NSS
Neutrosophic Sets and Systems, Vol. 72, 2024
University of New Mexico

Neutrosophic Quadruple H_v-modules and their fundamental module

Madeleine Al Tahan[1], Saba Al-Kaseasbeh [2,*], and Bijan Davvaz [3]
[1]College of Engineering, Abu Dhabi University, United Arab Emirates; altahan.madeleine@gmail.com
[2]Department of Mathematics, Tafila Technical University, Tafila, Jordan; saba.alkaseasbeh@gmail.com
[3] Department of Mathematical Sciences, Yazd University, Yazd, Iran; davvaz@yazd.ac.ir
[*]Correspondence:saba.alkaseasbeh@gmail.com

Abstract. This article aims to explore the deeper connections between neutrosophy and hyperstructures. Specifically, we introduce the concept of neutrosophic quadruple H_v-modules over neutrosophic quadruple H_v-rings, examine their properties, and identify their fundamental module.

Keywords: H_v-module; neutrosophic quadruple number; neutrosophic quadruple H_v-module; fundamental ring; fundamental module.

---

1. **Introduction**

In 1934, Marty [15] introduced the concept of hypergroups by generalizing the notion of a group. In a hypergroup, the binary operation is defined on a non-empty set and satisfies properties such as associativity, the existence of an identity element, and the presence of invertible elements. Hypergroups can be seen as a natural extension of groups and have applications in various fields, including physics, computer science, and combinatorics. Beyond hypergroups, other algebraic hyperstructures such as hyperrings and hyperfields also exist. These structures are utilized in algebraic geometry, number theory, and many areas of pure and applied mathematics (see [3–5, 9–12, 25]).

Vougiouklis introduced the notion of weak hyperstructures, also known as H_v-structures, at the Fourth Congress of the Algebraic Hyperstructures and its Applications (AHA) in 1990 [23, 24]. A weak hyperstructure is a generalization of a hyperstructure where the binary operation is not necessarily defined for all pairs of elements in the set, but only for certain subsets of pairs. This weakening of the definition of a hyperstructure allows for a wider range of structures to be studied, and has applications in fields such as topology and fuzzy logic. Vougiouklis's work on weak hyperstructures has also led to the development of other related concepts, such as soft sets and rough sets, which are used in data analysis and decision making.

Fuzzy set theory, introduced by Lotfi A. Zadeh [26] in 1965, deals with the concept of partial truth, where an element can belong to a set to a certain degree, rather than being either completely in or completely out of the set. While fuzzy set theory has many real-life applications, it has limitations when it comes to dealing with situations of indeterminacy. This is where neutrosophy comes in. Neutrosophy, introduced by Florentin Smarandache in 1995, is a generalization of fuzzy set theory that deals with situations of indeterminacy, inconsistency, and incomplete knowledge. It extends the concept of partial truth by introducing the concept of "neutrosophic truth," where a statement can be simultaneously true, false, and indeterminate to different degrees. Neutrosophy allows for a more flexible and nuanced way of modeling real-world situations, and it has applications in a variety of fields, including philosophy, mathematics, and artificial intelligence. Symbolic neutrosophic theory refers to the use of abstract symbols, such as the letters T, I, and F, to represent the neutrosophic components of truth, indeterminacy, and falsehood. This notation allows for a more concise and precise way of expressing neutrosophic statements and computations, and it has been used in a variety of applications, such as decision-making, image processing, and expert systems. Fore more detail about neutrosophy, we refer to [16, 18–21].

In 2015, Florentin Smarandache [17] introduced the concept of neutrosophic quadruple numbers, which are a generalization of real and complex numbers that incorporate the neutrosophic concept of partial truth. Neutrosophic quadruple numbers consist of four components: a real component, an imaginary component, a neutrosophic truth component, and a neutrosophic falsity component. Smarandache presented the arithmetic operations of addition, subtraction, multiplication, and scalar multiplication on the set of neutrosophic quadruple numbers. Agboola et al. [1] subsequently established a connection between neutrosophy and algebraic structures by considering the set of neutrosophic quadruple numbers and using the defined operations to discuss neutrosophic quadruple algebraic structures, such as neutrosophic groups and neutrosophic rings. Akinleye et al. [2] further generalized this work in 2016 by considering the set of neutrosophic quadruple numbers and defining hyperoperations on it, leading to the study of neutrosophic quadruple hyperstructures, such as neutrosophic hypergroups and neutrosophic hyperrings. Overall, the introduction of neutrosophic quadruple numbers and the study of neutrosophic algebraic structures and hyperstructures have opened up new avenues of research and applications in fields such as mathematics, computer science, and engineering. In neutrosophic mathematics, a neutrosophic quadruple H_v-group is a hypergroup with four binary operations, namely neutrosophic addition, neutrosophic multiplication, neutrosophic h-addition, and neutrosophic h-multiplication, which satisfy certain axioms. Similarly, a neutrosophic quadruple H_v-ring is a ring with four binary operations that satisfy additional axioms.

The study of neutrosophic quadruple H_v-groups and neutrosophic quadruple H_v-rings is a relatively new area of research in neutrosophic mathematics, and many of their properties and characteristics are still being explored. However, it is interesting to note that the concept of the fundamental group, which is a central concept in algebraic topology, has been extended to neutrosophic quadruple H_v-groups, and its properties have been studied.

In particular, it has been shown that the fundamental group of a neutrosophic quadruple H_v-group is also a neutrosophic quadruple group, which is a group-like structure with four components representing truth, falsehood, indeterminacy, and non-membership. This result provides a deeper understanding of the topological properties of neutrosophic quadruple H_v-groups and their relationship to algebraic structures.

The authors in [6,7] discussed neutrosophic quadruple H_v-groups and neutrosophic quadruple H_v-rings and studied their properties. Then in [8], they found the fundamental group of neutrosophic quadruple H_v-groups and proved that it is a neutrosphic quadruple group. Our paper extends their results to H_v-modules and it is constructed as follows: after an Introduction, Section 2 presents the basic concepts that are used throughout our paper. Section 3 defines neutrosophic H_v-modules and studies their basic properties. Finally, Section 4 characterizes the fundamental module of neutrosophic H_v-modules up to isomorphism.

2. Preliminaries

This section provides some hyperstructure theory, definitions and theorems that are used throughout the paper. (See [10,12,25].)

A hyperoperation on a non-empty set H is a binary operation $\circ : H \times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ is the set of all non-empty subsets of H. The operation $\circ$ is usually denoted by $\circ_n$ when it has arity n, that is, when it takes n arguments. For example, $\circ_2$ is the binary operation defined above.

A hyperoperation can be thought of as a generalization of a binary operation, which takes two elements of H and returns a single element of H. In contrast, a hyperoperation takes two elements of H and returns a non-empty subset of H.

Hyperoperations are used in the study of hypergroups, which are algebraic structures that generalize groups.

To clarify the definitions further, if we have given a hyperoperation $\circ$ on a non-empty set H and two non-empty subsets $A, B \subseteq H$, we define $A \circ B$ as the union of all elements of the form $a \circ b$ where $a \in A$ and $b \in B$. That is,

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b.$$

We also define $x \circ A$ as $\{x\} \circ A$ and $A \circ x$ as $A \circ \{x\}$.

Definition 2.1. Let $(H, \circ)$ be a hypergrupoid. Then:

- $(H, \circ)$ is a semihypergorup if $\circ$ is associative, meaning that for any $x, y, z \in H$, we have $(x \circ y) \circ z = x \circ (y \circ z)$.
- $(H, \circ)$ is a quasi-hypergroup if every element $x \in H$ reproduces the entire set H under the hyperoperation $\circ$, meaning that $x \circ H = H = H \circ x$.
- $(H, \circ)$ is a hypergroup if it is both a semihypergroup and a quasi-hypergroup.

Hypergroups generalize the notion of groups and provide a natural framework for studying non-associative algebraic structures.

The concept of H_v-structures were introduced by T. Vougiouklis [23, 24] is an important generalization of hyperstructures that allows for more flexible and weaker axioms, which can capture a wider range of algebraic systems. This makes H_v-structures a powerful tool for studying hyperstructures that arise in various areas of mathematics and science such as algebraic geometry, combinatorics, and physics, to model complex systems that cannot be described by traditional algebraic structures. However, Most of H_v-structures are used in representation theory.

Definition 2.2. A hypergroupoid $(H, \circ)$ is called an H_v-semigroup if it satisfies the following axiom:

$$(h_1 \circ (h_2 \circ h_3)) \cap ((h_1 \circ h_2) \circ h_3) \neq \emptyset \text{ for all } h_1, h_2, h_3 \in H.$$

This axiom is weaker than the associativity axiom of classical semigroups, which requires that $(h_1 \circ h_2) \circ h_3 = h_1 \circ (h_2 \circ h_3)$ for all $h_1, h_2, h_3 \in H$.

The H_v-semigroups are used to study non-associative algebraic structures that arise in representation theory.

An element $0 \in H$ is called an identity of $(H, \circ)$ if it satisfies the following axiom:

$$h \in 0 \circ h \cap h \circ 0 \text{ for all } h \in H.$$

Definition 2.3. A hypergroupoid $(H, \circ)$ is called an H_v-group if it is a quasi-hypergroup and an H_v-semigroup. A quasi-hypergroup is a hypergroupoid $(H, \circ)$ such that for every $h_1 \in H$, there exists $h_2 \in H$ such that $h_1 \in h_2 \circ h_1 \cap h_1 \circ h_2$.

An H_v-subgroup of an H_v-group $(H, \circ)$ is a non-empty subset $S \subseteq H$ such that $(S, \circ)$ is also an H_v-group.

Definition 2.4. A multivalued system $(R^{\sim}, +, \cdot)$ is a *hyperring* if (1) $(R^{\sim}, +)$ is a hypergroup; (2) $(R^{\sim}, \cdot)$ is a semihypergroup; (3) $\cdot$ is distributive with respect to $+$. And it is an *H_v-ring* if (1) $(R^{\sim}, +)$ is an H_v-group; (2) $(R^{\sim}, \cdot)$ is is an H_v-semigroup; (3) $\cdot$ is weak distributive with respect to $+$. $(R^{\sim}, +, \cdot)$ is said to be commutative if $\alpha + \beta = \beta + \alpha$ and $\alpha \cdot \beta = \beta \cdot \alpha$ for all $\alpha, \beta \in R^{\sim}$.

An element $1 \in R^{\sim}$ is called a *unit* if $\alpha \in 1 \cdot \alpha \cap \alpha \cdot 1$ for all $\alpha \in R^{\sim}$. A subset S of an H_v-ring $(R^{\sim}, +, \cdot)$ is called an *H_v-subring* of $R^{\sim}$ if $(S, +, \cdot)$ is an H_v-ring. To prove that $(S, +, \cdot)$ is an H_v-subring of $(R^{\sim}, +, \cdot)$, it suffices to show that $\alpha + S = S + \alpha = S$ and $\alpha \cdot \ \beta \subseteq S$ for all $\alpha, \beta \in R^{\sim}$.

Now, a homomorphism between two H_v-rings $(R^{\sim}, +, \star)$ and $(S, +', \star')$ is a function $f : R^{\sim} \to S$ that preserves the ring structure and the valuation function, meaning:

$f(\alpha + \beta) = f(\alpha) +' f(\beta)$ for all $\alpha, \beta \in R^{\sim}$. $f(\alpha \star \beta) = f(\alpha) \star' f(\beta)$ for all $\alpha, \beta \in R^{\sim}$. $v_S(f(\alpha)) = v_{R^{\sim}}(\alpha)$ for all $\alpha \in R^{\sim}$. Here, $v_R^{\sim}$ and v_S denote the valuation functions on $R^{\sim}$ and S, respectively.

Finally, two H_v-rings $R^{\sim}$ and S are isomorphic if there exists an H_v-ring homomorphism $f : R^{\sim} \to S$ that is bijective, and its inverse $f^{-1} : S \to R^{\sim}$ is also an H_v-ring homomorphism. In other words, $R^{\sim}$ and S are isomorphic if they have the same ring structure and the same valuation function, up to a bijective change of variables.

Definition 2.5. A non-empty set M is an H_v-module over an H_v-ring $R^{\sim}$, if $(M, +)$ is an H_v-group and there exists a map $\star : R \times M \longrightarrow \mathcal{P}^*(M)$, $(r, \alpha) \to r \star \alpha$, such that for all $\alpha, \beta \in M$, $r, s \in R$, the following conditions hold.

- $(r \star (\alpha + \beta)) \cap (r \star \alpha + r \star \beta) \neq \emptyset$;
- $((r + s) \star \alpha) \cap (r \star \alpha + s \star \alpha) \neq \emptyset$;
- $((rs) \star \alpha) \cap (r \star (s \star \alpha)) \neq \emptyset$.

Definition 2.6. A non-empty subset K of an H_v-module $(M, +, R, \cdot)$ over an H_v-ring R is called an *H_v-submodule* of M if $r \cdot x \subseteq K$ and $\alpha + K = K + \alpha = K$ for all $r \in R, \alpha, x \in K$.

These conditions ensure that K is a well-behaved subset of M that is stable under the action of R, and forms a group under addition that is closed under both vector addition and set addition. Note that if M is a finite-dimensional vector space over a field, then an H_v-submodule of M is simply a subspace of M.

Definition 2.7. An H_v-module homomorphism $f : M \to N$ is a structure-preserving map between H_v-modules $(M, +, R, \star)$ and $(N, +', S, \star')$ over H_v-rings R and S, respectively. It satisfies the following conditions:

- $f(x + y) = f(x) +' f(y)$ for all $x, y \in M$.
- There exists an H_v-ring homomorphism $g : R \to S$ such that $f(r \star x) = g(r) \star' f(x)$ for all $x \in M$ and $r \in R$. This means that f preserves the H_v-module structure of M and N, and also the ring structure of R and S.

Two H_v-modules $(M, +, R, \star)$ and $(N, +', S, \star')$ are isomorphic if there exists a bijective H_v-module homomorphism $f : M \to N$ and an H_v-ring isomorphism $g : R \to S$ such that $f(x + y) = f(x) +' f(y)$ and $f(r \star x) = g(r) \star' f(x)$ for all $x, y \in M$ and $r \in R$. This means that M and N have the same H_v-module structure and are related by an H_v-module homomorphism and an H_v-ring isomorphism.

Example 2.8. Let R be a ring, $(M, +, R, \star)$ be an R-module and N be a submodule of M. Then $(M, +, R, \cdot)$ is an R-H_v-module where "$\cdot$" is defined as follows: for all $(r, m) \in R \times M$, $r \cdot m = r \star m + N$.

For the special case $R = \mathbb{Z}$, $M = \mathbb{Z}_4$ (M is the $\mathbb{Z}$-module of integers under standard addition and multiplication modulo four.) and $N = 2\mathbb{Z}_4 = \{0, 2\}$. we get the following:

$$k \cdot 0 = k \cdot 2 = \{0, 2\},$$

$$k \cdot 1 = k \cdot 3 = \{k \mod 4, (k + 2) \mod 4\} = \begin{cases} \{0, 2\}, \text{if } k \text{ is an even integer;} \\ \{1, 3\}, \text{otherwise.} \end{cases}$$

Example 2.9. Let $F_2 = \{0, 1\}$ and define $(F_2, +)$ and $(F_2, \times)$ by the following tables:

$+$	0	1
0	0	1
1	1	F_2

$\times$	0	1
0	0	0
1	0	1

Define the map $\cdot : F_2^2 \times F_2 \to \mathcal{P}^*(F_2^2)$ as:

$$a \cdot (x, y) = (a \times x, a \times y)$$

for all $a, x, y, z \in F_2$.

It is clear that F_2^2 is an H_v-module over F_2. We present "$+_M$" by the following symmetric table after setting $m_0 = (0,0), m_1 = (1,0), m_2 = (0,1), m_3 = (1,1)$.

$+_M$	m_0	m_1	m_2	m_3
m_0	m_0	m_1	m_2	m_3
m_1		$\{m_0, m_1\}$	m_3	$\{m_2, m_3\}$
m_1			$\{m_0, m_2\}$	$\{m_1, m_3\}$
m_1				F_2^2

and "$\cdot$" as follows: $0 \cdot m_i = m_0$ and $1 \cdot m_i = m_i$ for $i = 0, 1, 2, 3$.

3. Neutrosophic quadruple H_v-modules and their properties

Neutrosophic quadruple algebraic structures are mathematical systems that generalize the concepts of groups, rings, and fields. They are defined based on the concept of a hyper-operation, which is a binary operation that can be extended to n-ary operations. In [1, 2], Agboola et al. and Akinleye et al. studied neutrosophic quadruple algebraic structures based on quadruple numbers over the set of real numbers. As an alternative to real or complex numbers, neutrosophic quadruple numbers are considered in this section, and they are used to define neutrosophic quadruple Hv-modules over neutrosophic quadruple numbers. Hv-rings.

Definition 3.1. [13] A neutrosophic quadruple X-number is an ordered quadruple (a_1, a_2T, a_3I, a_4F), where $a_1, a_2, a_3, a_4 \in X$ and a_2T, a_3I, a_4F are subsets of X such that $a_1 + a_2T + a_3I + a_4F = X$ and $a_2T \cap a_3I = a_3I \cap a_4F = a_4F \cap a_2T = \varnothing$.

Here, T, I, and F stand for the truth, indeterminacy, and falsity values, respectively. The neutrosophic quadruple X-number is used to represent the truth, indeterminacy, and falsity degrees of a statement with respect to a nonempty set X.

The element a_1 represents the true degree of the statement, while the sets a_2T, a_3I, and a_4F represent the indeterminate, uncertain, and false degrees of the statement, respectively. The sets a_2T, a_3I, and a_4F are mutually exclusive, meaning that a statement cannot have both a high degree of truth and a high degree of falsity. The set of all neutrosophic quadruple X-numbers is denoted by NQ(X), that is,

$$NQ(X) = \{(a_1, a_2T, a_3I, a_4F) : a_1, a_2, a_3, a_4 \in X\}.$$

The *Absorbance Law* for the multiplications of T, I, and F with respect to the preference law $T < I < F$ is defined as follows:

- $T \cdot T = T$
- $T \cdot I = I \times T = I$

- $T \cdot F = F \times T = F$
- $I \cdot I = I$
- $I \cdot F = F \times I = I$
- $F \cdot F = F$

This law governs how the neutrosophic values of truth, indeterminacy, and falsity interact with each other under multiplication. The law is called the Absorbance Law because it specifies that the values of indeterminacy and falsity are absorbed by the value of truth under multiplication.

For example, if a statement has a high degree of truth (T) and a moderate degree of indeterminacy (I), then the Absorbance Law implies that the overall degree of the statement should be closer to T than to I. Similarly, if a statement has a high degree of falsity (F) and a moderate degree of indeterminacy (I), then the overall degree of the statement should be closer to F than to I.

We recall some concepts and results related to neutrosophic quadruple H_v-groups and neutrosophic quadruple H_v-rings from [6, 7].

Let $(R, +, \times)$ be an H_v-ring with zero "0" and unit "1" and define "$\oplus$" and "$\otimes$" on $NQ(R)$ as follows:

$$(x_1, x_2T, x_3I, x_4F) \oplus (y_1, y_2T, y_3I, y_4F)$$
$$= \{(a, bT, cI, dF) : \ a \in x_1 + y_1, b \in x_2 + y_2, c \in x_3 + y_3, d \in x_4 + y_4\}.$$

and

$$(x_1, x_2T, x_3I, x_4F) \otimes (y_1, y_2T, y_3I, y_4F)$$
$$= \{(a, bT, cI, dF) : \ a \in x_1 \times y_1, b \in x_1 \times y_2 \cup x_2 \times y_1 \cup x_2 \times y_2,$$
$$jc \in x_1 \times y_3 \cup x_2 \times y_3 \cup x_3 \times y_1 \cup x_3 \times y_2 \cup x_3 \times y_3,$$
$$d \in x_1 \times y_4 \cup x_2 \times y_4 \cup x_3 \times y_4 \cup x_4 \times y_1 \cup x_4 \times y_2 \cup x_4 \times y_3 \cup x_4 \times y_4\}.$$

In the context of the paper, the symbols T, I, and F represent the truth, indeterminacy, and falsity values, respectively, with $T < I < F$ denoting the preference order of these values. Moreover, the mathematical structure under consideration is an H_v-ring denoted by $(R, +, \times)$, which satisfies the following conditions:

- R is a set equipped with two binary operations, $+$ (addition) and $\times$ (multiplication).
- $(R, +)$ is an abelian group with identity element 0.
- $(R, \times)$ is a monoid with identity element 1.
- The multiplication operation $\times$ distributes over the addition operation $+$, i.e., $a \times (b + c) = (a \times b) + (a \times c)$ and $(a + b) \times c = (a \times c) + (b \times c)$ for all $a, b, c \in R$.
- The element 0 is an absorbing element for the multiplication operation $\times$, i.e., $x \times 0 = 0 \times x = 0$ for all $x \in R$.

Theorem 3.2. *[6] Let H be a set with $0 \in H$. Then $(NQ(H), \oplus)$ is an H_v-group (called neutrosophic H_v-group) with identity $\overline{0} = (0, 0T, 0I, 0F)$ if and only if $(R, +)$ is an H_v-group with identity "0" and $0 + 0 = 0$.*

The following theorem characterizes the relationship between the hyperoperations on a set R and the hyperstructure of its neutrosophic quadruple numbers $NQ(R)$.

Theorem 3.3. *[7] $(NQ(R), \oplus, \otimes)$ is a neutrosophic H_v-ring with zero and unit as $\overline{0} = (0, 0T, 0I, 0F)$ and $\overline{1} = (1, 0T, 0I, 0F)$ respectively if and only if $(R, +, \times)$ is an H_v-ring with zero and unit as 0" and 1," respectively.*

The above theorem provides a useful criterion for determining whether a given hyperstructure of neutrosophic quadruple numbers is a neutrosophic H_v-ring. By checking whether the hyperoperations on R satisfy the conditions of an H_v-ring, we can determine whether the corresponding neutrosophic quadruple hyperstructure is also an H_v-ring.

Let $(M, \star, R, \cdot)$ be an R-H_v-module with $0_M \in M$, $0_M \star 0_M = 0_M$, $m \in 1 \cdot m$, $0_M \in 0_R \cdot m$, $0_M \in r \cdot 0_M$ for all $r \in R, m \in M$. We define the following hyperoperations on $NQ(M)$:

$$(m, nT, pI, qF) \boxplus (m', n\prime T, p\prime I, q\prime F) = (m \star m', (n \star n\prime)T, (p \star p\prime)I, (q \star q\prime)F)$$

and

$$(a, bT, cI, dF) \odot (m, nT, pI, qF) = (a \cdot m, (b \cdot n)T, (c \cdot p)I, (d \cdot q)F).$$

In what follows, $(M, \star, R, \cdot)$ is an R-H_v-module with identity $0_M \in M$, $0_M \star 0_M = 0_M$, $0_M \in 0_R \cdot m$ and $m \in 1 \cdot m$ for all $m \in M$ and $0_M \in r \cdot 0_M$ for all $r \in R$.

Theorem 3.4. *Let $(M, \star, R, \cdot)$ be an R-H_v-module. Then $(NQ(M), \boxplus, NQ(R), \odot)$ is an $NQ(R)$-H_v-module.*

Proof. Suppose that $(M, \star, R, \cdot)$ is an R-H_v-module. Then $(M, \star)$ is an H_v-group. Theorem 3.2 asserts that $(NQ(M), \boxplus)$ is an H_v-group. Since $R \cdot M \subseteq M$, it follows from the definition of "$\odot$" that $NQ(R) \odot NQ(M) \subseteq NQ(M)$. We prove now that the conditions of Definition 2.5 are satisfied. Let $\overline{r} = (r_1, r_2T, r_3I, r_4F), \overline{s} = (s_1, s_2T, s_3I, s_4F) \in NQ(R)$ and $\overline{m} = (m_1, m_2T, m_3I, m_4F), \overline{n} = (n_1, n_2T, n_3I, n_4F) \in NQ(M)$.

(1) We have: $(r_1, r_2T, r_3I, r_4F) \odot ((m_1, m_2T, m_3I, m_4F) \boxplus (n_1, n_2T, n_3I, n_4F)) = (r_1 \cdot (m_1 \star n_1), r_2 \cdot (m_2 \star n_2)T, r_3 \cdot (m_3 \star n_3)I, r_4 \cdot (m_4 \star n_4)F)$ and $((r_1, r_2T, r_3I, r_4F) \odot (m_1, m_2T, m_3I, m_4F)) \boxplus ((r_1, r_2T, r_3I, r_4F) \odot (n_1, n_2T, n_3I, n_4F))$
$= (r_1 \cdot m_1 \star r_1 \cdot n_1, (r_2 \cdot m_2 \star r_2 \cdot n_2)T, (r_3 \cdot m_3 \star r_3 \cdot n_3)I, (r_4 \cdot m_4 \star r_4 \cdot n_4)F)$. Having $(r_i \cdot (m_i \star n_i)) \cap (r_i \cdot m_i \star r_i \cdot n_i) \neq \emptyset$ for $i = 1, 2, 3, 4$ implies that $(r_1, r_2T, r_3I, r_4F) \odot ((m_1, m_2T, m_3I, m_4F) \boxplus) (n_1, n_2T, n_3I, n_4F)) \cap ((r_1, r_2T, r_3I, r_4F) \odot (m_1, m_2T, m_3I, m_4F)) \boxplus ((r_1, r_2T, r_3I, r_4F) \odot (n_1, n_2T, n_3I, n_4F)) \neq \emptyset$.

(2) We have: $((r_1, r_2T, r_3I, r_4F) \oplus (s_1, s_2T, s_3I, s_4F)) \odot (m_1, m_2T, m_3I, m_4F) = ((r_1 + s_1) \cdot m_1, ((r_2 + s_2) \cdot m_2)T, ((r_3 + s_3) \cdot m_3)I, ((r_4 + s_4) \cdot m_4)F)$ and $(r_1, r_2T, r_3I, r_4F) \odot (m_1, m_2T, m_3I, m_4F) \boxplus (s_1, s_2T, s_3I, s_4F)) \odot (m_1, m_2T, m_3I, m_4F) = ((r_1 \cdot m_1 \star s_1 \cdot m_1), (r_2 \cdot m_2 \star s_2 \cdot m_2)T, (r_3 \cdot m_3 \star s_3 \cdot m_3)I, (r_4 \cdot m_4 \star s_4 \cdot m_4)F)$. Having $((r_i + s_i) \cdot m_i) \cap (r_i \cdot m_i \star s_i \cdot m_i) \neq \emptyset$ for $i = 1, 2, 3, 4$ implies that $((r_1, r_2T, r_3I, r_4F) \oplus (s_1, s_2T, s_3I, s_4F)) \odot (m_1, m_2T, m_3I, m_4F) = ((r_1 + s_1) \cdot m_1, (r_2 + s_2) \cdot m_2T, (r_3 + s_3) \cdot m_3I, (r_4 + s_4) \cdot m_4F) \cap (r_1, r_2T, r_3I, r_4F) \odot (m_1, m_2T, m_3I, m_4F) \boxplus (s_1, s_2T, s_3I, s_4F)) \odot (m_1, m_2T, m_3I, m_4F) \neq \emptyset$.

(3) One can easily see that $(r_1 \star (s_1 \star m_1), (r_2 \star (s_2 \star m_2))T, (r_3 \star (s_3 \star m_3))I, (r_4 \star (s_4 \star m_4))F) \in \overline{r} \odot (\overline{s} \odot \overline{m})$ and that $((r_1 \times s_1) \star m_1, ((r_2 \times s_2) \star m_2)T, ((r_3 \times s_3) \star m_3)I, ((r_4 \times s_4) \star m_4)F) \in (\overline{r} \otimes \overline{s}) \odot \overline{m}$. Having $(r_i \star (s_i \star m_i)) \cap ((r_i \times s_i) \star m_i) \neq \emptyset$ for $i = 1, 2, 3, 4$ implies that $(\overline{r} \odot (\overline{s} \odot \overline{m})) \cap ((\overline{r} \otimes \overline{s}) \odot \overline{m})) \neq \emptyset$.

It is clear that $\overline{0_M} = (0_M, 0_MT, 0_MI, 0_MF)$ is an identity in $NQ(M)$ and that $\overline{0_M} \boxplus \overline{0_M} = \overline{0_M}$. Therefore, $NQ(M)$ is an H_v-module. □

Notation 1. *Let $(M, \star, R, \cdot)$ be an R-H_v-module. Then $(NQ(M), \boxplus, NQ(R), \odot)$ is called Neutrosophic quadruple H_v-module.*

Corollary 3.5. *Let $(M, \star, R, \cdot)$ be an R-H_v-module. Then we can construct infinite non-isomorphic Neutrosophic quadruple H_v-module.*

Proof. Suppose that $(M, \star, R, \cdot)$ is an R-H_v-module. Theorem 3.4 asserts that $(NQ(M), \boxplus, NQ(R), \odot)$ is a neutrosophic quadruple H_v-module over $NQ(R)$. Having $NQ(NQ(R))$ a neutrosophic quadruple H_v-ring and applying Theorem 3.4 on $(NQ(M), \boxplus, NQ(R), \odot)$, we get $(NQ(NQ(M)), \boxplus_1, NQ(NQ(R)), \odot_1)$ is a neutrosophic quadruple H_v-module over $NQ(NQ(R))$. Continuing on this pattern, we get $NQ(NQ(\ldots NQ(\ldots(M))\ldots)$ is a neutrosophic quadruple H_v-module over $NQ(NQ(\ldots NQ(\ldots(R))\ldots)$. □

Theorem 3.6. *Let M be a non-empty set with $0_M \in M$ and let $(NQ(M), \boxplus, NQ(R), \odot)$ be an R-H_v-module with identity $\overline{0_M}$ and $\overline{0_M} \boxplus \overline{0_M} = \overline{0_M}$. Then $(M, \star, R, \cdot)$ is an H_v-module.*

Proof. Theorems 3.2 and 3.3 assert that $(M, \star)$ is an H_v-group and $(R, +, \times)$ is an H_v-ring respectively. Let $r, s \in R$, $m, n \in M$. Then $\overline{r} = (r, 0T, 0I, 0F), \overline{s} = (s, 0T, 0I, 0F) \in NQ(R)$ and $\overline{m} = (m, 0_MT, 0_MI, 0_MF), \overline{n} = (n, 0_MT, 0_MI, 0_MF) \in NQ(M)$. Applying conditions of Definition 2.5 on $\overline{r}, \overline{s}, \overline{m}, \overline{n}$, we get the conditions of Definition 2.5 are satisfied for r, s, m, n. □

Using Theorem 3.4 and Theorem 3.6, we get the following Theorem.

Theorem 3.7. *Let M be a non-empty set with $0_M \in M$. Then $(NQ(M), \boxplus, NQ(R), \odot)$ is an $NQ(R)$-H_v-module if and only if $(M, \star, R, \cdot)$ is an R-H_v-module.*

Remark 3.8. Let $(M, \star, R, \cdot)$ be a finite R-H_v-module. Then $|NQ(M)| = |M|^4$.

Example 3.9. Let R be any ring with unit "1" and M be an R-module. Then $(NQ(M), \boxplus, NQ(R), \odot)$ is a neutrosophic quadruple H_v-module.

Example 3.10. Let $R = \mathbb{Z}$, $M = \mathbb{Z}_4$ be an R-module and $N = 2\mathbb{Z}_4 = \{0, 2\}$ in Example 2.8. Then $(NQ(\mathbb{Z}_4), \boxplus, NQ(\mathbb{Z}), \odot)$ is a neutrosophic quadruple H_v-module with 4^4 elements. We present how the hyperoperation "$\odot$" is defined: for all $m, n, p, q \in M$, $(0, 0T, 0I, 0F) \odot (m, nT, pI, qF) = \{(a, bT, cI, dF) : a, b, c, d \in \{0, 2\}\}$, $(1, 1T, 0I, 0F) \odot (m, nT, pI, qF) = \{(a, bT, cI, dF) : a, b \in \{1, 3\}, c, d \in \{0, 2\}\}$.

Example 3.11. Let $(F_2^2, +_M, F_2, \cdot)$ be the H_v-module defined in Example 2.9 over F_2. Then $(NQ(F_2^2), \boxplus, NQ(F_2), \odot)$ is a neutrosophic quadruple H_v-module with 4^4 elements.

Definition 3.12. Let $(M, \star, R, \cdot)$ be an R-H_v-module. A subset X of $NQ(M)$ with $\overline{0_M} = (0_M, 0_MT, 0_MI, 0_MF) \in X$ is called a *Neutrosophic H_v-submodule of $NQ(M)$* if there exists $S \subseteq M$ such that $X = NQ(S)$ and $(X, \boxplus, NQ(R), \odot)$ is a neutrosophic quadruple H_v-module.

Proposition 3.13. *Let $(M, \star, R, \cdot)$ be an R-H_v-module and $S \subseteq M$. A subset $X = NQ(S) \subseteq NQ(M)$ is a Neutrosophic H_v-submodule of $NQ(M)$ if the following conditions are satisfied:*

(1) $\overline{0_M} = (0_M, 0_MT, 0_MI, 0_MF) \in X$;
(2) $(m, nT, pI, qF) \boxplus X = X \boxplus (m, nT, pI, qF) = X$ *for all* $(m, nT, pI, qF) \in X$;
(3) $NQ(R) \odot X \subseteq X$.

Proof. The proof is straightforward. □

Theorem 3.14. *Let $(M, \star, R, \cdot)$ be an R-H_v-module, N a non-empty subset of M and $X = NQ(N)$. Then $(X, \boxplus, NQ(R), \odot)$ is a neutrosophic H_v-submodule of $(NQ(M), \boxplus, NQ(R), \odot)$ if and only if N is an H_v-submodule of M with $0_M \in N$.*

Proof. Let $(M, \star, R, \cdot)$ be an R-H_v-module and X be a neutrosophic H_v-submodule of $NQ(M)$. Then there exist $N \subseteq M$ with $0_M \in N$ such that $X = NQ(N)$. We need to show that N is a submodule of M. For all $n \in N$ and $r \in R$, we have $(n, 0_MT, 0_MI, 0_MF) \in NQ(N)$ and $(r, 0T, 0I, 0F) \in NQ(R)$. Since $NQ(N)$ is a neutrosophic H_v-submodule of $NQ(M)$, it follows that $(n, 0_MT, 0_MI, 0_MF) \boxplus NQ(N) = NQ(N) \boxplus (n, 0_MT, 0_MI, 0_MF) = NQ(N)$ and that $(r \cdot n, 0_MT, 0_MI, 0_MF) = (r, 0T, 0I, 0F) \odot (n, 0_MT, 0_MI, 0_MF) \subseteq NQ(N)$. The latter implies that $n \star N = N \star n = N$ and that $r \cdot N \subseteq N$.

Conversely, let N be an H_v-submodule M with $0_M \in N$. It is clear that $\overline{0_M} = (0_M, 0_MT, 0_MI, 0_MF) \in X = NQ(N)$. Let $(m_1, m_2T, m_3I, m_4F) \in NQ(N)$ and $(r_1, r_2T, r_3I, r_4F) \in NQ(R)$. Having $m_i \in N, r_i \in R$ for $i = 1, 2, 3, 4$ and N an H_v-submodule of M imply that $n_i \star N = N \star n_i = N$ and $r_i \cdot N \subseteq N$ for $i = 1, 2, 3, 4$. It is obvious now that $(m_1, m_2T, m_3I, m_4F) \boxplus NQ(N) = NQ(N) \boxplus (m_1, m_2T, m_3I, m_4F) = NQ(N)$ and that $(r_1, r_2T, r_3I, r_4F) \odot NQ(N) \subseteq NQ(N)$. □

Example 3.15. Let $(NQ(\mathbb{Z}_4), \boxplus, NQ(\mathbb{Z}), \odot)$ be the neutrosophic quadruple H_v-module defined in Example 3.11. Then $NQ(\mathbb{Z}_4), NQ(\{0\})$ and $NQ(\{0, 2\})$ are the only neutrosophic quadruple H_v-submodules of $NQ(\mathbb{Z}_4)$.

Example 3.16. Let $(NQ(F_2^2), \boxplus, NQ(F_2), \odot)$ be the neutrosophic quadruple H_v-module defined in Example 3.11. Then $NQ(F_2^2), NQ(\{m_0\}), NQ(\{m_0, m_1\}), NQ(\{m_0, m_2\})$ are the only neutrosophic quadruple H_v-submodules of $NQ(F_2^2)$.

Definition 3.17. Let $(NQ(M), \boxplus_1, NQ(R), \odot_1)$ and $(NQ(N), \boxplus_2, NQ(R), \odot_2)$ be neutrosophic quadruple H_v-modules. A function $\phi : NQ(M) \to NQ(N)$ is called *neutosophic homomorphism* if the following conditions are satisfied.

(1) $\phi(0_M, 0_MT, 0_MI, 0_MF) = (0_N, 0_NT, 0_NI, 0_NF)$;
(2) $\phi(x \boxplus_1 y) = \phi(x) \boxplus_2 \phi(y)$ for all $x, y \in NQ(M)$;
(3) $\phi(r \odot_1 x) = r \odot_2 \phi(x)$ for all $r \in NQ(R), x \in NQ(M)$.

If ϕ is a neutrosophic homomorphism and bijective then it is called *neutrosophic isomorphism* and we write $NQ(M) \cong NQ(N)$.

Proposition 3.18. *If $f : M \to N$ is an R-H_v-module homomorphism with $f(0_M) = 0_N$. Then there exist a neutrosophic homomorphism from $(NQ(M), \boxplus_1, NQ(R), \odot_1)$ to $(NQ(N), \boxplus_2, NQ(R), \odot_2)$.*

Proof. Let $\phi : NQ(M) \to NQ(N)$ be defined as follows:

$$\phi((m, nT, pI, qF)) = (f(m), f(n)T, f(p)I, f(q)F).$$

It is clear that ϕ is a neutrosophic homomorphism. □

Proposition 3.19. *Let $(M, +_1, R, \cdot_1)$ and $(N, +_2, R, \cdot_2)$ be isomorphic H_v-modules with our assumption on the conditions for $0_R, 1_R \in R$ and $0_M \in M, 0_N \in N$ and $f : (M, +_1, R, \cdot_1) \to (N, +_2, R, \cdot_2)$ be an isomorphism. Then $f(0_M) = 0_N$.*

Proof. Let $f(0_M) = a$. Since $a = f(0_M) = f(0_M +_1 0_M) = a +_2 a$ and $a +_2 y = f(0_M +_1 x) \ni f(x) = y$ for all $y \in N$, it follows that a is a zero of N satisfying $a +_2 a = a$. □

Corollary 3.20. *Let $(M, +_1, R, \cdot_1)$ and $(N, +_2, R, \cdot_2)$ be isomorphic H_v-modules. Then $(NQ(M), \boxplus_1, NQ(R), \odot_1)$ and $(NQ(N), \boxplus_2, NQ(R), \odot_2)$ are isomorphic neutrosophic quadruple H_v-modules.*

Proof. The proof is straightforward by using Proposition 3.18 and Proposition 3.19. □

Let $(M, \star, R, \cdot)$ be a commutative H_v-module, i.e. $(M, \star)$ is a commutative H_v-group, and S be an H_v-submodule of M containing 0_M. Then $(M/S, \star', R, \cdot')$ is an H_v-module under the following hyperoperations: For all $s, t \in S$, $r \in R$, we have:

$$(s \star S) \star' (t \star S) = (s \star t) \star S \text{ and } r \cdot' (s \star S) = (r \cdot s) \star S.$$

It is clear that S is a zero for M/S and that $S \star' S = S$.
We define "$\boxplus'$" and "$\odot'$" on $NQ(M/S)$ in the usual way as defining "$\boxplus$" and "$\odot$" on $NQ(M)$.

Proposition 3.21. *Let $(S, \star, R, \cdot)$ be an H_v-submodule of a commutative H_v-module $(M, \star, R, \cdot)$. Then $(NQ(M/S), \boxplus', NQ(R), \odot')$ is an H_v-module.*

Proof. The proof follows from having $(M/S, +', R, \cdot')$ an H_v-module with S as zero and from Theorem 3.7. □

Proposition 3.22. *[6] Let $(S, +)$ be an H_v-subgroup of a commutative H_v- group $(M, +)$. Then $(NQ(M/S), \oplus) \cong (NQ(M)/NQ(S), \oplus')$.*

The authors used in the proof of Proposition 3.22 a function $g : NQ(M)/NQ(S) \to NQ(M/S)$ and defined it as follows:

$$g((x_1, x_2T, x_3I, x_4F) \oplus NQ(S)) = (x_1 + S, (x_2 + S)T, (x_3 + S)I, (x_4 + S)F).$$

And they proved that it is an H_v-group isomorphism.

Proposition 3.23. *Let $(S, \star, R, \cdot)$ be an H_v-submodule of a commutative H_v-module $(M, \star, R, \cdot)$. Then $(NQ(M/S), \boxplus', NQ(R), \odot') \cong (NQ(M)/NQ(S), \boxplus'', NQ(R), \odot'')$.*

Proof. Let $g : NQ(M)/NQ(S) \to NQ(M/S)$ be defined as follows:

$$g((x_1, x_2T, x_3I, x_4F) \oplus NQ(S)) = (x_1 + S, (x_2 + S)T, (x_3 + S)I, (x_4 + S)F).$$

We claim that g is a neutrosophic isomorphism, that is, g is well defined, one-to-one, onto and neutrosophic homomorphism. The proof that g is well defined and one-to-one is the same as that in the proof of Proposition 3.22. (We refer to [6] for more details.) Moreover, it is clear that f is an onto function. We prove that g is neutrosophic homomorphism. Let $x_i \in M, r_i \in R$ for $i = 1, 2, 3, 4$.

- $g((0_M, 0_M T, 0_M I, 0_M F) \boxplus NQ(S)) = (S, ST, SI, SF)$,
- We have $g(((x_1, x_2T, x_3I, x_4F) \boxplus NQ(S)) \boxplus'' ((y_1, y_2T, y_3I, y_4F) \boxplus NQ(S))) = g((x_1 + y_1, (x_2 + y_2)T, (x_3 + y_3)I, (x_4 + y_4)F) \boxplus NQ(S)) = (x_1 + y_1 + S, (x_2 + y_2 + S)T, (x_3 + y_3 + S)I, (x_4 + y_4 + S)F)$.

  On the other hand, we have $g((x_1, x_2T, x_3I, x_4F) \boxplus NQ(S)) \boxplus' g((y_1, y_2T, y_3I, y_4F) \boxplus NQ(S)) = (x_1 + S, (x_2 + S)T, (x_3 + S)I, (x_4 + S)F) \boxplus (y_1 + S, (y_2 + S)T, (y_3 + S)I, (y_4 + S)F)$.
- We have:
  $(r_1, r_2T, r_3I, r_4F) \odot'' ((x_1, x_2T, x_3I, x_4F) \oplus NQ(S)) = (r_1 \cdot x_1, (r_2 \cdot x_2)T, (r_3 \cdot x_3)I, (r_4 \cdot x_4)F) \boxplus NQ(S)$ and $(r_1, r_2T, r_3I, r_4F) \odot' g((x_1, x_2T, x_3I, x_4F) \boxplus NQ(S)) = (r_1, r_2T, r_3I, r_4F) \odot' (x_1 + S, (x_2 + S)T, (x_3 + S)I, (x_4 + S)F) = (r_1 \cdot x_1 + S, (r_2 \cdot x_2 + S)T, (r_3 \cdot x_3 + S)I, (r_4 \cdot x_4 + S)F)$. It is clear that $g((r_1, r_2T, r_3I, r_4F) \odot'' ((x_1, x_2T, x_3I, x_4F) \oplus NQ(S))) = (r_1, r_2T, r_3I, r_4F) \odot' g((x_1, x_2T, x_3I, x_4F) \oplus NQ(S))$.

Therefore, $(NQ(M/S), \boxplus', NQ(R), \odot') \cong (NQ(M)/NQ(S), \boxplus'', NQ(R), \odot'')$. $\square$

Example 3.24. Using Examples 3.10 and 3.15, we get $NQ(\mathbb{Z}_4/\{0, 2\}) \cong NQ(\mathbb{Z}_4)/NQ(\{0, 2\})$.

4. Fundamental module of neutrosophic H_v-module

In this section, we find the fundamental ring of neutrosophic quadruple H_v-rings and we use it to find the fundamental module of neutrosophic quadruple H_v-modules.

Fundamental relations are a key tool in connecting hyperstructures with classical algebraic structures. They are a generalization of the concept of subalgebra from classical algebra, and they allow us to define algebraic operations on subsets of a hyperstructure in a way that preserves the hyperstructure properties. In 1936, Marty in his seminal paper [15] introduced the concept of fundamental relations in hyperstructure theory, Then Koskas [14] later contributed to the study of fundamental relations and their applications in hyperstructure theory.

Definition 4.1. [11] For all $n \geq 2$, we define the relation γ on an H_v-ring $(R, +, \cdot)$ as follows:

$$a\gamma b \Leftrightarrow \{a, b\} \subseteq u \text{ where } u \text{ is finite sum of finite products of elements in } R.$$

Note that The relation γ is reflexive and symmetric. To see that it is reflexive, note that for any $a \in R$, we have $a \in a$, which is a finite sum of finite products of elements in R. Therefore, $a\gamma a$. To see that it is symmetric, suppose $a\gamma b$ for some $a, b \in R$. Then there exists a finite sum $s = \sum_{i=1}^{k} a_i$ of finite products of elements in R such that $a, b \in s$. But then we also have $b, a \in s$, so $b\gamma a$.

The fundamental equivalence relation $\gamma^\star$ is defined as the transitive closure of γ. This means that $a\gamma^\star b$ if and only if there exist elements $a_0, \ldots, a_n \in R$ such that $a = a_0 \gamma a_1 \gamma \cdots \gamma a_n = b$. In other words, $a\gamma^\star b$ if and only if a and b are connected by a chain of elements related by γ. It is easy to see that $\gamma^\star$ is an equivalence relation on R.

The fundamental ring $R/\gamma^\star$ is the quotient of R by the equivalence relation $\gamma^\star$. Elements of $R/\gamma^\star$ are equivalence classes $[a]$ of elements in R that are related by $\gamma^\star$. The addition and multiplication operations on $R/\gamma^\star$ are defined by $[a] + [b] = [a + b]$ and $[a] \times [b] = [a \times b]$. It can be shown that $R/\gamma^\star$ is a well-defined ring.

Similarly, we can define the relation γ_N on $NQ(R)$ and its fundamental equivalence relation $\gamma_N^\star$, which gives rise to the fundamental ring $NQ(R)/\gamma_N^\star$. This ring is obtained by quotienting the neutrosophic quadruple ring $NQ(R)$ by the equivalence relation $\gamma_N^\star$. The addition and multiplication operations on $NQ(R)/\gamma_N^\star$ are defined in a similar way to those on $R/\gamma^\star$.

Proposition 4.2. *Let $(R, +, \times)$ be an H_v-ring and $a, b, c, d \in R$. Then*

(1) $\gamma_N(a, 0T, 0I, 0F) = \gamma_N(a, bT, cI, dF)$;
(2) $\gamma_N^\star(a, 0T, 0I, 0F) = \gamma_N^\star(a, bT, cI, dF)$;
(3) $\gamma_N(0, 0T, 0I, 0F) = \gamma_N(0, bT, cI, dF)$;
(4) $\gamma_N^\star(0, 0T, 0I, 0F) = \gamma_N^\star(0, bT, cI, dF)$.

Proof. Let $a, b, c, d \in R$.

(1) Having $(1, 0T, 0I, 0F) \otimes (0, 1T, 0I, 0F) = \{(0, 0T, 0I, 0F), (0, 1T, 0I, 0F)\}$ implies that

$$\{(0, 0T, 0I, 0F), (0, bT, cI, dF)\} \subseteq (1, 0T, 0I, 0F) \otimes (0, 1T, 0I, 0F) \otimes (0, bT, cI, dF).$$

It is obvious that $(a, 0T, 0I, 0F), (0, bT, cI, dF) \in ((1, 0T, 0I, 0F) \otimes (a, 0T, 0I, 0F)) \oplus ((1, 0T, 0I, 0F) \otimes (0, 1T, 0I, 0F) \otimes (0, bT, cI, dF))$.
(2) It is straightforward by using 1.
(3) By setting $a = 0$ in 1., we get the required.
(4) It is straightforward by using 3.□

Proposition 4.3. *Let $(R, +, \times)$ be an H_v-ring and $a, a' \in R$. Then $a\gamma a'$ if and only if $\gamma_N(a, 0T, 0I, 0F) = \gamma_N(a', 0T, 0I, 0F)$.*

Proof. Let $a\gamma a'$. Then there exist $x_{ij} \in R$ and $n, k_i \in \mathbb{N}$ such that $a, a' \in \sum\limits_{1 \leq i \leq n} (\prod\limits_{1 \leq j \leq k_i} x_{ij})$. It is easy to see that

$$(a, 0T, 0I, 0F), (a', 0T, 0I, 0F) \in \sum_{1 \leq i \leq n} (\prod_{1 \leq j \leq k_i} \overline{x_{ij}}),$$

where $\overline{x_{ij}} = (x_{ij}, 0T, 0I, 0F)$.

Let $\gamma_N(a, 0T, 0I, 0F) = \gamma_N(a', 0T, 0I, 0F)$. Then there exist $(x_{ij}, y_{ij}T, z_{ij}I, w_{ij}F) \in NQ(R)$, $n, k_i \in \mathbb{N}$ such that $(a, 0T, 0I, 0F), (a', 0T, 0I, 0F) \in \sum_{1\leq i\leq n}(\prod_{1\leq j\leq k_i}(x_{ij}, y_{ij}T, z_{ij}I, w_{ij}F))$. Then it is clear that $a, a' \in \sum_{1\leq i\leq n}(\prod_{1\leq j\leq k_i} x_{ij})$. Thus, $a\gamma a'$. □

Proposition 4.4. *Let $(R, +, \times)$ be an H_v-ring and $a, a', b, b', c, c', d, d' \in R$.*

If $\gamma_N(a, bT, cI, dF) = \gamma_N(a', b'T, c'I, d'F)$ then $a\gamma a'$.

Proof. Let $(a, bT, cI, dF)\gamma_N(a', b'T, c'I, d'F)$. Then there exist $(x_{ij}, y_{ij}T, z_{ij}I, w_{ij}F) \in NQ(R)$, $n, k_i \in \mathbb{N}$ such that

$$(a, bT, cI, dF), (a', b'T, c'I, d'F) \in \sum_{1\leq i\leq n}(\prod_{1\leq j\leq k_i}(x_{ij}, y_{ij}T, z_{ij}I, w_{ij}F)).$$

Then it is clear that $a, a' \in \sum_{1\leq i\leq n}(\prod_{1\leq j\leq k_i} x_{ij})$. Thus, $a\gamma a'$. □

Proposition 4.5. *Let $(R, +, \times)$ be an H_v-ring and $a, a' \in R$. Then $a\gamma^\star a'$ if and only if $\gamma_N^\star(a, 0T, 0I, 0F) = \gamma_N^\star(a', 0T, 0I, 0F)$.*

Proof. Let $a\gamma^\star a'$. Then there exist $a_i \in R$ and $k \in \mathbb{N}$ with $i = 1, 2, \ldots, k$ such that $a\gamma a_1$, $a_1\gamma a_2$,..., $a_k\gamma a'$. Using Proposition 4.3, we get $(a, 0T, 0I, 0F)\gamma_N(a_1, 0T, 0I, 0F)$, $(a_1, 0T, 0I, 0F)\gamma_N(a_2, 0T, 0I, 0F)$, ..., $(a_k, 0T, 0I, 0F)\gamma_N(a', 0T, 0I, 0F)$. Thus, $(a, 0T, 0I, 0F)\gamma_N^\star(a', 0T, 0I, 0F)$.

Suppose that $(a, 0T, 0I, 0F)\gamma_N^\star(a', 0T, 0I, 0F)$. Then there exist $k \in \mathbb{N}$ and $\overline{x_i} = (x_i, y_iT, z_iI, w_iF) \in NQ(R)$, $i = 1, 2, 3, 4$ such that $(a, 0T, 0I, 0F)\gamma_N\overline{x_1}$, $\overline{x_1}\gamma_N\overline{x_2}$, ..., $\overline{x_k}\gamma_N(a', 0T, 0I, 0F)$. Proposition 4.4 asserts that $a\gamma x_1$, $x_1\gamma x_2$, $\ldots x_k\gamma a'$. Thus, $a\gamma^\star a'$. □

Proposition 4.6. *Let $(R, +, \times)$ be an H_v-ring and $a, b, c, d \in R$. Then $a\gamma^\star a'$ if and only if $\gamma_N^\star(a, bT, cI, dF) = \gamma_N^\star(a', b'T, c'I, d'F)$.*

Proof. Suppose that $a\gamma^\star a'$. Proposition 4.5 asserts that $(a, 0T, 0I, 0F)\gamma_N^\star(a', 0T, 0I, 0F)$. By using Proposition 4.2, we get

$$(a, 0T, 0I, 0F)\gamma_N^\star(a, bT, cI, dF) \text{ and } (a', 0T, 0I, 0F)\gamma_N^\star(a', b'T, c'I, d'F).$$

Having $\gamma_N^\star$ an equivalence relation on $NQ(R)$ implies that

$$(a, bT, cI, dF)\gamma_N^\star(a', b'T, c'I, d'F).$$

Conversely, let $(a, bT, cI, dF)\gamma_N^\star(a', b'T, c'I, d'F)$. By using Proposition 4.2, we get $(a, 0T, 0I, 0F)\gamma_N^\star(a, bT, cI, dF)$ and $(a', 0T, 0I, 0F)\gamma_N^\star(a', b'T, c'I, d'F)$. Having $\gamma_N^\star$ an equivalence relation on $NQ(R)$ implies that $(a, 0T, 0I, 0F)\gamma_N^\star(a', 0T, 0I, 0F)$. The latter and using Proposition 4.5 imply that $a\gamma^\star a'$. □

We define "$\oplus_\gamma$", "$\otimes_\gamma$" on $R/\gamma^\star$ and "$\oplus_{\gamma_N}$", "$\otimes_{\gamma_N}$" on $NQ(R)/\gamma_N^\star$ as follows: For all $r_i, s_i \in R$ with $i = 1, 2, 3, 4$, we have:

$$\gamma^\star(r_1) \oplus_\gamma \gamma^\star(r_2) = \gamma^\star(r), \text{ where } r \in r_1 + r_2,$$

$$\gamma^\star(r_1) \otimes_\gamma \gamma^\star(r_2) = \gamma^\star(r), \text{ where } r \in r_1 \times r_2,$$

$$\begin{aligned}&\gamma_N^\star(r_1, r_T, r_3I, r_4F) \oplus_{\gamma_N} \gamma_N^\star(s_1, s_2T, s_3I, s_4F)\\ &= \gamma_N^\star(\overline{r}), \overline{r} \in (r_1, r_T, r_3I, r_4F) \oplus (s_1, s_2T, s_3I, s_4F),\end{aligned}$$

$$\begin{aligned}&\gamma_N^\star(r_1, r_T, r_3I, r_4F) \otimes_{\gamma_N} \gamma_N^\star(s_1, s_2T, s_3I, s_4F)\\ &= \gamma_N^\star(\overline{r}), \overline{r} \in (r_1, r_2T, r_3I, r_4F) \otimes (s_1, s_2T, s_3I, s_4F).\end{aligned}$$

Theorem 4.7. *Let $(R, +, \times)$ be an H_v-ring. Then $NQ(R)/\gamma_N^\star \cong R/\gamma^\star$.*

Proof. Suppose that $\psi : NQ(R)/\gamma_N^\star \to R/\gamma^\star$ be defined as follows:

$$\psi(\gamma_N^\star(a, bT, cI, dF)) = \gamma^\star(a).$$

Proposition 4.6 asserts that ψ is well defined and one-to-one. Moreover, it is clear that ψ is onto as $\psi(\gamma_N^\star(a, 0T, 0I, 0F)) = \gamma^\star(a)$ for all $a \in R$. We need to show that ψ is ring homomorphism. We have:

(1) $\psi(\gamma_N^\star(a, bT, cI, dF) \oplus_{\gamma_N} \gamma_N^\star(a', b'T, c'I, d'F)) = \psi(\gamma_N^\star(e, fT, gI, hF)) = \gamma^\star(e)$, where $(e, fT, gI, hF) \in (a, bT, cI, dF) \oplus (a', b'T, c'I, d'F) = (a+a', (b+b')T, (c+c')I, (d+d')F)$. On the other hand, we have

$$\psi(\gamma_N^\star(a, bT, cI, dF)) \oplus_\gamma \psi(\gamma_N^\star(a', b'T, c'I, d'F)) = \gamma^\star(a) \oplus_\gamma \gamma^\star(a') = \gamma^\star(e),$$

where $e \in a + a'$. Consequently, we obtain

$$\begin{aligned}&\psi(\gamma_N^\star(a, bT, cI, dF) \oplus_{\gamma_N} \gamma_N^\star(a', b'T, c'I, d'F))\\ &= \psi(\gamma_N^\star(a, bT, cI, dF)) \oplus_\gamma \psi(\gamma_N^\star(a', b'T, c'I, d'F)).\end{aligned}$$

(2) $\psi(\gamma_N^\star(a, bT, cI, dF) \otimes_{\gamma_N} \gamma_N^\star(a', b'T, c'I, d'F)) = \psi(\gamma_N^\star(e, fT, gI, hF)) = \gamma^\star(e)$ where $(e, fT, gI, hF) \in (a, bT, cI, dF) \otimes (a', b'T, c'I, d'F)$. It is clear that $e \in a \times a'$. On the other hand, we have $\psi(\gamma_N^\star(a, bT, cI, dF)) \otimes_\gamma \psi(\gamma_N^\star(a', b'T, c'I, d'F)) = \gamma^\star(a) \otimes_\gamma \gamma^\star(a') = \gamma^\star(e)$ where $e \in a \times a'$. Thus, $\psi(\gamma_N^\star(a, bT, cI, dF) \otimes_{\gamma_N} \gamma_N^\star(a', b'T, c'I, d'F)) = \psi(\gamma_N^\star(a, bT, cI, dF)) \otimes_\gamma \psi(\gamma_N^\star(a', b'T, c'I, d'F))$. □

Corollary 4.8. *Let $(R, +, \times)$ be an H_v-ring. Then $NQ(R)$ has a trivial fundamental ring if and only if R has a trivial fundamental ring.*

Proof. The proof follows from Theorem 4.7. □

The authors in [?,4] considered the set of arithmetic functions R and defined hyperoperations $+$ and $\times$ on it as follows:

$$\alpha + \beta(n) = \{\alpha(d) + \beta(\frac{n}{d}) : d|n\};$$

and

$$\alpha \times \beta(n) = \{\alpha(d)\beta(\frac{n}{d}) : d|n\}.$$

Let $0_\star(n) = 0$ for all $n \in \mathbb{N}$ and

$$\iota(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $0_\star + 0_\star = 0_\star$ and that $\iota \times \iota = \iota$. The authors proved that $(R, +, \times)$ is an H_v-ring with $0_\star$ as zero and ι as unit and found its fundamental ring as the ring of complex numbers $\mathbb{C}$ under standard addition and multiplication.
It is obvious that $0_\star$ is an absorbing element.

Proposition 4.9. *Let $(R, +, \times)$ be the H_v-ring of arithmetic functions under the above hyperoperations. Then $(NQ(R), \oplus, \otimes)$ is a neutrosophic quadruple H_v-ring.*

Proof. The proof follows from Theorem 3.3. □

Example 4.10. Let $(R, +, \times)$ be the H_v-ring of arithmetic functions. Then the ring of complex numbers $\mathbb{C}$ under standard addition and multiplication is the fundamental ring of $(NQ(R), \oplus, \otimes)$.

Definition 4.11. For all $n > 1$, we define the relation ε on an H_v-module $(M, +, R, \star)$ over an H_v-ring R as follows: $x\varepsilon y$ if and only if there exist $n, n_i \in \mathbb{N}$, $(m_1, \cdots, m_n) \in M^n$, $(k_1, \cdots, k_n) \in \mathbb{N}^n$, $(x_{i1}, \cdots, x_{ik}) \in R^{k_i}$ such that

$$x, y \in \sum_{i=1}^{n} m_i', m_i' = m_i \text{ or } m_i' = \sum_{j=1}^{n_i} (\prod_{k=1}^{k_i j} x_{ijk}) m_i.$$

Clearly, the relation ε is reflexive and symmetric. Denote by ε^* the transitive closure of ε. The $\varepsilon^\star$ is called the *fundamental equivalence relation* on M and M/ε^* is the *fundamental module*.

In what follows, $(M, \star, R, \cdot)$ is an R-H_v-module with identity $0_M \in M$, $0_M \star 0_M = 0_M$, $0_M \in 0_R \cdot m$ and $m \in 1 \cdot m$ for all $m \in M$ and $0_M \in r \cdot 0_M$ for all $r \in R$ and $\varepsilon, \varepsilon_N$ the relation on $R, NQ(R)$ and $\varepsilon^\star, \varepsilon_N^\star$ their fundamental relations respectively.

Proposition 4.12. *Let $(M, \star, R, \cdot)$ be an H_v-module and $m, n, p, q \in M$. Then*

(1) $\varepsilon_N(m, 0_M T, 0_M I, 0_M F) = \varepsilon_N(m, nT, pI, qF)$;

(2) $\varepsilon_N^\star(m, 0_M T, 0_M I, 0_M F) = \varepsilon_N^\star(m, nT, pI, qF)$;

(3) $\varepsilon_N(0_M, 0_MT, 0_MI, 0_MF) = \varepsilon_N(0_M, nT, pI, qF)$;

(4) $\varepsilon^\star_N(0_M, 0_MT, 0_MI, 0_MF) = \varepsilon^\star_N(0_M, nT, pI, qF)$.

Proof. We prove 1., the others are straightforward. One can easily see that

$$\begin{aligned}
&(m, 0_MT, 0_MI, 0_MF), (m, nT, pI, qF)\\
&\in ((1, 0T, 0I, 0F) \odot (m, 0_MT, 0_MI, 0_MF))\\
&\quad \boxplus(((1, 0T, 0I, 0F) \otimes (0, 1T, 0I, 0F)) \odot (0_M, nT, 0_MI, 0_MF))\\
&\quad \boxplus(((1, 0T, 0I, 0F) \otimes (0, 0T, 1I, 0F)) \odot (0_M, 0_MT, pI, 0_MF))\\
&\quad \boxplus(((1, 0T, 0I, 0F) \otimes (0, 0T, 0I, 1F)) \odot (0_M, 0_MT, 0_MI, qF)).
\end{aligned}$$

□

Proposition 4.13. *Let $(M, \star, R, \cdot)$ be an H_v-module and $m, m' \in M$. Then $m\varepsilon m'$ if and only if $\varepsilon_N(m, 0_MT, 0_MI, 0_MF) = \varepsilon_N(m', 0_MT, 0_MI, 0_MF)$.*

Proof. Let $m\varepsilon m'$. Then there exist $n, n_i \in \mathbb{N}$, $(m_1, \cdots, m_n) \in M^n$, $(k_1, \cdots, k_n) \in \mathbb{N}^n$, $(x_{i1}, \cdots, x_{ik}) \in R^{k_i}$ such that

$$m, m \in \sum_{i=1}^{n} m'_i, m'_i = m_i \text{ or } m'_i = \sum_{j=1}^{n_i}(\prod_{k=1}^{k_ij} x_{ijk})m_i.$$

It is easy to see that

$$(m, 0_MT, 0_MI, 0_MF), (m', 0_MT, 0_MI, 0_MF) \in \sum_{i=1}^{n} \overline{m'_i}, \overline{m'_i} = \overline{m_i} \text{ or } \overline{m'_i} = \sum_{j=1}^{n_i}(\prod_{k=1}^{k_ij} \overline{x_{ijk}})\overline{m_i}.$$

where $\overline{x_{ij}} = (x_{ij}, 0T, 0I, 0F)$ and $\overline{m_i} = (m_i, 0_MT, 0_MI, 0_MF)$. In a similar way, we prove the backward direction. □

Proposition 4.14. *Let $(M, \star, R, \cdot)$ be an H_v-module and $m, m', n, n', p, p', q, q' \in M$. Then $m\varepsilon^\star m'$ if and only if $\varepsilon^\star_N(m, nT, pI, qF) = \varepsilon^\star_N(m', n'T, p'I, q'F)$;*

Proof. The proof is similar to that of Proposition 4.6. □

Let $(M, \star, R, \cdot)$ be an H_v-module. Then $NQ(M)/\varepsilon^\star_N$ is an $NQ(R)/\gamma^\star_N$-module and that $M/\varepsilon^\star$ is an $R\gamma^\star$-module.

We define "$\boxplus_\varepsilon$", "$\odot_\varepsilon$" on $M/\varepsilon^\star$ and "$\boxplus_{\varepsilon_N}$", "$\odot_{\varepsilon_N}$" on $NQ(M)/\varepsilon^\star_N$ as follows: For all $m_i, n_i \in M$, $r_i \in R$ with $i = 1, 2, 3, 4$, we have:

$$\varepsilon^\star(m_1) \boxplus_\varepsilon \varepsilon^\star(m_2) = \varepsilon^\star(m) \text{ where } m \in m_1 + m_2,$$

$$\gamma^\star(r_1) \odot_\varepsilon \varepsilon^\star(m_1) = \varepsilon^\star(m) \text{ where } m \in r_1 \cdot m_1,$$

$$\varepsilon^\star_N(m_1, m_2T, m_3I, m_4F)) \boxplus_{\varepsilon_N} \varepsilon^\star_N(n_1, n_2T, n_3I, n_4F) = \varepsilon^\star_N(\overline{m}),$$

$$\gamma_N^\star(r_1, r_T, r_3I, r_4F) \odot_{\varepsilon_N} \varepsilon_N^\star(m_1, m_2T, m_3I, m_4F) = \varepsilon_N^\star(\overline{n}),$$

where $\overline{m} \in (m_1, m_2T, m_3I, m_4F) \boxplus ((n_1, n_2T, n_3I, n_4F)$ and $\overline{n} \in (r_1, r_2T, r_3I, r_4F) \odot (m_1, m_2T, m_3I, m_4F)$,

Theorem 4.15. *Let $(M, \star, R, \cdot)$ be an H_v-module. Then $NQ(M)/\varepsilon_N^\star \cong M/\varepsilon^\star$.*

Proof. Let $\chi : NQ(M)/\varepsilon_N^\star \to M/\varepsilon^\star$ be defined as follows:

$$\chi(\varepsilon_N^\star(a, bT, cI, dF)) = \varepsilon^\star(a).$$

Proposition 4.14 asserts that χ is well defined and one-to-one. Moreover, it is clear that χ is onto as $\chi(\varepsilon_N^\star(a, 0_MT, 0_MI, 0_MF)) = \varepsilon^\star(a)$ for all $a \in M$. We need to show that χ is module homomorphism. We have:

(1) $\chi(\varepsilon_N^\star(a, bT, cI, dF) \boxplus_{\varepsilon_N} \varepsilon_N^\star(a', b'T, c'I, d'F)) = \chi(\varepsilon_N^\star(e, fT, gI, hF)) = \varepsilon^\star(e)$ where $(e, fT, gI, hF) \in (a, bT, cI, dF) \boxplus (a', b'T, c'I, d'F) = (a \star a', (b \star b')T, (c \star c')I, (d \star d')F)$. On the other hand, we have $\chi(\varepsilon_N^\star(a, bT, cI, dF)) \boxplus_\varepsilon \chi(\varepsilon_N^\star(a', b'T, c'I, d'F)) = \varepsilon^\star(a) \boxplus_\varepsilon \varepsilon^\star(a') = \varepsilon^\star(e)$ where $e \in a \star a'$. Thus, we obtain $\chi(\varepsilon_N^\star(a, bT, cI, dF) \boxplus_{\varepsilon_N} \varepsilon_N^\star(a', b'T, c'I, d'F)) = \chi(\varepsilon_N^\star(a, bT, cI, dF)) \boxplus_\varepsilon \chi(\varepsilon_N^\star(a', b'T, c'I, d'F))$.

(2) $\chi(\gamma_N^\star(r_1, r_2T, r_3I, r_4F) \odot_{\varepsilon_N} \varepsilon_N^\star(a, bT, cI, dF)) = \chi(\varepsilon_N^\star(e, fT, gI, hF)) = \varepsilon^\star(e)$ where $(e, fT, gI, hF) \in (r_1, r_2T, r_3I, r_4F) \odot (a, bT, cI, dF)$. It is clear that $e \in r_1 \cdot a$. On the other hand, we have $\gamma^\star(r_1) \odot_\varepsilon \chi(\varepsilon_N^\star(a, bT, cI, dF)) = \gamma^\star(r_1) \odot_\varepsilon \varepsilon^\star(a) = \varepsilon^\star(e)$ where $e \in r_1 \cdot a$. Thus, $\chi(\gamma_N^\star(r_1, r_2T, r_3I, r_4F) \odot_{\varepsilon_N} \varepsilon_N^\star(a, bT, cI, dF)) = \psi(r_1, r_2T, r_3I, r_4F) \odot_\varepsilon \chi(\varepsilon_N^\star(a, bT, cI, dF))$. Here "$\psi$ is the ring isomorphism defined in the proof of Theorem 4.7. □

Corollary 4.16. *Let $(M, \star, R, \cdot)$ be an H_v-module. Then $NQ(M)$ has a trivial fundamental module if and only if M has a trivial fundamental module.*

Proof. The proof follows from Theorem 4.15. □

Example 4.17. Let $R = \mathbb{Z}$, $M = \mathbb{Z}_4$ and $N = 2\mathbb{Z}_4 = \{0, 2\}$ in Example 2.8. One can easily see that $M/\varepsilon^\star \cong M/N \cong \mathbb{Z}_2$ where $(\mathbb{Z}_2, +, \mathbb{Z}, \cdot)$ is the $\mathbb{Z}$-module of integers under standard addition and multiplication modulo two. Using Theorem 4.15, we get $(NQ(\mathbb{Z}_4/\varepsilon_N^\star)) \cong \mathbb{Z}_2$.

5. **Conclusion**

The paper presented a detailed study on neutrosophic quadruple H_v-modules. It established that if M and N are isomorphic H_v-modules, then their corresponding neutrosophic quadruple H_v-modules are also isomorphic. Furthermore, the paper provided significant insights into the properties and characteristics of neutrosophic quadruple H_v-modules, thereby contributing to the understanding and application of neutrosophic logic in the field of mathematics.

References

1. A. A. A. Agboola, B. Davvaz and F. Smarandache, *Neutrosophic quadruple algebraic hyperstructures*, Ann. Fuzzy Math. Inform., 14 (2017) 29-42.
2. S. A. Akinleye, F. Smarandache and A. A. A. Agboola, *On neutrosophic quadruple algebraic structures*, Neutrosophic Sets and Systems, 12 (2016) 122-126.
3. M. Al-Tahan, B. Davvaz, *On the existence of hyperrings associated to arithmetic functions*, J. Number Theory, 174 (2017) 136-149.
4. M. Al-Tahan and B. Davvaz, *Strongly regular relations of arithmetic functions*, J. Number Theory, 187 (2018) 391-402.
5. M. Al-Tahan and B. Davvaz, *Chemical Hyperstructures for Elements with Four Oxidation States*, Iranian Journal of Mathematical Chemistry, 13(2) (2022), 85-97.
6. M. Al-Tahan and B. Davvaz, *On neutrosophic quadruple H_v-groups and their properties*, Southeast Asian Bulletin of Mathematics, 44 (2020) 13–625.
7. M. Al-Tahan and B. Davvaz, *On neutrosophic quadruple H_v-rings and their properties*,Neutrosophic Sets and Systems, 36 (2020) 256-270.
8. M. Al-Tahan, B. Davvaz, F. Smarandache, S. Al-Kaseasbeh, *Fundamental group and complete parts of Neutrosophic Quadruple Hv-groups*, Neutrosophic Sets and Systems, 53 (2023) 107-116.
9. M. Al-Tahan and B. Davvaz, *H_v-module of functions over H_v-ring of arithmetics and it's fundamental module*, Analele Stiintifice ale Universitatii Ovidius Constanta: Seria Matematica, 32(1) (2024), 5-18.
10. P. Corsini and V. Leoreanu, *Applications of Hyperstructures Theory*, Advances in Mathematics, Kluwer Academic Publisher, 2003.
11. B. Davvaz and V. Leoreanu-Fotea, *Hyperring Theory and Applications*, International Academic Press, USA, 2007.
12. B. Davvaz, *Polygroup Theory and Related Systems*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013. viii+200 pp.
13. Y.B. Jun , S-Z. Song, F. Smarandache and H. Bordbar, *Neutrosophic quadruple BCK/BCI-algebras*, Axioms, 7(41) (2018) doi:10.3390/axioms7020041.
14. M. Koskas, *Groupoides, demi-hypergroupes et hypergroupes*, J. Math. Pure Appl., (9)49 (1970) 155-192.
15. F. Marty, *Sur une generalization de la notion de group*, In $8th$ Congress Math. Scandenaves, (1934), 45-49.
16. M. Muthuswamy, *Functions of Neu g ζ*H in NT Spaces*, Plithogenic Logic and Computation, 2 (2024), 10-19.
17. F. Smarandache, *Neutrosophic quadruple numbers, refined neutrosophic quadruple numbers, absorbance law, and the multiplication of neutrosophic quadruple numbers*, Neutrosophic Sets and Systems, 10 (2015) 96-98.
18. F. Smarandache, *Neutrosophy, neutrosophic probability, set, and logic*, American Research Press, USA, 1998.
19. F. Smarandache, *Neutrosophic set- A generalization of the intuitionistic fuzzy set.* Int. J. Pure Appl. Math., 24 (2005) 287-297.
20. F. Smarandache and M.Al-Tahan, Theory and Applications of NeutroAlgebras as Generalizations of Classical Algebras, IGI Global, 2022.
21. F. Smarandache and M.Al-Tahan, NeutroGeometry, NeutroAlgebra, and SuperHyperAlgebra in Today's World , IGI Global, 2023. DOI: 10.4018/978-1-6684-4740-6
22. T. Vougiouklis, *Cyclicity in a special class of hypergroups*, Acta Univ. Carolin. Math. Phys., 22(1) (1981) 3-6.
23. T. Vougiouklis, *Hyperstructures and their representations*, Hadronic Press Monographs in Mathematics, Hadronic Press, Inc., Palm Harbor, FL, 1994.

24. T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield*, Algebraic hyperstructures and applications (Xnthi, 1990), 203-211, World Sci. Publ., Teaneck, NJ, 1991.
25. T. Vougiouklis, *From H_v-rings to H_v-fields*. Int. J. Algebraic Hyperstructures Appl., 1(1) (2014) 1-13.
26. L. A. Zadeh, *Fuzzy sets*, Inform and control, 8 (1965) 338-353.

Received: April 10, 2024. **Accepted**: August 29, 2024.