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Circular Economy Strategies to Promote Sustainable Development using t-Neutrosophic Fuzzy graph

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Abstract: This study discusses the use of t-neutrosophic graphs to express and understand complex interactions. It also shows how these graphs might manage intricate relationships with a wide range of variables or scenario elements. Additionally, it offers fundamental t-neutrosophic graph operations and talks about concepts in these graphs such as homomorphism and isomorphism. Additionally, the study addresses how this approach may be applied in practical contexts to address circular economies and sustainability growth. Through the use of t-neutrosophic graphs, various components, and their connections, the research demonstrates that managing the several circular dimensions is feasible. financial system. This example shows how adaptable and practical t-neutrosophic graphs are as a tool for developing policies and making decisions about complicated social issues that involve some numbers principles.

Keywords: Graph Theory, t- Neutrosophic Fuzzy Graph, Decision Making, and Sustainability.

1. Introduction

One of the pioneering works that popularized fuzzy sets was Lotfi A. Zadeh [1]. Unlike typical sets with distinct membership criteria, fuzzy sets assign degrees of membership to items, allowing for the depiction of ambiguity and uncertainty in the data. The book by Abraham Kandel [2] offers a thorough overview of fuzzy mathematical methods and their real-world applications. It is an invaluable resource for comprehending the use of fuzzy sets in a variety of fields and covers a broad range of topics. Fuzzy sets were extended to intuitionistic fuzzy sets by Krassimir T. Atanassov [3]. More details about the extent of non-membership are incorporated into intuitionistic fuzzy sets, which offer a more complex depiction of uncertainty. The book by Lotfi A. Zimmermann [4] offers a thorough introduction to fuzzy set theory and all of its various uses. It examines the applications of fuzzy sets in decision-making, control systems, and other domains, providing a fundamental resource for scholars and professionals. The book by George J. Klir and Bo Yuan [5] provides a thorough introduction to fuzzy logic and fuzzy sets. In addition to covering the theoretical underpinnings of fuzzy sets and logic, it offers real-world applications in domains including pattern recognition, decision analysis, and control systems. Jerry M. Mendel [6] paper provides an overview of fuzzy logic systems, with a focus on engineering applications. It goes over the fundamentals of fuzzy logic and offers some ideas about how engineering systems might use it to handle imprecision and uncertainty. In [28-30] applied fuzzy set and their logic to find better decision making.

Mapari, and Naidu [7], Examines fuzzy set theory and its uses, offering a basic comprehension of fuzzy sets and their useful applications in various domains. Akrama M [8] focuses on neutrosophic graphs in the field of graph theory by concentrating on single-valued neutrosophic planar graphs. Nasir M. and Akram M. [9] examine certain bipolar neutrosophic competition graphs, probably with a focus on mathematical competition applications. Siddique S. and Akram M. [10] investigate applications of neutrosophic competition graphs and show how useful they are for fuzzy and intelligent systems. Waseem, Dudek, and Akram [11] explain specific kinds of edge m-polar fuzzy graphs, adding to our knowledge of fuzzy graphs with an emphasis on edge relativity. In 2020, Ajay, and Chellamani, [12] presented Pythagorean neutrosophic fuzzy graphs, presumably investigating the combination of neutrosophic sets and Pythagorean fuzzy sets in the context of graph theory. Dey, Pal, Long, Mohanta, and Son [13] carries out research on m-polar neutrosophic graphs with applications, perhaps looking at real-world uses for these graphs in intelligent systems. Mateen, Kausar, Gulzar, and Alghazzawi, [14] study a particular class of fuzzy subgroups that are t-intuitionistic, contributing to the study of subgroup theory and fuzzy algebra. P. Chellamani and D. Ajay [15] examine Pythagorean neutrosophic Dombi fuzzy graphs and apply them to multicriteria decision-making (MCDM), presumably demonstrating their usefulness in this context. Reviews the literature on intelligent traffic control systems using a variety of theoretical frameworks, such as fuzzy sets, rough sets, graph theory, and neutrosophic sets [16]. In 2016 [20-23] money researcher devolved fuzzy concept in graph theory.

In 2022, Broumi, Bakali, Shanmugapriya, Sundareswaran, and Talea [17] examine Fermatean neutrosophic graph theory and applications, possibly advancing knowledge of Fermatean neutrosophic structures in graph theory. Mohanta, Dey, Mondal, Pal, and Lakhwani [18] contribute to the study of operations and transformations in the field of Dombi neutrosophic graphs by looking into a few operations on them. Razaq, Alhamzi, Shuaib, Razzaque, Masmali, Latif, and Noor [19] offer a thorough examination of t-intuitionistic fuzzy graphs and investigates how to use them to combat poverty; this certainly contributes to both the theoretical and practical sides of fuzzy graph theory.

In [25] Victor Christianto and Florentin Smarandache discussed The Paradoxist Movement, Precursor of Neutrosophy, in the Shadow of a Totalitarian Regime. [26] Minxia Luo et al, Fuzzy Inference Quintuple Implication Method Based on introduced the Single Valued Neutrosophic t-representable t-norm. [27] Vázquez and Smarandache examines how postcolonial modernist ideas and indigenous philosophies may be integrated within the MultiAlism framework, showing how these disparate philosophical traditions in Latin America can interact and come together. Using areas of neutrality like cultural variety and criticisms of power structures, Florentin Smarandache's notion of MultiAlism is used to fill in the gaps and reconcile these systems. [31] Elsayed employ the Approach for Preference, Performance and Ranking Evaluation with SAtisfaction Level (APPRESAL) method to analyze the BRT system's features in accordance with user satisfaction. This will help address the shortcomings of multi-criteria decision-making (MCDM) methods, which are unable to adequately address user satisfaction or the impact of various factors on the analysis conducted in a type-2 neutrosophic numbers (T2NN) environment. Lastly, the method will help resolve any ambiguity or uncertainty that may arise from the data that has been gathered.

In [33] Ahmed El-Douh et al, presenting the technique of Multi-Attributive Border Approximation Area Comparisons. In order to manage untrue data, this MCDM approach is combined with a neutrosophic set and used to evaluate the sustainability of soil activities. [34] Zenat Mohamed et al. provides a comprehensive review of sustainable supplier selection and looks at the key parameters and elements that should be considered when evaluating suppliers' sustainability performance. From the above

It has been demonstrated that using t-neutrosophic fuzzy graphs can assist manage ambiguity and uncertainty by providing a flexible approach to decision-making. These models act as a bridge between traditional numerical engineering models and symbolic expert systems. The t-neutrosophic fuzzy theory manages complex and uncertain circumstances well, communicating imprecision and unpredictability. We introduced the concept of a t-neutrosophic fuzzy graph that uses linear operators to solve practical problems. The parameter "t" reduces ambiguity in the decision-making process and enables precise control and customized solutions. This parameter offers a personalized uncertainty management plan while enhancing flexibility. t-neutrosophic fuzzy graphs provide a powerful tool for interpreting complex decision-making situations, allowing for a holistic approach to handling decision-making challenges. This strategy reduces the limitations of binary logic and significantly increases the precision of decision-making.

After a brief introduction to t-neutrosophic fuzzy graphs, the work is organized as follows: To help readers understand the originality of the work being presented, some basic terminology is provided in the "Preliminaries" section. "t-Neutrosophic Fuzzy Graph" discusses the concept of t-neutrosophic fuzzy graphs and highlights some of its salient characteristics. The set-theoretical operations of t-neutrosophic fuzzy graphs are examined and graphical representations of these operations are provided in the section "Operations on the t-neutrosophic fuzzy graph". The section "Isomorphism of t-neutrosophic fuzzy graphs" establishes the concepts of homomorphism and isomorphism specific to t-neutrosophic fuzzy graphs. In the section "Complement of t-neutrosophic fuzzy graph", the notion of the complement of t-neutrosophic fuzzy graphs is established and its key characteristics are analyzed. In the "Application of t-neutrosophic fuzzy graph" section, a mechanism for Sustainable Development through a Circular Economy is created using the recently developed technology. Finally, the "Comparative Analysis" and "Conclusion" sections provide a comparison of the article's many components and particular results.

1.1 Motivation

1. The main motivation behind the selection of t-neutrosophic graphs is their capacity to manage intricate scenarios including ambiguous information and hesitant, fluctuating elemental interactions.
2. These graphs offer a mechanism to assess and model various degrees of uncertainty and confidence in connections with the addition of the "t" parameter.
3. The integration and separation of uncertain information are managed by introducing a strategy that incorporates t-norms and t-conorms. This approach is especially meant to be used for making decisions in practical scenarios with a variety of inputs and results.
4. Applications for this method can be found in many different domains, including systems optimization, risk assessment, and decision analysis. Its objective is to strike a balance between usefulness in many contexts and undiscovered connections.

1.2 Novelty

1. To help establish an orderly and understandable depiction of unclear connections, the "t" parameter serves as a threshold reflecting reluctance.
2. The relationship visualization is enhanced by the addition of the "t" option, which determines which edges and nodes to display based on a predetermined confidence level.

3. It is possible to distinguish between strong and delicate associations with more accuracy using this strategy, which facilitates the methodical handling of ambiguity.
4. A t-neutrosophic approach connects various parameter values "t" to different graph layers, enabling multi-layered analysis. This makes it possible to analyze the relationships within the graph in detail while accounting for different levels of confidence, which leads to a deeper understanding of the underlying framework.

1.3 Goal

1. Explain the t-neutrosophic framework concept. The flexibility with which this framework may be used to communicate the ambiguity and uncertainty inherent in decision-making processes makes it valuable. Numerous academic fields, including as economics, chemistry, computer science, medicine, and engineering, depend on it.

2. Examine a number of set-theoretical operations within the framework of t-neutrosophic theory and identify the salient features of these freshly developed operations. These features support the development of informed decision-making across a range of domains, the integration of data, and the exploration of linkages.
3. Discuss homomorphism and isomorphism to t-neutrosophic theory and provide recently defined characteristics. These ideas facilitate data transmission and comparison analysis, particularly when hesitant and unclear graph topologies are involved.
4. Introduce the notion of a t-neutrosophic complement and verify its essential characteristics. This idea of ambiguity has applications in mistake detection, system verification, and decision analysis by revealing inverse linkages that were not immediately obvious in the original graph.
5. Utilize the recently established t-neutrosophic approach to pinpoint critical elements for reducing poverty in a certain community. This methodology facilitates improved representation, discerns vulnerable groups, distributes resources, tracks and assesses advancements, and develops educated policies.
6. Examine the complexities and uncertainties surrounding poverty to evaluate its causes, evolution, and effects using the t-neutrosophic framework.

Symbols	Meaning
NS	Neutrosophic Set
NG	Neutrosophic Graph
t-NS	t- Neutrosophic Set
t-NG	t- Neutrosophic Graph
$\mathbf{T}_{\mathcal{G}_t}$	Truth Membership Function
$\mathbf{I}_{\mathcal{G}_t}$	Indeterminacy Membership Function
$\mathbf{F}_{\mathcal{G}_t}$	Falsity Membership Function

Table 1 List of symbols used

2. t-Neutrosophic Graph

2.1. Definition. Let G be the NS of a universal set U with t ∈ [0,1]. The $NS_{\mathcal{G}_t}$ of U, also known as a t-NS, is defined as $T_{\mathcal{G}_t}(u_1) = \min\{T_{\mathcal{G}}(u_1), t\}$, $I_{\mathcal{G}_t}(u_1) = \max\{I_{\mathcal{G}}(u_1), 1 - t\}$ and $F_{\mathcal{G}_t}(u_1) = \max\{F_{\mathcal{G}}(u_1), 1 - t\}, \forall\, u_1 \in U$. The form of t-NS is $\mathcal{G}_t = \{u_1, T_{\mathcal{G}}(u_1), I_{\mathcal{G}}(u_1), F_{\mathcal{G}}(u_1) \colon u_1 \in U\}$ where $T_{\mathcal{G}_t}, I_{\mathcal{G}_{t,}}$ and $F_{\mathcal{G}_t}$ are functions that assign to each degree. Moreover, the function $T_{\mathcal{G}_t}, I_{\mathcal{G}_t}$ *and* $F_{\mathcal{G}_t}$ satisfy the condition $0 \leq T_{\mathcal{G}}(u_1) + I_{\mathcal{G}}(u_1) + F_{\mathcal{G}}(u_1) \leq 1$.

2.2. Definition. Let $\mathcal{G} = (A, B)$ be a NG for a given simple graph G= (V,E). A t-NG is represented by the notation $\mathcal{G}_t = (A_t, B_t)$, where

$A_t = \{(u_i, T_{\mathcal{G}}(u_i), I_{\mathcal{G}}(u_i), F_{\mathcal{G}}(u_i)): u_i \in V\}$ is a t-NS on V and
$B_t = \{((u_i, u_j), T_{\mathcal{G}}(u_i, u_j), I_{\mathcal{G}}(u_i, u_j), F_{\mathcal{G}}(u_i, u_j)): (u_i, u_j) \in E\}$ is a t-NS on E⊆ $V \times V$, such that $\forall (u_i, u_j) \in E$

$$T_{B_t}(u_i, u_j) \leq \min\{T_{A_t}(u_i), T_{A_t}(u_j)\}$$
$$I_{B_t}(u_i, u_j) \leq \max\{I_{A_t}(u_i), I_{A_t}(u_j)\}$$
$$F_{B_t}(u_i, u_j) \leq \max\{F_{A_t}(u_i), F_{A_t}(u_j)\},$$

Satisfy the condition: $0 \leq T_{A_t}(u_i) + I_{A_t}(u_i) + F_{A_t}(u_i) \leq 1$ and $0 \leq T_{B_t}(u_i, u_j) + I_{B_t}(u_i, u_j) + F_{B_t}(u_i, u_j) \leq 1$.

2.3 Example Examine the $G' = (V, E)$ in which $V = \{a, b, c, d\}$ and $E = \{ab, ac, ad, bc, cd\}$. Given V, the node strengths of A *is* $\{(a, 0.2,0.4,0.6), (b, 0.5,0.4,0.7), (c, 0.3,0.6,0.9), (d, 0.8,0.7,0.1)\}$. The Edge strength of B *is* $\{(ab, 0.2,0.4,0.7), (ac, 0.2,0.6,0.9), (ad, 0.2,0.7,0.6), (bc, 0.3,0.6,0.9), (cd, 0.3,0.7,0.9)\}$. Applying the concept of t-NS to the two NS A and B that are provided, which correspond to the value t = 0.40, reveals that,

$A_{0.40} = \{(a, 0.2,0.6,0.6), (b, 0.4,0.6,0.7), (c, 0.3,0.6,0.9), (d, 0.4,0.7,0.6)\}$ and

$B_{0.40} = \{(ab, 0.2,0.6,0.7), (ac, 0.2,0.6,0.9), (ad, 0.2,0.6,0.7), (bc, 0.3,0.6,0.9), (cd, 0.3,0.7,0.9)\}$.

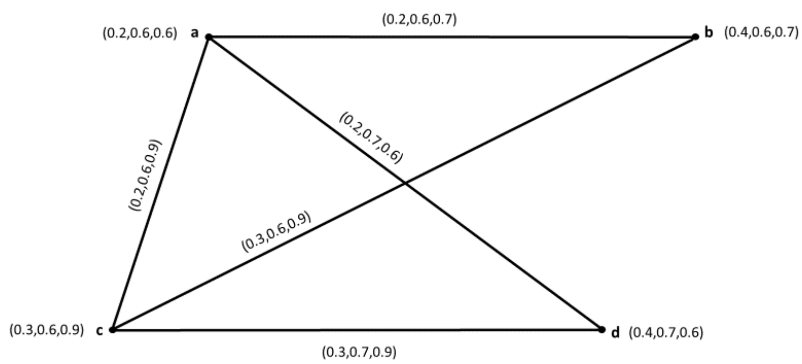

Figure 1. 0.4-NG $\mathcal{G}_{0.4}$

The graphical representation of t-NG, where t=0.4; $\mathcal{G}_{0.4} = (A_{0.40}, B_{0.40})$

*2.4. Definition***.** Let $\mathcal{G}_t = (A_t, B_t)$ be an t-NG then $\mathcal{H}_t = (A'_t, B'_t)$ is considered a t-neutrosophic subgraph if $A'_t \subseteq A_t$ and $B'_t \subseteq B_t$.
*2.5.Definition***.** A t-NG $\mathcal{G}_t$ is said to be a complete t-NG if it meets the requirements listed below:

$$T_{B_t}(u_1, u_2) = \min\{T_{A_t}(u_1), T_{A_t}(u_2)\}$$
$$I_{B_t}(u_1, u_2) = \max\{I_{A_t}(u_1), I_{A_t}(u_2)\}$$
$$F_{B_t}(u_1, u_2) = \max\{F_{A_t}(u_1), F_{A_t}(u_2)\}, \forall (u_1, u_2) \in E$$

2.6 Example. Figure 2 illustrates the entire 0.5-NG $\mathcal{G}_t$.

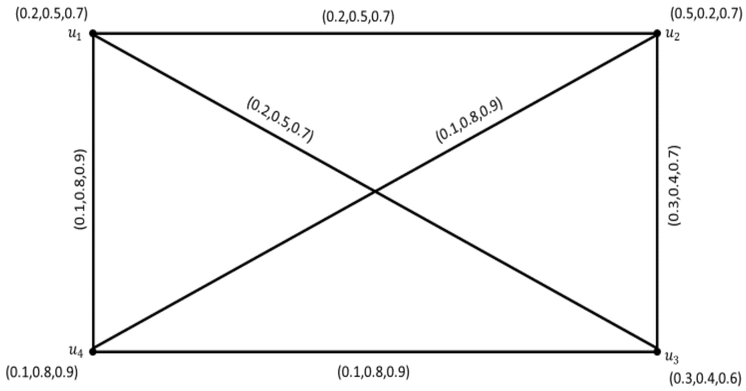

Figure 2. Complete 0.5-NG $\mathcal{G}_{0.5}$

2.7. Definition. In t-NG, the order is defined as follows

$$O(\mathcal{G}_t) = \left(\sum_{u_1 \in V} T_{A_t}(u_1), \sum_{u_1 \in V} I_{A_t}(u_1), \sum_{u_1 \in V} F_{A_t}(u_1) \right)$$

*2.8. Example***.** The order of t-NG $\mathcal{G}_t$ is (1.3, 2.5, 2.8) from example 1.
*2.9 Definition***.** The t-NG has a size defined by

$$S(\mathcal{G}_t) = \left(\sum_{(u_1,u_2) \in E} T_{B_t}(u_1, u_2), \sum_{(u_1,u_2) \in E} I_{B_t}(u_1, u_2), \sum_{(u_1,u_2) \in E} F_{B_t}(u_1, u_2) \right)$$

2.10. Definition. t-NG defines the degree of vertex $u_1$in $\mathcal{G}_t$ *as follows:*

$$deg_{\mathcal{G}_t}(u_1) = \left(deg_{T_{B_t}}(u_1), deg_{I_{B_t}}(u_1), deg_{F_{B_t}}(u_1) \right)$$

$$deg_{\mathcal{G}_t}(u_1) = \left(\sum_{(u_1,u_2) \in E} T_{B_t}(u_1, u_2), \sum_{(u_1,u_2) \in E} I_{B_t}(u_1, u_2), \sum_{(u_1,u_2) \in E} F_{B_t}(u_1, u_2) \right)$$

2.11. Example. Referring to Example 1,

1. The degree of vertex in $\mathcal{G}_t$ are, $deg_{\mathcal{G}_t}(a) = (0.6,1.9,2.2)$; $deg_{\mathcal{G}_t}(b) = (0.5,1.2,1.6)$; $deg_{\mathcal{G}_t}(c) = (0.8,1.9,2.7)$; $deg_{\mathcal{G}_t}(d) = (0.5,1.4,1.5)$.

2. The minimum degree $\delta(\mathcal{G}_t)$of t-NG is given by $\delta(\mathcal{G}_t) = (\delta_{T_{B_t}}(\mathcal{G}_t), \delta_{I_{B_t}}(\mathcal{G}_t). \delta_{F_{B_t}}(\mathcal{G}_t))$

$$\delta(\mathcal{G}_t) = \left(\min\left\{ deg_{T_{B_t}}(u_1): u_1 \in V \right\}, \min\left\{ deg_{I_{B_t}}(u_1): u_1 \in V \right\}, \min\left\{ deg_{F_{B_t}}(u_1): u_1 \in V \right\} \right)$$

3. The maximum degree $\Delta(\mathcal{G}_t)$ of t-NG is given by $\Delta(\mathcal{G}_t) = (\Delta_{T_{B_t}}(\mathcal{G}_t), \Delta_{I_{B_t}}(\mathcal{G}_t). \Delta_{F_{B_t}}(\mathcal{G}_t))$

$$\Delta(\mathcal{G}_t) = \left(\max\left\{ deg_{T_{B_t}}(u_1): u_1 \in V \right\}, \max\left\{ deg_{I_{B_t}}(u_1): u_1 \in V \right\}, \max\left\{ deg_{F_{B_t}}(u_1): u_1 \in V \right\} \right)$$

From Example 1: $\delta(\mathcal{G}_t)$ = (0.5, 1.2, 1.5); $\Delta(\mathcal{G}_t) = (0.8, 1.9, 2.7)$. In t-NG the following inequality holds, $\delta(\mathcal{G}_t) \leq \Delta(\mathcal{G}_t) \leq S(\mathcal{G}_t) \leq O(\mathcal{G}_t)$

3.2 Theorem. An t-NG is represented as G_t=(A_t,B_t), then

$$\sum deg_{\mathcal{G}t}(u_i) = \left(2\sum T_{\mathcal{G}t}(u_i,w), 2\sum I_{\mathcal{G}t}(u_i,w), 2\sum F_{\mathcal{G}t}(u_i,w)\right)$$

Proof. Considering t-NG represented by G_t=(A_t,B_t), let's investigate,

$$\sum deg_{\mathcal{G}t}(u_i) = \left(\sum deg_{T_{B_t}}(u_i), \sum deg_{I_{B_t}}(u_i), \sum deg_{F_{B_t}}(u_i)\right)$$

$$= \left(deg_{T_{B_t}}(u_1), deg_{I_{B_t}}(u_1), deg_{F_{B_t}}(u_1)) + \left(deg_{T_{B_t}}(u_2), deg_{I_{B_t}}(u_2), deg_{F_{B_t}}(u_2)\right) \dots + (deg_{T_{B_t}}(u_n), deg_{I_{B_t}}(u_n), deg_{F_{B_t}}(u_n)\right)$$

$$= (T_{B_t}(u_1,u_2), I_{B_t}(u_1,u_2), F_{B_t}(u_1,u_2)) + (T_{B_t}(u_1,u_3), I_{B_t}(u_1,u_3), F_{B_t}(u_1,u_3)) + \dots + (T_{B_t}(u_1,u_n), I_{B_t}(u_1,u_n), F_{B_t}(u_1,u_n)) + \cdots + (T_{B_t}(u_n,u_1), I_{B_t}(u_n,u_1), F_{B_t}(u_n,u_1)) + (T_{B_t}(u_n,u_2), I_{B_t}(u_n,u_2), F_{B_t}(u_n,u_2)) + \dots + (T_{B_t}(u_n,u_{n-1}), I_{B_t}(u_n,u_{n-1}), F_{B_t}(u_n,u_{n-1})))$$

$$= (2(T_{B_t}(u_1,u_2), I_{B_t}(u_1,u_2), F_{B_t}(u_1,u_2)) + 2(T_{B_t}(u_1,u_3), I_{B_t}(u_1,u_3), F_{B_t}(u_1,u_3)) + \dots + 2(T_{B_t}(u_1,u_n), I_{B_t}(u_1,u_n), F_{B_t}(u_1,u_n)))$$

$$= \left(2\sum T_{\mathcal{G}t}(u_i,w), 2\sum I_{\mathcal{G}t}(u_i,w), 2\sum F_{\mathcal{G}t}(u_i,w)\right).$$

Hence proved.

3. Operation on t-Neutrosophic Graph

*3.1. Definition***.** Consider any two t-NG of G = (V,E) and G′=(V′,E′) correspond to $\mathcal{G}_t = (A_t, B_t)$ and $\mathcal{G}'_t = (A'_t, B'_t)$, respectively. $(A_t \times A'_t, B_t \times B'_t)$ defines the Cartesian product $\mathcal{G}_t \times \mathcal{G}'_t$ of two t-NG. Where $A_t \times A'_t$ and $B_t \times B'_t$ are t-NS on $V \times V' = \{(\varphi_1,\varpi_1),(\varphi_2,\varpi_2): \varphi_1 \& \varphi_2 \in V; \varpi_1 \& \varpi_2 \in V'\}$ and $E \times E' = \{(\varphi_1,\varpi_1),(\varphi_2,\varpi_2): \varphi_1 = \varphi_2, \varphi_1 \& \varphi_2 \in V, (\varpi_1,\varpi_2) \in E'\}\ U\ \{(\varphi_1,\varpi_1),(\varphi_2,\varpi_2): \varpi_1 = \varpi_2, \varpi_1 \& \varpi_2 \in V', (\varphi_1,\varphi_2) \in E\}\ U\ \{(\varphi_1,\varpi_1),(\varphi_2,\varpi_2): \varpi_1 \neq \varpi_2, \varphi_1 \neq \varphi_2, (\varpi_1,\varpi_2) \in E', (\varphi_1,\varphi_2) \in E\}$ respectively, which fulfills the given requirement.

1. $\forall((\varphi_1,\varpi_1) \in V \times V'$
   a) $T_{A_t \times A'_t}(\varphi_1,\varpi_1) = \min\left\{T_{A_t}(\varphi_1), T_{A_{t'}}(\varpi_1)\right\}$
   b) $I_{A_t \times A'_t}(\varphi_1,\varpi_1) = \max\left\{I_{A_t}(\varphi_1), I_{A_{t'}}(\varpi_1)\right\}$
   c) $F_{A_t \times A'_t}(\varphi_1,\varpi_1) = \max\left\{F_{A_t}(\varphi_1), F_{A_{t'}}(\varpi_1)\right\}$

2. If $\varphi_1 = \varphi_2$ and $\forall\ (\varpi_1,\varpi_2) \in E'$
   a) $T_{B_t \times B'_t}((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)) = \min\left\{T_{A_t}(\varphi_1), T_{B_{t'}}(\varpi_1,\varpi_2)\right\}$
   b) $I_{B_t \times B'_t}((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)) = \max\left\{I_{A_t}(\varphi_1), I_{B_{t'}}(\varpi_1,\varpi_2)\right\}$
   c) $F_{B_t \times B'_t}((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)) = \max\left\{F_{A_t}(\varphi_1), F_{B_{t'}}(\varpi_1,\varpi_2)\right\}$
3. If $\varpi_1 = \varpi_2$ and $\forall\ (\varphi_1,\varphi_2) \in E$
   a) $T_{B_t \times B'_t}((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)) = \min\left\{T_{B_t}(\varphi_1,\varphi_2), T_{A_{t'}}(\varpi_1)\right\}$
   b) $I_{B_t \times B'_t}((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)) = \max\left\{I_{B_t}(\varphi_1,\varphi_2), I_{A_{t'}}(\varpi_1)\right\}$
   c) $F_{B_t \times B'_t}((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)) = \max\left\{F_{B_t}(\varphi_1,\varphi_2), F_{A_{t'}}(\varpi_1)\right\}$
4. If $\varpi_1 \neq \varpi_2\ and \varphi_1 \neq \varphi_2,\ \forall\ (\varpi_1,\varpi_2) \in E', (\varphi_1,\varphi_2) \in E$
   a) $T_{B_t \times B'_t}((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)) = \min\left\{T_{B_t}(\varphi_1,\varphi_2), T_{B_{t'}}(\varpi_1,\varpi_2)\right\}$

b) $I_{B_t \times B_t'}((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)) = \max\left\{I_{B_t}(\varphi_1, \varphi_2), I_{B_{t'}}(\varpi_1, \varpi_2)\right\}$

c) $F_{B_t \times B_t'}\big((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)\big) = \max\left\{F_{B_t}(\varphi_1, \varphi_2), F_{B_{t'}}(\varpi_1, \varpi_2)\right\}$

*3.2 Example***.** The two 0.5-NG $\mathcal{G}_t$and$\mathcal{G}_t'$, which are the elements to be taken into consideration, are shown in Figures 3 and 4. Figure 5 displays the corresponding Cartesian product $\mathcal{G}_{0.5} \times \mathcal{G}_{0.5}'$.

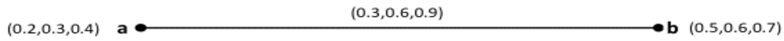

Figure 3. 0.5-NG $\mathcal{G}_{0.5}$

Figure 4. 0.5-NG $\mathcal{G}_{0.5}'$

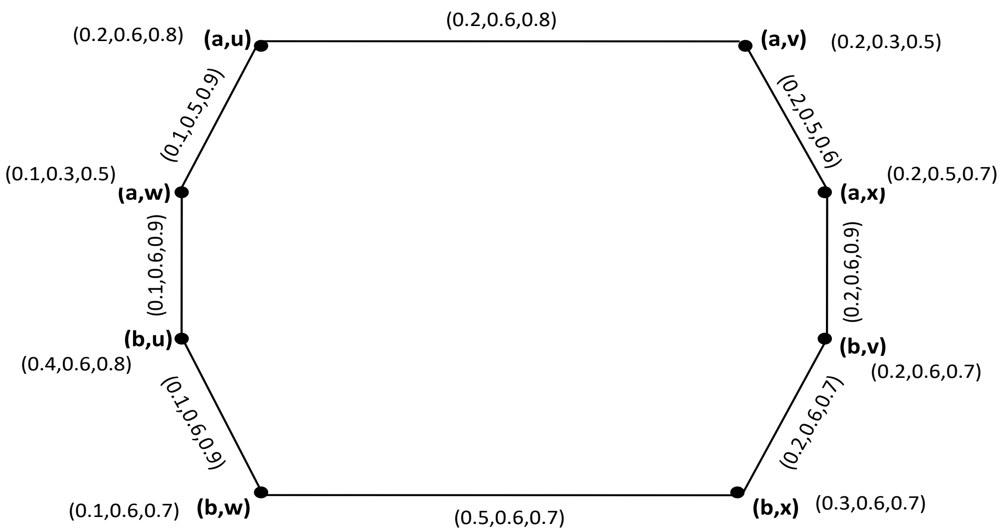

Figure 5. 0.5-NG $\mathcal{G}_{0.5} \times \mathcal{G}_{0.5}'$

3.3 Definition $deg_{\mathcal{G}_t \times \mathcal{G}_t'}(\varphi_1, \varpi_1) =$
$$\left(\deg\left\{T_{B_t \times B_t'}\big((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)\big)\right\}, \deg\left\{I_{B_t \times B_t'}\big((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)\big)\right\}, \deg\left\{F_{B_t \times B_t'}\big((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)\big)\right\}\right)$$

Where,

$$\deg\left\{T_{B_t\times B_t'}\big((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)\big)\right\} = \sum_{\varphi_1=\varphi_2,(\varpi_1,\varpi_2)\in E'} \min\left\{T_{A_t}(\varphi_1),T_{B_{t'}}(\varpi_1,\varpi_2)\right\}$$

$$+\sum_{\varpi_1=\varpi_2,(\varphi_1,\varphi_2)\in E} \min\left\{T_{B_t}(\varphi_1,\varphi_2),T_{A_{t'}}(\varpi_1)\right\}$$

$$+\sum_{\varpi_1\neq\varpi_2,\varphi_1\neq\varphi_2} \min\left\{T_{B_t}(\varphi_1,\varphi_2),T_{B_{t'}}(\varpi_1,\varpi_2)\right\}$$

$$\deg\left\{I_{B_t\times B_t'}\big((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)\big)\right\} = \varphi_1=\varphi_2,(\varpi_1,\varpi_2)\in E'$$

$$+\sum_{\varpi_1=\varpi_2,(\varphi_1,\varphi_2)\in E} \max\left\{I_{B_t}(\varphi_1,\varphi_2),I_{A_{t'}}(\varpi_1)\right\}$$

$$+\sum_{\varpi_1\neq\varpi_2,\varphi_1\neq\varphi_2} \max\left\{T_{B_t}(\varphi_1,\varphi_2),I_{B_{t'}}(\varpi_1,\varpi_2)\right\}$$

$$\deg\left\{F_{B_t\times B_t'}\big((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)\big)\right\} = \sum_{\varphi_1=\varphi_2,(\varpi_1,\varpi_2)\in E'} \max\left\{F_{A_t}(\varphi_1),F_{B_{t'}}(\varpi_1,\varpi_2)\right\}$$

$$+\sum_{\varpi_1=\varpi_2,(\varphi_1,\varphi_2)\in E} \max\left\{F_{B_t}(\varphi_1,\varphi_2),F_{A_{t'}}(\varpi_1)\right\}$$

$$+\sum_{\varpi_1\neq\varpi_2,\varphi_1\neq\varphi_2} \max\left\{F_{B_t}(\varphi_1,\varphi_2),F_{B_{t'}}(\varpi_1,\varpi_2)\right\}$$

3.4 Example. Every vertex in $\mathcal{G}_t \times \mathcal{G}_t'$ has the following degree, per example 5.
$deg_{\mathcal{G}_t\times\mathcal{G}_t'}(a,u) = (0.3,1.1,1.7)$, $deg_{\mathcal{G}_t\times\mathcal{G}_t'}(a,v) = (0.4,1.1,1.4)$, $deg_{\mathcal{G}_t\times\mathcal{G}_t'}(a,w) = (0.2,1.1,1.8)$, $deg_{\mathcal{G}_t\times\mathcal{G}_t'}(a,x) = (0.4,1.1,1.5)$, $deg_{\mathcal{G}_t\times\mathcal{G}_t'}(b,u) = (0.2,1.2,1.8)$, $deg_{\mathcal{G}_t\times\mathcal{G}_t'}(b,v) = (0.4,1.2,1.6)$, $deg_{\mathcal{G}_t\times\mathcal{G}_t'}(b,w) = (0.6,1.2,1.6)$, $deg_{\mathcal{G}_t\times\mathcal{G}_t'}(b,x) = (0.7,1.2,1.4)$

3.5. Theorem. The Cartesian products of two t-NGs, result is a new t-NG

Proof: The requirement is clear for $A_t \times A_t'$. Considering that $(\varpi_1,\varpi_2) \in E'$ and $\varphi_1 \in V$,

$T_{B_t\times B_t'}\big((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)\big) = \min\left\{T_{A_t}(\varphi_1),T_{B_{t'}}(\varpi_1,\varpi_2)\right\}$

$T_{B_t\times B_t'}\big((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)\big) \leq \min\left\{T_{A_t}(\varphi_1),\min\{T_{A_{t'}}(\varpi_1),T_{A_{t'}}(\varpi_2)\}\right\}$

$T_{B_t\times B_t'}\big((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)\big) \leq \min\left\{\min\{T_{A_t}(\varphi_1),T_{A_{t'}}(\varpi_1)\},\min\{T_{A_t}(\varphi_1),T_{A_{t'}}(\varpi_2)\}\right\}$

$T_{B_t\times B_t'}\big((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)\big) = \min\left\{T_{A_t\times A_t'}(\varphi_1,\varpi_1),T_{A_t\times A_t'}(\varpi_1,\varpi_2)\right\}$

Consequently,

$T_{B_t\times B_t'}\big((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)\big) \leq \min\left\{T_{A_t\times A_t'}(\varphi_1,\varpi_1),T_{A_t\times A_t'}(\varpi_1,\varpi_2)\right\}$ if $u_1 \in V$,

$(\varpi_1,\varpi_2) \in E'$. Similarly for,

$I_{B_t\times B_t'}\big((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)\big) \leq \max\left\{I_{A_t\times A_t'}(\varphi_1,\varpi_1),I_{A_t\times A_t'}(\varpi_1,\varpi_2)\right\}$ if $u_1 \in V$,

$(\varpi_1,\varpi_2) \in E'$ and

$F_{B_t\times B_t'}\big((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)\big) \leq \max\left\{F_{A_t\times A_t'}(\varphi_1,\varpi_1),F_{A_t\times A_t'}(\varpi_1,\varpi_2)\right\}$ if $u_1 \in V$,

$(\varpi_1,\varpi_2) \in E'$. Likewise we can demonstrate it for$\varpi_1 \in V'$,$(\varphi_1,\varphi_2) \in E$.

3.7. Definition. Consider any two t-NG of G = (V,E) and G′=(V′,E′) correspond to $\mathcal{G}_t = (A_t, B_t)$ and $\mathcal{G}'_t = (A'_t, B'_t)$, respectively. $(A_t \circ A'_t, B_t \circ B'_t)$ defines the composition $\mathcal{G}_t \circ \mathcal{G}'_t$ of two t-NG. Where $A_t \circ A'_t$ and $B_t \circ B'_t$ are t-NS on $V \times V' = \{(\varphi_1, \varpi_1), (\varphi_2, \varpi_2): \varphi_1 \& \varphi_2 \in V; \varpi_1 \& \varpi_2 \in V'\}$ and $E \times E' = \{(\varphi_1, \varpi_1), (\varphi_2, \varpi_2): \varphi_1 = \varphi_2, \varphi_1 \& \varphi_2 \in V, (\varpi_1, \varpi_2) \in E'\}\ U\ \{(\varphi_1, \varpi_1), (\varphi_2, \varpi_2): \varpi_1 = \varpi_2, \varpi_1 \& \varpi_2 \in V', (\varphi_1, \varphi_2) \in E\}\ U\ \{(\varphi_1, \varpi_1), (\varphi_2, \varpi_2): \varpi_1 \neq \varpi_2, \varphi_1 \neq \varphi_2, (\varpi_1, \varpi_2) \in E', (\varphi_1, \varphi_2) \in E\}$ respectively, which fulfills the given requirement.

1. $\forall((\varphi_1, \varpi_1) \in V \circ V'$

   a) $T_{A_t \circ A'_t}(\varphi_1, \varpi_1) = \min\left\{T_{A_t}(\varphi_1), T_{A_{t'}}(\varpi_1)\right\}$

   b) $I_{A_t \circ A'_t}(\varphi_1, \varpi_1) = \max\left\{I_{A_t}(\varphi_1), I_{A_{t'}}(\varpi_1)\right\}$

   c) $F_{A_t \circ A'_t}(\varphi_1, \varpi_1) = \max\left\{F_{A_t}(\varphi_1), F_{A_{t'}}(\varpi_1)\right\}$

2. If $\varphi_1 = \varphi_2$ and $\forall\ (\varpi_1, \varpi_2) \in E'$

   a) $T_{B_t \circ B'_t}((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)) = \min\left\{T_{A_t}(\varphi_1), T_{B_{t'}}(\varpi_1, \varpi_2)\right\}$

   b) $I_{B_t \circ B'_t}((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)) = \max\left\{I_{A_t}(\varphi_1), I_{B_{t'}}(\varpi_1, \varpi_2)\right\}$

   c) $F_{B_t \circ B'_t}\big((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)\big) = \max\left\{F_{A_t}(\varphi_1), F_{B_{t'}}(\varpi_1, \varpi_2)\right\}$

3. If $\varpi_1 = \varpi_2$ and $\forall\ (\varphi_1, \varphi_2) \in E$

   a) $T_{B_t \circ B'_t}((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)) = \min\left\{T_{B_t}(\varphi_1, \varphi_2), T_{A_{t'}}(\varpi_1)\right\}$

   b) $I_{B_t \circ B'_t}((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)) = \max\left\{I_{B_t}(\varphi_1, \varphi_2), I_{A_{t'}}(\varpi_1)\right\}$

   c) $F_{B_t \circ B'_t}\big((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)\big) = \max\left\{F_{B_t}(\varphi_1, \varphi_2), F_{A_{t'}}(\varpi_1)\right\}$

4. If $\varpi_1 \neq \varpi_2$ and $\forall\ (\varphi_1, \varphi_2) \in E$

   a) $T_{B_t \circ B'_t}((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)) = \min\left\{T_{B_t}(\varphi_1, \varphi_2), T_{A_{t'}}(\varpi_1), T_{A_{t'}}(\varpi_2)\right\}$

   b) $I_{B_t \circ B'_t}((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)) = \max\left\{I_{B_t}(\varphi_1, \varphi_2), I_{A_{t'}}(\varpi_1), I_{A_{t'}}(\varpi_2)\right\}$

   c) $F_{B_t \circ B'_t}\big((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)\big) = \max\left\{F_{B_t}(\varphi_1, \varphi_2), F_{A_{t'}}(\varpi_1), F_{A_{t'}}(\varpi_2)\right\}$

5. If $\varphi_1 \neq \varphi_2$ and $\forall\ (\varpi_1, \varpi_2) \in E'$

   a) $T_{B_t \circ B'_t}((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)) = \min\left\{T_{A_t}(\varphi_1), T_{A_t}(\varphi_2), T_{B_{t'}}(\varpi_1, \varpi_2)\right\}$

   b) $I_{B_t \circ B'_t}((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)) = \max\left\{I_{A_t}(\varphi_1), I_{A_t}(\varphi_2), I_{B_{t'}}(\varpi_1, \varpi_2)\right\}$

   c)$F_{B_t \circ B'_t}\big((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)\big) = \max\left\{F_{A_t}(\varphi_1), F_{A_t}(\varphi_2), F_{B_{t'}}(\varpi_1, \varpi_2)\right\}$

*3.8. Example***.** Consider the two 0.6-NG $\mathcal{G}_t$ and $\mathcal{G}'_t$ illustrated in figure 6 and 7.

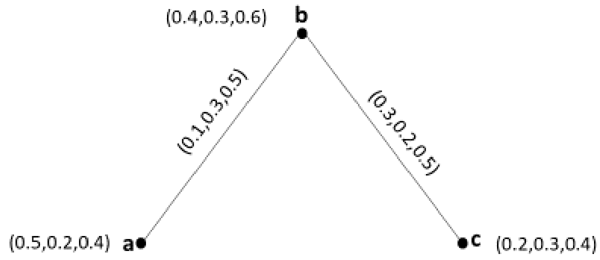

Figure 6. 0.6-NG $\mathcal{G}_{0.6}$

Figure 7. 0.6-NG $\mathcal{G}'_{0.6}$

Figure 8. Shows their corresponding Cartesian product $\mathcal{G}_{0.6} \circ \mathcal{G}'_{0.6}$

Figure 8. 0.6-NG $\mathcal{G}_{0.6} \circ \mathcal{G}'_{0.6}$

3.9. Definition. The degree of a vertex in $\mathcal{G}_t \circ \mathcal{G}'_t$ is defined as follows for any

$$(\varphi_1, \varpi_1) \in V \times V'; deg_{\mathcal{G}_t \circ \mathcal{G}'_t}(\varphi_1, \varpi_1)$$
$$= (\deg\left\{T_{B_t \circ B'_t}\big((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)\big)\right\}, \deg\left\{I_{B_t \circ B'_t}\big((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)\big)\right\}, \deg\{F_{B_t \circ B'_t}\big((\varphi_1, \varpi_1), (\varphi_2, \varpi_2)\big)\})$$

Where, $\deg\left\{T_{B_t\circ B_t'}\big((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)\big)\right\}=\sum_{\varphi_1=\varphi_2,(\varpi_1,\varpi_2)\in E'}\min\left\{T_{A_t}(\varphi_1),T_{B_{t'}}(\varpi_1,\varpi_2)\right\}$

$$+\sum_{\varpi_1=\varpi_2,(\varphi_1,\varphi_2)\in E}\min\left\{T_{B_t}(\varphi_1,\varphi_2),T_{A_{t'}}(\varpi_1)\right\}$$

$$+\sum_{\varpi_1\neq\varpi_2,(\varphi_1,\varphi_2)\in E}\min\left\{T_{B_t}(\varphi_1,\varphi_2),T_{A_{t'}}(\varpi_1),T_{A_{t'}}(\varpi_2)\right\}$$

$$+\sum_{\varphi_1\neq\varphi_2,(\varpi_1,\varpi_2)\in E'}\min\left\{T_{A_t}(\varphi_1),T_{A_t}(\varphi_2),T_{B_{t'}}(\varpi_1,\varpi_2)\right\}$$

$$\deg\left\{I_{B_t\circ B_t'}\big((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)\big)\right\}=\sum_{\varphi_1=\varphi_2,(\varpi_1,\varpi_2)\in E'}\max\left\{I_{A_t}(\varphi_1),I_{B_{t'}}(\varpi_1,\varpi_2)\right\}$$

$$+\sum_{\varpi_1=\varpi_2,(\varphi_1,\varphi_2)\in E}\max\left\{I(\varphi_1,\varphi_2),I_{A_{t'}}(\varpi_1)\right\}$$

$$+\sum_{\varpi_1\neq\varpi_2,(\varphi_1,\varphi_2)\in E}\max\left\{I_{B_t}(\varphi_1,\varphi_2),I_{A_{t'}}(\varpi_1),I_{A_{t'}}(\varpi_2)\right\}$$

$$+\sum_{\varphi_1\neq\varphi_2,(\varpi_1,\varpi_2)\in E'}\max\left\{I_{A_t}(\varphi_1),I_{A_t}(\varphi_2),I_{A_{t'}}(\varpi_1,\varpi_2)\right\}$$

$$\deg\left\{F_{B_t\circ B_t'}\big((\varphi_1,\varpi_1),(\varphi_2,\varpi_2)\big)\right\}=\sum_{\varphi_1=\varphi_2,(\varpi_1,\varpi_2)\in E'}\max\left\{F_{A_t}(\varphi_1),F_{B_{t'}}(\varpi_1,\varpi_2)\right\}$$

$$+\sum_{\varpi_1=\varpi_2,(\varphi_1,\varphi_2)\in E}\max\left\{F(\varphi_1,\varphi_2),F_{A_{t'}}(\varpi_1)\right\}$$

$$+\sum_{\varpi_1\neq\varpi_2,(\varphi_1,\varphi_2)\in E}\max\left\{F_{B_t}(\varphi_1,\varphi_2),F_{A_{t'}}(\varpi_1),F_{A_{t'}}(\varpi_2)\right\}$$

$$+\sum_{\varphi_1\neq\varphi_2,(\varpi_1,\varpi_2)\in E'}\max\left\{F_{A_t}(\varphi_1),F_{A_t}(\varphi_2),F_{A_{t'}}(\varpi_1,\varpi_2)\right\}$$

*3.10. Example***.** In Example 7 shows that each vertex in $\mathcal{G}_t\circ\mathcal{G}_t'$ has the following degree
$deg_{\mathcal{G}_t\times\mathcal{G}_t'}(a,d)=(0.6,0.5,0.9)$, $deg_{\mathcal{G}_t\times\mathcal{G}_t'}(a,e)=(0.6,0.7,0.9)$, $deg_{\mathcal{G}_t\times\mathcal{G}_t'}(b,d)=(0.4,0.5,1.0)$, $deg_{\mathcal{G}_t\times\mathcal{G}_t'}(b,e)=(0.4,1.0,1.0)$, $deg_{\mathcal{G}_t\times\mathcal{G}_t'}(c,d)=(0.8,0.9,1.0)$, $deg_{\mathcal{G}_t\times\mathcal{G}_t'}(c,e)=(0.8,1.2,1.0)$

3.11. Definition. Consider any two t-NG of G = (V,E) and G′=(V′,E′) correspond to $\mathcal{G}_t=(A_t,B_t)$ and $\mathcal{G}_t'=(A_t',B_t')$, respectively. $(A_t\cup A_t',B_t\cup B_t')$ defines the union $\mathcal{G}_t\cup\mathcal{G}_t'$ of two t-NG, where $A_t\cup A_t'$ and $B_t\cup B_t'$ respectively, represent t-NS on $V\cup V'$ and $E\cup E'$, which fulfills the given requirement,

1) If $\varphi_1\in V$ and $\varphi_1\notin V'$
   a) $T_{A_t\cup A_t'}(\varphi_1)$=$T_{A_t}(\varphi_1)$
   b) $I_{A_t\cup A_t'}(\varphi_1)$=$I_{A_t}(\varphi_1)$
   c) $F_{A_t\cup A_t'}(\varphi_1)$=$F_{A_t}(\varphi_1)$
2) If $\varphi_1\notin V$ and $\varphi_1\in V'$
   a) $T_{A_t\cup A_t'}(\varphi_1)$=$T_{A_t'}(\varphi_1)$
   b) $I_{A_t\cup A_t'}(\varphi_1)$=$I_{A_t'}(\varphi_1)$
   c) $F_{A_t\cup A_t'}(\varphi_1)$=$F_{A_t'}(\varphi_1)$
3) If $\varphi_1\in V\cap V'$
   a) $T_{A_t\cup A_t'}(\varphi_1)$= $max\{T_{A_t}(\varphi_1),T_{A_t'}(\varphi_1)\}$
   b) $I_{A_t\cup A_t'}(\varphi_1)$= $min\{I_{A_t}(\varphi_1),I_{A_t'}(\varphi_1)\}$
   c) $F_{A_t\cup A_t'}(\varphi_1)$= $min\{F_{A_t}(\varphi_1),F_{A_t'}(\varphi_1)\}$
4) If $(\varphi_1,\varpi_1)\in E$ and $(\varphi_1,\varpi_1)\notin E'$
   a) $T_{B_t\cup B_t'}(\varphi_1,\varpi_1)$=$T_{B_t}(\varphi_1,\varpi_1)$
   b) $I_{B_t\cup B_t'}(\varphi_1,\varpi_1)$=$I_{B_t}(\varphi_1,\varpi_1)$

c) $F_{B_t \cup B_t'}(\varphi_1, \varpi_1)$=$F_{B_t}(\varphi_1, \varpi_1)$

5) If $(\varphi_1, \varpi_1) \notin E$ and $(\varphi_1, \varpi_1) \in E'$

a) $T_{B_t \cup B_t'}(\varphi_1, \varpi_1)$= $T_{B_t'}(\varphi_1, \varpi_1)$

b) $I_{B_t \cup B_t'}(\varphi_1, \varpi_1)$=$I_{B_t'}(\varphi_1, \varpi_1)$

c) $F_{B_t \cup B_t'}(\varphi_1, \varpi_1)$=$F_{B_t'}(\varphi_1, \varpi_1)$

6) If $(\varphi_1, \varpi_1) \in E \cap E'$

a) $T_{B_t \cup B_t'}(\varphi_1, \varpi_1) = max\{T_{B_t}(\varphi_1, \varpi_1), T_{B_t'}(\varphi_1, \varpi_1)\}$

b) $I_{B_t \cup B_t'}(\varphi_1, \varpi_1)$= $min\{I_{B_t}(\varphi_1, \varpi_1), I_{B_t'}(\varphi_1, \varpi_1)\}$

c) $F_{B_t \cup B_t'}(\varphi_1, \varpi_1)$= $min\{F_{B_t}(\varphi_1, \varpi_1), F_{B_t'}(\varphi_1, \varpi_1)\}$

*3.12. Example***.** Figures 9 and 10 show 0.7-NG $\mathcal{G}_t$ and $\mathcal{G}_t'$.

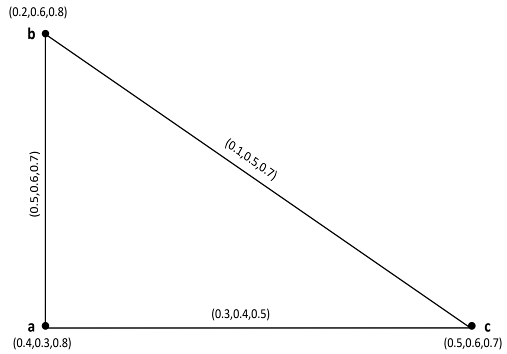

Figure 9. $0.7 - NG\mathcal{G}_{0.7}$

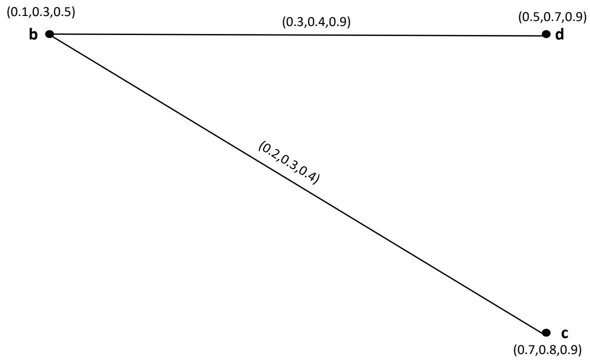

Figure 10. $0.7 - NG\mathcal{G}_{0.7}'$

Figure 11 shows their corresponding Cartesian product $\mathcal{G}_{0.5} \cup \mathcal{G}_{0.5}'$

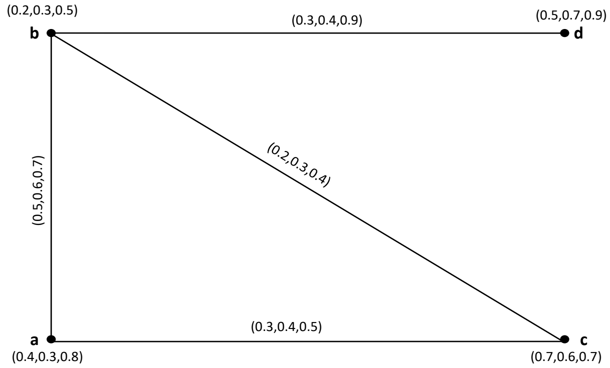

Figure 11. $\mathcal{G}_{0.7} \cup \mathcal{G}'_{0.7}$

*3.13. Definition***.** The degree of vertex (φ_1, ϖ_1) at a t-NG for any $(\varphi_1, \varpi_1) \in V \cup V'$

$$deg_{\mathcal{G}_t \cup \mathcal{G}'_t}(\varphi_1, \varpi_1) = \left(\deg\left\{T_{B_t \cup B'_t}(\varphi_1, \varpi_1)\right\}, \deg\left\{I_{B_t \cup B'_t}(\varphi_1, \varpi_1)\right\}, \deg\left\{F_{B_t \cup B'_t}(\varphi_1, \varpi_1)\right\}\right)$$

Where

$$deg\left\{T_{B_t \cup B'_t}(\varphi_1, \varpi_1)\right\} = \sum_{(\varphi_1,\varpi_1)\in E\,,(\varphi_1,\varpi_1)\notin E'} T_{B_t}(\varphi_1, \varpi_1) + \sum_{(\varphi_1,\varpi_1)\notin E,(\varphi_1,\varpi_1)\in E'} T_{B'_t}(\varphi_1, \varpi_1) + \sum_{(\varphi_1,\varpi_1)\in E\cap E'} max\{T_{B_t}(\varphi_1, \varpi_1), T_{B'_t}(\varphi_1, \varpi_1)\}$$

$$deg\left\{I_{B_t \cup B'_t}(\varphi_1, \varpi_1)\right\} = \sum_{(\varphi_1,\varpi_1)\in E\,,(\varphi_1,\varpi_1)\notin E'} I_{B_t}(\varphi_1, \varpi_1) + \sum_{(\varphi_1,\varpi_1)\notin E,(\varphi_1,\varpi_1)\in E'} I_{B'_t}(\varphi_1, \varpi_1) + \sum_{(\varphi_1,\varpi_1)\in E\cap E'} min\{I_{B_t}(\varphi_1, \varpi_1), I_{B'_t}(\varphi_1, \varpi_1)\}$$

$$deg\left\{F_{B_t \cup B'_t}(\varphi_1, \varpi_1)\right\} = \sum_{(\varphi_1,\varpi_1)\in E\,,(\varphi_1,\varpi_1)\notin E'} F_{B_t}(\varphi_1, \varpi_1) + \sum_{(\varphi_1,\varpi_1)\notin E,(\varphi_1,\varpi_1)\in E'} F_{B'_t}(\varphi_1, \varpi_1) + \sum_{(\varphi_1,\varpi_1)\in E\cap E'} min\{F_{B_t}(\varphi_1, \varpi_1), F_{B'_t}(\varphi_1, \varpi_1)\}$$

3.14. Definition. . Consider two t-NGs $\mathcal{G}_t = (A_t, B_t)$ and $\mathcal{G}'_t = (A'_t, B'_t)$. These t-NGs join operation $\mathcal{G}_t + \mathcal{G}'_t$ is described as $(A_t + A'_t, B_t + B'_t)$, where $A_t + A'_t$ produces a t-NG on $V \cup V'$ and $B_t + B'_t$ forms a t-NG on $E \cup E' \cup E''$, subject to certain requirements.

1) If $\varphi_1 \in V$ and $\varphi_1 \notin V'$

a) $T_{A_t+A'_t}(\varphi_1)$=$T_{A_t}(\varphi_1)$
b) $I_{A_t+A'_t}(\varphi_1)$=$I_{A_t}(\varphi_1)$
c) $F_{A_t+A'_t}(\varphi_1)$=$F_{A_t}(\varphi_1)$

2) If $\varphi_1 \notin V$ and $\varphi_1 \in V'$

a) $T_{A_t+A'_t}(\varphi_1)$=$T_{A'_t}(\varphi_1)$
b) $I_{A_t+A'_t}(\varphi_1)$=$I_{A'_t}(\varphi_1)$
c)$F_{A_t+A'_t}(\varphi_1)$=$F_{A'_t}(\varphi_1)$

3) If $\varphi_1 \in V \cap V'$
a) $T_{A_t+A'_t}(\varphi_1)$= $max\{T_{A_t}(\varphi_1), T_{A'_t}(\varphi_1)\}$
b) $I_{A_t+A'_t}(\varphi_1)$= $min\{I_{A_t}(\varphi_1), I_{A'_t}(\varphi_1)\}$
c) $F_{A_t+A'_t}(\varphi_1)$= $min\{F_{A_t}(\varphi_1), F_{A'_t}(\varphi_1)\}$

4) If $(\varphi_1, \varpi_1) \in E$ and $(\varphi_1, \varpi_1) \notin E'$
a) $T_{B_t+B'_t}(\varphi_1, \varpi_1)$=$T_{B_t}(\varphi_1, \varpi_1)$
b) $I_{B_t+B'_t}(\varphi_1, \varpi_1)$=$I_{B_t}(\varphi_1, \varpi_1)$
c) $F_{B_t+B'_t}(\varphi_1, \varpi_1)$=$F_{B_t}(\varphi_1, \varpi_1)$

5) If $(\varphi_1, \varpi_1) \notin E$ and $(\varphi_1, \varpi_1) \in E'$
a) $T_{B_t+B'_t}(\varphi_1, \varpi_1)$= $T_{B'_t}(\varphi_1, \varpi_1)$
b) $I_{B_t+B'_t}(\varphi_1, \varpi_1)$=$I_{B'_t}(\varphi_1, \varpi_1)$
c) $F_{B_t+B'_t}(\varphi_1, \varpi_1)$=$F_{B'_t}(\varphi_1, \varpi_1)$

6) If $(\varphi_1, \varpi_1) \in E \cap E'$
a) $T_{B_t+B'_t}(\varphi_1, \varpi_1) = max\{T_{B_t}(\varphi_1, \varpi_1), T_{B'_t}(u_1, w_1)\}$
b) $I_{B_t+B'_t}(\varphi_1, \varpi_1)$= $min\{I_{B_t}(\varphi_1, \varpi_1), I_{B'_t}(\varphi_1, \varpi_1)\}$
c) $F_{B_t+B'_t}(\varphi_1, \varpi_1)$= $min\{F_{B_t}(\varphi_1, \varpi_1), F_{B'_t}(\varphi_1, \varpi_1)\}$

7) If $(\varphi_1, \varpi_1) \in E''$
a) $T_{B_t+B'_t}(\varphi_1, \varpi_1) = max\{T_{B_t}(\varphi_1), T_{B'_t}(\varpi_1)\}$
b) $I_{B_t+B'_t}(\varphi_1, \varpi_1)$= $min\{I_{B_t}(\varphi_1), I_{B'_t}(\varpi_1)\}$
c) $F_{B_t+B'_t}(\varphi_1, \varpi_1)$= $min\{F_{B_t}(\varphi_1), F_{B'_t}(\varpi_1)\}$

*3.15. Example***.** From example 9, the graphical representation of 0.9-NG$\mathcal{G}_t + \mathcal{G}'_t$ is displayed in the image below.

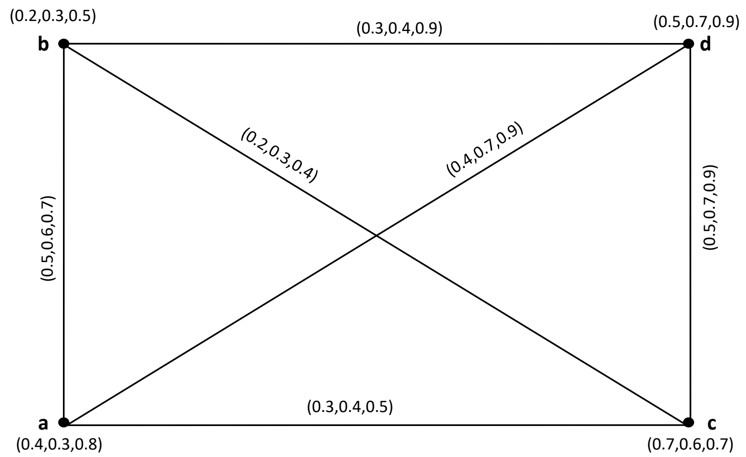

Figure 12. $\mathcal{G}_{0.7} + \mathcal{G}'_{0.7}$

3.16. Definition. Examine the subsequent pair of t-NGs, $\mathcal{G}_t$ and $\mathcal{G}_t'$. The following describes the vertex degree in the t-NG $\mathcal{G}_t + \mathcal{G}_t'$. If $(\varphi_1, \varpi_1) \in V + V'$, then

$$deg_{\mathcal{G}_t+\mathcal{G}_t'}(\varphi_1, \varpi_1) = \left(deg\left\{T_{B_t+B_t'}(\varphi_1, \varpi_1)\right\}, deg\left\{I_{B_t+B_t'}(\varphi_1, \varpi_1)\right\}, deg\left\{F_{B_t+B_t'}(\varphi_1, \varpi_1)\right\}\right)$$

Where

$$deg\left\{T_{B_t\cup B_t'}(\varphi_1, \varpi_1)\right\} = \left(\sum_{(\varphi_1,\varpi_1)\in E\,,(\varphi_1,\varpi_1)\notin E'} T_{B_t}(\varphi_1, \varpi_1) + \sum_{(\varphi_1,\varpi_1)\notin E,(\varphi_1,\varpi_1)\in E'} T_{B_t'}(\varphi_1, \varpi_1) + \sum_{(\varphi_1,\varpi_1)\in E\cap E'} max\{T_{B_t}(\varphi_1, \varpi_1), T_{B_t'}(\varphi_1, \varpi_1)\} + \sum_{(\varphi_1,\varpi_1)\in E''} max\{T_{B_t}(\varphi_1), T_{B_t'}(\varpi_1)\}\right)$$

$$deg\left\{I_{B_t\cup B_t'}(\varphi_1, \varpi_1)\right\} = \left(\sum_{(\varphi_1,\varpi_1)\in E\,,(\varphi_1,\varpi_1)\notin E'} I_{B_t}(\varphi_1, \varpi_1) + \sum_{(\varphi_1,\varpi_1)\notin E,(\varphi_1,\varpi_1)\in E'} I_{B_t'}(\varphi_1, \varpi_1) + \sum_{(\varphi_1,\varpi_1)\in E\cap E'} min\{I_{B_t}(\varphi_1, \varpi_1), I_{B_t'}(\varphi_1, \varpi_1)\} + \sum_{(\varphi_1,\varpi_1)\in E''} min\{I_{B_t}(\varphi_1), I_{B_t'}(\varpi_1)\}\right)$$

$$deg\left\{F_{B_t\cup B_t'}(\varphi_1, \varpi_1)\right\} = \left(\sum_{(\varphi_1,\varpi_1)\in E\,,(\varphi_1,\varpi_1)\notin E'} F_{B_t}(\varphi_1, \varpi_1) + \sum_{(\varphi_1,\varpi_1)\notin E,(\varphi_1,\varpi_1)\in E'} F_{B_t'}(\varphi_1, \varpi_1) + \sum_{(\varphi_1,\varpi_1)\in E\cap E'} min\{F_{B_t}(\varphi_1, \varpi_1), F_{B_t'}(\varphi_1, \varpi_1)\} + \sum_{(\varphi_1,\varpi_1)\in E''} min\{F_{B_t}(\varphi_1), F_{B_t'}(\varpi_1)\}\right)$$

3.17. Theorem. For any two t-NGs, their union is a t-NG.

Proof. Let us assume a t-NG, $\mathcal{G}_t \cup \mathcal{G}_t'$. Let $(\varphi_1, \varpi_1) \in E$, $(\varphi_1, \varpi_1) \notin E'$, and $(\varphi_1, \varpi_1) \in V - V'$

Consider

$$T_{B_t}(\varphi_1, \varpi_1) = T_{B_t\cap B_t'}(\varphi_1, \varpi_1)$$

$$T_{B_t}(\varphi_1, \varpi_1) \leq \min\{T_{A_t\cup A_t'}(\varphi_1), T_{A_t\cup A_t'}(\varpi_1)\}$$

$$T_{B_t}(\varphi_1, \varpi_1) = \min\{T_{A_t}(\varphi_1), T_{A_t}(\varpi_1)\}$$

Consequently $T_{B_t}(\varphi_1, \varpi_1) \leq \min\{T_{A_t}(\varphi_1), T_{A_t}(\varpi_1)\}$.

Also

$$I_{B_t}(\varphi_1, \varpi_1) = I_{B_t\cap B_t'}(\varphi_1 \varpi_1)$$

$$I_{B_t}(\varphi_1, \varpi_1) \leq \max\{I_{A_t\cup A_t'}(\varphi_1), I_{A_t\cup A_t'}(\varpi_1)\}$$

$$I_{B_t}(\varphi_1, \varpi_1) = \max\{I_{A_t}(\varphi_1), I_{A_t}(\varpi_1)\}$$

$$F_{B_t}(\varphi_1, \varpi_1) = F_{B_t \cap B'_t}(\varphi_1, \varpi_1)$$

$$F_{B_t}(\varphi_1, \varpi_1) \leq \max\{F_{A_t \cup A'_t}(\varphi_1), F_{A_t \cup A'_t}(\varpi_1)\}$$

$$F_{B_t}(\varphi_1, \varpi_1) = \max\{F_{A_t}(\varphi_1), F_{A_t}(\varpi_1)\}$$

Consequently $I_{B_t}(\varphi_1, \varpi_1) \leq \max\{I_{A_t}(\varphi_1), I_{A_t}(\varpi_1)\}$, $F_{B_t}(\varphi_1, \varpi_1) \leq \max\{F_{A_t}(\varphi_1), F_{A_t}(\varpi_1)\}$.

Hence, $\mathcal{G}_t = (A_t, B_t)$ is established as a t-NG. Likewise, we deduce that $\mathcal{G}'_t = (A'_t, B'_t)$ is a t-NG of G''. Given that $\mathcal{G}_t$ and $\mathcal{G}'_t$ are assumed, and since two t-NGs together produce a t-NG, we may deduce that $\mathcal{G}_t \cup \mathcal{G}'_t$.

4. Isomorphism of t-Neutrosophic Graph

4.1. Definition. Let $\mathcal{G}_t$ and $\mathcal{G}'_t$ be any two t-NG. A homomorphism θ from $\mathcal{G}_t$ and $\mathcal{G}'_t$ is a mapping $V to\, V'$, satisfies the following requirements:

1. $T_{A_t}(\varphi_1) \leq T_{A'_t}(\theta(\varphi_1))$,
   $I_{A_t}(\varphi_1) \leq I_{A'_t}(\theta(\varphi_1))$ and
   $F_{A_t}(\varphi_1) \leq F_{A'_t}(\theta(\varphi_1))$; $\forall \varphi_1 \in V$.
2. $T_{B_t}(\varphi_1, \varpi_1) \leq T_{B'_t}(\theta(\varphi_1), \theta(\varpi_1))$,
   $I_{B_t}(\varphi_1, \varpi_1) \leq I_{B'_t}(\theta(\varphi_1), \theta(\varpi_1))$ and
   $F_{B_t}(\varphi_1, \varpi_1) \leq F_{B'_t}(\theta(\varphi_1), \theta(\varpi_1))$; $\forall (\varphi_1, \varpi_1) \in E$.

4.2. Definition. A weak isomorphism $\theta: V \to V'$, from t-NG $\mathcal{G}_t$ to $\mathcal{G}'_t$ must meet the following conditions. the following condition: $T_{A_t}(\varphi_1) = T_{A'_t}(\theta(\varphi_1))$, $I_{A_t}(\varphi_1) \leq I_{A'_t}(\theta(\varphi_1))$ and $F_{A_t}(\varphi_1) \leq F_{A'_t}(\theta(\varphi_1))$; $\forall \varphi_1 \in V$.

4.3. Definition. A bijective mapping $\theta: V \to V'$ between any two t-NGs, $\mathcal{G}_t = (A_t, B_t)$ and $\mathcal{G}'_t = (A'_t, B'_t)$ of G=(V,E) and G′=(V′,E′) respectively, that satisfies the following criteria is called a strong co-isomorphism.

$T_{A_t}(\varphi_1) \leq T_{A'_t}(\theta(\varphi_1))$, $I_{A_t}(\varphi_1) \leq I_{A'_t}(\theta(\varphi_1))$ and $F_{A_t}(\varphi_1) \leq F_{A'_t}(\theta(\varphi_1))$; $\forall \varphi_1 \in V$.

1. $T_{B_t}(\varphi_1, \varpi_1) \leq T_{B'_t}(\theta(\varphi_1), \theta(\varpi_1))$, $I_{B_t}(\varphi_1, \varpi_1) \leq I_{B'_t}(\theta(\varphi_1), \theta(\varpi_1))$ and
   $F_{B_t}(\varphi_1, \varpi_1) \leq F_{B'_t}(\theta(\varphi_1), \theta(\varpi_1))$; $\forall (\varphi_1, \varpi_1) \in E$.
2. $T_{B_t}(\varphi_1, \varpi_1) = T_{B'_t}(\theta(\varphi_1), \theta(\varpi_1))$, $I_{B_t}(\varphi_1, \varpi_1) = I_{B'_t}(\theta(\varphi_1), \theta(\varpi_1))$ and
   $F_{B_t}(\varphi_1, \varpi_1) = F_{B'_t}(\theta(\varphi_1), \theta(\varpi_1))$; $\forall (\varphi_1, \varpi_1) \in E$.

4.4. Definition. An isomorphism between t-NGs $\mathcal{G}_t = (A_t, B_t)$ and $\mathcal{G}'_t = (A'_t, B'_t)$ is a bijective homomorphism mapping $\theta: V \to V'$ (written as $\mathcal{G}_t \approx \mathcal{G}'_t$) which satisfies the following condition:

1. $T_{A_t}(\varphi_1) = T_{A'_t}(\theta(\varphi_1))$, $I_{A_t}(\varphi_1) \leq I_{A'_t}(\theta(\varphi_1))$ and $F_{A_t}(\varphi_1) \leq F_{A'_t}(\theta(\varphi_1))$; $\forall \varphi_1 \in V$.
2. $T_{B_t}(\varphi_1, \varpi_1) = T_{B'_t}(\theta(\varphi_1), \theta(\varpi_1))$, $I_{B_t}(\varphi_1, \varpi_1) = I_{B'_t}(\theta(\varphi_1), \theta(\varpi_1))$ and
   $F_{B_t}(\varphi_1, \varpi_1) = F_{B'_t}(\theta(\varphi_1), \theta(\varpi_1))$; $\forall (\varphi_1, \varpi_1) \in E$.

4.5. Example. According to the following figures, take the two 0.7-$\mathcal{G}_t$ and $\mathcal{G}'_t$;

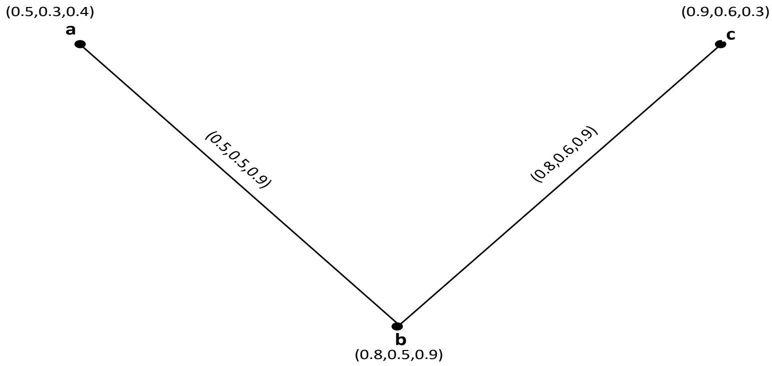

Figure 13. $0.7 - NG\ \mathcal{G}_{0.7}$

Figure 14. $0.7 - NG\ \mathcal{G}'_{0.7}$

According to definition (22), the mapping ζ(a)=g, ζ(b)=f, and ζ(c)=e gives us $\mathcal{G}_{0.7} \approx \mathcal{G}'_{0.7}$

4.6. Theorem. The isomorphism between t-NGs satisfies the properties of an equivalence relation.

Proof: It is evident that there is symmetry and reflexivity. $\varphi: V \to V'$ and $\theta: V' \to V''$ indicate the isomorphisms of $\mathcal{G}_t$ onto $\mathcal{G}'_t$ and $\mathcal{G}'_t$ onto $\mathcal{G}''_t$, respectively. A bijective map fromV' to V'' is therefore $\theta \circ \varphi: V \to V''$, and it is defined as follows:

$$(\theta \circ \varphi)(\varphi_1) = \theta(\varphi(\varphi_1)), \forall \varphi_1 \in V$$

For a map $\varphi: V \to V'$defined by $\varphi(\varphi_1) = \varpi_1, \forall \varphi_1 \in V$, it is an isomorphism. Considering definition (22) we have

$$T_{A_t}(\varphi_1) = T_{A'_t}(\varphi(\varphi_1)) = T_{A'_t}(\varpi_1), \forall \varphi_1 \in V$$
$$I_{A_t}(\varphi_1) = I_{A'_t}(\varphi(\varphi_1)) = I_{A'_t}(\varpi_1), \forall \varphi_1 \in V$$
$$F_{A_t}(\varphi_1) = F_{A'_t}(\varphi(\varphi_1)) = F_{A'_t}(\varpi_1), \forall \varphi_1 \in V$$

and

$$T_{B_t}(\varphi_1, \varphi_2) = T_{B'_t}(\varphi(\varphi_1), \varphi(\varphi_2)) = T_{B'_t}(\varpi_1, \varpi_2), \forall (\varphi_1, \varphi_2) \in E$$
$$I_{B_t}(\varphi_1, \varphi_2) = I_{B'_t}(\varphi(\varphi_1), \varphi(\varphi_2)) = I_{B'_t}(\varpi_1, \varpi_2), \forall (\varphi_1, \varphi_2) \in E$$
$$F_{B_t}(\varphi_1, \varphi_2) = F_{B'_t}(\varphi(\varphi_1), \varphi(\varphi_2)) = F_{B'_t}(\varpi_1, \varpi_2), \forall (\varphi_1, \varphi_2) \in E$$

In the same way, we obtained that

$$T_{A'_t}(\varpi_1) = T_{A''_t}(v_1), \forall \varpi_1 \in V'$$
$$I_{A'_t}(\varpi_1) = I_{A''_t}(v_1), \forall \varpi_1 \in V'$$

$$F_{A_t'}(\varpi_1) = F_{A_t''}(v_1), \forall \varpi_1 \in V'$$

and

$$T_{B_t'}(\varpi_1, \varpi_2) = T_{B_t''}(v_1, v_2), \forall (\varpi_1, \varpi_2) \in E'$$
$$I_{B_t'}(\varpi_1, \varpi_2) = I_{B_t''}(v_1, v_2), \forall (\varpi_1, \varpi_2) \in E'$$
$$F_{B_t'}(\varpi_1, \varpi_2) = F_{B_t''}(v_1, v_2), \forall (\varpi_1, \varpi_2) \in E'$$

By using the relations above and $\varphi(\varphi_1) = \varpi_1$, $\forall \varphi_1 \in V$, we have

$$T_{A_t}(\varphi_1) = T_{A_t'}(\varphi(\varphi_1)) = T_{A_t'}(\varpi_1) = T_{A_t''}(\theta(\varpi_1)) = T_{A_t''}\left(\theta(\varphi(\varphi_1))\right)$$
$$I_{A_t}(\varphi_1) = I_{A_t'}(\varphi(\varphi_1)) = I_{A_t'}(\varpi_1) = I_{A_t''}(\theta(\varpi_1)) = I_{A_t''}\left(\theta(\varphi(\varphi_1))\right)$$
$$F_{A_t}(\varphi_1) = F_{A_t'}(\varphi(\varphi_1)) = F_{A_t'}(\varpi_1) = F_{A_t''}(\theta(\varpi_1)) = F_{A_t''}\left(\theta(\varphi(\varphi_1))\right)$$

and

$$T_{B_t}(\varphi_1, \varphi_2) = T_{B_t'}(\varpi_1, \varpi_2) = T_{B_t''}(\theta(\varpi_1), \theta(\varpi_2)) = T_{B_t''}\left(\theta(\varphi(\varphi_1)), \theta(\varphi(\varphi_2))\right)$$
$$I_{B_t}(\varphi_1, \varphi_2) = I_{B_t'}(\varpi_1, \varpi_2) = I_{B_t''}(\theta(\varpi_1), \theta(\varpi_2)) = I_{B_t''}\left(\theta(\varphi(\varphi_1)), \theta(\varphi(\varphi_2))\right)$$
$$F_{B_t}(\varphi_1, \varphi_2) = F_{B_t'}(\varpi_1, \varpi_2) = F_{B_t''}(\theta(\varpi_1), \theta(\varpi_2)) = F_{B_t''}\left(\theta(\varphi(\varphi_1)), \theta(\varphi(\varphi_2))\right)$$

Thus, $\mathcal{G}_t$ and $\mathcal{G}_t''$ are isomorphic using $\theta \circ \varphi$. As a consequence, the proof was completed.

5. Complement of t-Neutrosophic Graph

5.1. Definition. The t-NG of G=(V,E) is $\mathcal{G}_t = (A_t, B_t)$ .A t-NG $\overline{\mathcal{G}_t}$ on $\overline{G} = (\overline{V}, \overline{E})$ is the complement of a t-NG $\mathcal{G}_t$ and is defined as follows.

1. $\overline{V} = V$
2. If $u_1 \in V$ then $T_{\overline{A_t}}(\varphi_1) = T_{A_t}(\varphi_1), I_{\overline{A_t}}(\varphi_1) = I_{A_t}(\varphi_1)$ *and* $F_{\overline{A_t}}(\varphi_1) = F_{A_t}(\varphi_1)$
3. If $T_{B_t}(\varphi_1, \varphi_2) \neq 0, I_{B_t}(\varphi_1, \varphi_2) \neq 0$ *and* $F_{B_t}(\varphi_1, \varphi_2) \neq 0$ then $T_{\overline{B_t}}(\varphi_1, \varphi_2) \neq 0, I_{\overline{B_t}}(\varphi_1, \varphi_2) \neq 0$ *and* $F_{\overline{B_t}}(\varphi_1, \varphi_2) \neq 0$
4. If $T_{B_t}(\varphi_1, \varphi_2) = 0, I_{B_t}(\varphi_1, \varphi_2) = 0$ *and* $F_{B_t}(\varphi_1, \varphi_2) = 0$ then $T_{\overline{B_t}}(\varphi_1, \varphi_2) = \min\{T_{A_t}(\varphi_1), T_{A_t}(\varphi_2)\}, I_{\overline{B_t}}(\varphi_1, \varphi_2) = \max\{I_{A_t}(\varphi_1), I_{A_t}(\varphi_2)\}$ *and* $F_{\overline{B_t}}(\varphi_1, \varphi_2) = \max\{F_{A_t}(\varphi_1), F_{A_t}(\varphi_2)\}$.

5.2. Example. Consider a 0.7-NG$\mathcal{G}_t$ as shown in the below figure 15. Then the complement $\overline{\mathcal{G}_t}$ of 0.7-NG$\overline{\mathcal{G}_t}$ is shown in figure 16

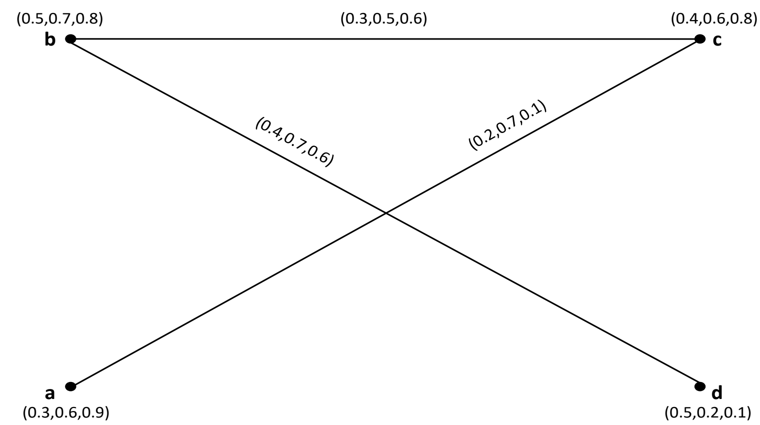

Figure 15. 0.7-NG$\mathcal{G}_{0.7}$

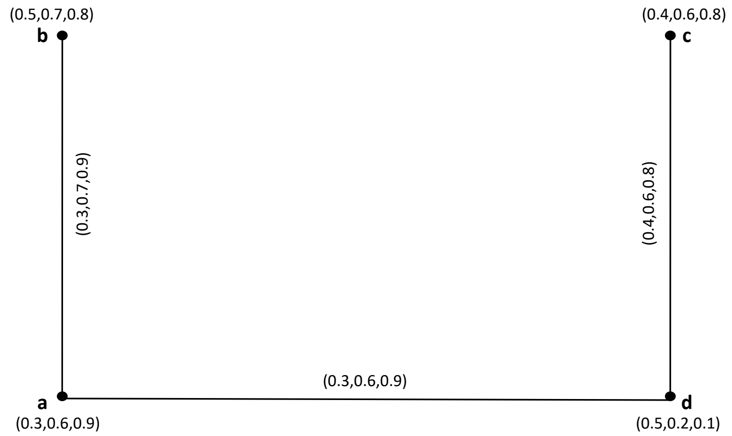

Figure 16. 0.7-NG$\overline{\mathcal{G}_{0.7}}$

5.3. Definition. A t-NG$\mathcal{G}_t$ is called self-complementary t-NG if $\overline{\mathcal{G}_t} \approx \mathcal{G}_t$.

5.4. Preposition. Let $\mathcal{G}_t = (A_t, B_t)$ be a self-complementary t-NG. Then

$$\sum_{\varphi_1 \neq \varphi_2} T_{B_t}(\varphi_1, \varphi_2) = \sum_{\varphi_1 \neq \varphi_2} min\{T_{A_t}(\varphi_1), T_{A_t}(\varphi_2)\}$$

$$\sum_{\varphi_1 \neq \varphi_2} I_{B_t}(\varphi_1, \varphi_2) = \sum_{\varphi_1 \neq \varphi_2} max\{I_{A_t}(\varphi_1), I_{A_t}(\varphi_2)\}$$

$$\sum_{\varphi_1 \neq \varphi_2} F_{B_t}(\varphi_1, \varphi_2) = \sum_{\varphi_1 \neq \varphi_2} max\{F_{A_t}(\varphi_1), F_{A_t}(\varphi_2)\}$$

5.5. Preposition. Let $\mathcal{G}_t = (A_t, B_t)$ t-NG. If

$$\sum_{\varphi_1 \neq \varphi_2} T_{B_t}(\varphi_1, \varphi_2) = \sum_{\varphi_1 \neq \varphi_2} min\{T_{A_t}(\varphi_1), T_{A_t}(\varphi_2)\}$$

$$\sum_{\varphi_1 \neq \varphi_2} I_{B_t}(\varphi_1, \varphi_2) = \sum_{\varphi_1 \neq \varphi_2} max\{I_{A_t}(\varphi_1), I_{A_t}(\varphi_2)\}$$

$$\sum_{\varphi_1 \neq \varphi_2} F_{B_t}(\varphi_1, \varphi_2) = \sum_{\varphi_1 \neq \varphi_2} max\{F_{A_t}(\varphi_1), F_{A_t}(\varphi_2)\} \; \forall \; \varphi_1, \varphi_2 \in V$$

Then $\mathcal{G}_t$ is a self-complementary t-NG

5.6. Preposition. For any two t-NG $\mathcal{G}_t$ and $\mathcal{G}_t'$. If $\mathcal{G}_t$ and $\mathcal{G}_t'$ have a strong homomorphism, then $\overline{\mathcal{G}_t}$ and $\overline{\mathcal{G}_t'}$ also have a strong isomorphism.

Proof. Let $\mathcal{G}_t$ and $\mathcal{G}_t'$ have a strong isomorphism, denoted by φ. Given that φ is a bijective map, φ^{-1} is thus a bijective map, with $\varphi^{-1}(\varpi_1) = \varphi_1, \forall \varpi_1 \in V'$. Thus

$$T_{A_t}(\varphi^{-1}(\varpi_1)) = T_{A_t'}(\varphi_1), I_{A_t}(\varphi^{-1}(\varpi_1)) = I_{A_t'}(\varphi_1) \; and \; F_{A_t}(\varphi^{-1}(\varpi_1)) = F_{A_t'}(\varphi_1), \forall \varpi_1 \in V'$$

Applying definition 23 makes it clear that:

$$T_{\overline{B_t}}(\varphi_1, \varpi_1) = \min\{T_{A_t}(\varphi_1), T_{A_t}(\varpi_1)\}$$

$$T_{\overline{B_t}}(\varphi_1, \varpi_1) \le \min\left\{T_{A_t'}(\varphi(\varphi_2)), T_{A_t'}(\varphi(\varpi_2))\right\}$$

$$T_{\overline{B_t}}(\varphi_1, \varpi_1) \le \min\left\{T_{A_t'}(\varphi_2), T_{A_t'}(\varpi_2)\right\}$$

$$T_{\overline{B_t}}(\varphi_1, \varpi_1) = T_{\overline{B_t}}(\varphi_2, \varpi_2)$$

Thus $T_{\overline{B_t}}(\varphi_1, \varpi_1) \le T_{\overline{B_t}}(\varphi_2, \varpi_2)$

Also

$$I_{\overline{B_t}}(\varphi_1, \varpi_1) = \max\{I_{A_t}(\varphi_1), I_{A_t}(\varpi_1)\}$$
$$I_{\overline{B_t}}(\varphi_1, \varpi_1) \leq \max\left\{I_{A_t'}(\varphi(\varphi_2)), I_{A_t'}(\varphi(\varpi_2))\right\}$$
$$I_{\overline{B_t}}(\varphi_1, \varpi_1) \leq \max\left\{I_{A_t'}(\varphi_2), I_{A_t'}(\varpi_2)\right\}$$
$$I_{\overline{B_t}}(\varphi_1, \varpi_1) = I_{\overline{B_t}}(\varphi_2, \varpi_2)$$

And

$$F_{\overline{B_t}}(\varphi_1, \varpi_1) = \max\{F_{A_t}(\varphi_1), F_{A_t}(\varpi_1)\}$$
$$F_{\overline{B_t}}(\varphi_1, \varpi_1) \leq \max\left\{F_{A_t'}(\varphi(\varphi_2)), F_{A_t'}(\varphi(\varpi_2))\right\}$$
$$F_{\overline{B_t}}(\varphi_1, \varpi_1) \leq \max\left\{F_{A_t'}(\varphi_2), F_{A_t'}(\varpi_2)\right\}$$
$$F_{\overline{B_t}}(\varphi_1, \varpi_1) = F_{\overline{B_t}}(\varphi_2, \varpi_2)$$

Consequently $I_{\overline{B_t}}(\varphi_1, \varpi_1) \leq I_{\overline{B_t}}(\varphi_2, \varpi_2), F_{\overline{B_t}}(\varphi_1, \varpi_1) \leq F_{\overline{B_t}}(\varphi_2, \varpi_2)$

It follows from this that $\overline{\mathcal{G}_t}$ and $\overline{\mathcal{G}_t'}$ are strongly isomorphic.

6. Application: Sustainable Development through a Circular Economy:

The idea of a circular economy has gained popularity in the fight against poverty and for sustainable development. Regenerative systems, reduced waste, and increased resource efficiency are the goals of a circular economy. Let's examine a case where the seven vertices stand for important elements in the shift to a circular economy. Resource Efficiency (f_1) is the term used to describe how best to use resources while reducing waste production. Waste Management (f_2) this field focuses on effective methods for disposing of waste and recycling it. Green innovation (f_3) Promotes the creation and uptake of eco-friendly activities and technologies. The concept of Sustainable Consumption (f_4) involves encouraging conscientious and responsible consumption habits in both individuals and industries. Renewable Energy Adoption (f_5) Signifies the transition towards sustainable and renewable energy sources. Eco-friendly Production (f_6) Promotes the use of sustainable and environmentally friendly production techniques in industry.

Let $V = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ be the vertex of set of factors that are Sustainable Development through a Circular Economy. The edges are the level of connections between two factors. The graphical representation of the factor of poverty reduction is given below;

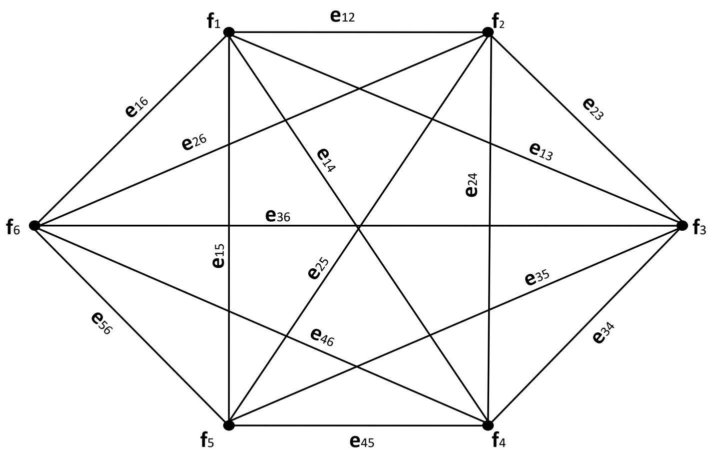

Figure 17.Neutrosophic Graph

Using the t-neutrosophic fuzzy model, decision-makers can examine the degree of correlation between these components. To illustrate how encouraging responsible consumer has a major impact on eco-friendly production practices, consider a high membership degree from Sustainable consumer (f_4) to Eco-friendly Production (f_6). This might indicate a substantial positive influence. Conversely, the degree of non-membership may suggest any felt disconnection or lack of significance. This illustration shows how a circular economy might function as a comprehensive plan that addresses a number of issues, including waste management, resource efficiency, green innovation, and community involvement. In order to negotiate the uncertainties associated with moving to a circular economy, decision-makers can utilize the t-neutrosophic fuzzy model to customize their strategy. They can do this by modifying the parameter't' based on their domain expertise and priorities. The edges of truth membership, indeterminacy membership and falsity membership degrees of factors as t-neutrosophic values, the truth membership value represents the possible way of two factors, the indeterminacy membership represents either connected or disconnected of two factors both can occur and the falsity membership values represents the disconnections of two factors. For example, the edge e_{12} represents the connection of 'Resource Efficiency' and 'Waste Management' indicates that the industrial properties creating the job opportunities. ie, $e_{12} = (0.7,0.3,0.3)$, the truth membership value 0.7 indicates the strong connection, the indeterminacy membership value 0.3 indicates the connection is either strong or weak or else average strength and the falsity membership value 0.3 indicates the weak connection between the two factors. The decision-makers can customize the t-NG according to their background and situation by using the stated values of't'. Furthermore, various't' values correspond to varied attitudes toward risk and uncertainty. Table 1 displays the NS and 0.7-NS highlighted for the edges.

Edges	**NS**	**0.7-NS**	**Edges**	**NS**	**0.7-NS**
$\boldsymbol{e_{12}}$=$(\boldsymbol{f_1},\boldsymbol{f_2})$	(0.8,0.2,0.1)	(0.7,0.3,0.3)	e_{26}=(f_2,f_6)	(0.8,0.4,0.4)	(0.7,0.4,0.4)
$\boldsymbol{e_{13}}$=$(\boldsymbol{f_1},\boldsymbol{f_3})$	(0.6,0.4,0.3)	(0.6,0.4,0.3)	e_{34}=(f_3,f_4)	(0.6,0.3,0.1)	(0.6,0.3,0.3)
$\boldsymbol{e_{14}}$=$(\boldsymbol{f_1},\boldsymbol{f_4})$	(0.7,0.3,0.2)	(0.7,0.3,0.3)	e_{35}=(f_3,f_5)	(0.3,0.1,0.3)	(0.3,0.3,0.3)
$\boldsymbol{e_{15}}$=$(\boldsymbol{f_1},\boldsymbol{f_5})$	(0.6,0.2,0.3)	(0.6,0.3,0.3)	e_{36}=(f_3,f_6)	(0.5,0.3,0.6)	(0.5,0.3,0.6)
$\boldsymbol{e_{16}}$=$(\boldsymbol{f_1},\boldsymbol{f_6})$	(0.8,0.1,0.2)	(0.7,0.3,0.3)	e_{45}=(f_4,f_5)	(0.8,0.2,0.7)	(0.7,0.3,0.7)
$\boldsymbol{e_{23}}$=$(\boldsymbol{f_2},\boldsymbol{f_3})$	(0.9,0.4,0.5)	(0.7,0.4,0.5)	e_{46}=(f_4,f_6)	(0.9,0.6,0.5)	(0.7,0.6,0.5)
$\boldsymbol{e_{24}}$=$(\boldsymbol{f_2},\boldsymbol{f_4})$	(0.5,0.2,0.3)	(0.5,0.3,0.3)	e_{56}=(f_5,f_6)	(0.8,0.7,0.4)	(0.7,0.7,0.4)
$\boldsymbol{e_{25}}$=$(\boldsymbol{f_2},\boldsymbol{f_5})$	(0.4,0.1,0.2)	(0.4,0.3,0.3)			

Table 2: Edges of NS and 0.7-NS

The table of truth membership, indeterminacy membership and falsity membership degree of each factor is given below;

factors	**Degree of each factor**
$\boldsymbol{f_1}$	$\deg(f_1) = (3.3,1.6,1.5)$
$\boldsymbol{f_2}$	$\deg(f_2) = (3.0,1.7,1.8)$
$\boldsymbol{f_3}$	$\deg(f_3) = (2.7,1.7,2.0)$
$\boldsymbol{f_4}$	$\deg(f_4) = (3.2,1.8,2.1)$
$\boldsymbol{f_5}$	$\deg(f_5) = (2.7,1.9,2.0)$

$\boldsymbol{f_6}$	$\deg(f_6) = (3.3, 2.3, 2.2)$

Table 3: Degree of each Factor

The score function of edges are defined as:

$$\mathfrak{T} = \sqrt{T_j^2 + (1 - I_j)^2 + (1 - F_j)^2}\ , 1 \leq j \leq 6$$

By using this definition in the above table in order to find the optimal factor for reducing the poverty is given below;

factors	Score function $\mathfrak{T}(\boldsymbol{f_j})$
$\boldsymbol{f_1}$	3.38673
$\boldsymbol{f_2}$	3.18276
$\boldsymbol{f_3}$	2.96310
$\boldsymbol{f_4}$	3.47706
$\boldsymbol{f_5}$	3.01662
$\boldsymbol{f_6}$	3.97663

Table 4: Score function

Comparatively, $\mathfrak{T}(f_6)$ =3.97663 is the greatest value. So, by our assumption f_6 factor has the high potential to Sustainable Development through a Circular Economy.

6.1 Comparative Analysis

The specific requirements of the decision-making issue dictate the form of fuzzy graph to use: t-neutrosophic or t-intuitionistic. Both approaches employ the 't' parameter to fine-tune uncertainty, but when a more extensive, exact, and detailed representation of uncertainty is necessary, t-neutrosophic fuzzy graphs excel. t-neutrosophic fuzzy graphs are particularly beneficial in complicated choice situations such as pattern recognition, medical diagnosis, and sophisticated decision support systems due to their ability to independently judge truth, indeterminacy, and falsehood. Because of their exceptional flexibility and meticulous treatment of uncertainty factors, t-neutrosophic fuzzy graphs are the preferred choice for high levels of precision and granularity in uncertainty modeling.

7. Conclusion

This study concludes by introducing and examining the use of t-neutrosophic graphs as an effective tool for comprehending and visualizing elaborate scenarios and complex interactions. The study provides a strong theoretical framework for t-neutrosophic graphs by introducing ideas like homomorphism and isomorphism. These graphs can be used to effectively describe relationships involving multiple variables thanks to the suggested core procedures. The ability of t-neutrosophic graphs to handle various aspects of complex systems is demonstrated by their practical application in addressing sustainable development through a circular economy. Through the consideration of several components and their interdependencies, t-neutrosophic graphs are an adaptable and useful tool for policy formulation and well-informed decision making about complex social issues.

This study demonstrates how flexible and helpful t-neutrosophic graphs are when handling complex situations in the real world. Their proven ability to tackle the circular economy demonstrates how they can support the objectives of sustainable development. t-neutrosophic graphs are a visual and analytical tool that shed light on complex linkages and serve as a foundation for well-informed policy formation and decision-making. All things considered, this study advances our knowledge of t-neutrosophic graphs and their useful applications, opening us new avenues for investigation in a variety of contexts and challenging problem-solving situations.

Future work: We planned to extend this work in following

1. Super hyper t-neutrsophic graphs
2. Maximum product of t- Super hyper t-neutrsophic graphs
3. Complements of t- Super hyper t-neutrsophic graphs

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