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The extended study of 2-refined neutrosophic numbers

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Abstract: Numerical roots play a crucial role in real or complex equations, as well as in various mathematical models involving differentiation, integration, and other mathematical relationships. In the realm of mathematics, there is a growing trend of utilizing new formulas that incorporate non-classical numbers, such as neutrosophic and refined neutrosophic numbers. The objective of this research is to establish precise and comprehensive mathematical procedures for dealing with refined neutrosophic roots within mathematical formulas, be it equations or other mathematical constructs. This paper presents an extensive study on 2-refined neutrosophic numbers, focusing on the square root of a 2-refined neutrosophic real or complex number. Additionally, this work introduces the concept of 2-refined neutrosophic real or complex polynomials and explores the process of finding the refined neutrosophic roots to solve 2-refined neutrosophic equations. To illustrate these concepts, several examples have been provided.

Keywords: refined neutrosophic; square root; complex polynomial; real polynomial.

1. Introduction and Preliminaries

To describe a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, and contradiction, Smarandache suggested the neutrosophic Logic as an alternative to the current logics. Smarandache made refined neutrosophic numbers available in the following form: $(a, b_1I_1, b_2I_2, \dots, b_nI_n)$ where $a, b_1, b_2, \dots, b_n \in R \text{ or } C$ [1]

Agboola introduced the concept of refined neutrosophic algebraic structures [2]. Also, the refined neutrosophic rings I was studied in paper [3], where it assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. It then follows logically that: [3]

$$I_1I_1 = {I_1}^2 = I_1 \qquad (1)$$

$$I_2I_2 = {I_2}^2 = I_2 \qquad (2)$$

$$I_1I_2 = I_2I_1 = I_1 \qquad (3)$$

In addition, there are many papers presenting studies on refined neutrosophic numbers [4-5-6-7-8-12-13-14-15] and Mehmet Celik and Ahmed Hatip presented a study on the refined ah-isometry and its applications in refined neutrosophic surfaces. Smarandache discussed neutrosophic indefinite integral (Refined Indeterminacy) [11]

Alhasan also presented several papers on calculus, in which he discussed neutrosophic definite and indefinite integrals. He also presented the most important applications of definite integrals in neutrosophic logic [9-10].

This paper dealt with several topics, in the first part of which introduction and preliminaries were presented, and in the main discussion part the 2-refined neutrosophic numbers that contain two part of indeterminacy (I_1, I_2). In the last part, a conclusion to the paper is given.

2. Main Discussion

2.1 The square root of a 2-refined neutrosophic real number

First, let's compute the square root: $\sqrt{a + bI_1 + cI_2}$ whereas $a \geq 0, a + c \geq 0, a + b + c \geq 0$, while $I_1, I_2 =$ indeterminacy

$$\sqrt{a + bI_1 + cI_2} = x + yI_1 + zI_2$$

Where x, y *and* z are real unknowns.

By raise both sides to the second power, we get:

$$a + bI_1 + cI_2 = (x + yI_1 + zI_2)^2$$

$$a + bI_1 + cI_2 = (x + yI_1)^2 + 2(x + yI_1)(zI_2) + (zI_2)^2$$

$$a + bI_1 + cI_2 = x^2 + 2xyI_1 + (yI_1)^2 + 2(x + yI_1)(zI_2) + (zI_2)^2$$

$$a + bI_1 + cI_2 = x^2 + 2xyI_1 + y^2I_1 + 2xzI_2 + 2yzI_1 + z^2I_2$$

$$a + bI_1 + cI_2 = x^2 + (y^2 + 2xy + 2yz)I_1 + (z^2 + 2xz)I_2$$

Whence:

$$\begin{cases} x^2 = a \\ y^2 + 2xy + 2yz = b \\ z^2 + 2xz = c \end{cases}$$

$$\Rightarrow \begin{cases} x = \pm\sqrt{a} \\ y^2 + 2xy + 2yz = b \quad (1) \\ z^2 + 2xz = c \quad (2) \end{cases}$$

Case1: $x = \sqrt{a}$ by substitution in (2)

$$z^2 + 2\sqrt{a}\ z - c = 0$$

$$\Delta = 4a + 4c$$

Then:

$$\begin{cases} z = \dfrac{-2\sqrt{a} + 2\sqrt{a+c}}{2} = -\sqrt{a} + \sqrt{a+c} \\ z = \dfrac{-2\sqrt{a} - 2\sqrt{a+c}}{2} = -\sqrt{a} - \sqrt{a+c} \end{cases}$$

- For $z = -\sqrt{a} + \sqrt{a+c}$ we substitute in (1)

$$y^2 + 2\sqrt{a}y + 2\left(-\sqrt{a} + \sqrt{a+c}\right)y - b = 0$$

$$y^2 + 2\sqrt{a+c}\ y - b = 0$$

$$\Delta = 4(a+c) + 4b$$

Then:

$$\begin{cases} y = \dfrac{-2\sqrt{a+c} + 2\sqrt{a+c+b}}{2} = -\sqrt{a+c} + \sqrt{a+c+b} \\ y = \dfrac{-2\sqrt{a+c} - 2\sqrt{a+c+b}}{2} = -\sqrt{a+c} - \sqrt{a+c+b} \end{cases}$$

Hence:

$$\sqrt{a+bI_1+cI_2} = \sqrt{a} + \left(-\sqrt{a+c} + \sqrt{a+c+b}\right)I_1 + \left(-\sqrt{a} + \sqrt{a+c}\right)I_2$$

Or:

$$= \sqrt{a} + \left(-\sqrt{a+c} - \sqrt{a+c+b}\right)I_1 + \left(-\sqrt{a} + \sqrt{a+c}\right)I_2$$

- For $z = -\sqrt{a} - \sqrt{a+c}$ we substitute in (1)

$$y^2 + 2\sqrt{a}y + 2\left(-\sqrt{a} - \sqrt{a+c}\right)y - b = 0$$

$$y^2 - 2\sqrt{a+c}\ \ y - b = 0$$

$$\Delta = 4(a+c) + 4b$$

Then:

$$\begin{cases} y = \dfrac{2\sqrt{a+c} + 2\sqrt{a+c+b}}{2} = \sqrt{a+c} + \sqrt{a+c+b} \\ y = \dfrac{2\sqrt{a+c} - 2\sqrt{a+c+b}}{2} = \sqrt{a+c} - \sqrt{a+c+b} \end{cases}$$

Hence:

$$\sqrt{a+bI_1+cI_2} = \sqrt{a} + \left(\sqrt{a+c} + \sqrt{a+c+b}\right)I_1 + \left(-\sqrt{a} - \sqrt{a+c}\right)I_2$$

Or:

$$= \sqrt{a} + \left(\sqrt{a+c} - \sqrt{a+c+b}\right)I_1 + \left(-\sqrt{a} - \sqrt{a+c}\right)I_2$$

Case2: $x = -\sqrt{a}$ by substitution in (2)

$$z^2 - 2\sqrt{a}\ \ z - c = 0$$

$$\Delta = 4a + 4c$$

Then:

$$\begin{cases} z = \dfrac{2\sqrt{a} + 2\sqrt{a+c}}{2} = \sqrt{a} + \sqrt{a+c} \\ z = \dfrac{2\sqrt{a} - 2\sqrt{a+c}}{2} = \sqrt{a} - \sqrt{a+c} \end{cases}$$

- For $z = \sqrt{a} + \sqrt{a+c}$ we substitute in (1)

$$y^2 - 2\sqrt{a}y + 2\left(\sqrt{a} + \sqrt{a+c}\right)y - b = 0$$

$$y^2 + 2\sqrt{a+c}\ \ y - b = 0$$

$$\Delta = 4(a+c) + 4b$$

Then:

$$\begin{cases} y = \dfrac{-2\sqrt{a+c} + 2\sqrt{a+c+b}}{2} = -\sqrt{a+c} + \sqrt{a+c+b} \\ y = \dfrac{-2\sqrt{a+c} - 2\sqrt{a+c+b}}{2} = -\sqrt{a+c} - \sqrt{a+c+b} \end{cases}$$

Hence:

$$\sqrt{a+bI_1+cI_2} = -\sqrt{a} + \left(-\sqrt{a+c} + \sqrt{a+c+b}\right)I_1 + \left(\sqrt{a} + \sqrt{a+c}\right)I_2$$

Or:

$$= -\sqrt{a} + \left(-\sqrt{a+c} - \sqrt{a+c+b}\right)I_1 + \left(\sqrt{a} + \sqrt{a+c}\right)I_2$$

- For $z = \sqrt{a} - \sqrt{a+c}$ we substitute in (1)

$$y^2 - 2\sqrt{a}y + 2\left(\sqrt{a} - \sqrt{a+c}\right)y - b = 0$$

$$y^2 - 2\sqrt{a+c}\;\; y - b = 0$$

$$\Delta = 4(a+c) + 4b$$

Then:

$$\begin{cases} y = \dfrac{2\sqrt{a+c} + 2\sqrt{a+c+b}}{2} = \sqrt{a+c} + \sqrt{a+c+b} \\ y = \dfrac{2\sqrt{a+c} - 2\sqrt{a+c+b}}{2} = \sqrt{a+c} - \sqrt{a+c+b} \end{cases}$$

Hence:

$$\sqrt{a+bI_1+cI_2} = -\sqrt{a} + \left(\sqrt{a+c} + \sqrt{a+c+b}\right)I_1 + \left(\sqrt{a} - \sqrt{a+c}\right)I_2$$

Or:

$$= -\sqrt{a} + \left(\sqrt{a+c} - \sqrt{a+c+b}\right)I_1 + \left(\sqrt{a} - \sqrt{a+c}\right)I_2$$

The eight results are:

$$(x,y,z) = \left(\sqrt{a}\;, -\sqrt{a+c} + \sqrt{a+c+b}\;, -\sqrt{a} + \sqrt{a+c}\right), \left(\sqrt{a}\;, -\sqrt{a+c} - \sqrt{a+c+b}\;, -\sqrt{a} + \sqrt{a+c}\right)$$

$$\left(\sqrt{a}\;, \sqrt{a+c} + \sqrt{a+c+b}\;, -\sqrt{a} - \sqrt{a+c}\right), \left(\sqrt{a}\;, \sqrt{a+c} - \sqrt{a+c+b}\;, -\sqrt{a} - \sqrt{a+c}\right)$$

$$\left(-\sqrt{a}\;, -\sqrt{a+c} + \sqrt{a+c+b}\;, \sqrt{a} + \sqrt{a+c}\right), \left(-\sqrt{a}\;, -\sqrt{a+c} - \sqrt{a+c+b}\;, \sqrt{a} + \sqrt{a+c}\right)$$

$$\left(-\sqrt{a}\;, \sqrt{a+c} + \sqrt{a+c+b}\;, \sqrt{a} - \sqrt{a+c}\right), \left(-\sqrt{a}\;, \sqrt{a+c} - \sqrt{a+c+b}\;, \sqrt{a} - \sqrt{a+c}\right)$$

Because we are now calculating the square root of a 2-refined neutrosophic number (according to classical analysis), we only take the result with a positive value, hence:

$$\sqrt{a+bI_1+cI_2} = \sqrt{a} + \left(-\sqrt{a+c} + \sqrt{a+c+b}\right)I_1 + \left(-\sqrt{a} + \sqrt{a+c}\right)I_2$$

Clearly: $\sqrt{a} \geq 0\;\;, -\sqrt{a+c} + \sqrt{a+c+b} \geq 0\;\; and\;\; -\sqrt{a} + \sqrt{a+c} \geq 0$

However, when we solve the 2-refined neutrosophic equation, we take all eight results.

Example 1

Let's find: $\sqrt{9+9I_1+7I_2}$

$$\sqrt{9+9I_1+7I_2}=x+yI_1+zI_2$$

By raise both sides to the second power, we get:

$$9+9I_1+7I_2=(x+yI_1+zI_2)^2$$

$$9+9I_1+7I_2=(x+yI_1)^2+2(x+yI_1)(zI_2)+(zI_2)^2$$

$$9+9I_1+7I_2=x^2+2xyI_1+(yI_1)^2+2(x+yI_1)(zI_2)+(zI_2)^2$$

$$9+9I_1+7I_2=x^2+2xyI_1+y^2I_1+2xzI_2+2yzI_1+z^2I_2$$

$$9+9I_1+7I_2=x^2+(y^2+2xy+2yz)I_1+(z^2+2xz)I_2$$

Whence:

$$\begin{cases} x^2=9 \\ y^2+2xy+2yz=9 \\ z^2+2xz=7 \end{cases}$$

$$\Rightarrow \begin{cases} x=\pm 3 \\ y^2+2xy+2yz=9 \quad (1) \\ z^2+2xz=7 \quad (2) \end{cases}$$

Case1: $x=3$ by substitution in (2)

$$z^2+6z-7=0$$
$$(z-1)(z+7)=0$$

Then:

$$\begin{cases} z=1 \\ z=-7 \end{cases}$$

- For $z=1$ we substitute in (1)

$$y^2+8y-9=0$$
$$(y+9)(y-1)=0$$

Then:

$$\begin{cases} y=1 \\ y=-9 \end{cases}$$

Hence:

$$\sqrt{9+9I_1+7I_2}=3+I_1+I_2 \quad \text{(Accepted)}$$

Or:

$$=3-9I_1+I_2 \quad \text{(Rejected)}$$

- For $z=-7$ we substitute in (1)

$$y^2-8y-9=0$$
$$(y-9)(y+1)=0$$

Then:

$$\begin{cases} y=9 \\ y=-1 \end{cases}$$

Hence:

$$\sqrt{9+9I_1+7I_2}=-3+9I_1-7I_2 \quad \text{(Rejected)}$$

Or:

$$=-3-I_1-7I_2 \quad \text{(Rejected)}$$

Case2: $x = -3$ by substitution in (2)

$$z^2 - 6z - 7 = 0$$
$$(z-7)(z+1) = 0$$

Then:

$$\begin{cases} z = 7 \\ z = -1 \end{cases}$$

- For $z = 7$ we substitute in (1)

$$y^2 + 8y - 9 = 0$$

$$(y-1)(y+9) = 0$$

Then:

$$\begin{cases} y = 1 \\ y = -9 \end{cases}$$

Hence:

$$\sqrt{9 + 9I_1 + 7I_2} = -3 + I_1 + 7I_2 \text{ (Rejected)}$$

Or:

$$= -3 - 9I_1 + 7I_2 \text{ (Rejected)}$$

- For $z = -1$ we substitute in (1)

$$y^2 - 8y - 9 = 0$$
$$(y-9)(y+1) = 0$$

$$\begin{cases} y = 9 \\ y = -1 \end{cases}$$

Hence:

$$\sqrt{9 + 9I_1 + 7I_2} = -3 + 9I_1 - I_2 \text{ (Rejected)}$$

Or:

$$= -3 - I_1 - I_2 \text{ (Rejected)}$$

Remarks

As a particular case we can compute $\sqrt{I_1 + I_2}$

$$\sqrt{I_1 + I_2} = x + yI_1 + zI_2$$

By raise both sides to the second power, we get:

$$I_1 + I_2 = (x + yI_1 + zI_2)^2$$

$$I_1 + I_2 = (x + yI_1)^2 + 2(x + yI_1)(zI_2) + (zI_2)^2$$

$$I_1 + I_2 = x^2 + 2xyI_1 + (yI_1)^2 + 2(x + yI_1)(zI_2) + (zI_2)^2$$

$$I_1 + I_2 = x^2 + 2xyI_1 + y^2I_1 + 2xzI_2 + 2yzI_1 + z^2I_2$$

$$I_1 + I_2 = x^2 + (y^2 + 2xy + 2yz)I_1 + (z^2 + 2xz)I_2$$

Whence:

$$\begin{cases} x^2 = 0 \\ y^2 + 2xy + 2yz = 1 \\ z^2 + 2xz = 1 \end{cases}$$

$$\Rightarrow \begin{cases} x = 0 \\ y^2 + 2yz = 1 \\ z^2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} x = 0 \\ y^2 + 2yz = 1 \quad (1) \\ z = \pm 1 \end{cases}$$

- For $z = 1$ we substitute in (1)

$$y^2 + 2y - 1 = 0$$
$$\Delta = 8$$

Then:

$$\begin{cases} y = \frac{-2 + 2\sqrt{2}}{2} = -1 + \sqrt{2} \\ y = \frac{-2 - 2\sqrt{2}}{2} = -1 - \sqrt{2} \end{cases}$$

Hence:

$$\sqrt{I_1 + I_2} = \left(-1 + \sqrt{2}\right)I_1 + I_2$$

Or:

$$= \left(-1 - \sqrt{2}\right)I_1 + I_2$$

- For $z = -1$ we substitute in (1)

$$y^2 - 2y - 1 = 0$$
$$\Delta = 8$$

Then:

$$\begin{cases} y = \frac{2 + 2\sqrt{2}}{2} = 1 + \sqrt{2} \\ y = \frac{2 - 2\sqrt{2}}{2} = 1 - \sqrt{2} \end{cases}$$

Hence:

$$\sqrt{I_1 + I_2} = \left(1 + \sqrt{2}\right)I_1 - I_2$$

Or:

$$= \left(1 - \sqrt{2}\right)I_1 - I_2$$

The four results are:

$$(x, y, z) = \left(0 \ , -1 + \sqrt{2} \ , 1\right), \left(0 \ , -1 - \sqrt{2} \ , 1\right)$$

$$\left(0 \ , 1 + \sqrt{2} \ , -1\right), \left(0 \ , 1 - \sqrt{2} \ , -1\right)$$

Because we are now calculating the square root of a 2-refined neutrosophic number (according to classical analysis), we only take the result with a positive value, hence:

$$\sqrt{I_1 + I_2} = \left(-1 + \sqrt{2}\right)I_1 + I_2$$

However, when we solve the 2-refined neutrosophic equation, we take all four results.

2.3 2-Refined neutrosophic real or complex polynomial

Definition:

A polynomial whose coefficients (at least one of them containing I_1 or I_2 or both I_1, I_2) are 2-refined neutrosophic numbers is called 2-Refined Neutrosophic Polynomials.

Similarly, we may have 2-Refined Neutrosophic Real Polynomials if its coefficients are refined neutrosophic real numbers and 2-Refined Neutrosophic Complex Polynomials if its coefficients are 2-refined neutrosophic complex numbers.

Example 2

- $P(x) = 10x^2 + (5 - 7I_1 + 3I_2)x - 4 + 2I_2$
  Is a 2-refined refined neutrosophic real polynomial
- $Q(x) = x^4 + (2 + 3I_2 - 4i + 3I_1 i)x^3 + 6I_2 x^2 + (9 - 8I_1 + 3I_2 i)x - 4 + 2I_2 + 4I_1 i$
  Is a 2-refined refined neutrosophic complex polynomial.

Example 3

Solve the equation: $x^2 - 1 - 5I_1 - 3I_2 = 0$

Solution:

$$x^2 - 1 - 5I_1 - 3I_2 = 0$$

$$x^2 = 1 + 5I_1 + 3I_2$$

Let's find: $\sqrt{1 + 5I_1 + 3I_2}$

$$\sqrt{1 + 5I_1 + 3I_2} = x + yI_1 + zI_2$$

By raise both sides to the second power, we get:

$$1 + 5I_1 + 3I_2 = (x + yI_1 + zI_2)^2$$

$$1 + 5I_1 + 3I_2 = (x + yI_1)^2 + 2(x + yI_1)(zI_2) + (zI_2)^2$$

$$1 + 5I_1 + 3I_2 = x^2 + 2xyI_1 + (yI_1)^2 + 2(x + yI_1)(zI_2) + (zI_2)^2$$

$$1 + 5I_1 + 3I_2 = x^2 + 2xyI_1 + y^2 I_1 + 2xzI_2 + 2yzI_1 + z^2 I_2$$

$$1 + 5I_1 + 3I_2 = x^2 + (y^2 + 2xy + 2yz)I_1 + (z^2 + 2xz)I_2$$

Whence:

$$\begin{cases} x^2 = 1 \\ y^2 + 2xy + 2yz = 5 \\ z^2 + 2xz = 3 \end{cases}$$

$$\Rightarrow \begin{cases} x = \pm 1 \\ y^2 + 2xy + 2yz = 5 \quad (1) \\ z^2 + 2xz = 3 \quad (2) \end{cases}$$

Case1: $x = 1$ by substitution in (2)

$$z^2 + 2z - 3 = 0$$

$$(z - 1)(z + 3) = 0$$

Then:

$$\begin{cases} z = 1 \\ z = -3 \end{cases}$$

- For $z = 1$ we substitute in (1)

$$y^2+4y-5=0$$
$$(y+5)(y-1)=0$$

Then:

$$\begin{cases} y=1 \\ y=-5 \end{cases}$$

Hence:

$$\sqrt{1+5I_1+3I_2}=1+I_1+I_2$$

Or:

$$=1-5I_1+I_2$$

- For $z=-3$ we substitute in (1)

$$y^2-4y-5=0$$
$$(y-5)(y+1)=0$$

Then:

$$\begin{cases} y=5 \\ y=-1 \end{cases}$$

Hence:

$$\sqrt{1+5I_1+3I_2}=1+5I_1-3I_2$$

Or:

$$=1-I_1-3I_2$$

Case2: $x=-1$ by substitution in (2)

$$z^2-2z-3=0$$
$$(z-3)(z+1)=0$$

Then:

$$\begin{cases} z=3 \\ z=-1 \end{cases}$$

- For $z=3$ we substitute in (1)

$$y^2+4y-5=0$$

$$(y-1)(y+5)=0$$

Then:

$$\begin{cases} y=1 \\ y=-5 \end{cases}$$

Hence:

$$\sqrt{1+5I_1+3I_2}=-1+I_1+3I_2$$

Or:

$$=-1-5I_1+3I_2$$

- For $z=-1$ we substitute in (1)

$$y^2-4y-5=0$$
$$(y-5)(y+1)=0$$

$$\begin{cases} y=5 \\ y=-1 \end{cases}$$

Hence:

$$\sqrt{1+5I_1+3I_2}=-1+5I_1-I_2$$

Or:

$$= -1 - I_1 - I_2$$

Hence, we got eight 2-refined neutrosophic solutions:

$$x = \sqrt{1 + 5I_1 + 3I_2} = 1 + I_1 + I_2$$

Or:

$$= 1 - 5I_1 + I_2$$

Or:

$$= 1 + 5I_1 - 3I_2$$

Or:

$$= 1 - I_1 - 3I_2$$

Or:

$$= -1 + I_1 + 3I_2$$

Or:

$$= -1 - 5I_1 + 3I_2$$

Or:

$$= -1 + 5I_1 - I_2$$

Or:

$$= -1 - I_1 - I_2$$

First 2-refined neutrosophic factoring:

$$P(x) = x^2 - 1 - 5I_1 - 3I_2 = (x - 1 - I_1 - I_2)(x + 1 + I_1 + I_2)$$

Second 2-refined neutrosophic factoring:

$$P(x) = x^2 - 1 - 5I_1 - 3I_2 = (x - 1 + 5I_1 - I_2)(x + 1 - 5I_1 + I_2)$$

Third 2-refined neutrosophic factoring:

$$P(x) = x^2 - 1 - 5I_1 - 3I_2 = (x - 1 - 5I_1 + 3I_2)(x + 1 + 5I_1 - 3I_2)$$

Fourth 2-refined neutrosophic factoring:

$$P(x) = x^2 - 1 - 5I_1 - 3I_2 = (x - 1 + I_1 + 3I_2)(x + 1 - I_1 - 3I_2)$$

Differently from the classical polynomial with real or complex coefficients, the 2-refined neutrosophic polynomials do not have a unique factoring! If we check each solution, we get:

$$P(x_1) = P(x_2) = P(x_3) = P(x_4) = P(x_5) = P(x_6) = P(x_7) = P(x_8) = 0$$

Let's compute:

$$P(x_5) = P(-1 + I_1 + 3I_2) = (-1 + I_1 + 3I_2)^2 - 1 - 5I_1 - 3I_2$$

$$= 1 - 2I_1 + I_1 - 6I_2 + 6I_1 + 9I_2 - 1 - 5I_1 - 3I_2 = 0 \text{ (True)}$$

$$P(x_8) = P(-1 - I_1 - I_2) = (-1 - I_1 - I_2)^2 - 1 - 5I_1 - 3I_2$$

$$= 1 + 2I_1 + I_1 + 2I_2 + 2I_1 + I_2 - 1 - 5I_1 - 3I_2 = 0 \text{ (True)}$$

And so on for the rest of the results.

2.3 The square root of a 2-refined neutrosophic complex number

Let $a + bi + (c + di)I_1 + (e + fi)I_2$ be a neutrosophic complex number, where a, b, c, d, e, f are reals. Let's compute square root of it:

$$\sqrt{a + bi + (c + di)I_1 + (s + ti)I_2} = x + yi + (z + wi)I_1 + (e + fi)I_2$$

$$a+bi+(c+di)I_1+(s+ti)I_2=(x+yi+(z+wi)I_1+(e+fi)I_2)^2$$

$$=(x+yi+(z+wi)I_1)^2+2(x+yi+(z+wi)I_1)(e+fi)I_2+((e+fi)I_2)^2$$

$$=(x+yi)^2+2(x+yi)(z+wi)I_1+((z+wi)I_1)^2+2(x+yi+(z+wi)I_1)(e+fi)I_2+((e+fi)I_2)^2$$

$$=x^2+2xyi-y^2+2xzI_1+2xwiI_1+2yziI_1-2ywI_1+z^2I_1+2zwiI_1-w^2I_1+2xeI_2+2xfiI_2+2yeiI_2-2yfI_2+2zeI_1+2zfiI_1+2weiI_1-2wfI_1+e^2I_2+2efiI_2-f^2I_2$$

$$=x^2-y^2+2xyi+(2xz-2yw+z^2+2ze-2wf-w^2)I_1+(2xw+2yz+2zw+2zf+2we)iI_1+(2xeI_2-2yf+e^2)I_2+(2xf+2ye+2ef-f^2)iI_2$$

Then we get a non-linear algebraic system in six variables (x,y,z,w,e,f) and six equations:

$$\begin{cases} x^2-y^2=a \\ 2xy=b \\ 2xz-2yw+z^2+2ze-2wf-w^2=c \\ 2xw+2yz+2zw+2zf+2we=d \\ 2xeI_2-2yf+e^2=s \\ 2xf+2ye+2ef-f^2=t \end{cases}$$

That we need to solve.

5. Conclusions

This paper presents an extensive study on 2-refined neutrosophic numbers. The research explores the process of finding the square root of a 2-refined neutrosophic real or complex number. One of the key contributions of this study is that it provides all possible solutions when solving the 2-refined neutrosophic equation, rather than just one solution as in the case of finding the square root of a 2-refined neutrosophic number. Additionally, the obtained results were tested by substitution into the basic equation presented. We highly recommend that researchers take advantage of this study, as it greatly simplifies calculations in various mathematical formulas for future applications.

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