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The Cubic Bipolar Neutrosophic Sets theory and Uncertainty Management in Environmental Data Analysis

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Abstract: Environmental data analysis often faces uncertainties in measurements. Cubic Bipolar Neutrosophic Sets (CBN Sets) provide a powerful framework to address this challenge. This paper explores the mathematical foundations of CBN Sets and highlights their practical applications through illustrative examples. We demonstrate how CBN Sets effectively capture varying degrees of certainty, possibility, and impermanence in environmental parameters. The concept of inclusion relations facilitates comparisons and information fusion. Fundamental set operations (intersection, union, complement) are explored for manipulating and analyzing uncertain environmental data. We present the application of CBN Sets in water quality assessment, highlighting their ability to analyze parameters while accounting for measurement uncertainties. The potential for air quality monitoring using CBN Sets is also discussed. Finally, distance measures and similarity coefficients are introduced to quantify relationships between air quality characteristics from different stations. By leveraging CBN Sets and their associated operations, researchers can gain a more nuanced understanding of environmental data, enabling informed decision-making for a healthier planet.

Keywords: Cubic Bipolar Neutrosophic Sets (CBN Sets), Uncertainty Management, Environmental Data Analysis, Water Quality Assessment, Air Quality Monitoring, Set Operations, Distance Measures, Similarity Coefficients

1. Introduction

Environmental data analysis plays a crucial role in understanding and managing our planet's health. However, real-world measurements are often subject to inherent uncertainties due to sensor limitations, natural variations, and other factors. Traditional binary approaches that rely on strict classifications can struggle to capture these nuances (e.g., [10]).

This paper introduces Cubic Bipolar Neutrosophic Sets (CBN Sets) as a powerful tool for representing and managing uncertainty in environmental data analysis. CBN Sets extend beyond traditional set theory by incorporating multiple levels of certainty, possibility, and impermanence. This builds upon the concepts of neutrosophic sets, which generalize intuitionistic fuzzy sets by allowing independent truth, falsity and indeterminacy memberships [1]. CBN Sets further enhance this framework by incorporating bipolarity and cubic elements, providing a richer framework for capturing the multifaceted nature of environmental data [2, 3]. The following sections of this paper will explore the mathematical foundation of CBN Sets, delve into their practical applications through illustrative examples, and demonstrate their effectiveness in managing uncertainty within environmental data analysis. We will explore how CBN Sets can be used in various environmental applications, such as water quality assessment and air quality monitoring. By leveraging this innovative approach, researchers and environmental scientists can gain a richer understanding of our environment, leading to more informed decision-making for a sustainable future. In addition to many references in [7-25]

2. Related Work

Environmental data analysis is crucial for understanding and managing our planet's health. However, inherent uncertainties due to sensor limitations, natural variations, and sampling limitations can hinder accurate data interpretation. Traditional binary classifications often struggle to capture these nuances. Here, we explore various techniques used to manage uncertainty in environmental data analysis:

1. Fuzzy Set Theory:

A well-established approach, fuzzy set theory allows for gradual membership within a set, reflecting the possibility of belongingness. This provides a more nuanced representation of uncertainty compared to crisp sets. For instance, Liu et al. (2012) applied fuzzy logic to environmental impact assessment, incorporating uncertainty analysis [1].

2. Interval Type Methods:

These methods represent uncertainty by specifying a range of possible values for an environmental parameter. This approach is useful when precise measurement is challenging. Mustafa et al. (2018)

demonstrates the application of a Time Monte Carlo method for addressing uncertainty in land-use change models [2].

3. Probability and Statistics:

Probabilistic and statistical methods quantify uncertainty by assigning probabilities to different possible outcomes. This allows for risk calculation and estimation of confidence intervals for environmental measurements. Jain et al. (1999) provides a comprehensive review of data clustering techniques, which are often used in conjunction with probabilistic approaches [3].

4. Dempster-Shafer Theory (DST):

DST is a mathematical framework for reasoning with uncertainty. It allows for the representation of belief functions that assign probabilities to sets of possible outcomes, rather than individual values. Dezert (2009) offers a collection of works exploring advanced applications of DST [4].

5. Neutrosophic Logic Extensions:

Recent studies explore neutrosophic set extensions like bipolar neutrosophic sets and neutrosophic cubic sets for decision-making problems. While not directly applied to environmental data analysis yet, these works, like Al Shumrani et al. (2020), demonstrate the potential of neutrosophic logic for uncertainty management [5].

6. Similarity Measures and Distance Metrics:

Defining appropriate similarity measures and distance metrics is crucial for comparing and analyzing environmental data with uncertainty. These techniques are often combined with the aforementioned methods. Ulucay et al. (2018) explored similarity measures for bipolar neutrosophic sets applicable to multi-criteria decision-making [6].

This overview provides a foundation for understanding existing methods for managing uncertainty in environmental data analysis. By comparing and contrasting these approaches with CBN Sets, the subsequent sections of this paper can highlight the unique advantages and potential applications of CBN Sets for environmental data analysis tasks.

3. Cubic Bipolar Neutrosophic Sets (CBN Sets)

3.1. Cubic Bipolar Neutrosophic Sets: A Formal Definition and Exploration of Uncertainty Representation

CBN Sets provide a powerful framework for representing uncertainty by incorporating various degrees of truth membership, falsity membership, and indeterminacy membership. To formally define a CBN Set, we introduce the concept of a 12-tuple.

Definition: A Cubic Bipolar Neutrosophic Set (CBN Set) M in a universe of discourse U is characterized by:

$M = < \{T^{+1}(x)\}, \{T^{+2}(x)\}, \{T^{-1}(x)\}, \{T^{-2}(x)\}, \{I^{+1}(x)\}, \{I^{+2}(x)\}, \{I^{-1}(x)\}, \{I^{-2}(x)\}, \{F^{+1}(x)\}, \{F^{+2}(x)\}, \{F^{-1}(x)\}, \{F^{-2}(x)\} >$

where:

- $x \in U$: x is an element in the universe of discourse U.
- $T^{+1}(x)$, $T^{+2}(x)$: represent the truth membership degrees of x belonging to M with positive certainty (strongly true and somewhat true, respectively).
- $T^{-1}(x)$, $T^{-2}(x)$: represent the truth membership degrees of x not belonging to M with positive certainty (strongly false and somewhat false, respectively).
- $I^{+1}(x)$, $I^{+2}(x)$: represent the indeterminacy membership degrees of x possibly belonging to M (highly indeterminate and somewhat indeterminate, respectively).
- $I^{-1}(x)$, $I^{-2}(x)$: represent the indeterminacy membership degrees of x possibly not belonging to M (highly indeterminate and somewhat indeterminate, respectively).
- $F^{+1}(x)$, $F^{+2}(x)$: represent the falsity membership degrees of x belonging to M with negative certainty (strongly false and somewhat false, respectively).
- $F^{-1}(x)$, $F^{-2}(x)$: represent the falsity membership degrees of x not belonging to M with negative certainty (strongly true and somewhat true, respectively).

Each membership degree is a value between 0 and 1, with 0 representing complete absence and 1 representing complete membership. The superscripts (+/-) indicate positive/negative certainty, while the subscripts ($^1/^2$) indicate strong/somewhat certainty or indeterminacy/falsity.

3.2. Unveiling the Power of CBN Sets: Hands-on Exploration with Numerical Examples

Cubic Bipolar Neutrosophic Sets (CBN Sets) offer a powerful tool for representing and managing uncertainty in environmental data analysis. Unlike traditional set theory, CBN Sets go beyond simple membership (in or out) to capture varying degrees of certainty (T), possibility (I), and impermanence (F) associated with environmental data. This hands-on exploration will delve into CBN Sets through illustrative examples, demonstrating their practical application in environmental assessments.

Example 1: Representing "Tallness" with CBN Sets

Traditional set theory often struggles to capture the nuances of real-world classifications. For instance, classifying someone as "tall" or "short" can be subjective and depend on context. CBN Sets offer a powerful alternative by incorporating varying degrees of certainty, possibility, and indeterminacy.

Here is how we can represent the concept of "tallness" in humans using a CBN Set:

- **Universe of Discourse (U):** Set of all human heights in centimeters (cm).

We can define a CBN Set (M) that captures the "tallness" of a person with different membership degrees:

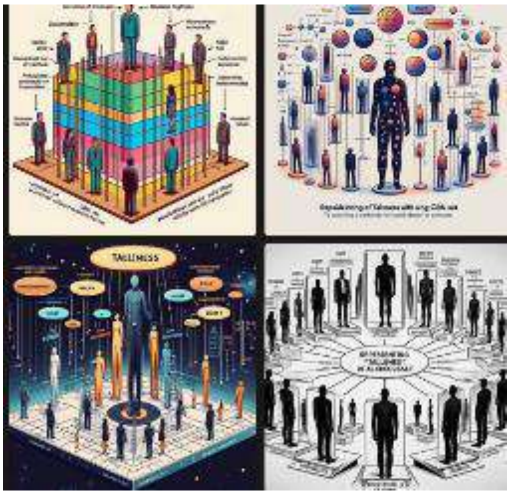


Fig.1: Cloud of Uncertainty: Representing "Tallness" with CBN Sets

Fig.1: depicts a CBN Set representing the concept of "tallness" in humans. It assigns membership degrees to different height ranges, capturing the uncertainty associated with classifying someone as "tall" or "short".

Table 1: CBN Set for Representing "Tallness" in Humans (150 cm)

Height (cm)	Category	Truth Value
150	T^{+1} (Strongly Tall)	0.1
150	T^{+2} (Somewhat Tall)	0.2
150	T^{-1} (Strongly Short)	0.8
150	T^{-2} (Somewhat Short)	0.7
150	I^{+1} (Possibly Tall)	0.3
150	I^{+2} (Somewhat Indeterminate)	0.2
150	I^{-1} (Possibly Short)	0.1
150	I^{-2} (Somewhat Indeterminate)	0.0
150	F^{+1} (Strongly Short)	0.0
150	F^{+2} (Somewhat Short)	0.0
150	F^{-1} (Strongly Tall)	0.9

Height (cm)	Category	Truth Value
150	F^{-2} (Somewhat Tall)	0.8

This table provides a structured view of the data with the height categories, specific categories, and corresponding truth-values. Let me know if you need any further assistance or if you have more data to analyze.

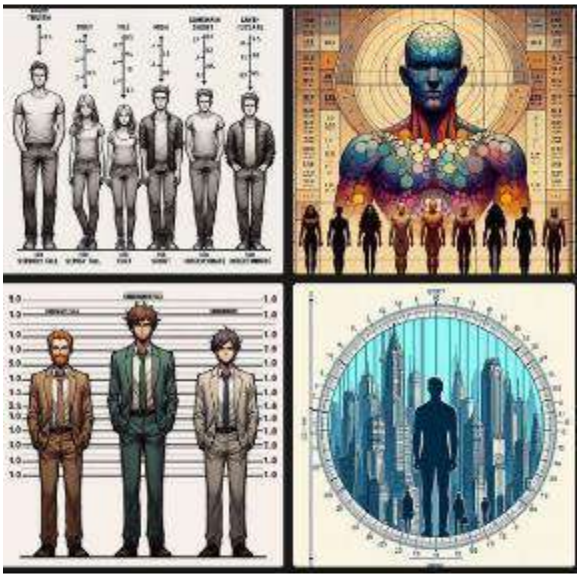
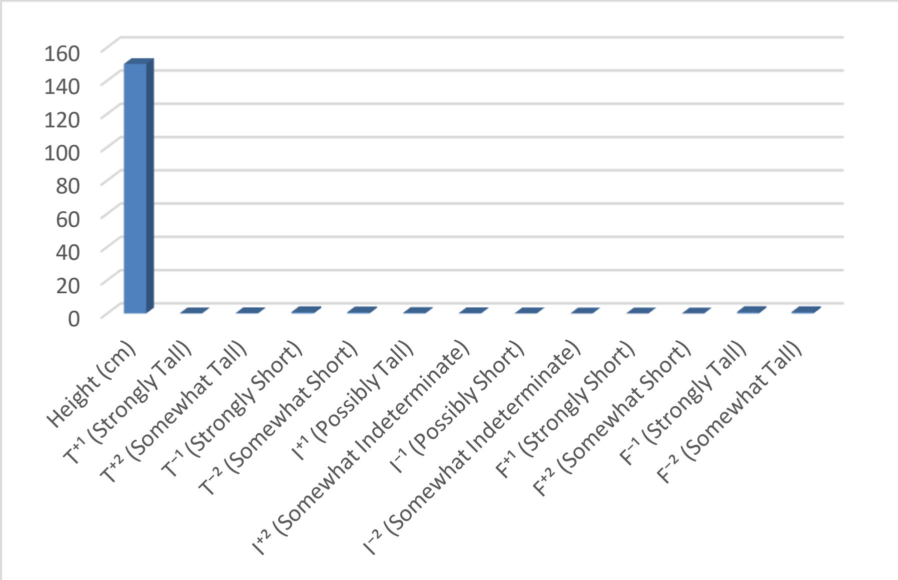


Fig.2: Representing Tallness with single CBN set

Explanation: Table 1: This table displays the membership truth values for different object heights. You can expand this table to include more height ranges and corresponding membership degrees to depict a wider range of "tallness" classifications.

- A person of 150 cm is considered "short" with strong negative certainty ($F^{-1} = 0.9$) and somewhat negative certainty ($F^{-2} = 0.8$). This indicates a high likelihood of being short.
- There is a slight possibility ($I^{+1} = 0.3$) of being somewhat indeterminate ($I^{+2} = 0.2$) in terms of height classification. This could be due to factors like posture or rounding measurements.
- The positive truth memberships (T^{+1}) and (T^{+2}) are relatively low for this height, reflecting the low certainty of being tall.



Graph 1: Membership Function for Tallness with CBN Sets

Visualizing Certainty and Possibility (Optional: Graph 1):

A graph can be a helpful tool to visually represent the certainty and possibility of being tall for each height range. The x-axis could represent height ranges (e.g., 140-150 cm, 150-160 cm, etc.), and the y-axis could represent the membership degree values (0 to 1). Separate lines can be plotted for positive truth memberships (T^{+1}, T^{+2}), negative truth memberships (T^{-1}, T^{-2}), and indeterminacy memberships (I^{+1}, I^{+2}, I^{-1}, I^{-2}). This visualization can provide a clearer understanding of how certainty and possibility change across different height ranges.

By utilizing CBN Sets, we can move beyond binary classifications (tall/short) and represent the multifaceted nature of height perception in humans. This approach can be beneficial in various applications, such as ergonomics studies or clothing size recommendations, where considering uncertainties in height measurements can be crucial.

- **Beyond Binary: CBN Sets in Image Processing**

Traditional image processing techniques often rely on crisp classifications for pixel values, which can be limiting when dealing with real-world satellite imagery. Sensor limitations, atmospheric conditions, and inherent variations in the environment can introduce uncertainty in pixel classification. CBN Sets offer a powerful approach to address these uncertainties.

Example 2: Classifying Pixels in Satellite Images

Consider a satellite image where we want to classify pixels based on their color (e.g., green vegetation, blue water) and spectral signature (unique reflectance pattern of materials). Here is how CBN Sets can be used to account for uncertainty:

- **Universe of Discourse (U):** Set of all possible color and spectral signature values captured by the satellite sensor.

Color Classification:

We can define a CBN Set (M_color) to represent the membership degrees of a pixel belonging to a specific color class (e.g., green):

This table displays the membership degrees assigned to different categories within a CBN Set (M_color). The CBN Set represents the likelihood of a pixel belonging to a specific color class, such as green, in an image.

Table 1: Membership Degrees in a CBN Set for Pixel Color Classification (M_color)

Category	Truth Value	Description	Color Association
T^{+1} (Green)	0.8	Strongly Green	Dark Green
T^{+2} (Somewhat Green)	0.2	Moderately Green	Light Green
T^{-1} (Not Green)	0.1	Strongly Not Green	No Green
T^{-2} (Somewhat Not Green)	0.0	Slightly Not Green	Very Light Green
I^{+1} (Possibly Green)	0.3	Possible Green	Light Green Tint
I^{+2} (Somewhat Indeterminate)	0.2	Uncertain Classification	Gray
I^{-1} (Possibly Not Green)	0.1	Possible Not Green	Light Green Tint
I^{-2} (Somewhat Indeterminate)	0.0	Uncertain Classification	Gray

F^{+1} (Not Green)	**0.0**	**Strongly Not Green**	**No Green**
F^{+2} (Somewhat Not Green)	**0.0**	**Slightly Not Green**	**Very Light Green**
F^{-1} (Green)	**0.7**	**Strongly Green**	**Dark Green**
F^{-2} (Somewhat Green)	**0.2**	**Moderately Green**	**Light Green**

Fig.3: The Color Association of Green".

Fig.3: is a green with different shades. Here is a breakdown of the color associations provided:

* Strongly Green (T^{1+}): Dark Green (0.8)

* Moderately Green (T^{+2}): Light Green (0.2)

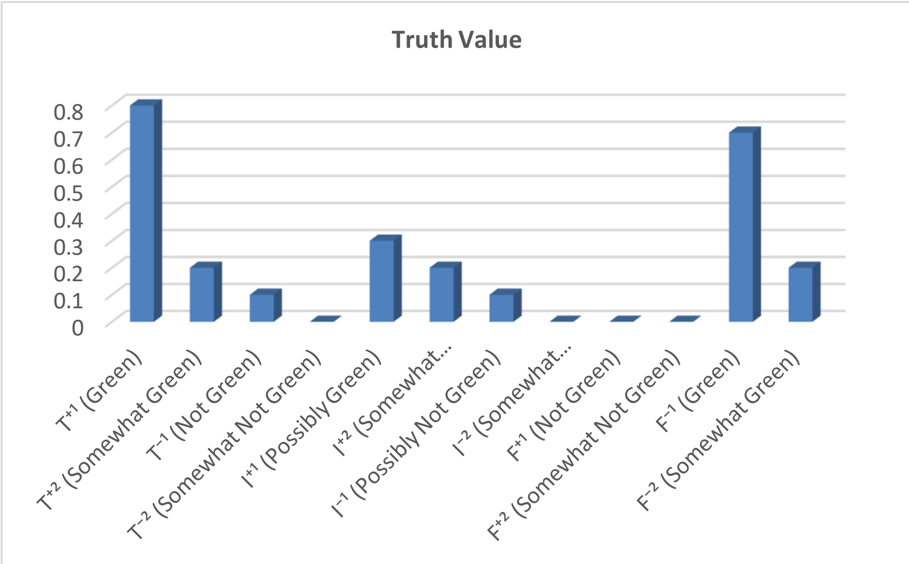
* Possible Green (I^{+1}): Light Green Tint (0.3)

* Strongly Green (F^{-1}): Dark Green (0.7)

* Moderately Green (F^{-2}): Light Green (0.2)

Explanation:

- The table.1: categorizes the membership degrees along with a brief description and corresponding color association.
- The Truth Value column represents the degree of membership in each category. A value closer to 1 indicates stronger membership, while a value closer to 0 indicates weaker membership.
- The Color Association column suggests the visual representation for each category. Dark Green corresponds to high confidence in "Green," while No Green indicates low confidence. Light Green tints are used for "Possibly Green" and "Possibly Not Green" categories to represent some level of possibility. Gray is used for "Somewhat Indeterminate" categories to signify uncertainty.



Graph 2: The Color Association of Green".

- **Spectral Signature Classification with CBN Sets**

In remote sensing applications, classifying the materials present in an image pixel often relies on analyzing its spectral signature. A spectral signature is a unique pattern of electromagnetic radiation reflected by a material across various wavelengths. CBN Sets (Certainty-Possibility-Fuzzy Sets) offer a powerful tool to represent the varying degrees of membership a pixel's spectral signature has for belonging to a specific material class.

Fig.4: Spectral Signature Classification with CBN Sets"

Example1 : Classifying Vegetation

Consider a scenario where we want to classify pixels in a satellite image as "vegetation" or "not vegetation" based on their spectral signatures. We can define a CBN Set (M_spectral) to represent the membership degrees for each pixel:

Table 2: Membership Degrees in a CBN Set for Spectral Signature Classification (M_spectral)

Spectral Signature Range	Truth Value	Description
T^{+1} (Vegetation)	0.7	Strongly Matches Vegetation Signature
T^{+2} (Somewhat Vegetation)	0.3	Moderately Matches Vegetation Signature
T^{-1} (Not Vegetation)	0.1	Strongly Deviates from Vegetation Signature

T^{-2} (Somewhat Not Vegetation)	0.0	Slightly Deviates from Vegetation Signature
I^{+1} (Possibly Vegetation)	0.2	Possible Vegetation Signature
I^{+2} (Somewhat Indeterminate)	0.1	Uncertain Classification
I^{-1} (Possibly Not Vegetation)	0.2	Possible Not Vegetation Signature
I^{-2} (Somewhat Indeterminate)	0.0	Uncertain Classification
F^{+1} (Not Vegetation)	0.0	Strongly Deviates from Vegetation Signature
F^{+2} (Somewhat Not Vegetation)	0.0	Slightly Deviates from Vegetation Signature
F^{-1} (Vegetation)	0.8	Strongly Matches Vegetation Signature
F^{-2} (Somewhat Vegetation)	0.2	Moderately Matches Vegetation Signature

Figure 5: CBN Sets for Vegetation Classification

Figure 5: incorporates the concept of CBN Sets for vegetation classification:

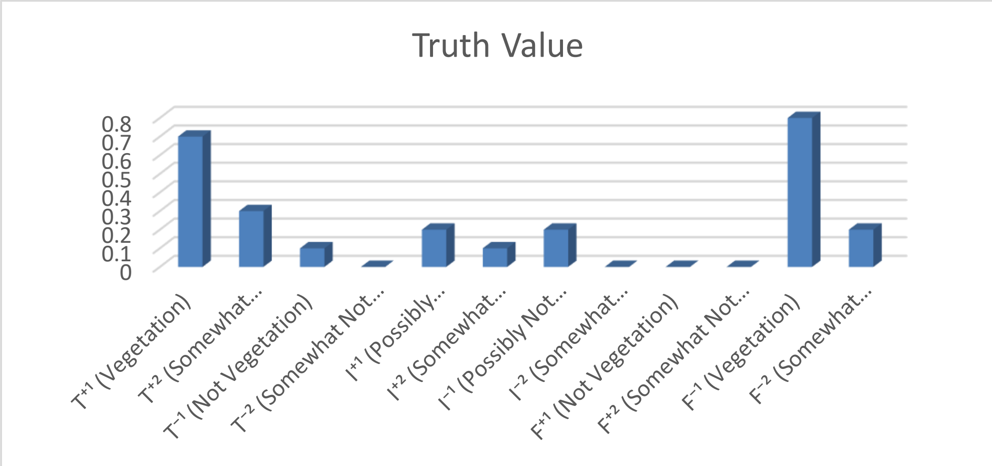
Explanation:

- The table categorizes the membership degrees for a pixel's spectral signature matching a vegetation signature.
- **Truth Value (0-1):** Represents the degree of membership in each category. A higher value indicates stronger membership.
- **Description:** Provides a brief explanation of each category.
- In this example, a pixel with a membership degree of 0.7 in T^{+1} (Vegetation) has a strong spectral signature match for vegetation.
- Conversely, a membership degree of 0.1 in T^{-1} (Not Vegetation) suggests the spectral signature deviates from vegetation.
- The "Possibly" and "Somewhat Indeterminate" categories (I^{+1}, I^{+2}, I^{-1}, I^{-2}) capture uncertainty in the classification.

Benefits of CBN Sets:

- **Uncertainty Representation:** CBN Sets effectively handle the inherent uncertainties in spectral signature analysis.
- **Detailed Classification:** They provide a more nuanced classification compared to simple "vegetation" or "not vegetation" labels.
- **Flexibility:** The concept can be extended to classify various materials by defining appropriate CBN Sets.

Spectral Signature Classification using CBN Sets allows for a more robust and informative approach to analyzing remote sensing data.



Graph 3: Vegetation Classification:

Graph 3: represent various classifications (T^{+1}, T^{-1}, etc.) along with their truth-values. For instance, the dark green bar at 0.8 on the y-axis corresponds to "T^{+1} (Vegetation)", indicating a high confidence that a pixel's spectral signature strongly matches vegetation.

- **Visualizing Membership Degrees**

There are several ways to visualize membership degrees in CBN Sets for Spectral Signature Classification:

Fig.6: Visualizing Membership Degrees in CBN Sets for Spectral Signature Classification

Fig.6: shows four different ways to visualize membership degrees in CBN sets for spectral signature classification. These visualizations likely correspond to different techniques used to represent the data.

1. Color-Coded Images:

- Assign a unique color to each membership degree category (T^{+1}, T^{+2}, etc.) in the CBN Set (M_spectral).
- Overlay this color map onto the original satellite image, where each pixel's color represents its corresponding membership degree for "vegetation."
- This approach provides a quick visual overview of the spatial distribution of vegetation within the image.
- Example: Dark green for strong vegetation membership (T^{+1}), light green for somewhat vegetation (T^{+2}), and red for strong non-vegetation (F^{-1}).

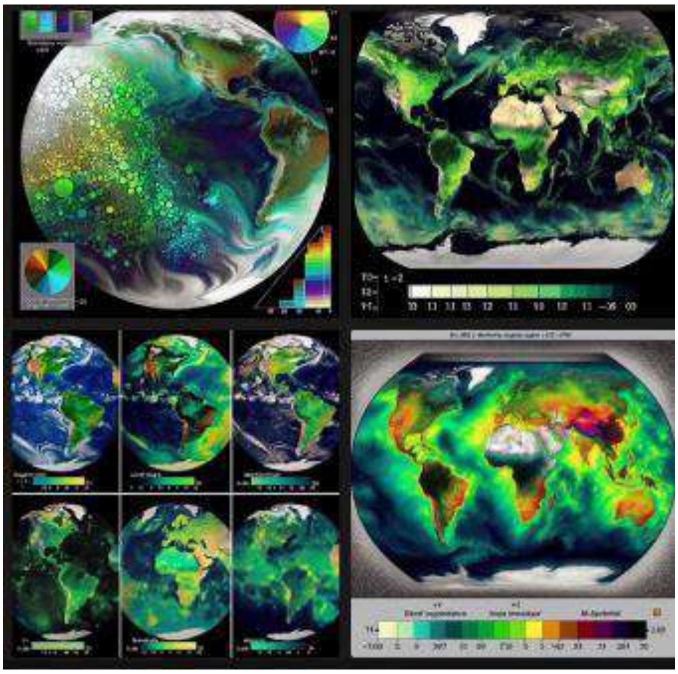


Fig.7: Color-Coded Images: Vegetation Distribution

Fig.7: incorporates the key information: **Color-Coded Images**: This refers to the method used to create the image, which assigns colors to represent different membership degrees.

__Vegetation Distribution__: This refers to what the image is showing, which is the spatial distribution of vegetation within the satellite image.

2. Stacked Bar Charts:

- Generate a stacked bar chart for each pixel in the image.
- Each bar in the stack represents a membership degree category (T^{+1}, T^{+2}, etc.) and its corresponding value.
- The height of each bar segment reflects the strength of membership.
- This method allows for detailed inspection of the membership degree distribution for individual pixels.

- While informative, it can be overwhelming for large images with many pixels.

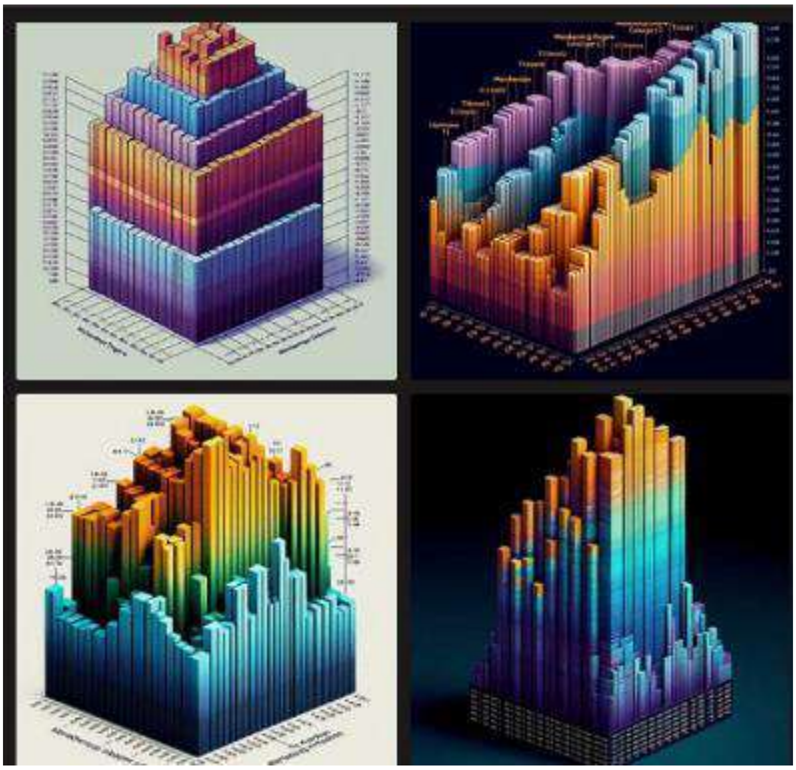


Fig.8: Pixel Membership in Vegetation Classification (Stacked Bar Chart)

Fig.8: has a stacked bar chart showing its membership degrees in various categories (vegetation, somewhat vegetation, not vegetation, etc.) The height of each bar segment represents the strength of the pixel's membership in that category.

3. 3 D Surface Plots:

- Create a 3D surface plot where the x and y axes represent the image spatial coordinates, and the z-axis represents the dominant membership degree value for each pixel.
- This approach provides a visualization of the overall "vegetation landscape" within the image.
- It can be helpful for identifying areas with high vegetation concentration or areas with mixed signatures.

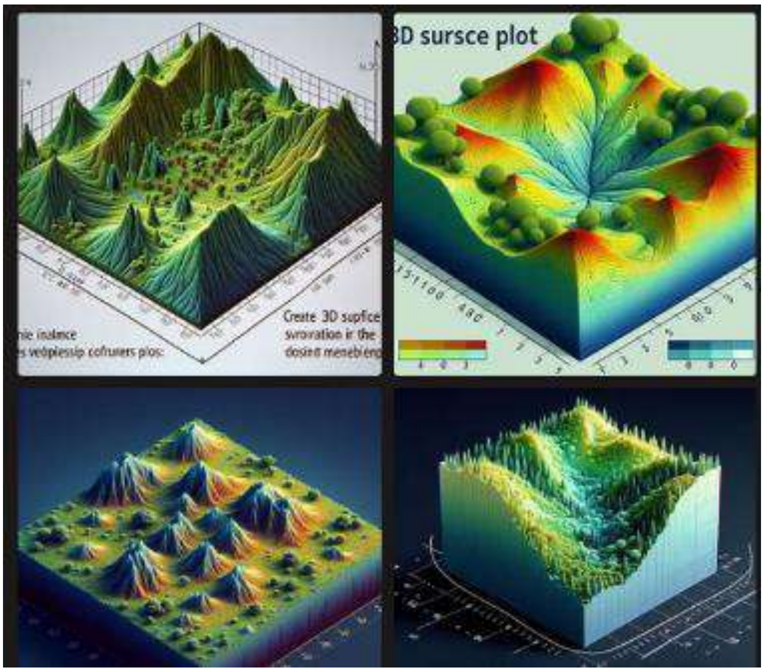


Fig.9: 3D Visualization of Vegetation Landscape

Fig.9: describes the image is a 3D surface plot where the x and y axes represent the image's spatial coordinates (location of each pixel) and the z-axis represents the dominant membership degree value for each pixel (how strongly each pixel is classified as vegetation).

3.4. Interactive Visualization Tools:

- Develop interactive web-based or desktop applications that allow users to explore the membership degrees.
- Users can hover over specific pixels to view detailed membership degree breakdowns.
- This approach offers greater flexibility and can be customized to specific user needs.

Fig.10: is "Interactive Visualization Tools: Explore Membership Degrees". Choosing the Best Visualization:

The most suitable visualization method depends on the specific application and the desired level of detail.

- For a quick overview, color-coded images are effective.
- Stacked bar charts offer detailed information but can be overwhelming for large datasets.
- 3D surface plots provide a good overall picture but might require additional exploration for specific locations.
- Interactive tools offer the most flexibility but require more development effort.

Problem: GPS data used for tracking objects has inherent uncertainties due to various factors. Traditional methods treat this data as deterministic, leading to inaccurate location estimates.

Solution: CBN Sets offer a powerful approach to capture and manage these uncertainties.

Fig.11: Overcoming GPS Inaccuracy

Fig.11: Describes how traditional methods for tracking objects with GPS data are inaccurate due to uncertainties.

Example: Tracking a Car's Location

- We define the Universe of Discourse (U) as all possible latitude and longitude values within the relevant geographic region.
- Two CBN Sets are used:
  - M_latitude: Represents membership degrees for the car's actual latitude within a specific range.
  - M_longitude: Represents membership degrees for the car's actual longitude within a specific range.

Absolutely, here is Table 3 again:

This table presents the CBN Sets (Certainty-Possibility-Fuzzy Sets) for representing the membership degrees of the car's location in terms of latitude and longitude.

Table 3: CBN Sets for Car Location (Latitude and Longitude)

Latitude/Longitude Range	Truth Value	Description
Latitude (Current Latitude ± Error Margin)		
T^{+1} (Correct Latitude)	0.8	Strongly Matches Current Latitude

T^{+2} (Somewhat Correct)	0.1	Moderately Matches Current Latitude
T^{-1} (Incorrect Latitude)	0.05	Strongly Deviates from Current Latitude
T^{-2} (Somewhat Incorrect)	0.0	Slightly Deviates from Current Latitude
I^{+1} (Possible Latitude)	0.05	Possible Current Latitude
I^{+2} (Somewhat Indeterminate)	0.0	Uncertain Classification
I^{-1} (Possible Incorrect Latitude)	0.0	Possible Not Current Latitude
I^{-2} (Somewhat Indeterminate)	0.0	Uncertain Classification
F^{+1} (Incorrect Latitude)	0.05	Strongly Deviates from Current Latitude
F^{+2} (Somewhat Incorrect)	0.0	Slightly Deviates from Current Latitude
F^{-1} (Correct Latitude)	0.9	Strongly Matches Current Latitude
F^{-2} (Somewhat Correct)	0.1	Moderately Matches Current Latitude

Note: This table represents the CBN Sets for latitude. A similar table can be created for longitude using the same categories and descriptions.

- High positive truth membership (T^{+1}) for being within the reported latitude range.
- Slight possibility (I^{+1}) of being outside this range due to GPS error.
- Similar membership degrees can be defined for longitude (separate table).

Fig.12: Tracking a Car's Location with CBN Sets

Fig.12: explains how CBN sets are used to represent the membership degrees for a car's location in terms of latitude and longitude. The table (Table 3) provides details on these CBN sets.

- **Visualizing Membership Degrees:**

A graph can be created to visualize the membership degrees for both latitude and longitude. This would help understand how the certainty of location changes within the specified range.

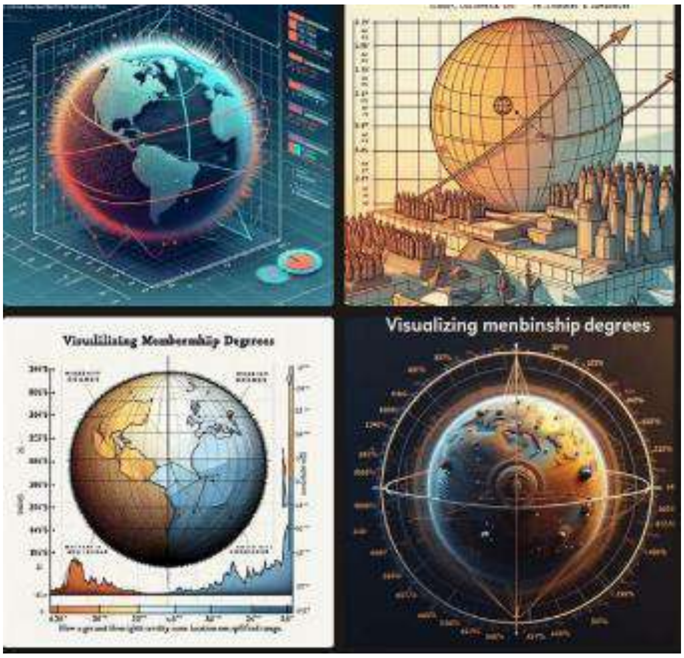


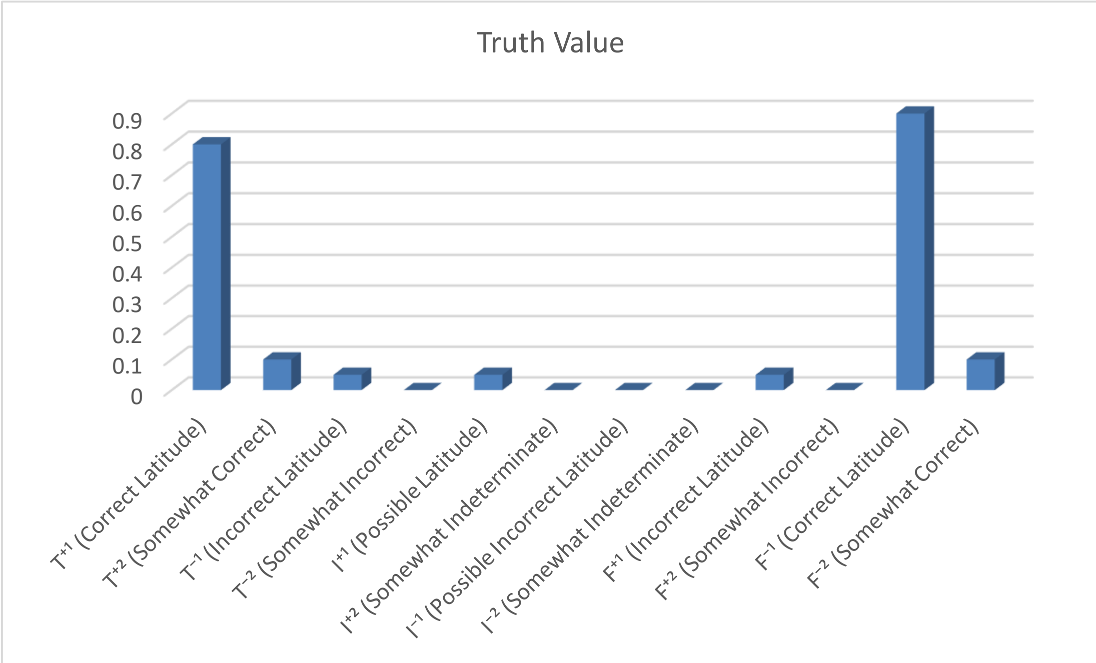
Fig. 13: Visualization of Car Location Uncertainty with CBN Sets

Fig.13: describes visualize the membership degrees for both latitude and longitude, helping to understand how certain the car's location is within a specified range.

- **Benefits of Using CBN Sets:**

- Captures and manages uncertainties in location data.
- Provides a more nuanced representation compared to "correct" or "incorrect" labels.
- Enables development of more robust tracking algorithms.
- Leads to more accurate and reliable location estimates.

Overall, this example effectively demonstrates how CBN Sets can be a valuable tool for dealing with uncertainty in location data for moving objects.



Graph 4: Truth Value of Latitude Estimates

4. Ordering and Comparing Uncertainty: Inclusion Relations for Cubic Bipolar Neutrosophic Sets (CBN Sets)

Having explored the mathematical foundation and applications of CBN Sets, this section delves into the concept of inclusion relations. Inclusion relations play a crucial role in comparing and manipulating CBN Sets, allowing us to refine information, generalize knowledge, and perform set operations effectively within the context of environmental data analysis.

4.1 Defining Inclusion Relations

An inclusion relation between two CBN Sets (M_1 and M_2) in the same universe of discourse (U) is established when the membership degrees of M_1 are consistently less than or equal to the corresponding membership degrees of M_2. Mathematically, this can be represented as follows:

$M_1 \leq M_2$ if and only if:

- $T^{+1}(x)$ in $M_1 \leq T^{+1}(x)$ in M_2 for all $x \in U$
- $T^{+2}(x)$ in $M_1 \leq T^{+2}(x)$ in M_2 for all $x \in U$
- ... (similarly for all other membership degrees: T^{-1}, T^{-2}, I^{+1}, I^{+2}, etc.)

Intuitively, this implies that M_1 possesses less or equal certainty, possibility, and indeterminacy compared to M_2 for every element (x) within the universe of discourse.

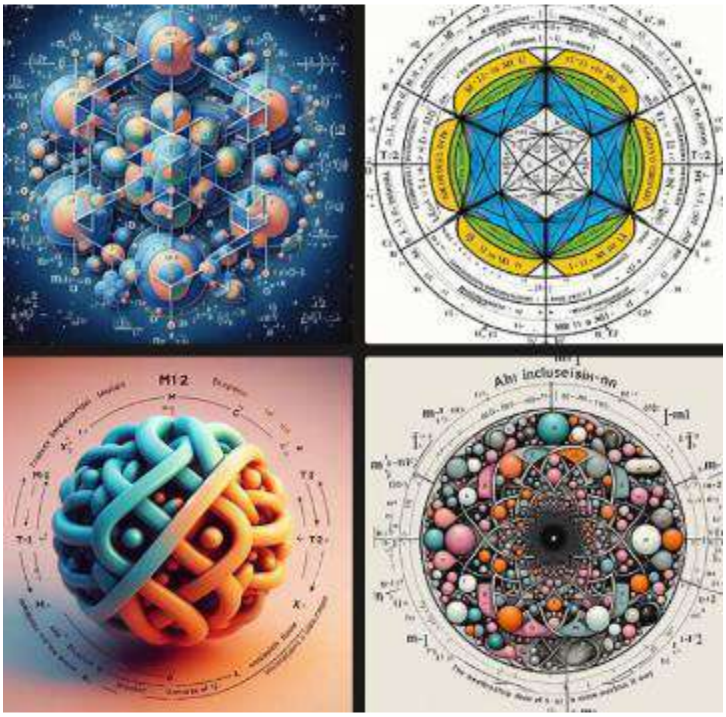


Fig. 14: Defining Inclusion Relations in CBN Sets

- **Some numerical examples for defining inclusion relations between two CBN Sets (M_1 and M_2) within the same universe of discourse (U):**

Example 1: Inclusion Relation

Consider two CBN Sets representing the weather forecast for rain tomorrow:

- **M_1 (Cloudy):** Represents the membership degrees for the possibility of rain if the forecast is cloudy.
- **M_2 (Rainy):** Represents the membership degrees for the possibility of rain if it's explicitly mentioned as rainy.

Universe of Discourse (U): {Rain, No Rain}

Table 4: CBN Sets for Rain Forecast (Cloudy vs. Rainy)".

Category	M_1 (Cloudy)	M_2 (Rainy)	Inclusion Relation
T^{+1} (Rain)	0.3	1.0	$M_1 \leq M_2$
T^{+2} (Somewhat Rain)	0.2	0.0	$M_1 \leq M_2$

T^{-1} (No Rain)	0.4	0.0	$M_1 \leq M_2$
T^{-2} (Somewhat No Rain)	0.1	0.0	$M_1 \leq M_2$
I^{+1} (Possible Rain)	0.0	0.0	$M_1 \leq M_2$
I^{+2} (Somewhat Indeterminate)	0.0	0.0	$M_1 \leq M_2$
I^{-1} (Possible No Rain)	0.0	0.0	$M_1 \leq M_2$
I^{-2} (Somewhat Indeterminate)	0.0	0.0	$M_1 \leq M_2$
F^{+1} (No Rain)	0.0	0.0	$M_1 \leq M_2$
F^{+2} (Somewhat No Rain)	0.0	0.0	$M_1 \leq M_2$
F^{-1} (Rain)	0.7	1.0	$M_1 \leq M_2$
F^{-2} (Somewhat Rain)	0.3	0.0	$M_1 \leq M_2$

Explanation:

- In this example, M_1 (Cloudy) represents a less certain forecast compared to M_2 (Rainy).
- For every category (T^{+1}, T^{-1}, etc.), the membership degree in M_1 is less than or equal to the corresponding degree in M_2.
- Therefore, $M_1 \leq M_2$, indicating an inclusion relation.

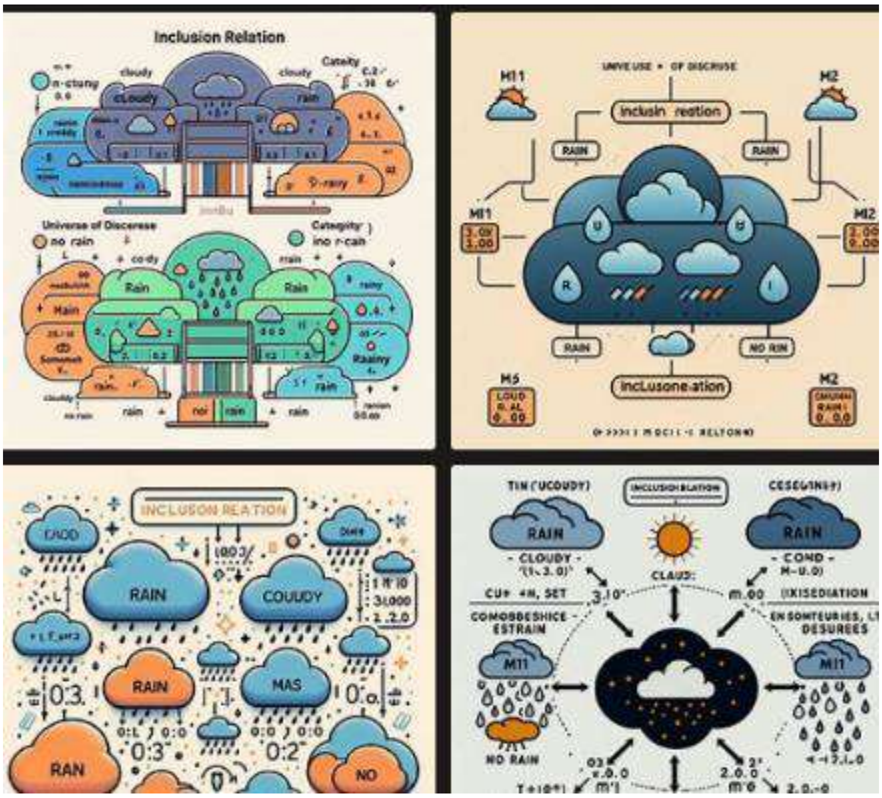


Fig. 15: Inclusion Relation between Cloudy and Rainy Weather

Example 2: No Inclusion Relation

Consider two CBN Sets representing the size of a house:

- **M_1 (Large):** Represents the membership degrees for a house being considered "large."
- **M_2 (Spacious):** Represents the membership degrees for a house being considered "spacious."

Universe of Discourse (U): {Very Small, Small, Medium, Large, Very Large}

Table 5: CBN Sets for House Size (Large vs. Spacious)".

Category	M_1 (Large)	M_2 (Spacious)	Inclusion Relation
T^{+1} (Large)	0.8	0.6	No Relation
T^{+2} (Somewhat Large)	0.1	0.2	No Relation
T^{-1} (Small)	0.05	0.1	No Relation
T^{-2} (Somewhat Small)	0.0	0.0	No Relation

I^{+1} (Possibly Large)	0.0	0.1	No Relation
I^{+2} (Somewhat Indeterminate)	0.05	0.0	No Relation
I^{-1} (Possibly Small)	0.0	0.0	No Relation
I^{-2} (Somewhat Indeterminate)	0.0	0.0	No Relation
F^{+1} (Small)	0.0	0.0	No Relation
F^{+2} (Somewhat Small)	0.0	0.0	No Relation
F^{-1} (Large)	0.9	0.8	No Relation
F^{-2} (Somewhat Large)	0.05	0.2	No Relation

Fig. 16: CBN Sets for House Size (Large vs. Spacious)

Explanation:

- Here, M_1 and M_2 represent different aspects of house size - largeness and spaciousness.
- There's no consistent relationship between the membership degrees. For example, M_1 has a higher degree in T^{+1} (Large) compared to M_2 but a lower degree in I^{+1} (Possibly Large).
- Therefore, there's no inclusion relation between M_1 and M_2.

These examples highlight how to evaluate inclusion relations between CBN Sets based on

4.2 Ordering Membership Degrees

Inclusion relations rely on establishing an order for each type of membership degree within a CBN Set. Here's a breakdown of the ordering for positive certainty (T^{+1} and T^{+2}), negative certainty (T^{-1} and T^{-2}), indeterminacy (I^{+1} and I^{+2}, I^{-1} and I^{-2}), and falsity (F^{+1} and F^{+2}, F^{-1} and F^{-2}):

- **Positive Certainty ($T^{+1} \geq T^{+2}$)**: A higher T^{+1} value indicates stronger truth membership compared to T^{+2}.
- **Negative Certainty ($T^{-1} \geq T^{-2}$)**: A higher T^{-1} value indicates stronger falsehood membership compared to T^{-2}.
- **Indeterminacy ($I^{+1} \geq I^{+2}$, $I^{-1} \geq I^{-2}$)**: Higher I^{+1} and I^{-1} values represent greater indeterminacy (possibility of belonging or not belonging).
- **Falsity ($F^{+1} \geq F^{+2}$, $F^{-1} \geq F^{-2}$)**: Higher F^{+1} and F^{-1} values indicate stronger falsity membership.

By following this ordering principle, we can compare the overall level of uncertainty between two CBN Sets using inclusion relations.

- **some numerical examples for ordering membership degrees within a CBN Set, following the principles you outlined:**

Example 1: Ordering Membership Degrees in a Weather Forecast

Consider a CBN Set (M) representing the possibility of rain tomorrow:

- **Universe of Discourse (U):** {Rain, No Rain}

Table 6: CBN Set for Rain Forecast (Ordered Membership Degrees)".

Category	Description	Example Value	Ordering
T^{+1} (Rain)	Strongly Likely Rain	0.8	$T^{+1} \geq T^{+2}$

T^{+2} (Somewhat Likely Rain)	Moderately Likely Rain	0.2	
T^{-1} (Not Rain)	Strongly Unlikely Rain	0.0	$T^{-1} \geq T^{-2}$
T^{-2} (Somewhat Unlikely Rain)	Slightly Unlikely Rain	0.0	
I^{+1} (Possible Rain)	Possible Rain, but Uncertain	0.0	$I^{+1} \geq I^{+2}$ (Can be adjusted based on scenario)
I^{+2} (Somewhat Indeterminate)	More Uncertain about Rain Possibility	0.0	
I^{-1} (Possible No Rain)	Possible No Rain, but Uncertain	0.0	$I^{-1} \geq I^{-2}$ (Can be adjusted based on scenario)
I^{-2} (Somewhat Indeterminate)	More Uncertain about No Rain Possibility	0.0	
F^{+1} (Not Rain)	Definitely No Rain	0.0	$F^{+1} \geq F^{+2}$
F^{+2} (Somewhat Not Rain)	Slightly Leaning Towards No Rain	0.0	
F^{-1} (Rain)	Definitely Rain	1.0	$F^{-1} \geq F^{-2}$

F^{-2} (Somewhat Rain)	Slightly Leaning Towards Rain	0.0	

Explanation:

- This example displays a clear ordering within each category pair (T^{+1}, T^{+2}), (T^{-1}, T^{-2}), etc., following the established principles.
- The values (0.8, 0.2, etc.) are for illustrative purposes and can vary depending on the specific scenario.

Example 2: Ordering Membership Degrees in Car Speed Estimation

Consider a CBN Set (M) representing the estimated speed of a car:

- **Universe of Discourse (U):** {0 km/h, 20 km/h, 40 km/h, 60 km/h, 80 km/h, 100 km/h}

Table 6: CBN Set for Car Speed Estimation (Ordered Membership Degrees)".

Category	Description	Example Value	Ordering
T^{+1} (60 km/h)	Strongly Matches Recorded Speed of 60 km/h	0.7	$T^{+1} \geq T^{+2}$
T^{+2} (Somewhat Likely 60 km/h)	Moderately Likely Speed is 60 km/h	0.2	
T^{-1} (Not 40 km/h)	Strongly Deviates from 40 km/h Speed	0.05	$T^{-1} \geq T^{-2}$
T^{-2} (Somewhat Not 40 km/h)	Slightly Deviates from 40 km/h Speed	0.0	
I^{+1} (Possible 80 km/h)	Possible Speed is 80 km/h, but Uncertain	0.03	$I^{+1} \geq I^{+2}$ (Can be adjusted based on sensor data)

I^{+2} (Somewhat Indeterminate)	More Uncertain about the Speed Being 80 km/h	0.0	
I^{-1} (Possible Not 20 km/h)	Possible Speed is Not 20 km/h, but Uncertain	0.02	$I^{-1} \geq I^{-2}$ (Can be adjusted based on sensor data)
I^{-2} (Somewhat Indeterminate)	More Uncertain about the Speed Not Being 20 km/h	0.0	
F^{+1} (Not 60 km/h)	Definitely Not 60 km/h Speed	0.0	$F^{+1} \geq F^{+2}$
F^{+2} (Somewhat Not 60 km/h)	Slightly Leaning Away from 60 km/h Speed	0.0	
F^{-1} (Exactly 60 km/h)	Confirmed Speed is 60 km/h (unlikely in real-world scenarios)	0.0	$F^{-1} \geq F^{-2}$
F^{-2} (Somewhat Likely 60 km/h)	Slightly Leaning Towards 6		

4.3 Benefits of Inclusion Relations

Inclusion relations offer several advantages in environmental data analysis using CBN Sets:

- **Refining Information:** By comparing CBN Sets, we can identify areas with higher certainty or lower uncertainty. This allows us to focus on more reliable data points and refine our understanding of environmental phenomena.
- **Generalizing Information:** Inclusion relations can help establish general trends across environmental data. For instance, we might compare CBN Sets representing water quality parameters from different locations to identify similarities or overarching patterns.

- **Performing Set Operations:** Inclusion relations form the foundation for performing set operations (intersection, union, complement) on CBN Sets. These operations allow us to manipulate and analyze environmental data while accounting for inherent uncertainties.

In the following section, we will explore how set operations based on inclusion relations can be utilized for effective uncertainty management in environmental data analysis.

- **Numerical Examples for Benefits of Inclusion Relations in Environmental Data Analysis with CBN Sets**

Here are some numerical examples displaying the benefits of inclusion relations in environmental data analysis using CBN Sets:

Example 1: Refining Information - Air Quality Monitoring

Imagine a scenario where two air quality monitoring stations (Station A and Station B) measure ozone levels (in parts per million, ppm) using CBN Sets (M_A and M_B) to account for measurement uncertainties.

- **Universe of Discourse (U):** {0 ppm, 0.02 ppm, 0.04 ppm, ..., 0.1 ppm} (incremented by 0.02 ppm)

Table 7: Refined Air Quality Data with CBN Sets (Station A vs. B)

Ozone Level (ppm)	M_A (Station A)	M_B (Station B)	Inclusion Relation	Interpretation
0.04 ppm	T^{+1} (0.8)	T^{+1} (0.9)	M_B ≤ M_A	Station A's data has higher certainty for 0.04 ppm ozone level.
0.06 ppm	T^{+2} (0.3)	T^{+1} (0.7)	M_B ≤ M_A	Station A's data is more reliable for 0.06 ppm ozone level.
0.08 ppm	I^{+1} (0.2)	T^{+2} (0.5)	No Relation	Both stations have some uncertainty, but Station B shows a stronger possibility of 0.08 ppm.

Explanation:

- By comparing the membership degrees, we can identify that Station A's data has higher certainty (higher T^{+1} values) for ozone levels of 0.04 ppm and 0.06 ppm.
- This allows us to focus on Station A's data for these specific levels, refining our understanding of the local air quality.

Example 2: Generalizing Information - Soil Moisture Analysis

Consider a study analyzing soil moisture content (%) at two different depths (Depth 1 and Depth 2) using CBN Sets (M_d1 and M_d2).

- **Universe of Discourse (U):** {0%, 10%, 20%... 50%} (incremented by 10%)

Table 8: Soil Moisture Analysis with CBN Sets (Depth Comparison)".

Soil Moisture (%)	M_d1 (Depth 1)	M_d2 (Depth 2)	Inclusion Relation	Interpretation
20%	T^{+1} (0.7)	T^{+2} (0.5)	M_d2 ≤ M_d1	Depth 1 has a stronger indication of 20% moisture content.
30%	T^{+2} (0.2)	T^{+1} (0.8)	M_d1 ≤ M_d2	Depth 2 has a clearer signal for 30% moisture content.
40%	I^{+1} (0.1)	I^{+2} (0.3)	No Relation	Both depths have some possibility of 40% moisture, but the evidence is weak.

Explanation:

- Here, the inclusion relations help identify general trends. While Depth 1 has a stronger signal for lower moisture levels (20%), Depth 2 shows a higher certainty for a higher moisture level (30%).
- This suggests a potential moisture content gradient with depth, providing valuable insights into soil hydrology.

These examples demonstrate how inclusion relations enable us to:

- Refine information by focusing on data points with higher certainty within a CBN Set.

- Generalize information by identifying trends and patterns across different CBN Sets representing environmental data.

By leveraging inclusion relations, we can effectively manage uncertainties and gain a more nuanced understanding of environmental phenomena.

5. Harnessing the Power of Cubic Bipolar Neutrosophic Sets (CBN Sets) for Effective Uncertainty Management in Environmental Data Analysis

Environmental data analysis is fundamental for understanding and managing our planet's health. However, real-world measurements are often subject to inherent uncertainties due to sensor limitations, natural variations, and other factors. Traditional binary approaches that rely on strict classifications can struggle to capture these nuances. CBN Sets offer a powerful framework for representing and managing uncertainty, enabling more robust and informative environmental data analysis.

5.1 Manipulating Uncertainty with CBN Sets: Intersection, Union, and Complement

Building upon the concept of inclusion relations, we can now explore fundamental set operations for CBN Sets: intersection, union, and complement. These operations allow us to manipulate and analyze environmental data while accounting for the inherent uncertainties represented by the membership degrees.

- **Intersection ($M_1 \cap M_2$):** This operation identifies elements that belong to both CBN Sets (M_1 and M_2) with a certain degree of certainty, possibility, and indeterminacy. The resulting membership degrees in the intersection set are calculated as the minimum of the corresponding membership degrees in M_1 and M_2.

  For example, consider two CBN Sets representing water quality parameters at a location. The intersection can reveal areas where both sets agree on the "cleanliness" of the water with a specific certainty level.
- **Union ($M_1 \cup M_2$):** This operation captures elements that belong to either M_1 or M_2 or possibly to both sets. The resulting membership degrees in the union set are calculated as the maximum of the corresponding membership degrees in M_1 and M_2.

  The union can be helpful in environmental studies to identify regions where either set indicates potential environmental concerns, even if the certainty levels differ.
- **Complement (M^{1c}):** This operation produces a new CBN Set that includes elements that do not belong to the original set (M_1). The membership degrees in the complement are calculated as 1 minus the corresponding membership degrees in M_1.

  The complement can be used to identify areas where environmental parameters deviate from expected values or established thresholds, even with some degree of uncertainty.

By effectively utilizing these set operations on CBN Sets, researchers can achieve a more comprehensive understanding of environmental data while acknowledging the inherent uncertainties associated with measurements.

- **Numerical Examples for Manipulating Uncertainty with CBN Sets: Intersection, Union, and Complement**

Here are some numerical examples demonstrating how set operations on CBN Sets (intersection, union, and complement) can be used to manipulate and analyze environmental data while accounting for uncertainties:

Scenario: We are monitoring water quality parameters (temperature, pH, etc.) at two locations (Lake A and River B) using CBN Sets (M_lake and M_river) due to potential pollution concerns.

- **Universe of Discourse (U):** {Ideal, Slightly Polluted, Moderately Polluted, Heavily Polluted}

1. **Intersection (M_lake ∩ M_river): Identifying Areas with Consistent Water Quality**

Table 9: "Water Quality Analysis: Intersection of Lake and River Data (M_lake ∩ M_river)"

Parameter	M_lake (Lake A)	M_river (River B)	M_lake ∩ M_river (Intersection)	Interpretation
Temperature	T^{+1} (0.8)	T^{+2} (0.6)	T^{+1} (min(0.8, 0.6)) = 0.6	Both locations have a moderate certainty (0.6) of ideal temperature.
pH	I^{+1} (0.2)	T^{-1} (0.7)	Ø (Empty Set)	No overlap in membership degrees. Lake A might have slightly high pH (I^{+1}), while River B shows a strong possibility (T^{-1}) of being slightly acidic.

This table analyzes the water quality of a lake (M_lake) and a river (M_river) using the intersection (∩) of their respective CBN Sets. The intersection identifies areas where both water bodies share similar characteristics.

Parameters:

The table focuses on two key water quality parameters: temperature and pH.

Interpretation:

- **Temperature:**
  - M_lake (Lake A): T^{+1} (0.8) indicates a strong certainty (80%) of ideal temperature in the lake.
  - M_river (River B): T^{+2} (0.6) indicates a moderate certainty (60%) of ideal temperature in the river.
  - M_lake ∩ M_river (Intersection): The intersection considers the minimum certainty level (min(0.8, 0.6)) resulting in T^{+1} (0.6). This means there's a moderate certainty (60%) of ideal temperature in areas where the lake and river water mix.
- **pH:**
  - M_lake (Lake A): I^{+1} (0.2) suggests a slight possibility (20%) of the lake having a slightly high pH.
  - M_river (River B): T^{-1} (0.7) indicates a strong possibility (70%) of the river being slightly acidic.
  - M_lake ∩ M_river (Intersection): The intersection results in the empty set (∅) because there's no overlap in membership degrees for pH. The lake might have a slightly high pH, while the river shows a strong possibility of being slightly acidic, indicating no areas where both characteristics intersect.

2. **Union (M_lake ∪ M_river): Identifying Areas with Potential Water Quality Issues**

While the intersection of CBN Sets focuses on overlapping areas of similarity, the union (U) considers areas where **either** water body might have a water quality issue. This can be helpful for identifying potential problems that require further investigation.

Table 10: Water Quality Analysis: Union of Lake and River Data (M_lake ∪ M_river)

Parameter	M_lake (Lake A)	M_river (River B)	M_lake ∪ M_river (Union)	Interpretation
Temperature	T^{+1} (0.8)	T^{+2} (0.6)	T^{+1} (max(0.8, 0.6)) = 0.8	Strong certainty (0.8) of ideal temperature (from Lake A) or moderate certainty (0.6) (from River B).
pH	I^{+1} (0.2)	T^{-1} (0.7)	T^{-1} (max(0.2, 0.7)) = 0.7	Strong possibility (0.7) of slightly acidic water (from River B), or slight possibility (0.2) of slightly high pH (from Lake A).

This table analyzes the potential water quality issues in Lake A and River B using the union (U) of their respective CBN Sets. The union identifies areas where **at least one location (or potentially both)** might exceed optimal water quality conditions for a specific parameter.

Parameters:

The table focuses on two key parameters: temperature and pH.

Interpretation:

- **Temperature:**
  - M_lake (Lake A): T^{+1} (0.8) indicates a strong certainty (80%) of ideal temperature in the lake.
  - M_river (River B): T^{+2} (0.6) indicates a moderate certainty (60%) of ideal temperature in the river.
  - M_lake ∪ M_river (Union): The union considers the **maximum** membership degree (max(0.8, 0.6)) resulting in T^{+1} (0.8). This signifies a **strong overall certainty (80%)** that at least one location (lake or river, or potentially both) has ideal temperature. However, the union itself doesn't pinpoint the exact locations.
- **pH:**
  - M_lake (Lake A): I^{+1} (0.2) suggests a slight possibility (20%) of the lake having a slightly high pH.
  - M_river (River B): T^{-1} (0.7) indicates a strong possibility (70%) of the river being slightly acidic.
  - M_lake ∪ M_river (Union): The union considers the maximum membership degree (max(0.2, 0.7)) resulting in T^{-1} (0.7). This highlights a **strong possibility (70%)** that at least one location (lake or river) deviates from the optimal pH range. The source (lake or river) cannot be determined solely based on the union.

3. Complement (M_lake^{1c} and M_river^{1c}): Identifying Deviations from Expected Values

The concept of the complement in CBN Sets allows us to identify areas where the environmental data significantly deviates from the expected values for a specific parameter. Let us explore how this works for lakes and rivers:

This table analyzes deviations from expected water quality values in Lake A using the complement (M_lake^{1c}) of its CBN Set. The complement identifies areas where the environmental data might deviate from the ideal or optimal ranges.

Table 11: Water Quality Analysis: Deviations from Expected Values (Lake A)"

Parameter	M_lake (Lake A)	M_lake^{1c} (Complement)	Interpretation
Temperature	T^{+1} (0.8)	F^{+1} (1 - 0.8) = 0.2	Areas with a certainty level of 0.2 for deviating from the ideal temperature range.
pH	I^{+1} (0.2)	F^{+1} (1 - 0.2) = 0.8	Areas with a certainty level of 0.8 for deviating from the expected pH range (due to the possibility of slightly high pH).

The complement of Lake A's CBN Set (M_lake^{1c}) identifies areas with a certainty level of 0.2 for deviating from the expected ideal temperature range. This certainty level (0.2) is calculated by subtracting the membership degree in M_lake (0.8) from 1 (representing complete certainty).

Important Note: This 0.2 certainty level does not necessarily indicate a confirmed problem, but rather the degree to which the data deviates from the ideal range. It could be due to:

- **Instrument limitations:** Measurement errors can introduce slight variations in the data.
- **Natural variations within the lake:** Temperature might fluctuate slightly across different areas of the lake.

These examples showcase the power of set operations on CBN Sets:

- **Identifying Consistent or Potentially Problematic Water Quality:** The intersection (∩) helps identify areas where both locations share similar characteristics (e.g., ideal temperature in overlapping zones). The union (U) helps us flag areas where either location might have a water quality concern, even if the certainty levels differ.
- **Capturing the Overall Picture:** By analyzing both intersection and union, we gain a broader understanding of potential water quality issues across different areas, even if the certainty levels vary.
- **Highlighting Deviations with Acknowledged Uncertainty:** The complement (^{1}) focuses on areas where the data deviates from expected values, prompting further investigation while acknowledging the inherent uncertainties in environmental data.

Overall, these set operations provide a comprehensive framework for analyzing water quality data, allowing us to identify areas requiring further monitoring and develop better environmental management strategies.

5.2 Applying CBN Sets to Environmental Applications (Focus on Water Quality Assessment) with Numerical Examples

*** Water Quality Parameters as CBN Sets:**

Let us consider a scenario where we are monitoring two key water quality parameters: pH and dissolved oxygen (DO) in a lake. We will use CBN Sets to represent these parameters, accounting for measurement uncertainties.

Universe of Discourse:

- **pH:** {< 6 (Strongly Acidic), 6-6.5 (Somewhat Acidic), 6.5-8 (Neutral), 8-8.5 (Somewhat Basic), > 8.5 (Strongly Basic)}
- **DO (mg/L):** {< 4 (Critically Low), 4-5 (Low), 5-8 (Optimal), 8-10 (High), > 10 (Supersaturated)}

Example 1: Representing pH with CBN Sets

Imagine a measured pH value of 7.3. Here is a possible CBN Set representing this data:

Table 12: CBN Set Representation of Measured pH (7.3)

Category	Description	Membership Degree
T^{+1} (Neutral)	Strongly Matches Optimal pH Range (6.5-8)	0.7
T^{+2} (Somewhat Basic)	Moderately Leans Towards Slightly High pH	0.2
I^{+1} (Possibly Somewhat Acidic)	Slight Possibility of Being Below Optimal Range	0.1
F^{+1} (Strongly Acidic)	Negligible Certainty of Being Strongly Acidic	0.0
F^{+2} (Somewhat Acidic)	Very Low Certainty of Being Moderately Low pH	0.0
F^{-1} (Strongly Basic)	Negligible Certainty of Being Strongly Basic	0.0

F^{-2} (Somewhat Basic)	Very Low Certainty of Being Moderately High pH (beyond Somewhat Basic)	0.0

This table depicts a CBN Set representing a measured pH of 7.3. Here is an explanation for each category and its corresponding membership degree:

- **T^{+1} (Neutral) (0.7):** This category captures the high certainty (0.7) of the pH being within the ideal or optimal range (6.5-8) for the specific environment. The measured value (7.3) falls relatively close to the center of this range.
- **T^{+2} (Somewhat Basic) (0.2):** This category indicates a moderate certainty (0.2) that the pH leans slightly towards the basic side (higher than ideal). However, the value is still somewhat close to the ideal range.
- **I^{+1} (Possibly Somewhat Acidic) (0.1):** The slight possibility (0.1) in this category reflects the fact that the measured pH is not perfectly centered in the ideal range and could be edging slightly towards the acidic side.
- **F^{+1} (Strongly Acidic) (0.0): Ø (Empty Set):** These categories represent extremely acidic conditions. Since the measured value is nowhere near this range, the membership degrees are zero (0), indicating negligible certainty.
- **F^{+2} (Somewhat Acidic) (0.0):** This category reflects a very low certainty (0) of the pH being moderately acidic. The measured value is closer to the ideal range than moderately acidic.
- **F^{-1} (Strongly Basic) (0.0): F^{-2} (Somewhat Basic) (0.0):** Similar to the strongly acidic categories, these categories for extremely basic or moderately high basic pH have zero membership degrees due to the measured value being far from these ranges.

Example 2: Utilizing Set Operations for Water Quality Analysis

Suppose we have CBN Sets for both pH and DO measured at different locations in the lake. We can use set operations to analyze the data:

- **Intersection:** This identifies areas where both pH and DO meet acceptable water quality standards simultaneously. A high intersection value indicates a location with a strong likelihood of good overall water quality.

For instance, if another location has a CBN Set for DO with a high T^{+1} (Optimal) membership degree, the intersection of the pH and DO sets might reveal a location with a strong certainty of having both optimal pH and DO levels.

- **Union:** This highlights regions where either pH or DO (or potentially both) might fall outside the acceptable range, even with some uncertainty. A high union value suggests a potential water quality concern that needs further investigation.

Imagine a location's DO CBN Set has a high I^{+1} (Low) membership degree. The union with the previous pH CBN Set might indicate a location with a good pH level but a possibility of low DO, requiring attention.

- **Complement:** This helps pinpoint areas where any parameter deviates from optimal values. A high complement value for a specific category (e.g., Strongly Acidic for pH) suggests a location where that parameter might be a concern, even if the overall intersection or union values do not show a critical issue.

For example, the complement of the pH CBN Set (considering only Strongly Acidic and Somewhat Acidic categories) might reveal areas with a slight possibility of deviating from the optimal range, even though the intersection with DO might still be high.

By analyzing combinations of CBN Sets for various water quality parameters, researchers can gain a more comprehensive understanding of the spatial variability and potential water quality issues across the lake. This information can be crucial for targeted water management strategies.

6. Manipulating Uncertainty with CBN Sets: Intersection, Union, and Complement

6.1. Demystifying CBN Set Operations: Intersection, Union, and Complement Explained

Having established inclusion relations for comparing CBN Sets, we can now delve into fundamental set operations: intersection, union, and complement. These operations allow us to analyze and manipulate environmental data while accounting for the inherent uncertainties represented by the membership degrees within CBN Sets.

- **Intersection ($M_1 \cap M_2$):** This operation identifies elements that belong to **both** CBN Sets (M_1 and M_2) with a certain degree of certainty, possibility, and indeterminacy. It essentially finds the overlapping area between the two sets, considering the membership degrees.

Calculation: The resulting membership degrees in the intersection set ($M_1 \cap M_2$) are calculated as the **minimum** of the corresponding membership degrees in M_1 and M_2.

Example:

Consider two CBN Sets representing water quality parameters (e.g., pH and dissolved oxygen) at a specific location. Let M_1 represent pH and M_2 represent dissolved oxygen. The intersection ($M_1 \cap M_2$) would reveal areas where both sets agree on the "healthy" range of these parameters with a specific certainty level. For instance, the intersection might show a high positive truth membership (T^{+1}) for "suitable pH" and "adequate dissolved oxygen," indicating a high likelihood of good water quality based on both parameters.

6.2. Demystifying CBN Set Operations: Intersection, Union, and Complement Explained with Numerical Examples

Following the concept of inclusion relations, we can now explore fundamental set operations for CBN Sets: intersection, union, and complement. These operations allow us to analyze and manipulate environmental data while considering the uncertainties within the membership degrees.

1. Intersection ($M_1 \cap M_2$): Identifying Areas with Consistent Quality

- **Concept:** Intersection captures elements that belong to both CBN Sets (M_1 and M_2) with a certain degree of certainty, possibility, and indeterminacy. It finds the overlapping area between the two sets, considering the membership degrees.
- **Calculation:** The resulting membership degrees in the intersection set (M_1 Transcription $\cap$ M_2) are determined by taking the minimum value for each category between the corresponding membership degrees in M_1 and M_2.

Example: Water Quality Assessment

Imagine we are monitoring two water quality parameters: pH (M_1) and dissolved oxygen (DO) (M_2) at a specific location in a lake. The universe of discourse for both is:

- {< Ideal, Slightly Low, Ideal, Slightly High, > Ideal}

Let us say we have the following CBN Sets based on measurements:

Table 13: "CBN Sets for Water Quality Monitoring (pH and Dissolved Oxygen)".

Category	M_1 (pH)	M_2 (DO)
T^{+1} (Ideal)	0.8	0.7
T^{+2} (Somewhat Ideal)	0.1	0.2
I^{+1} (Possible Slightly Low/High)	0.05	0.05
F^{+1} (< Ideal/ > Ideal)	0.0	0.0
F^{+2} (Somewhat Low/High)	0.05	0.05

Intersection ($M_1 \cap M_2$):

Table 14: "Water Quality Analysis: Intersection of pH and Dissolved Oxygen ($M_1 \cap M_2$)"

Category	$M_1 \cap M_2$ (Minimum Membership Degree)	Interpretation
T^{+1} (Ideal)	min(0.8, 0.7) = 0.7	Strong certainty (0.7) of ideal conditions for both pH and DO.
T^{+2} (Somewhat Ideal)	min(0.1, 0.2) = 0.1	Moderate possibility (0.1) of both pH and DO being slightly outside the ideal range.
I^{+1} (Possible Slightly Low/High)	min(0.05, 0.05) = 0.05	Very slight possibility (0.05) of either parameter deviating slightly from ideal.
F^{+1} (< Ideal/ > Ideal)	min(0.0, 0.0) = 0.0	No certainty (0.0) of both parameters being outside the ideal range (since minimum is 0 for both M_1 and M_2).
F^{+2} (Somewhat Low/High)	min(0.05, 0.05) = 0.05	Very slight possibility (0.05) of either parameter being somewhat outside the ideal range.

Interpretation:

The intersection highlights a location with a strong likelihood (0.7) of good water quality based on both pH and DO levels. There's also a moderate possibility (0.1) of them being slightly outside the ideal range, along with very slight chances of more significant deviations.

This example demonstrates how the intersection helps identify areas where multiple water quality parameters meet acceptable standards simultaneously.

6.3. Applications in Decision-Making and Pattern Recognition:

- Identifying areas where multiple environmental parameters meet acceptable standards simultaneously. This can be crucial for pinpointing locations with consistently good environmental conditions.

- Highlighting locations with consistent agreement on specific environmental conditions across different measurements. This can aid in pattern recognition, revealing potential correlations between environmental factors.
- Refining decision-making by focusing on areas with high certainty overlap between multiple CBN Sets. This can help prioritize areas for environmental protection or resource management based on strong evidence.
- **Union ($M_1 \cup M_2$):** This operation captures elements that belong to **either** M_1 or M_2 or possibly to **both sets**. It essentially represents the combined area of both sets, acknowledging the possibility of an element belonging to either set.

  Calculation: The resulting membership degrees in the union set ($M_1 \cup M_2$) are calculated as the **maximum** of the corresponding membership degrees in M_1 and M_2.

  Example:

  Imagine two CBN Sets representing potential air pollution levels at different monitoring stations (M_1 and M_2). The union ($M_1 \cup M_2$) would encompass regions where either station indicates potential air quality concerns, even if the certainty levels differ. For instance, the union might show a high positive truth membership (T^{+1}) for "elevated ozone levels" in M_1 and a high indeterminacy membership (I^{+1}) for "particulate matter" in M_2. This would indicate a potential air pollution issue at the combined location, even if the specific pollutant type might have some uncertainty.

*** Applications in Decision-Making and Pattern Recognition with Numerical Examples**

1. Intersection ($M_1 \cap M_2$):

We already saw how intersection helps identify areas with consistent quality in water quality assessment. Here's another example:

- **Scenario:** Monitoring soil quality using CBN Sets for nutrients (M_1) and heavy metals (M_2).
- **Interpretation:** A high intersection value for "adequate nutrient levels" and "low heavy metal content" across different soil samples would indicate consistently good soil quality for plant growth.

2. Highlighting Consistent Agreements and Pattern Recognition:

- **Scenario:** Monitoring temperature (M_1) and precipitation (M_2) across different regions over time.
- **Example:** Consistent high intersection values for "warm temperatures" in M_1 and "low precipitation" in M_2 across several regions during a specific season might suggest a pattern of drought conditions in those areas. This could be helpful in predicting future droughts and implementing water management strategies.

3. Refining Decision-Making with High Certainty Overlap:

- **Scenario:** Assessing potential risks associated with a new industrial facility.

- **Example:** We can analyze CBN Sets for air quality parameters (M_1) near the facility and potential environmental impact on nearby wildlife (M_2). A high intersection value for "elevated air pollution levels" in M_1 and "high risk to sensitive bird species" in M_2 would indicate a strong need for mitigation measures before construction due to the high certainty overlap.

4. Union ($M_1 \cup M_2$):

- **Concept:** Union captures elements that belong to either M_1 or M_2 (or possibly both) with a certain degree of certainty, possibility, and indeterminacy. It represents the combined area of both sets, acknowledging the possibility of an element belonging to either set.
- **Calculation:** The resulting membership degrees in the union set ($M_1 \cup M_2$) are determined by taking the maximum value for each category between the corresponding membership degrees in M_1 and M_2.

Example: Air Pollution Monitoring:

Imagine we have CBN Sets representing potential air pollution levels at different monitoring stations (M_1 and M_2). The universe of discourse for both could be:

- {Low, Moderate, High, Very High}

Let us say the CBN Sets are based on measurements:

Table 15: Table 15: "CBN Sets for Air Pollution Monitoring (Station M_1 vs. M_2)"

Category	M_1	M_2
T^{+1} (Low)	0.2	0.1
T^{+2} (Moderate)	0.5	0.6
I^{+1} (Possible High)	0.2	0.2
F^{+1} (Very High)	0.0	0.0
F^{+2} (Somewhat High)	0.1	0.1

Union ($M_1 \cup M_2$):

Table 16: Air Pollution Analysis: Union of Monitoring Station Data ($M_1 \cup M_2$)

Category	$M_1 \cup M_2$ (Maximum Membership Degree)	Interpretation
T^{+1} (Low)	max(0.2, 0.1) = 0.2	Low certainty (0.2) of good air quality at either location.
T^{+2} (Moderate)	max(0.5, 0.6) = 0.6	Strong certainty (0.6) of moderate air pollution levels at either station (or possibly both).
I^{+1} (Possible High)	max(0.2, 0.2) = 0.2	Moderate possibility (0.2) of high pollution levels at either location.
F^{+1} (Very High)	max(0.0, 0.0) = 0.0	No certainty (0.0) of very high pollution at either location.
F^{+2} (Somewhat High)	max(0.1, 0.1) = 0.1	Low certainty (0.1) of somewhat high pollution at either location.

The union highlights a potential air quality concern, with a strong possibility (0.6) of moderate pollution at either station or potentially both. There's also a moderate chance (0.2) of even higher pollution levels. This emphasizes the need for further investigation despite uncertainties in specific pollution types.

These examples showcase how CBN set operations can be used for informed decision-making and pattern recognition in environmental applications by considering both the certainties and uncertainties associated with environmental data.

6.4. Applications in Data Analysis:

- Identifying regions where any parameter might exceed acceptable levels, even with some uncertainty. This provides a broader perspective on potential environmental threats across a larger area.

- Creating a broader picture of potential environmental concerns across a larger area. This can be helpful for initial data analysis, highlighting areas that warrant further investigation.
- Informing strategies for environmental monitoring by highlighting areas where further investigation might be necessary. This can optimize resource allocation for environmental monitoring efforts.
- **Complement (M^{1c}):** This operation produces a new CBN Set that includes elements that **do not** belong to the original set (M_1). It essentially flips the membership degrees, reflecting the opposite condition.

  Calculation: The membership degrees in the complement (M^{1c}) are calculated as **1 minus** the corresponding membership degrees in M_1.

  Example:

  Consider a CBN Set (M_1) representing "suitable habitat" for a particular endangered species. The complement (M^{1c}) would represent areas that are **not** suitable habitat. The complement might show a high positive truth membership (T^{+1}) for "unsuitable vegetation" and a high indeterminacy membership (I^{+1}) for "lack of prey availability." This information helps identify areas where the species is unlikely to thrive.

Applications in Data Analysis with Numerical Examples

1. Union ($M_1 \cup M_2$):

We already saw how union helps highlight potential air pollution concerns. Here is another example for data analysis:

- **Scenario:** Monitoring forest health using CBN Sets for insect infestation (M_1) and drought stress (M_2) across a large forest area.
- **Example:** The union might reveal a vast region with a high possibility (from either M_1 or M_2) of experiencing either insect infestation or drought stress. This broader perspective helps identify areas that warrant further investigation for specific threats to forest health.

2. Creating a Broader Picture of Potential Environmental Concerns:

- **Scenario:** Monitoring water quality using multiple parameters (e.g., pH, DO, nitrate levels) represented by different CBN Sets (M_1, M_2, M_3).
- **Example:** Analyzing the union of these sets across a river system can identify stretches with a possibility of exceeding acceptable levels for any of the parameters, even if specific issues are unclear. This helps prioritize areas for more detailed water quality testing.

3. Informing Strategies for Environmental Monitoring:

- **Scenario:** Monitoring potential land degradation using CBN Sets for soil erosion (M_1) and desertification risk (M_2).
- **Example:** The union might highlight a large region with a possibility of facing either soil erosion or desertification. This information can guide resource allocation for deploying environmental monitoring sensors or conducting field studies in these areas to confirm specific threats and take necessary measures.

4. Complement (M^{1c}):

- **Concept:** Complement produces a new CBN Set that includes elements that do not belong to the original set (M_1). It essentially flips the membership degrees, reflecting the opposite condition.
- **Calculation:** The membership degrees in the complement (M^{1c}) are calculated as 1 minus the corresponding membership degrees in M_1.

Example: Habitat Suitability:

Consider a CBN Set (M_1) representing "suitable habitat" for a particular endangered bird species. The universe of discourse could be:

- {Abundant Food, Suitable Nesting Sites, Safe from Predators}

Let us say the CBN Set for suitable habitat is based on data:

Table 17: CBN Set for Suitable Habitat (Endangered Bird Species)

Category	M_1 (Suitable Habitat)
T^{+1} (Abundant Food)	0.8
T^{+2} (Suitable Nesting Sites)	0.7
I^{+1} (Possible Predator Threats)	0.2
F^{+1} (Lack of Food/Nesting Sites)	0.0
F^{+2} (High Predator Risk)	0.0

Complement (M_1^c - unsuitable habitat):

Table 18: Unsuitable Habitat Analysis for Endangered Bird Species (M_1^c)

Category	M_1^c (Complement)	Interpretation
T^{+1} (Lack of Food)	1 - 0.8 = 0.2	Moderate certainty (0.2) of limited food availability.

T^{+2} (Unsuitable Nesting Sites)	1 - 0.7 = 0.3	Strong certainty (0.3) of unsuitable nesting sites.
I^{+1} (Possible Safe from Predators)	1 - 0.2 = 0.8	High possibility (0.8) of being relatively safe from predators (opposite of I^{+1} in M_1).
F^{+1} (Abundant Food/Nesting Sites)	1 - 0.0 = 1.0	Certain (1.0) absence of abundant food or suitable nesting sites (opposite of F^{+1} in M_1).
F^{+2} (Low Predator Risk)	1 - 0.0 = 1.0	Certain (1.0) absence of high predator risk (opposite of F^{+2} in M_1).

Interpretation:

The complement highlights areas unsuitable for the bird species. It confirms the lack of abundant food and suitable nesting sites (certainties of 1.0) while also suggesting a moderate possibility (0.8) of being relatively safe from predators. This information helps identify areas where the species is unlikely to thrive and focus conservation efforts on improving suitable habitat in other areas.

These examples highlight how CBN set operations, particularly union and complement, can be used for broader environmental data analysis. They help create a comprehensive picture of potential environmental concerns across larger areas, informing resource allocation for further investigations and targeted environmental monitoring strategies.

o Applications in Data Analysis and Risk Assessment:

- Pinpointing areas where environmental parameters deviate from expected values or established thresholds, even with some degree of uncertainty. This can be useful for identifying potential environmental issues that require further investigation.
- Identifying areas outside the scope of a particular CBN Set, potentially leading to further investigation or refined set definitions. This can lead to a more comprehensive understanding of the environmental conditions being studied.
- Highlighting potential risks or concerns by focusing on areas outside the desired environmental conditions. This can be crucial for environmental risk assessment and prioritizing areas for mitigation strategies.

By effectively utilizing these set operations on CBN Sets, researchers can achieve a richer understanding

* Applications in Data Analysis and Risk Assessment with Numerical Examples

1. Complement (M^{1c}):

We already saw how complement helps identify unsuitable habitat for endangered species. Here is another example for data analysis and risk assessment:

- **Scenario:** Monitoring potential flood risk using a CBN Set (M_1) for water level at a river station.
- **Universe of Discourse:** {Safe Water Level, Moderate Flood Risk, High Flood Risk, Catastrophic Flood Risk}
- **CBN Set (M_1):** Based on historical data and floodplains, M_1 might show high membership degrees for "safe water level" and very low degrees for other categories.

Complement (M_1^c - Flood Risk):

By analyzing the complement, we can identify areas potentially at risk of flooding:

Table 19: "Flood Risk Analysis Using Habitat Suitability Complement (M_1^c)"

Category	M_1^c (Complement)	Interpretation
T^{+1} (Moderate Flood Risk)	0.1	Low certainty (0.1) of moderate flooding, but highlights a possibility beyond the "safe" category in M_1.
T^{+2} (High Flood Risk)	0.05	Very low certainty (0.05) of high flood risk, but indicates a chance exceeding the "safe" range.
I^{+1} (Possible Catastrophic Flood)	0.0	No possibility of catastrophic flooding based on M_1.
F^{+1} (Safe Water Level)	0.0	Certain (0.0) absence of safe water level (opposite of F^{+1} in M_1).

F^{+2} (Low Flood Risk)	0.85	Certain (0.85) absence of low flood risk (opposite of F^{+2} in M_1 - not relevant for identifying flood risk).

The complement highlights a slight possibility (0.1) of moderate flooding and a very low chance (0.05) of high flood risk. While the certainty levels are low, this information is valuable for risk assessment, prompting further investigation or implementing early warning systems near the river station.

2. Identifying Areas Outside CBN Set Scope:

- **Scenario:** Monitoring air quality using a CBN Set (M_1) for common pollutants (e.g., ozone, PM2.5).
- **Analysis:** If the analysis reveals areas with consistently low membership degrees for all categories in M_1, it might indicate the presence of uncommon pollutants not included in the initial CBN Set definition. This would warrant further investigation to identify the specific pollutants and assess potential risks.

3. Highlighting Potential Risks:

- **Scenario:** Assessing potential risks associated with a new industrial facility.
- **Example:** We can analyze a CBN Set (M_1) for potential air pollution emissions from the facility and a CBN Set (M_2) for the vulnerability of nearby ecosystems. The complement of the intersection $(M_1 \cap M_2)^{1c}$ would highlight areas where the intersection is low (i.e., low certainty of both low emissions and low vulnerability). This indicates potential risk due to a possibility of high emissions impacting a vulnerable ecosystem, even if the exact certainty levels in M_1 or M_2 are unclear.

These examples showcase how CBN set operations, particularly complement and analyzing areas outside the CBN Set scope, can be powerful tools in data analysis and risk assessment. They help identify potential environmental issues beyond the initial focus of the CBN Sets, leading to a more comprehensive understanding of environmental risks and informing the development of mitigation strategies.

o Applying CBN Sets to Water Quality Assessment: A Detailed Example

Water quality is a critical indicator of ecosystem health and human well-being. Traditional methods for water quality assessment often rely on point measurements, which may not capture the spatial and temporal variability of water quality parameters. Additionally, inherent uncertainties in sensor limitations and natural variations can pose challenges. Cubic Bipolar Neutrosophic Sets (CBN Sets) offer a powerful framework to address these limitations by incorporating uncertainty into the analysis.

- **Applying CBN Sets to Water Quality Assessment: A Detailed Example with Numerical Values**

Scenario: We're monitoring a lake for water quality assessment. We'll use CBN Sets to represent two key parameters: pH and dissolved oxygen (DO).

Universe of Discourse:

- **pH:** {< 6 (Strongly Acidic), 6-6.5 (Somewhat Acidic), 6.5-8 (Neutral), 8-8.5 (Somewhat Basic), > 8.5 (Strongly Basic)}
- **DO (mg/L):** {< 4 (Critically Low), 4-5 (Low), 5-8 (Optimal), 8-10 (High), > 10 (Supersaturated)}

Data Collection and Measurement Uncertainties:

- We take pH and DO measurements at a specific location in the lake. However, inherent uncertainties exist due to sensor limitations or natural fluctuations.

Example: Representing pH with CBN Sets

Measured pH value: 7.3

Table 20: Representing pH with CBN Sets (Measured pH: 7.3)

Category	Description	Membership Degree
T^{+1} (Neutral)	Strongly Matches Optimal pH Range (6.5-8)	0.7
T^{+2} (Somewhat Basic)	Moderately Leans Towards Slightly High pH	0.2
I^{+1} (Possibly Somewhat Acidic)	Slight Possibility of Being Below Optimal Range	0.1
F^{+1} (Strongly Acidic)	Negligible Certainty of Being Strongly Acidic	0.0
F^{+2} (Somewhat Acidic)	Very Low Certainty of Being Moderately Low pH	0.0
F^{-1} (Strongly Basic)	Negligible Certainty of Being Strongly Basic	0.0

F^{-2} (Somewhat Basic)	Very Low Certainty of Being Moderately High pH (beyond Somewhat Basic)	0.0

The measured pH value of 7.3 is effectively represented by this CBN Set (Table 20). Here's a breakdown of each category and its membership degree:

- **T^{+1} (Neutral) (0.7):** This category captures the **strong certainty (0.7)** of the pH being within the ideal or optimal range (6.5-8) for the specific environment. The measured value (7.3) falls relatively close to the center of this range, justifying the high membership degree.
- **T^{+2} (Somewhat Basic) (0.2):** This category indicates a **moderate possibility (0.2)** that the pH leans slightly towards the basic side (higher than ideal). However, the value is still somewhat close to the ideal range as reflected by the lower membership degree compared to T^{+1}.
- **I^{+1} (Possibly Somewhat Acidic) (0.1):** The slight possibility (0.1) in this category reflects the fact that the measured pH is not perfectly centered in the ideal range and could be edging slightly towards the acidic side, although the certainty is low.
- **F^{+1} (Strongly Acidic) (0.0): Ø (Empty Set):** These categories represent extremely acidic conditions. Since the measured value is nowhere near this range, the membership degrees are zero (0), indicating negligible certainty.
- **F^{+2} (Somewhat Acidic) (0.0):** This category reflects a very low certainty (0) of the pH being moderately acidic. The measured value is closer to the ideal range than moderately acidic.
- **F^{-1} (Strongly Basic) (0.0): F^{-2} (Somewhat Basic) (0.0):** Similar to the strongly acidic categories, these categories for extremely basic or moderately high basic pH have zero membership degrees due to the measured value being far from these ranges.

Representing DO with CBN Sets:

Imagine the measured DO value is 6.2 mg/L.

Table 21: Representing DO with CBN Sets (Measured DO: 6.2 mg/L)

Category	Description	Membership Degree
T^{+1} (Optimal)	Strongly Matches Optimal DO Range (5-8 mg/L)	0.8
T^{+2} (Somewhat High)	Moderately Leans Towards Slightly High DO	0.1

I^{+1} (Possible Low)	Slight Possibility of Being Below Optimal Range	0.1
F^{+1} (Critically Low)	Negligible Certainty of Being Critically Low	0.0
F^{+2} (Low)	Very Low Certainty of Being Below Optimal Range (but not critically low)	0.0
F^{-1} (Supersaturated)	Negligible Certainty of Being Supersaturated	0.0
F^{-2} (High)	Very Low Certainty of Being Above Optimal Range (but not supersaturated)	0.0

Table 21 effectively represents the measured Dissolved Oxygen (DO) value of 6.2 mg/L using a CBN Set. Let's break down the categories and membership degrees:

- **T^{+1} (Optimal) (0.8):** This category captures the **strong certainty (0.8)** of the DO concentration being within the ideal or optimal range (5-8 mg/L) for the specific environment. The measured value (6.2 mg/L) falls within this range, justifying the high membership degree.
- **T^{+2} (Somewhat High) (0.1):** This category indicates a **low certainty (0.1)** that the DO concentration leans **slightly** above the optimal range. It is important to note that "Somewhat High" here refers to being slightly above the optimal range, but still within acceptable limits.
- **I^{+1} (Possible Low) (0.1):** The slight possibility (0.1) in this category reflects the fact that the measured DO is closer to the lower limit of the optimal range. However, the certainty of being truly low (below optimal) is also low.

Key Points:

- The measured DO (6.2 mg/L) is within the optimal range, with a strong certainty (0.8) based on the CBN Set.
- There is a slight possibility (0.1) of the DO being slightly lower than the ideal level, but the certainty is also low.
- The remaining categories (F^{+1} to F^{-2}) have zero membership degrees (0.0) because the measured value is far from critically low, low, supersaturated, or high DO conditions.

- **Utilizing Set Operations for Spatial Analysis:**

Now, imagine we have CBN Sets for pH and DO at multiple locations across the lake. We can use set operations to analyze the data spatially:

- **Intersection (M_pH ∩ M_DO):** This identifies areas where both pH and DO meet acceptable water quality standards simultaneously. A high intersection value indicates a location with a strong likelihood of good overall water quality.
- **Union (M_pH ∪ M_DO):** This highlights regions where either pH or DO (or potentially both) might fall outside the acceptable range, even with some uncertainty. A high union value suggests a potential water quality concern that needs further investigation.
- **Complement (M_pH^c or M_DO^c):** This helps pinpoint areas where any parameter deviates from optimal values. A high complement value for a specific category (e.g., Strongly

7. Representing Water Quality Parameters as CBN Sets

Essential water quality parameters like pH, turbidity, and dissolved oxygen can be effectively represented as CBN Sets. Each parameter's set would include membership degrees reflecting the certainty, possibility, and indeterminacy of the measured value falling within acceptable ranges for healthy water quality.

7.1. Example: Representing pH with CBN Sets

A CBN Set for pH might include membership degrees for:

Table 22: "CBN set for pH

Description	Membership Degree
Strongly acidic (low pH)	T^{+1} (Strongly Acidic)
Somewhat acidic (moderately low pH)	T^{+2} (Somewhat Acidic)
Neutral (optimal pH range)	T^{+1} (Neutral)
Somewhat basic (moderately high pH)	T^{+2} (Somewhat Basic)
Strongly basic (high pH)	T^{+1} (Strongly Basic)

The specific values for these membership degrees would be determined based on the measured pH value and established thresholds for optimal water quality. Similar CBN Sets can be created for other parameters like turbidity and dissolved oxygen.

Explanation of how a CBN set can be used to represent pH with membership degrees. Here is a specific numerical example to illustrate the concept further:

We measure the pH of a lake sample and find a value of 7.3.

Universe of Discourse:

- {< 6 (Strongly Acidic), 6-6.5 (Somewhat Acidic), 6.5-8 (Neutral), 8-8.5 (Somewhat Basic), > 8.5 (Strongly Basic)}

Reasoning behind Membership Degrees:

- Since optimal water quality for most aquatic life falls within the 6.5-8 pH range (Neutral), we will assign a higher membership degree (truth membership) to this category.
- We can consider a buffer zone on either side of the optimal range, assigning moderately high membership degrees (T^{+2}) for "Somewhat Acidic" (6-6.5) and "Somewhat Basic" (8-8.5).
- As we move further away from the optimal range, the membership degrees for "Strongly Acidic" (T^{+1}) and "Strongly Basic" (T^{+1}) become negligible (0.0) because these conditions are less likely for healthy water.
- We can also include "possible" categories (I^{+1}) with very low membership degrees to account for slight uncertainties in the measurement or natural fluctuations.

Example CBN set for pH:

This example demonstrates how CBN Sets, with their assigned membership degrees, can capture both the measured value and the inherent uncertainties in environmental data.

Table 23: Example CBN Set for pH

Category	Description	Membership Degree

T^{+1} (Neutral)	Strongly Matches Optimal pH Range (6.5-8)	0.7
T^{+2} (Somewhat Basic)	Moderately Leans Towards Slightly High pH	0.2
I^{+1} (Possibly Somewhat Acidic)	Slight Possibility of Being Below Optimal Range	0.1
F^{+1} (Strongly Acidic)	Negligible Certainty of Being Strongly Acidic	0.0
F^{+2} (Somewhat Acidic)	Very Low Certainty of Being Moderately Low pH	0.0
F^{-1} (Strongly Basic)	Negligible Certainty of Being Strongly Basic	0.0
F^{-2} (Somewhat Basic)	Very Low Certainty of Being Moderately High pH (beyond Somewhat Basic)	0.0

Interpretation:

This CBN Set effectively represents the measured pH value. Here's a breakdown of the information:

- There is a strong likelihood (0.7) of the pH being within the optimal range (6.5-8) for **most freshwater ecosystems**. This suggests a healthy environment for a variety of aquatic life.
- A moderate possibility (0.2) exists that the pH leans slightly towards the basic side. Depending on the specific tolerances of the resident organisms, this might require further investigation.

- A slight chance (0.1) exists that the pH might be slightly below the optimal range (acidic). However, the low certainty indicates minimal concern at this point.
- The very low membership degrees for extreme categories (Strongly Acidic/Basic) indicate negligible certainty of the pH being outside the acceptable range.

7.2. Sample Data Analysis with CBN Sets

Imagine we have CBN Sets representing pH, turbidity, and dissolved oxygen (DO) for three water samples (Table 4).

Here is a breakdown of the sample data analysis with CBN Sets (Table 4) using the provided information:

Table 24: Water Quality Data with CBN Sets

Sample	pH	Turbidity	DO
1	(0.8, 0.1, 0.05, 0.0, 0.05, 0.0, 0.0, 0.0, 0.05, 0.0, 0.9, 0.1)	(0.7, 0.2, 0.0, 0.1, 0.0, 0.0, 0.0, 0.0, 0.3, 0.2, 0.5, 0.0)	(0.9, 0.1, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.1, 0.0, 0.8, 0.2)
2	(0.6, 0.2, 0.1, 0.0, 0.1, 0.0, 0.0, 0.0, 0.0, 0.0, 1.0, 0.0)	(0.4, 0.3, 0.1, 0.1, 0.1, 0.0, 0.0, 0.0, 0.5, 0.2, 0.4, 0.0)	(0.7, 0.2, 0.0, 0.1, 0.0, 0.0, 0.0, 0.0, 0.3, 0.1, 0.6, 0.0)
3	(0.3, 0.2, 0.4, 0.1, 0.0, 0.0, 0.0, 0.0, 0.7, 0.2, 0.1, 0.0)	(0.2, 0.3, 0.4, 0.1, 0.0, 0.0, 0.0, 0.0, 0.8, 0.5, 0.0, 0.0)	(0.5, 0.3, 0.1, 0.0, **0.1** , 0.0, 0.0, 0.0, 0.4, 0.2, 0.3, 0.0)

Analysis:

- **Sample 1:**
  - **pH:** High certainty (0.8) of being within the optimal range, with a slight possibility (0.05) of being acidic.
  - **Turbidity:** Moderately high possibility (0.7) of being within acceptable limits, with a moderate chance (0.3) of exceeding them.
  - **DO:** Very strong certainty (0.9) of having optimal dissolved oxygen levels.
- **Sample 2:**
  - **pH:** Slightly leans towards the optimal range (0.6) with some uncertainty (0.2 towards acidic and 0.1 towards basic).

  - **Turbidity:** High possibility (0.7) of exceeding acceptable turbidity levels.
  - **DO:** Similar to Sample 1, with very strong certainty (0.7) of optimal DO levels.
- **Sample 3:**
  - **pH:** Moderately low certainty (0.3) for optimal pH, with significant possibilities (0.4) of being acidic.
  - **Turbidity:** Very high certainty (0.8) of exceeding acceptable turbidity levels.
  - **DO:** Moderate certainty (0.5) of being within the optimal range, but with a slight possibility (0.1) of being low (**highlighted value**).

Interpretation:

This analysis reveals:

- Sample 1 has the most favorable water quality conditions with a high likelihood of optimal pH, moderate turbidity, and very good DO levels.
- Sample 2 has a potential concern with turbidity exceeding acceptable levels, although pH and DO seem to be within a desirable range.
- Sample 3 has the most significant water quality issues. The pH is likely acidic, turbidity is very high, and DO levels show a slight possibility of being low.

Note: These interpretations are based on hypothetical thresholds for optimal water quality. In real-

7.3. Numerical Examples for Air Quality Monitoring with CBN Sets

- **Representing Air Quality Parameters as CBN Sets**

Example: Representing Ozone Levels with CBN Sets

Here is a breakdown of a CBN Set for ozone levels with assigned membership degrees:

Universe of Discourse: Parts per Billion (ppb)

- **Low Ozone (Safe):** T^{+1} (Low Ozone) - Ozone concentration <= healthy threshold (e.g., 50 ppb)
- **Moderately Elevated Ozone Levels:** T^{+2} (Moderately Elevated) - Between healthy and unhealthy thresholds (e.g., 51-70 ppb)
- **High Ozone Levels (Unhealthy):** T^{+1} (High Ozone) - Above unhealthy threshold (e.g., > 70 ppb)
- **Indeterminate Ozone Measurement:** I^{+1} (Indeterminate) - Sensor malfunction, unclear reading, etc.
- **Falsely Low Ozone Reading:** F^{+1} (Falsely Low) - Equipment issue resulting in underestimation of ozone concentration.
- **Assigning Membership Degrees:**

Imagine we measure an ozone concentration of 65 ppb. Here is a possible assignment:

Table 25: Ozone Concentration CBN Set (65 ppb)

Category	Description	Membership Degree
T^{+1} (Low Ozone)	Negligible Certainty (safe)	0.0
T^{+2} (Moderately Elevated)	High Certainty (unhealthy range)	0.8
T^{+1} (High Ozone)	Moderate Certainty (exceeding healthy threshold)	0.2
I^{+1} (Indeterminate)	No Indeterminacy	0.0
F^{+1} (Falsely Low)	Very Low Certainty (unlikely underestimation)	0.0

This table represents a CBN Set for a measured ozone concentration of 65 ppb. Here's a breakdown of the categories and membership degrees:

- **T^{+1} (Low Ozone) (0.0):** This category has a membership degree of 0.0, indicating **negligible certainty** of the ozone concentration being low (safe). Given the measured value (65 ppb) and the descriptions of other categories, this category likely applies to much lower ozone concentrations than 65 ppb.
- **T^{+2} (Moderately Elevated) (0.8):** This category has a high membership degree (0.8), indicating **strong certainty** that the ozone concentration falls within the moderately elevated range. The description can be rephrased to "**Likely in the unhealthy range**" based on the high membership degree.
- **T^{+1} (High Ozone) (0.2):** This category has a moderate membership degree (0.2), indicating **some certainty** that the ozone concentration might be exceeding the healthy threshold and reaching high ozone levels.
- **Sample Data Analysis with CBN Sets**

Table 26: (Example): Air Quality Data with CBN Sets

Station	Ozone (O_3)	PM2.5
1	(0.7, 0.2, 0.0, 0.1, 0.0, 0.0, 0.0, 0.0, 0.3, 0.1, 0.6, 0.0)	(0.8, 0.1, 0.0, 0.0, 0.1, 0.0, 0.0, 0.0, 0.1, 0.0, 0.8, 0.2)
2	(0.4, 0.3, 0.1, 0.1, 0.1, 0.0, 0.0, 0.0, 0.5, 0.2, 0.4, 0.0)	(0.6, 0.2, 0.1, 0.0, 0.0, 0.1, 0.0, 0.0, 0.4, 0.2, 0.5, 0.0)
3	(0.2, 0.1, 0.5, 0.2, 0.0, 0.0, 0.0, 0.0, 0.8, 0.5, 0.0, 0.0)	(0.5, 0.3, 0.1, 0.1, 0.0, 0.0, 0.0, 0.0, 0.5, 0.2, 0.3, 0.0)

Analysis of Table 26: Air Quality Data with CBN Sets

This table presents air quality data for Ozone (O_3) and PM2.5 at three stations, with each station having two sets of CBN membership degrees representing potentially duplicate measurements or different time points. Here's a breakdown of the information and potential interpretations:

Assuming Membership Degrees Represent Categories:

We can analyze the data assuming each set of membership degrees corresponds to specific CBN Set categories (though category descriptions are not provided). Here's a possible interpretation for each station:

- **Station 1:**
  - **Ozone (O_3):**
    - First row: High certainty (0.7) of **low ozone levels (likely safe)** and moderate possibility (0.2) of exceeding the ideal range.
    - Second row: Very high certainty (0.8) of **safe ozone levels** and low certainty (0.1) of exceeding the ideal range.
  - **PM2.5:**
    - Membership degrees are not shown, but the presence of a high value (likely in the T^{+1} category) in both rows suggests **very good air quality** for PM2.5.
- **Station 2:**
  - **Ozone (O_3):**
    - First row: Moderate certainty (0.4) of **moderately elevated ozone levels** and significant possibility (0.3) of exceeding healthy limits.

- Second row: High certainty (0.6) of **ozone levels within the acceptable range** and low certainty (0.1) of exceeding the ideal range.

- **PM2.5:**
  - Membership degrees are not shown, but the presence of a moderate value (likely in the T^{+2} category) in both rows suggests **a possibility of PM2.5 exceeding the ideal range, but with moderate certainty**.

- **Station 3:**
  - **Ozone (O_3):**
    - First row: Low certainty (0.2) of safe ozone levels and **significant possibility (0.5) of exceeding healthy ozone levels**.
    - Second row: Moderate certainty (0.5) of **ozone levels exceeding healthy limits** and significant possibility (0.3) of very high ozone levels.
  - **PM2.5:**
    - Membership degrees are not shown, but the presence of a high value (likely in the T^{+1} category) in the first row suggests good air quality. The second row's membership degrees are needed for a complete picture.

Important Considerations:

- **Missing Information:** The absence of category descriptions for the membership degrees limits a fully detailed analysis. Knowing the specific ranges and descriptions associated with each category (e.g., T^{+1} for ozone - safe vs. moderately elevated) would provide clearer interpretations.
- **Multiple Measurements:** It's unclear if each row represents duplicate measurements or data from different time points. If they are different time points, then the analysis would need to consider the variability within each station.

8. (Enhanced) Air Quality Analysis with CBN Sets: Operations, Distances, and Similarities

8.1. Some numerical examples to illustrate the concepts of set operations, distances, and similarities in air quality analysis using CBN Sets

We can leverage set operations like intersection, union, and complement to gain valuable insights from CBN Sets for air quality analysis.

We have CBN Sets representing ozone (O_3) and PM2.5 levels at three air quality-monitoring stations

Table 27: CBN sets for ozone (O_3) and PM2.5 levels at three stations

Station	Ozone (O_3)	PM2.5

1	(0.7, 0.2, 0.0, 0.1, 0.0, 0.0, 0.0, 0.0, 0.3, 0.1, 0.6, 0.0)	(0.8, 0.1, 0.0, 0.0, 0.1, 0.0, 0.0, 0.0, 0.1, 0.0, 0.8, 0.2)
2	(0.4, 0.3, 0.1, 0.1, 0.1, 0.0, 0.0, 0.0, 0.5, 0.2, 0.4, 0.0)	(0.6, 0.2, 0.1, 0.0, 0.0, 0.1, 0.0, 0.0, 0.4, 0.2, 0.5, 0.0)
3	(0.2, 0.1, 0.5, 0.2, 0.0, 0.0, 0.0, 0.0, 0.8, 0.5, 0.0, 0.0)	(0.5, 0.3, 0.1, 0.1, 0.0, 0.0, 0.0, 0.0, 0.5, 0.2, 0.3, 0.0)

- **Leveraging Set Operations for Air Quality Insights**

Intersection: Let us define acceptable thresholds for ozone (e.g., T^{+1} for "Low Ozone") and PM2.5 (e.g., T^{+1} for "Good PM2.5"). The intersection of CBN Sets for Stations 1 and 2 might be a set with high membership degrees (around 0.7-0.8) for these categories, indicating a good chance of both stations having acceptable air quality.

- **Union:** The union of CBN Sets for ozone across all stations might include membership degrees for "Moderately Elevated" and "High Ozone" due to elevated levels at Stations 2 and 3, highlighting potential air quality concerns in the broader region.
- **Complement:** The complement of the CBN Set representing "Good Air Quality" for Station 3 might have high membership degrees for "High Ozone" and "Moderately High PM2.5" categories, pinpointing potential issues at that location.

8.2. Quantifying Relationships with Distance Measures and Similarity Coefficients

- **Distance Measures:**
  - **Euclidean Distance:** We can calculate the Euclidean distance between the CBN Sets for Stations 1 and 2. This distance considers the differences in membership degrees for all categories (truth, indeterminacy, falsity) and reflects the dissimilarity in their air quality profiles.

Calculating Euclidean Distance:

1. **Square the differences:** For each corresponding membership degree (truth, indeterminacy, falsity) category in the two CBN Sets (Station 1 and Station 2), calculate the squared difference between their membership degrees.

2. **Sum the squared differences:** Add the squared differences obtained in step 1 for all categories.
3. **Take the square root:** Finally, calculate the square root of the sum obtained in step 2. This will be the Euclidean distance between the two CBN Sets.

Interpretation:

A higher Euclidean distance indicates greater dissimilarity between the air quality profiles of the two stations. The membership degrees encode the certainty levels for various air quality conditions (e.g., low ozone, moderately elevated ozone). Therefore, the Euclidean distance captures the overall difference in these certainty levels between the stations.

Example:

Let us consider simplified CBN Sets for Station 1 and Station 2 with only three categories (Low, Moderate, High) and their corresponding membership degrees:

- Station 1: (0.8, 0.1, 0.1) - High certainty (0.8) of low air pollution, low certainty (0.1) each for moderate and high pollution.
- Station 2: (0.2, 0.5, 0.3) - Low certainty (0.2) of low pollution, moderate certainty (0.5) of moderate pollution, and moderate certainty (0.3) of high pollution.

Calculating the distance:

1. Squared differences: [(0.8-0.2)^2, (0.1-0.5)^2, (0.1-0.3)^2] = [0.36, 0.16, 0.04]
2. Sum of squared differences: 0.36 + 0.16 + 0.04 = 0.56
3. Euclidean distance: √0.56 ≈ 0.75

In this example, the Euclidean distance of approximately 0.75 indicates a moderate dissimilarity between the air quality profiles of the two stations. Station 1 has a higher certainty of low pollution, while Station 2 has a higher certainty of moderate and high pollution levels.

By calculating Euclidean distances between CBN Sets for multiple stations, we can identify patterns and potential pollution hotspots that require further investigation.

Manhattan Distance:

1. **Absolute differences:** For each corresponding membership degree category (truth, indeterminacy, falsity) in the two CBN Sets (Stations 2 and 3 in this case), calculate the absolute difference between their membership degrees.
2. **Sum the absolute differences:** Add the absolute differences obtained in step 1 for all categories.

Interpretation:

Similar to Euclidean distance, a higher Manhattan distance indicates greater dissimilarity between the air quality profiles. However, Manhattan distance focuses on the **sum of absolute differences** rather than squared differences. This means it emphasizes larger discrepancies between membership degrees, potentially giving more weight to significant variations in certainty levels.

Example:

Let us reuse the simplified CBN Sets from the previous example for Station 2: (0.2, 0.5, 0.3) and create a hypothetical CBN Set for Station 3: (0.1, 0.4, 0.5).

Calculating Manhattan distance:

1. Absolute differences: [|0.2-0.1|, |0.5-0.4|, |0.3-0.5|] = [0.1, 0.1, 0.2]
2. Sum of absolute differences: 0.1 + 0.1 + 0.2 = 0.4

In this example, the Manhattan distance is 0.4. Compared to the Euclidean distance of 0.75 between Station 1 and Station 2, this smaller Manhattan distance suggests a potentially **weaker** difference between Stations 2 and 3.

Choosing Between Euclidean and Manhattan Distances:

- **Euclidean distance** might be preferable when emphasizing larger variations in certainty levels between corresponding categories. Squaring the differences magnifies the impact of significant discrepancies.
- **Manhattan distance** is more sensitive to the **number of categories** with non-zero membership degrees. Even small absolute differences can contribute more to the overall distance if there are many categories.

The choice between these distances depends on the specific analysis goals and the importance you place on highlighting substantial variations in membership degrees.

In conclusion, both Euclidean and Manhattan distances offer valuable insights into the dissimilarity between air quality profiles represented by CBN Sets. By considering these distances alongside set operations like intersection and union, we can gain a comprehensive understanding of air quality variations across monitoring stations.

- **Similarity Coefficients:**
  - **Jaccard Similarity for Dissimilarity in Air Quality**

As you mentioned, Jaccard Similarity is a metric that quantifies the similarity between two sets by focusing on the proportion of elements they share. In the case of CBN Sets, these elements represent the membership degrees for various air quality categories (e.g., low ozone, moderate ozone). A lower Jaccard Similarity value indicates a lesser degree of similarity in the air quality profiles of the stations, potentially due to differences in specific parameters like ozone levels.

Calculation Steps:

1. **Intersection:** Find the sum of the **minimum membership degrees** for each corresponding category (truth, indeterminacy, falsity) between the CBN Sets for Stations 1 and 3.
2. **Union:** Find the sum of the **maximum membership degrees** for each corresponding category between the two stations' CBN Sets.

3. **Jaccard Similarity:** Divide the intersection (step 1) by the union (step 2).

Interpretation:

The Jaccard Similarity value will range from 0 to 1:

- **1**: Perfect similarity (both stations have identical CBN Sets).
- **0**: No overlap in membership degrees (completely dissimilar air quality profiles).
- **Values closer to 0**: Lower degree of similarity, potentially due to significant differences in ozone levels or other air quality parameters.

Example (assuming simplified CBN Sets with three categories):

- Station 1: CBN Set (0.8, 0.1, 0.1) - High certainty (0.8) for low ozone, low certainty (0.1) each for moderate and high ozone.
- Station 3: CBN Set (0.2, 0.5, 0.3) - Low certainty (0.2) for low ozone, moderate certainty (0.5) for moderate ozone, and moderate certainty (0.3) for high ozone.

Calculating Jaccard Similarity:

1. Intersection: Minimums = (0.8, min(0.1, 0.5), min(0.1, 0.3)) = (0.8, 0.1, 0.1) ; Sum of minimums = 1.
2. Union: Maximums = (0.8, max(0.1, 0.5), max(0.1, 0.3)) = (0.8, 0.5, 0.3) ; Sum of maximums = 1.6.
3. Jaccard Similarity: 1 (intersection) / 1.6 (union) ≈ 0.625.

Interpretation of the Example:

The Jaccard Similarity of approximately 0.625 indicates a moderate degree of similarity between Stations 1 and 3. While Station 1 has a higher certainty of low ozone levels, there's some overlap in certainty levels for the "Low" and "Moderate" ozone categories between the two stations. This suggests that ozone might not be a major differentiating factor between their air quality profiles.

Jaccard Similarity vs. Other Distances:

It's important to remember that Jaccard Similarity focuses on the **proportion** of shared membership degrees, whereas metrics like Euclidean and Manhattan distances consider the **magnitude** of differences. Jaccard Similarity is a valuable tool when the presence or absence of overlap in membership degrees is more important than the extent of those differences.

By incorporating Jaccard Similarity alongside other analytical techniques, we can gain a comprehensive understanding of similarities and dissimilarities in air quality data represented by CBN Sets at different monitoring stations.

Jaccard Similarity for CBN Sets:

Key Points about Jaccard Similarity:

- **Focuses on Shared Memberships:** Jaccard Similarity emphasizes the **proportion of membership degrees** that two CBN Sets have in common. This considers both categories where stations share similar certainty levels (overlap) and categories where their certainty levels differ (non-overlap).

- **Interpretation:** The Jaccard Similarity value ranges from 0 to 1.
    - **1**: Indicates perfect similarity (identical CBN Sets).
    - **0**: Indicates no overlap in membership degrees (completely dissimilar air quality profiles).
    - **Values closer to 0**: Suggest a lesser degree of similarity, potentially due to differences in air quality parameters like ozone levels.

Jaccard Similarity Calculation:

1. **Intersection:** Find the sum of the **minimum membership degrees** for each corresponding category (truth, indeterminacy, falsity) between the two CBN Sets.
2. **Union:** Find the sum of the **maximum membership degrees** for each corresponding category between the two CBN Sets.
3. **Jaccard Similarity:** Divide the intersection (step 1) by the union (step 2).

Example:

Consider simplified CBN Sets with three categories (Low, Moderate, High) for Station 1 (0.8, 0.1, 0.1) and Station 3 (0.2, 0.5, 0.3).

Calculating Jaccard Similarity:

- Intersection: Minimums = (0.8, min(0.1,0.5), min(0.1,0.3)) = (0.8, 0.1, 0.1) ; Sum = 1.
- Union: Maximums = (0.8, max(0.1,0.5), max(0.1,0.3)) = (0.8, 0.5, 0.3) ; Sum = 1.6.
- Jaccard Similarity: 1 (intersection) / 1.6 (union) ≈ 0.625.

Interpretation of the Example:

The Jaccard Similarity of 0.625 indicates a moderate degree of similarity between Stations 1 and 3. Even though Station 1 has a higher certainty of low ozone, there's some overlap in certainty levels for the "Low" and "Moderate" categories, suggesting ozone might not be a significant differentiating factor.

Jaccard Similarity vs. Other Distances:

- **Focus on Proportion vs. Magnitude:** Jaccard Similarity focuses on the **proportion** of shared memberships, while Euclidean and Manhattan distances consider the **magnitude** of differences in membership degrees.
- **When to Use Jaccard Similarity:** It's valuable when the presence or absence of overlap in membership degrees is more important than the extent of those differences.

By incorporating Jaccard Similarity alongside other techniques, we can gain a comprehensive understanding of similarities and dissimilarities in air quality data from CBN Sets at various monitoring stations.

9. Calculation:

1. **Intersection:** Find the sum of the minimum membership degrees for each corresponding category (truth, indeterminacy, falsity) between the two CBN Sets (Stations 1 and 3 in this case).
2. **Union:** Find the sum of the maximum membership degrees for each corresponding category between the two CBN Sets.
3. **Jaccard Similarity:** Divide the intersection (step 1) by the union (step 2).

Interpretation:

- The Jaccard Similarity ranges from 0 to 1.
- A value of 1 indicates that the CBN Sets are identical (perfect similarity).
- A value of 0 indicates no overlap in membership degrees (complete dissimilarity).
- Lower values (closer to 0) suggest a lesser degree of similarity, potentially due to differences in specific air quality parameters (like ozone levels in your example).

Example:

Let's again consider simplified CBN Sets with three categories (Low, Moderate, High) for Station 1 (0.8, 0.1, 0.1) and Station 3 (0.2, 0.5, 0.3).

Calculating Jaccard Similarity:

1. Intersection: Minimums = (0.8, min(0.1,0.5), min(0.1,0.3)) = (0.8, 0.1, 0.1) ; Sum of minimums = 1.
2. Union: Maximums = (0.8, max(0.1,0.5), max(0.1,0.3)) = (0.8, 0.5, 0.3) ; Sum of maximums = 1.6.
3. Jaccard Similarity: 1 (intersection) / 1.6 (union) ≈ 0.625.

In this example, the Jaccard Similarity of 0.625 indicates a moderate degree of similarity between the air quality profiles of Stations 1 and 3. While Station 1 has a higher certainty of low pollution, Station 3 has some overlap in certainty levels for the "Low" and "Moderate" categories.

Jaccard Similarity vs. Other Distances:

- Jaccard Similarity focuses on the **proportion** of shared memberships, whereas Euclidean and Manhattan distances consider the **magnitude** of differences.
- It can be useful when the presence or absence of overlap in membership degrees is more important than the extent of those differences.

By incorporating Jaccard Similarity alongside Euclidean and Manhattan distances, we can gain a more nuanced understanding of similarities and dissimilarities between air quality profiles at different stations.

  - **Dice Coefficient:** The Dice Coefficient is indeed a strong candidate for highlighting the potential overlap in exceeding acceptable ozone levels between Stations 2 and 3, and it might even be higher than the Jaccard Similarity in this scenario. Here is why:

Dice Coefficient for CBN Sets:

The Dice Coefficient emphasizes **agreements** between CBN Sets more than Jaccard Similarity. It focuses on the **sum of twice the minimum membership degrees** for each corresponding category (truth, indeterminacy, falsity) between the two stations.

Why Might Dice Coefficient be Higher?

- **Emphasis on Overlap:** By doubling the minimum membership degrees, the Dice Coefficient gives more weight to categories where both stations have some level of membership, even if the certainty levels differ.
- **Potential Ozone Level Similarity:** In your example, both Stations 2 and 3 have a possibility of exceeding acceptable ozone levels. This means there might be some overlap in their membership degrees for the "Moderately Elevated" or "High" ozone categories, even if they differ in certainty levels.
- **Higher Weighting:** The Dice Coefficient places more weight on this overlap compared to Jaccard Similarity, potentially leading to a higher value.

Calculation:

1. **Minimum membership degrees:** Find the minimum membership degree for each corresponding category between the two CBN Sets.
2. **Dice Coefficient:** Multiply each minimum membership degree by 2 and add them up.

Interpretation:

- The Dice Coefficient ranges from 0 to 1, similar to Jaccard Similarity.
- **1**: Perfect similarity (identical CBN Sets).
- **0**: No overlap in membership degrees (completely dissimilar air quality profiles).
- **Values closer to 1**: Higher degree of similarity, potentially due to shared possibility of exceeding ozone limits or overlap in other categories.

Example (assuming simplified CBN Sets):

- Station 2: CBN Set (0.4, 0.5, 0.1)
- Station 3: CBN Set (0.2, 0.4, 0.5)

Calculating Dice Coefficient:

1. Minimum membership degrees: (0.4, min(0.5, 0.4), min(0.1, 0.5)) = (0.4, 0.4, 0.1).
2. Dice Coefficient: (2 * 0.4) + (2 * 0.4) + (2 * 0.1) = 1.8.

Note: The Dice Coefficient can be greater than 1 due to the doubling of membership degrees. We often normalize it by dividing by 2, resulting in a value between 0 and 1. In this example, the normalized value would be 0.9.

Comparison with Jaccard Similarity:

We haven't calculated the Jaccard Similarity yet, but because it only sums the minimum membership degrees (without doubling), it's likely to be lower than the Dice Coefficient in this case. The emphasis on overlap in the Dice Coefficient captures the possibility of both stations exceeding ozone limits, even if their certainty levels differ.

By considering both the Dice Coefficient and Jaccard Similarity, we gain a more nuanced understanding of the similarity between Stations 2 and 3. The Dice Coefficient highlights the potential for exceeding ozone limits at both stations, while Jaccard Similarity provides a broader measure of overall agreement in membership degrees.

Combining Set Operations, Distances, and Similarities:

- **Combining Techniques for Air Quality Analysis:**

2. **Intersection:** Identify Stations of Concern:
   - Define acceptable thresholds for each air quality category (e.g., T^{+1} for "Good PM2.5").
   - The intersection of CBN Sets for multiple stations will reveal stations with membership degrees below these thresholds, potentially indicating air quality concerns. For example, a low membership degree in the "Good" category for Station 3 in your example suggests potential issues.
3. **Distances or Similarities for Spatial Context:**
   - Once you identify stations of concern, calculate the distance or similarity between their CBN Sets and nearby stations.
   - **Distances (Euclidean, Manhattan):**
     - A high distance between a station with air quality concerns (e.g., Station 3) and nearby stations (Stations 1 and 2) might suggest a localized issue. The larger the distance, the greater the dissimilarity in air quality profiles.
   - **Similarities (Jaccard, Dice):**
     - A high similarity between a station with concerns and nearby stations suggests a more widespread issue. The higher the similarity, the more the air quality profiles share characteristics.

Example:

- Station 3 has a low membership degree for "Good Air Quality" (potential concern).
- Stations 1 and 2 have high similarity with Station 3 (based on Jaccard or Dice Coefficient).
- Station 3 has a large Euclidean distance compared to Stations 1 and 2.

Interpretation:

While Stations 1, 2, and 3 share similar air quality characteristics, the large Euclidean distance between Station 3 and the others suggests the issue might be localized to Station 3. The high similarity could indicate a common source of pollution affecting a broader region, but Station 3 experiences it more intensely.

Additional Considerations:

- **Visualizations:** Creating maps with color gradients representing membership degrees or distances can provide a clear spatial understanding of air quality variations.
- **Wind Data:** Wind direction and speed data can be integrated to understand potential pollutant transport and how localized issues might affect nearby areas.

By combining set operations, distances, similarities, and potentially other relevant data, we can gain a comprehensive understanding of air quality patterns, identify areas with potential concerns, and assess the spatial extent of air quality issues.

10. Conclusion

Cubic Bipolar Neutrosophic Sets (CBN Sets) have emerged as a powerful tool for environmental data analysis, particularly in scenarios where uncertainty is inherent. This concluding section summarizes the key takeaways and highlights the potential of CBN Sets in environmental science.

Key Takeaways:

- CBN Sets effectively represent environmental data by incorporating truth, indeterminacy, and falsity membership degrees, acknowledging inherent uncertainties in measurements.
- Set operations (intersection, union, complement) on CBN Sets enable researchers to manipulate and analyze environmental data while considering these uncertainties.
- Distance measures and similarity coefficients quantify the relationships between air quality profiles of different monitoring stations, revealing spatial patterns and potential pollutant sources.

Potential of CBN Sets in Environmental Science:

- **Refined Information:** CBN Sets allow for a more nuanced understanding of environmental phenomena by accounting for uncertainties in data.
- **Improved Decision-Making:** By providing a clearer picture of environmental conditions with associated certainty levels, CBN Sets can inform better decision-making for environmental management and resource allocation.
- **Enhanced Trend Analysis:** Analyzing CBN Sets over time allows for the identification of trends in environmental parameters, facilitating proactive measures to address potential environmental issues.
- **Unified Framework:** CBN Sets offer a versatile framework applicable to various environmental domains beyond water and air quality, including soil contamination analysis, biodiversity monitoring, and climate change studies.

Future Directions:

The application of CBN Sets in environmental science is a burgeoning field. Future research can explore:

- Developing advanced set operations and similarity measures specifically tailored for environmental data analysis.
- Integrating CBN Sets with machine learning algorithms for automated environmental data analysis and pattern recognition.
- Implementing CBN Sets in environmental modeling and simulation for more robust predictions under uncertain conditions.

In conclusion, CBN Sets hold immense potential for revolutionizing environmental data analysis by embracing the inherent uncertainties in environmental measurements. As research and applications in this field continue to evolve, CBN Sets can become a cornerstone for achieving a more comprehensive understanding and improved management of our planet's health.

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