - 9I_2}\right| + C$$

Hence:

$$\int \frac{5I}{x^2 - 4 + 3I} dx = \begin{cases} \left(\frac{1}{15}I_1 + \frac{1}{10}I_2\right) ln\left|\frac{x - 4 - I_1 - I_2}{x + 4 + I_1 + I_2}\right| + C \\ \left(-\frac{4}{15}I_1 + \frac{1}{10}I_2\right) ln\left|\frac{x - 4 + 11I_1 - I_2}{x + 4 - 11I_1 + I_2}\right| + C \\ \left(\frac{4}{15}I_1 - \frac{1}{10}I_2\right) ln\left|\frac{x - 4 - 11I_1 + 9I_2}{x + 4 + 11I_1 - 9I_2}\right| + C \\ \left(-\frac{1}{15}I_1 - \frac{1}{10}I_2\right) ln\left|\frac{x - 4 + I_1 + 9I_2}{x + 4 - I_1 - 9I_2}\right| + C \end{cases}$$

Whereas $C = a + bI_1 + cI_2$ *and* a, b, c are real numbers.

Example 6

Evaluate:

$$\int \frac{4 + I_1 + 2I_2}{(x + 1 + I_1 + 2I_2)(x^2 + 4 + I_2)} dx$$

Solution:

We note that $(x^2 + 4 + I_2)$ cannot be analyzing, because:

$$x^2 + 4 + I_2 = x^2 - \left(\sqrt{-4 - I_2}\right)^2$$

Let's find $\sqrt{-4 - I_2}$

$$\sqrt{-4 - I_2} = \alpha + \beta I_2 + \gamma I_2$$

$$-4 - I_2 = \alpha^2 + (\beta^2 + 2\alpha\beta + 2\beta\gamma)I_1 + (\gamma^2 + 2\alpha\gamma)I_2$$

then:

$$\alpha^2 = -4 \ \text{(Impossible in real number)}$$

So:

$$\frac{4 + I_1 + 2I_2}{(x + 1 + I_1 + 2I_2)(x^2 + 4 + I_2)} = \frac{A}{x + 1 + I_1 + 2I_2} + \frac{Bx + D}{x^2 + 4 + I_2} \qquad (*)$$

To find value of A, We multiply both sides $(*)$ by $(x + 1 + I_1 + 2I_2)$:

$$\frac{4 + I_1 + 2I_2}{x^2 + 4 + I_2} = A + \frac{(x + 1 + I_1 + 2I_2)(Bx + D)}{x^2 + 4 + I_2} \qquad (1)$$

by substituting $x = -1 - I_1 - 2I_2$ in (1), we get:

$$A = \frac{4 + I_1 + 2I_2}{(-1 - I_1 - 2I_2)^2 + 4 + I_2} = \frac{4 + I_1 + 2I_2}{5 + 7I_1 + 9I_2}$$

$$\Longrightarrow \qquad A = \frac{4}{5} - \frac{2}{21}I_1 - \frac{13}{35}I_2$$

To find value of B, We multiply both sides $(*)$ by x:

$$\frac{(4 + I_1 + 2I_2)x}{(x + 1 + I_1 + 2I_2)(x^2 + 4 + I_2)} = \frac{Ax}{x + 1 + I_1 + 2I_2} + \frac{Bx^2 + D}{x^2 + 4 + I_2} \qquad (2)$$

By take limit both sides in (2), when $x \longrightarrow \infty$, we get:

$$A + B = 0 \qquad \Longrightarrow \qquad B = -A = -\frac{4}{5} + \frac{2}{21}I_1 + \frac{13}{35}I_2$$

To find value of D, we substitute value of A, B and let be $x = 0$, in (*), we get:

$$\frac{4 + I_1 + 2I_2}{(0 + 1 + I_1 + 2I_2)(0^2 + 4 + I_2)} = \frac{\frac{4}{5} - \frac{2}{21}I_1 - \frac{13}{35}I_2}{0 + 1 + I_1 + 2I_2} + \frac{\left(-\frac{4}{5} + \frac{2}{21}I_1 + \frac{13}{35}I_2\right)(0) + D}{0^2 + 4 + I_2}$$

$$\frac{4 + I_1 + 2I_2}{4 + 5I_1 + 11I_2} = \frac{\frac{4}{5} - \frac{2}{21}I_1 - \frac{13}{35}I_2}{1 + I_1 + 2I_2} + \frac{D}{4 + I_2}$$

$$1 - \frac{1}{20}I_1 - \frac{3}{5}I_2 = \frac{4}{5} - \frac{5}{84}I_1 - \frac{23}{35}I_2 + \frac{D}{4 + I_2}$$

$$\frac{D}{4 + I_2} = \frac{1}{5} + \frac{1}{105}I_1 + \frac{2}{35}I_2$$

$$D = \frac{4}{5} + \frac{1}{21}I_1 + \frac{17}{35}I_2$$

By substituting in (*) we get:

$$\frac{4 + I_1 + 2I_2}{(x + 1 + I_1 + 2I_2)(x^2 + 4 + I_2)} = \frac{\frac{4}{5} - \frac{2}{21}I_1 - \frac{13}{35}I_2}{x + 1 + I_1 + 2I_2} + \frac{\left(-\frac{4}{5} + \frac{2}{21}I_1 + \frac{13}{35}I_2\right)x + \frac{4}{5} + \frac{1}{21}I_1 + \frac{17}{35}I_2}{x^2 + 4 + I_2}$$

$$\Rightarrow \int \frac{4+I_1+2I_2}{(x+1+I_1+2I_2)(x^2+4+I_2)}dx$$

$$= \int \left(\frac{\frac{4}{5}-\frac{2}{21}I_1-\frac{13}{35}I_2}{x+1+I_1+2I_2} + \frac{\left(-\frac{4}{5}+\frac{2}{21}I_1+\frac{13}{35}I_2\right)x+\frac{4}{5}+\frac{1}{21}I_1+\frac{17}{35}I_2}{x^2+4+I_2} \right)dx$$

$$= \int \frac{\frac{4}{5}-\frac{2}{21}I_1-\frac{13}{35}I_2}{x+1+I_1+2I_2}dx + \int \frac{\left(-\frac{4}{5}+\frac{2}{21}I_1+\frac{13}{35}I_2\right)x}{x^2+4+I_2}dx + \int \frac{\frac{4}{5}+\frac{1}{21}I_1+\frac{17}{35}I_2}{x^2+4+I_2}dx$$

$$= \left(\frac{4}{5}-\frac{2}{21}I_1-\frac{13}{35}I_2\right)ln|x+1+I_1+2I_2| - \left(\frac{2}{5}-\frac{1}{21}I_1-\frac{13}{70}I_2\right)ln|x^2+4+I_2| + \int \frac{\frac{4}{5}+\frac{1}{21}I_1+\frac{17}{35}I_2}{x^2+\left(\sqrt{4+I_2}\right)^2}dx$$

$$= \left(\frac{4}{5}-\frac{2}{21}I_1-\frac{13}{35}I_2\right)ln|x+1+I_1+2I_2| - \left(\frac{2}{5}-\frac{1}{21}I_1-\frac{13}{70}I_2\right)ln|x^2+4+I_2| + \int \frac{\frac{4}{5}+\frac{1}{21}I_1+\frac{17}{35}I_2}{x^2+\left(2+(\sqrt{5}-2)I_2\right)^2}dx$$

$$= \left(\frac{4}{5}-\frac{2}{21}I_1-\frac{13}{35}I_2\right)ln|x+1+I_1+2I_2| - \left(\frac{2}{5}-\frac{1}{21}I_1-\frac{13}{70}I_2\right)ln|x^2+4+I_2| + \left(\frac{\frac{4}{5}+\frac{1}{21}I_1+\frac{17}{35}I_2}{2+(\sqrt{5}-2)I_2}\right)tan^{-1}\left(\frac{x}{2+(\sqrt{5}-2)I_2}\right)$$

$$= \left(\frac{4}{5}-\frac{2}{21}I_1-\frac{13}{35}I_2\right)ln|x+1+I_1+2I_2| - \left(\frac{2}{5}-\frac{1}{21}I_1-\frac{13}{70}I_2\right)ln|x^2+4+I_2| + \left(\frac{2}{5}+\frac{\sqrt{5}}{105}I_1+\left(\frac{9}{7\sqrt{5}}-\frac{2}{5}\right)I_2\right)tan^{-1}\left(\frac{1}{2}+\left(-\frac{1}{2}+\frac{1}{\sqrt{5}}\right)I_2\right)x + C$$

Whereas $\sqrt{4+I_2} = \sqrt{4} + (-\sqrt{5}+\sqrt{5})I_1 + (-\sqrt{4}+\sqrt{5})I_2 = 2 + (\sqrt{5}-2)I_2$ and $C = a + bI_1 + cI_2$ and a, b, c are real numbers.

2.2. Integral of the 2- refined neutrosophic improper rational functions

To facilitate the integration process, we can apply the long division method or the synthetic division method if the degree of the numerator is higher than the degree of the denominator.

Example 7

$$\int \frac{x^3+(1+I_1+2I_2)x^2+(2+I_2)x-3+2I_1-4I_2}{x-5-I_1-6I_2}dx$$

By using synthetic division method, we get:

$5+I_1+6I_2$	1	$1+I_1+2I_2$	$2+I_2$	$-3+2I_1-4I_2$
		$5+I_1+6I_2$	$30+38I_1+124I_2$	$160+613I_1+1567I_2$
	1	$6+2I_1+8I_2$	$32+38I_1+125I_2$	$157+615I_1+1563I_2$

Then:

$$\frac{x^3+(1+I_1+2I_2)x^2+(2+I_2)x-3+2I_1-4I_2}{x-5-I_1-6I_2}$$

$$=x^2+(6+2I_1+8I_2)x+(32+38I_1+125I_2)+\frac{157+615I_1+1563I_2}{x-5-I_1-6I_2}$$

$$\Rightarrow \int \frac{x^3+(1+I_1+2I_2)x^2+(2+I_2)x-3+2I_1-4I_2}{x-5-I_1-6I_2}dx$$

$$=\int\left(x^2+(6+2I_1+8I_2)x+(32+38I_1+125I_2)+\frac{157+615I_1+1563I_2}{x-5-I_1-6I_2}\right)dx$$

$$=\frac{x^3}{3}+(3+I_1+4I_2)x^2+(32+38I_1+125I_2)x+(157+615I_1+1563I_2)ln|x-5-I_1-6I_2|+C$$

Whereas $C=a+bI_1+cI_2$ *and* a,b,c are real numbers.

Example 8

Evaluate:

$$\int \frac{(5+I_1-3I_2)x^2+(-4-24I_1-10I_2)x-6-12I_1-10I_2}{x-1-6I_1-7I_2}dx$$

By using synthetic division method, we get:

$1+6I_1+7I_2$	$5+I_1-3I_2$	$-4-24I_1-10I_2$	$-6-12I_1-10I_2$
		$5+26I_1+11I_2$	$1+40I_1+15I_2$
	$5+I_1-3I_2$	$1+2I_1+I_2$	$-5+28I_1+5I_2$

Then:

$$\frac{(5+I_1-3I_2)x^2+(-4-24I_1-10I_2)x-6-12I_1-10I_2}{x-1-6I_1-7I_2}$$

$$=(5+I_1-3I_2)x+(1+2I_1+I_2)+\frac{-5+28I_1+5I_2}{x-1-6I_1-7I_2}$$

$$\Rightarrow \int \frac{(5+I_1-3I_2)x^2+(-4-24I_1-10I_2)x-6-12I_1-10I_2}{x-1-6I_1-7I_2}dx$$

$$=\int\left((5+I_1-3I_2)x+(1+2I_1+I_2)+\frac{-5+28I_1+5I_2}{x-1-6I_1-7I_2}\right)dx$$

$$=\left(\frac{5}{2}+\frac{1}{2}I_1-\frac{3}{2}I_2\right)x^2+(1+2I_1+I_2)x+(-5+28I_1+5I_2)ln|x-1-6I_1-7I_2|+C$$

Whereas $C = a + bI_1 + cI_2$ and a, b, c are real numbers.

4. Conclusions

The 2-refined neutrosophic partial fraction is important in neutrosophic logic. We searched for a way to integrate it by discussing all the cases using the division of 2-refined neutrosophic numbers, and we obtained accurate results that were verified. In addition, illustrated through a set of examples.

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