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Modified Non-Linear Triangular Neutrosophic Numbers: Theory and Applications in Integral Equation

A. Mohammed shapique[1,*] and E. Mathivadhana[2]

[1]IFET college of engineering, Villupuram, India; shapique@gmail.com

[2]IFET college of engineering, Villupuram, India; mathivadhana99@gmail.com

[*]Correspondence: shapique@gmail.com

Abstract. Existing methods for handling uncertainty and imprecision often fall short in addressing complex real-world problems. To overcome these limitations, this paper introduces a novel Generalized Non-Linear Triangular Neutrosophic Number (GNLTNN) that effectively captures uncertainty, indeterminacy, and falsity. By analysing GNLTNN through (α, β, γ)-cuts and defining arithmetic operations using the max-min principle, we provide a robust framework for handling neutrosophic information. The proposed neutrosophic Laplace transform method enables efficient solutions to integral equations involving non-linear neutrosophic numbers. The efficacy of our approach is demonstrated through graphical representations.

Keywords: Generalised neutrosophic number; Non-linearity; (α, β, γ)-cuts; Neutrosophic Laplace transform; Integral equation

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1. Introduction

The inherent ambiguity and vagueness of real-world data often pose challenges for traditional mathematical modeling techniques. To address this limitation, Zadeh [31] introduced fuzzy set theory, a revolutionary approach that empowers the representation of imprecise and subjective information. By assigning membership degrees ranging from 0 to 1 , fuzzy sets provide a powerful framework for incorporating the nuances of human perception and subjective experiences into mathematical models. Over the past few decades, fuzzy sets and their applications have been extensively studied across various domains, playing crucial roles in fields like control and engineering [17], medical diagnosis [27], and computer vision [32]. In fuzzy set theory, the membership degree of an element ranges between 0 and 1, representing its degree of belongingness to the set. However, it's crucial to note that the non-membership degree is not always simply the complement of the membership degree. This deviation occurs due to the presence of hesitation or uncertainty. Hesitation arises when there's doubt or ambiguity about an element's inclusion in the set, leading to a non-membership degree that may not directly correlate with its complement. Therefore, in fuzzy sets, non-membership degrees can deviate from being the exact inverse of membership degrees due to factors like uncertainty and hesitation.

Building upon this foundation, intuitionistic fuzzy set theory, proposed by Atanassov [4], expanded upon fuzzy set theory by introducing dual membership functions, capturing both membership and non-membership degrees. This dual representation enriched the characterization of uncertainty and ambiguity, offering researchers enhanced tools for modeling complex systems. Further advancing these concepts, neutrosophic set theory, developed by Smarandache [25], introduced a triadic membership structure accommodating truth, indeterminacy, and falsehood. This comprehensive framework enables neutrosophic sets to address a broader range of complexities and uncertainties, providing a versatile and robust approach for modeling and analyzing systems in the presence of indeterminacy, vagueness, and contradiction.

The significance of neutrosophic theory lies in its capacity to model and analyze systems characterized by ambiguity and inconsistency. Unlike traditional mathematical approaches, neutrosophic theory offers a more flexible and robust methodology for managing uncertainties, enabling researchers to develop accurate and reliable computational models. Wang et al. [30] proposed the notion of single-valued neutrosophic sets as a specific subset of neutrosophic sets tailored for applications in real-world economics and finance. Neutrosophic numbers find applications in various fields by providing a robust framework to handle uncertainty, vagueness, and imprecision. Recently Neutrosophic numbers are applied in solving neutrosophic data envelopment analysis model [16], multiple attribute decision-making problem [24], healthcare services [8], water management system [2], smart cities energy system [8], architecture selection for 5G-radio access network [23] and so on. For detailed studies on neutrosophic sets, see Peng [22] and Wang [29], and references therein.

Fuzzy integral equations constitute a significant branch of mathematics blending fuzzy set theory with integral equations, offering a powerful framework for modeling and solving problems involving uncertainty and imprecision prevalent in various real-world applications. Integrating fuzzy set theory into integral equations provides a flexible and robust approach to tackle problems where precise information is lacking or uncertain. Numerous studies have addressed the solution of fuzzy integral equations, including fuzzy Fredholm integral equations [1, 9] and fuzzy Volterra integral equations [3, 6, 11, 14, 21], as documented in various literature sources.

While linear membership functions have been a popular tool for solving differential and integral equations, a review of the literature reveals a potential gap in addressing non-linear scenarios. Linear membership functions can be considered a special case of nonlinear membership functions. Mondal et al. [19] discussed various arithmetic operations using nonlinear triangular intuitionistic fuzzy numbers (NLTIFN) and solved integral equations with NLTIFN by applying the intuitionistic fuzzy Laplace transform method. Mullor and Molina [18] utilized NLNN to assess supply chain risk management within the automotive sector. Chakraborty et al. [5] discussed various forms of linear and non-linear neutrosophic numbers and applied the concept of NLNN to solve sequencing problems. From the literature survey, it is evident that only a few authors have discussed the concept of NLNN. This paper investigate a modified form of generalised nonlinear triangular neutrosophic number (GNLTNN). Further, we apply GNLTNN to solve Fredholm integral equations. The significance of the paper is as follows. A modified form of an GNLTNN is investigated. The (α, β, γ)-cuts of GNLTNN are obtained. Furthermore, arithmetic operations on GNLTNN are discussed. Neutrosophic valued function are also considered. By employing the neutrosophic Laplace transform method, we derive solutions to integral equations involving non-linear neutrosophic numbers. Finally, a graphical illustration of the solution is provided. This study contributes to the advancement of neutrosophic set theory and nonlinear membership functions, potentially opening new avenues for further research and application.

The remainder of the paper is structured as follows: Section 2 provides an overview of neutrosophic preliminaries and the properties of the neutrosophic Laplace transform. The concept of the modified GNLTNN is introduced in Section 3. Further, the $(\alpha,\ \beta,\ \gamma)$-cuts and its arithmetic operations are discussed. Section 4 presents the method of computing the values and ambiguities of NLTNN. Section 5 discusses the application of neutrosophic methods in solving linear Fredholm integral equations. Results discussion is presented in Section 6. Conclusions drawn from the research and potential areas for further investigation is presented in Section 7.

2. **Preliminaries**

Definition 1 [26] Let X denote a space of points. A neutrosophic set A in X is characterized by a truth $T_A(x)$, indeterminacy $I_A(x)$ and a falsity $F_A(x)$ membership function. These membership functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$ and satisfies $0^- \leqslant S(x) \leqslant 3^+$, where $S(x) = T_A\left(x\right) + I_A\left(x\right) + F_A\left(x\right).$

In practical applications involving scientific and engineering problems, utilizing neutrosophic sets whose values are derived from real standard or non-standard subsets of $]0^-, 1^+[$ can be challenging. Consequently, the single-valued neutrosophic set (SVNS)is often preferred, as it confines its values to the interval $[0, 1]$. This simplification facilitates easier application and analysis in real-world scenarios.

TABLE 1. Comparison with the existing solution approach

Reference	Application	Uncertainty used	Methodology
[20]	Fredholm integral equation	Fuzzy	Integration theory of Fuzzy function
[1]	Fredholm integral equation	Fuzzy	Homotopy Perturbation Method
[10]	Fredholm integral equation	Fuzzy	Artificial neural networks
[19]	Fredholm integral equation	Non-linear Intuitionistic fuzzy number	(α, β)-cuts
Proposed	Fredholm integral equation	Modified non-linear neutrosophic number	(α, β, γ)-cuts

Definition 2 [30] Let X denote a set where each element $x \in X$ represents a point in space. A SVNS A in X is characterized by $T_A(x)$, $I_A(x)$, and $F_A(x)$. Each function assigns a value in the interval $[0,1]$ to every element such that $0 \leqslant S(x) \leqslant 3$. Within the context of a SVNS A in X, an element x represented as $x = (T_x, F_x, I_x)$ is referred to as a SVNN.

Definition 3 [28] Let $x_1 = (T_{x_1}, F_{x_1}, I_{x_1})$ and $x_2 = (T_{x_2}, F_{x_2}, I_{x_2})$ be any two SVNNs. The operations for SVNNs is defined as follows:

(1) $x_1 \oplus x_2 = (T_{x_1} + T_{x_2} - T_{x_1} \times T_{x_2}, I_{x_1} \times I_{x_2}, F_{x_1} \times F_{x_2})$
(2) $x_1 \otimes x_2 = (T_{x_1} + T_{x_2}, I_{x_1} + I_{x_2} - I_{x_1} \times I_{x_2}, F_{x_1} + F_{x_2} - F_{x_1} \times F_{x_2})$
(3) $ax = (1 - (1 - T_x)^a, (I_x)^a, (F_x)^a)$, where $a > 0$

Definition 4 [15]A SVNN $\tilde{A} = \left(\underline{c}, c, \bar{c}; u_{\tilde{A}}, v_{\tilde{A}}, w_{\tilde{A}}\right)$ is termed as Single valued Triangular neutrosophic number (SVTNN)when its membership functions are linear.

$$T_{\tilde{A}}(x) = \begin{cases} \dfrac{(x - \underline{c})\, u_{\tilde{A}}}{c - \underline{c}}, & \text{if } \underline{c} \leqslant x < c \\ u_{\tilde{A}}, & \text{if } x = c \\ \dfrac{(\bar{c} - x)\, u_{\tilde{A}}}{\bar{c} - c}, & \text{if } c < x \leqslant \bar{c} \end{cases} \tag{2.1}$$

$$I_{\tilde{A}}(x) = \begin{cases} \dfrac{(c - x) + (x - \underline{c})\, v_{\tilde{A}}}{c - \underline{c}}, & \text{if } \underline{c} \leqslant x < c \\ v_{\tilde{A}}, & \text{if } x = c \\ \dfrac{(x - c) + (\bar{c} - x)\, v_{\tilde{A}}}{\bar{c} - c}, & \text{if } c < x \leqslant \bar{c} \\ 1, & \text{otherwise} \end{cases} \tag{2.2}$$

$$F_{\tilde{A}}(x) = \begin{cases} \dfrac{(c-x)+(x-\underline{c})\,w_{\tilde{A}}}{c-\underline{c}}, & \text{if } \underline{c} \leqslant x < c \\ w_{\tilde{A}}, & \text{if } x = c \\ \dfrac{(x-c)+(\bar{c}-x)\,w_{\tilde{A}}}{\bar{c}-c}, & \text{if } c < x \leqslant \bar{c} \\ 1, & \text{otherwise} \end{cases} \tag{2.3}$$

The graph of the above functions is presented in Figure 1.

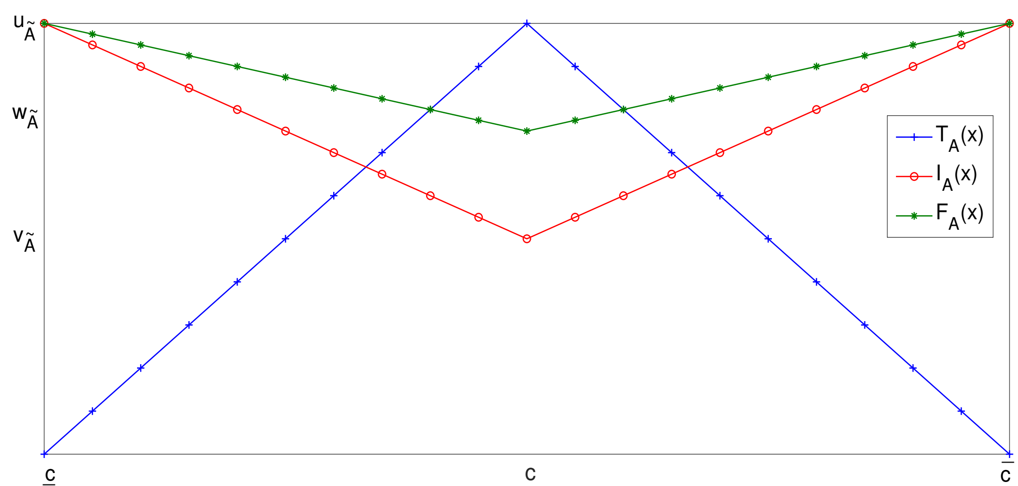

FIGURE 1. Graph of SVTNN

Definition 5 [15] Let $\tilde{A} = \left(\underline{c}, c, \bar{c}; u_{\tilde{A}}, v_{\tilde{A}}, w_{\tilde{A}}\right)$ represents SVTNN then the (α, β, γ)-cut set is defined as:

$$\tilde{A}_{(\alpha,\beta,\gamma)} = \left\{\left\langle x, T_{\tilde{A}}(x) \geqslant \alpha, I_{\tilde{A}}(x) \leqslant \beta, F_{\tilde{A}}(x) \leqslant \gamma \right\rangle : x \in X\right\}, \tag{2.4}$$

satisfies

$$0 \leqslant \alpha \leqslant u_{\tilde{A}}; v_{\tilde{A}} \leqslant \beta \leqslant 1; w_{\tilde{A}} \leqslant \gamma \leqslant 1 \tag{2.5}$$

and

$$\alpha + \beta + \gamma \leqslant 3. \tag{2.6}$$

The (α, β, γ)-cut set of SVTNN is represented as $\tilde{A}_{(\alpha,\beta,\gamma)} = \left\langle \tilde{A}_{(\alpha)}; \tilde{A}_{(\beta)}; \tilde{A}_{(\gamma)} \right\rangle$ which is a crisp subset of X. $\tilde{A}_{(\alpha)}$ represents the (α)-cut set of the SVTNN which is defined as follows.

$$\tilde{A}_{(\alpha)} = \left[T^L_{\tilde{A}}(\alpha), T^U_{\tilde{A}}(\alpha)\right].$$

Using Definition 4, the above set is expressed as

$$\tilde{A}_{(\alpha)} = \left[\underline{c}_{\tilde{A}} + \frac{\alpha\left(c_{\tilde{A}} - \underline{c}_{\tilde{A}}\right)}{u_{\tilde{A}}}, \bar{c}_{\tilde{A}} - \frac{\alpha\left(\bar{c}_{\tilde{A}} - c_{\tilde{A}}\right)}{u_{\tilde{A}}}\right]. \tag{2.7}$$

Similarly, the (β)-cut set and (γ)-cut set of the SVTNN are as follows

$$\tilde{A}_{(\beta)} = \left[I^L_{\tilde{A}}(\beta), I^U_{\tilde{A}}(\beta)\right] = \left[\frac{c_{\tilde{A}} - v_{\tilde{A}}\underline{c}_{\tilde{A}} - \beta\left(c_{\tilde{A}} - \underline{c}_{\tilde{A}}\right)}{1 - v_{\tilde{A}}}, \frac{c_{\tilde{A}} - v_{\tilde{A}}\bar{c}_{\tilde{A}} + \beta\left(\bar{c}_{\tilde{A}} - c_{\tilde{A}}\right)}{1 - v_{\tilde{A}}}\right]. \tag{2.8}$$

$$\tilde{A}_{(\gamma)} = \left[F^L_{\tilde{A}}(\gamma), F^U_{\tilde{A}}(\gamma)\right] = \left[\frac{c_{\tilde{A}} - w_{\tilde{A}}\underline{c}_{\tilde{A}} - \gamma\left(c_{\tilde{A}} - \underline{c}_{\tilde{A}}\right)}{1 - w_{\tilde{A}}}, \frac{c_{\tilde{A}} - w_{\tilde{A}}\bar{c}_{\tilde{A}} + \gamma\left(\bar{c}_{\tilde{A}} - c_{\tilde{A}}\right)}{1 - w_{\tilde{A}}}\right]. \tag{2.9}$$

Definition 6 [5] A SVNN $\tilde{A} = \left(\underline{c}, c, \bar{c}; u_{\tilde{A}}, v_{\tilde{A}}, w_{\tilde{A}}\right)$ is termed as NLTNN, when its membership functions are non-linear.

$$T_{\tilde{A}}(x) = \begin{cases} \left(\frac{x-a}{a-\underline{a}}\right)^{p_1}\omega & \text{if } \underline{a} \leq x < a, \\ \omega & \text{if } x = a, \\ \left(\frac{\bar{a}-x}{\bar{a}-a}\right)^{p_2}\omega & \text{if } a < x \leqslant \bar{a}, \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

$$I_{\tilde{A}}(x) = \begin{cases} \left(\frac{b-x}{b-\underline{b}}\right)^{q_1}\rho & \text{if } \underline{b} \leqslant x < b, \\ 0 & \text{if } x = b, \\ \left(\frac{x-\bar{b}}{\bar{b}-b}\right)^{q_2}\rho & \text{if } b < x \leqslant \bar{b}, \\ \rho & \text{otherwise} \end{cases} \tag{2.11}$$

$$F_{\tilde{A}}(x) = \begin{cases} \left(\frac{c-x}{c-\underline{c}}\right)^{r_1}\lambda & \text{if } \underline{c} \leqslant x < c, \\ 0 & \text{if } x = c, \\ \left(\frac{x-\bar{c}}{\bar{c}-c}\right)^{r_2}\lambda & \text{if } c < x \leqslant \bar{c}, \\ \lambda & \text{otherwise} \end{cases} \tag{2.12}$$

The above definition is derived from Chakraborty et al. [5]. These parameters, ω, ρ and λ scale the membership values in their respective functions.

Definition 7 The parametric form of neutrosophic valued function $f(t)$ is defined as follows :

$$[f(t)]_{(\alpha,\beta,\gamma)} = \left[\underline{f}(t,\alpha), \bar{f}(t,\alpha); \underline{f}(t,\beta), \bar{f}(t,\beta); \underline{f}(t,\gamma), \bar{f}(t,\gamma)\right] \tag{2.13}$$

Definition 8 If $f(t)$ is a neutrosophic valued function then the neutrosophic Laplace transform of $f(t)$ is defined as follows.

$$L\{f(t)\} = F(s) = \lim_{\kappa\to\infty}\int_0^{\kappa} e^{-st}\odot f(t)\,dt. \tag{2.14}$$

The lower and upper neutrosophic Laplace transform of the function $f(t)$ is as follows:

$$L\{f(t)\}_{(\alpha,\beta,\gamma)} = \Bigg[\lim_{\kappa\to\infty}\int_0^{\kappa} e^{-st}\underline{f}_T(t,\alpha)dt,\quad \lim_{\kappa\to\infty}\int_0^{\kappa} e^{-st}\bar{f}_T(t,\alpha)dt;\ \lim_{\kappa\to\infty}\int_0^{\kappa} e^{-st}\underline{f}_I(t,\beta)dt,$$
$$\lim_{\kappa\to\infty}\int_0^{\kappa} e^{-st}\bar{f}_I(t,\beta)dt;\ \lim_{\kappa\to\infty}\int_0^{\kappa} e^{-st}\underline{f}_F(t,\gamma)dt,\quad \lim_{\kappa\to\infty}\int_0^{\kappa} e^{-st}\bar{f}_F(t,\gamma)dt\Bigg].$$

Theorem 1 Let $f(t)$ be a continuous neutrosophic valued function defined on $[0,\infty)$ and exponential order p, then

(1) $L\{f'(t)\} = sF(s)\ominus f(0)$ if $f(t)$ is (i) differentiable
(2) $L\{f'(t)\} = -f(0)\ominus\{-sF(s)\}$ if $f(t)$ is (ii) differentiable

3. Modified form of generalized non-linear triangular neutrosophic number (GNLTNN)

Although Definition 6 outlines NLTNN, it lacks precision in representing NLTNN adequately. Specifically, the functions for indeterminacy-membership and falsity-membership do not associate parameters ρ and λ with their minimum values, but instead with their maximum values. We contend that this aspect hinders the accurate expression of the degree of indeterminacy and falsity within a decision-making framework. To refine the characterization of these functions, a modified form of NLNN is studied. This revised definition considers sets of parameters $p = (p_T, p_I, p_F)$ and $q = (q_T, q_I, q_F)$, where all elements fall within the range $[1, \infty)$.

Definition 9 [18] Let $\tilde{A}_{(p,q)} = \left(\underline{c}, c, \bar{c}; u_{\tilde{A}}, v_{\tilde{A}}, w_{\tilde{A}}\right)$ be a generalised non-linear triangular neutrosophic number whose membership functions are as follows

$$T_{\tilde{A}}(x) = \begin{cases} u_{\tilde{A}}\left\{1-\left(\dfrac{c-x}{c-\underline{c}}\right)^{p_T}\right\}, & \text{if } \underline{c}\leqslant x < c \\ u_{\tilde{A}}, & \text{if } x = c \\ u_{\tilde{A}}\left\{1-\left(\dfrac{c-x}{c-\bar{c}}\right)^{q_T}\right\}, & \text{if } c < x\leqslant \bar{c} \\ 0, & \text{otherwise}\end{cases} \tag{3.1}$$

$$I_{\tilde{A}}(x)=\begin{cases} v_{\tilde{A}}+(1-v_{\tilde{A}})\left(\dfrac{c-x}{\underline{c}}\right)^{p_I}, & \text{if } \underline{c}\leqslant x<c \\ v_{\tilde{A}}, & \text{if } x=c \\ v_{\tilde{A}}+(1-v_{\tilde{A}})\left(\dfrac{c-x}{c-\bar{c}}\right)^{q_I}, & \text{if } c<x\leqslant \bar{c} \\ 1, & \text{otherwise} \end{cases} \tag{3.2}$$

$$F_{\tilde{A}}(x)=\begin{cases} w_{\tilde{A}}+(1-w_{\tilde{A}})\left(\dfrac{c-x}{\underline{c}}\right)^{p_F}, & \text{if } \underline{c}\leqslant x<c \\ w_{\tilde{A}}, & \text{if } x=c \\ w_{\tilde{A}}+(1-w_{\tilde{A}})\left(\dfrac{c-x}{c-\bar{c}}\right)^{q_F}, & \text{if } c<x\leqslant \bar{c} \\ 1, & \text{otherwise} \end{cases} \tag{3.3}$$

The above definition also suggests that the membership functions are centered on parameter c. This scenario appears reasonable within a context where uncertainty is articulated around a specific reference point. Figure 2 and 3 presents various forms of NLTNN.

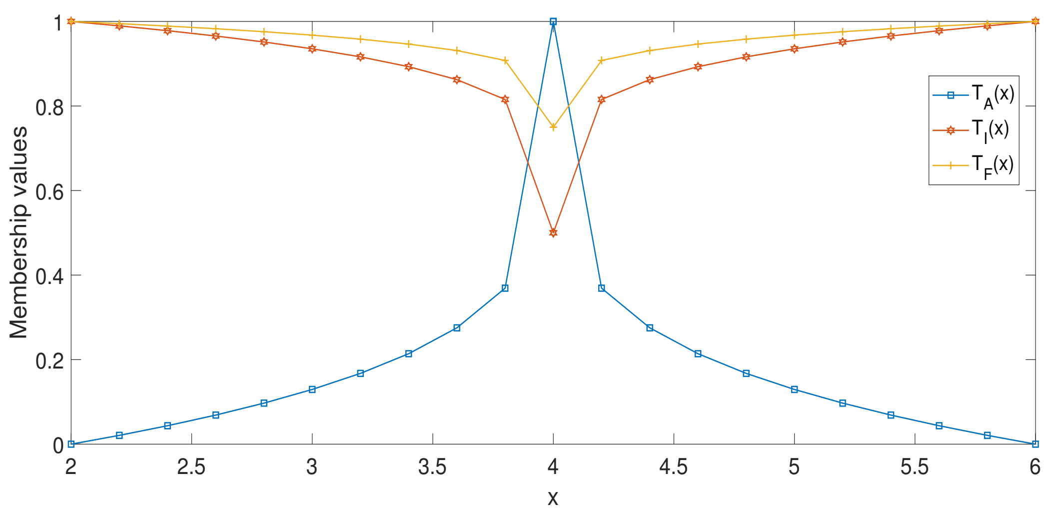

FIGURE 2. NLTNN with $p_T = p_I = p_F$=0.2, $q_T = q_I = q_F$=0.2

3.1. *(α, β, γ)-cuts of GNLTNN*

Definition 10 [18] For a GNLTNN $\tilde{A}_{(p.q)} = \left(\underline{c}, c, \bar{c}; u_{\tilde{A}}, v_{\tilde{A}}, w_{\tilde{A}}\right)$, the ($\alpha$, β, γ)-cut sets is defined as follows:

$$\tilde{A}_{(\alpha,\beta,\gamma)} = \left\{\left\langle x, T_{\tilde{A}}(x) \geqslant \alpha, I_{\tilde{A}}(x) \leqslant \beta, F_{\tilde{A}}(x) \leqslant \gamma\right\rangle : x \in X\right\} \tag{3.4}$$

and satisfies the conditions $0 \leqslant \alpha \leqslant u_{\tilde{A}}; v_{\tilde{A}} \leqslant \beta \leqslant 1; w_{\tilde{A}} \leqslant \gamma \leqslant 1$ and $\alpha+\beta+\gamma \leqslant 3$. The (α, β, γ) -cut sets of GNLTNN is represented as $\tilde{A}_{(p,q,\alpha,\beta,\gamma)}$ which is a crisp subset of X:

$$\tilde{A}_{(p,q,\alpha,\beta,\gamma)} = \left\langle \tilde{A}_{(p_T,q_T,\alpha)}; \tilde{A}_{(p_I,q_I,\beta)}; \tilde{A}_{(p_F,q_F,\gamma)} \right\rangle.$$

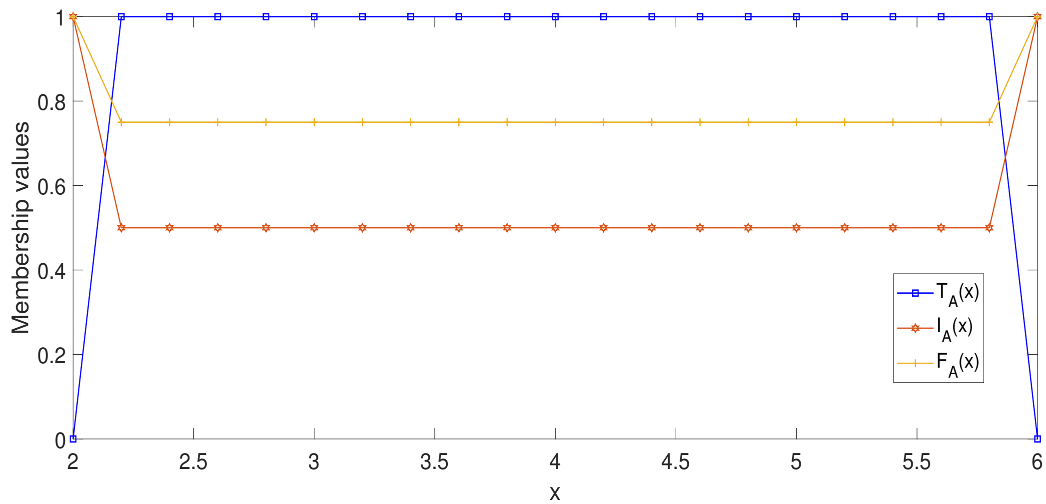

FIGURE 3. NLTNN with $p_T = p_I = p_F$=1000, $q_T = q_I = q_F$=1000

$\tilde{A}_{(p_T,q_T,\alpha)}$ represents the -cut set of the GNLTNN which is a closed interval defined as follows.

$$\tilde{A}_{(p_T,q_T,\alpha)} = \left[T^L_{\tilde{A}}\left(\alpha, p_T\right), T^U_{\tilde{A}}\left(\alpha, q_T\right)\right].$$

Using Definition 9, the above set is expressed as

$$\tilde{A}_{(p_T,q_T,\alpha)} = \left[c-(c-\underline{c})\left(1-\frac{\alpha}{u_{\tilde{A}}}\right)^{\frac{1}{p_T}}, c-(c-\bar{c})\left(1-\frac{\alpha}{u_{\tilde{A}}}\right)^{\frac{1}{q_T}}\right]. \tag{3.5}$$

Similarly,

$$\begin{aligned}\tilde{A}_{(p_I,q_I,\beta)} &= \left[I^L_{\tilde{A}}\left(\beta, p_I\right), I^U_{\tilde{A}}\left(\beta, q_I\right)\right] \\ &= \left[c-\underline{c}\left(\frac{\beta-v_{\tilde{A}}}{1-v_{\tilde{A}}}\right)^{\frac{1}{p_I}}, c-(c-\bar{c})\left(\frac{\beta-v_{\tilde{A}}}{1-v_{\tilde{A}}}\right)^{\frac{1}{q_I}}\right].\end{aligned} \tag{3.6}$$

$$\begin{aligned}\tilde{A}_{(p_F,q_F,\gamma)} &= \left[F^L_{\tilde{A}}\left(\gamma, p_F\right), F^U_{\tilde{A}}\left(\gamma, q_F\right)\right] \\ &= \left[c-\underline{c}\left(\frac{\gamma-w_{\tilde{A}}}{1-w_{\tilde{A}}}\right)^{\frac{1}{p_F}}, c-(c-\bar{c})\left(\frac{\gamma-w_{\tilde{A}}}{1-w_{\tilde{A}}}\right)^{\frac{1}{q_F}}\right].\end{aligned} \tag{3.7}$$

3.2. *Arithmetic operation on GNLTNN*

Let $\tilde{A}_{(p.q)} = \left(\underline{c}_1, c_1, \bar{c}_1; u_{\tilde{A}}, v_{\tilde{A}}, w_{\tilde{A}}\right)$ and $\tilde{B}_{(p.q)} = \left(\underline{c}_2, c_2, \bar{c}_2; u_{\tilde{A}}, v_{\tilde{A}}, w_{\tilde{A}}\right)$ be two GNLTNN with membership function as follows

$$T_{\tilde{A}}(x) = \begin{cases} u_{\tilde{A}}\left\{1 - \left(\dfrac{c_1 - x}{c_1 - \underline{c}_1}\right)^{p_{T1}}\right\}, & \text{if } \underline{c}_1 \leqslant x < c_1 \\ u_{\tilde{A}}, & \text{if } x = c_1 \\ u_{\tilde{A}}\left\{1 - \left(\dfrac{c_1 - x}{c_1 - \bar{c}_1}\right)^{q_{T1}}\right\}, & \text{if } c_1 < x \leqslant \bar{c}_1 \\ 0, & \text{otherwise} \end{cases} \tag{3.8}$$

$$I_{\tilde{A}}(x) = \begin{cases} v_{\tilde{A}} + (1 - v_{\tilde{A}})\left(\dfrac{c_1 - x}{\underline{c}_1}\right)^{p_{I1}}, & \text{if } \underline{c}_1 \leqslant x < c_1 \\ v_{\tilde{A}}, & \text{if } x = c_1 \\ v_{\tilde{A}} + (1 - v_{\tilde{A}})\left(\dfrac{c_1 - x}{c_1 - \bar{c}_1}\right)^{q_{I1}}, & \text{if } c_1 < x \leqslant \bar{c}_1 \\ 1, & \text{otherwise} \end{cases} \tag{3.9}$$

$$F_{\tilde{A}}(x) = \begin{cases} w_{\tilde{A}} + (1 - w_{\tilde{A}})\left(\dfrac{c_1 - x}{\underline{c}_1}\right)^{p_{F_1}}, & \text{if } \underline{c}_1 \leqslant x < c_1 \\ w_{\tilde{A}}, & \text{if } x = c_1 \\ w_{\tilde{A}} + (1 - w_{\tilde{A}})\left(\dfrac{c_1 - x}{c_1 - \bar{c}_1}\right)^{q_{F_1}}, & \text{if } c_1 < x \leqslant \bar{c}_1 \\ 1, & \text{otherwise} \end{cases} \tag{3.10}$$

and

$$T_{\tilde{B}}(x) = \begin{cases} u_{\tilde{A}}\left\{1 - \left(\dfrac{c_2 - x}{c_2 - \underline{c}_2}\right)^{p_{T2}}\right\}, & \text{if } \underline{c}_2 \leqslant x < c_2 \\ u_{\tilde{A}}, & \text{if } x = c_2 \\ u_{\tilde{A}}\left\{1 - \left(\dfrac{c_2 - x}{c_2 - \bar{c}_2}\right)^{q_{T2}}\right\}, & \text{if } c_2 \leqslant x \leqslant \bar{c}_2 \\ 0, & \text{otherwise} \end{cases} \tag{3.11}$$

$$I_{\tilde{B}}(x) = \begin{cases} v_{\tilde{A}} + (1 - v_{\tilde{A}})\left(\dfrac{c_2 - x}{\underline{c}_2}\right)^{p_{I2}}, & \text{if } \underline{c}_2 \leqslant x \leqslant c_2 \\ v_{\tilde{A}}, & \text{if } x = c_2 \\ v_{\tilde{A}} + (1 - v_{\tilde{A}})\left(\dfrac{c_2 - x}{c_2 - \bar{c}_2}\right)^{q_{I2}}, & \text{if } c_2 \leqslant x \leqslant \bar{c}_2 \\ 1, & \text{otherwise} \end{cases} \tag{3.12}$$

$$F_{\tilde{B}}(x) = \begin{cases} w_{\tilde{A}} + (1 - w_{\tilde{A}})\left(\dfrac{c_2 - x}{\underline{c}_2}\right)^{p_{F_2}}, & \text{if } \underline{c}_2 \leqslant x \leqslant c_2 \\ w_{\tilde{A}}, & \text{if } x = c_2 \\ w_{\tilde{A}} + (1 - w_{\tilde{A}})\left(\dfrac{c_2 - x}{c_2 - \bar{c}_1}\right)^{q_{F_2}}, & \text{if } c_2 \leqslant x \leqslant \bar{c}_2 \\ 1, & \text{otherwise} \end{cases} \tag{3.13}$$

The (α, β, γ)-cut is given by

$$\tilde{A}_{(p_1,q_1,\alpha,\beta,\gamma)} = \Bigg[c_1 - (c_1 - \underline{c}_1)\left(1 - \frac{\alpha}{u_{\tilde{A}}}\right)^{\frac{1}{p_{T_1}}}, c_1 - (c_1 - \bar{c}_1)\left(1 - \frac{\alpha}{u_{\tilde{A}}}\right)^{\frac{1}{q_{T_1}}}; c_1 - \underline{c}_1\left(\frac{\beta - v_{\tilde{A}}}{1 - v_{\tilde{A}}}\right)^{\frac{1}{p_{I_1}}}, c_1 - (c_1 - \bar{c}_1)\left(\frac{\beta - v_{\tilde{A}}}{1 - v_{\tilde{A}}}\right)^{\frac{1}{q_{I_1}}}; c_1 - \underline{c}_1\left(\frac{\gamma - w_{\tilde{A}}}{1 - w_{\tilde{A}}}\right)^{\frac{1}{p_{F_1}}}, c_1 - (c_1 - \bar{c}_1)\left(\frac{\gamma - w_{\tilde{A}}}{1 - w_{\tilde{A}}}\right)^{\frac{1}{q_{F_1}}}\Bigg] \tag{3.14}$$

and

$$\tilde{B}_{(p_2,q_2,\alpha,\beta,\gamma)} = \Bigg[c_2 - (c_2 - \underline{c}_2)\left(1 - \frac{\alpha}{u_{\tilde{A}}}\right)^{\frac{1}{p_{T_2}}}, c_2 - (c_2 - \bar{c}_2)\left(1 - \frac{\alpha}{u_{\tilde{A}}}\right)^{\frac{1}{q_{T_2}}}; c_2 - \underline{c}_2\left(\frac{\beta - v_{\tilde{A}}}{1 - v_{\tilde{A}}}\right)^{\frac{1}{p_{I_2}}}, c_2 - (c_2 - \bar{c}_2)\left(\frac{\beta - v_{\tilde{A}}}{1 - v_{\tilde{A}}}\right)^{\frac{1}{q_{I_2}}}; c_2 - \underline{c}_2\left(\frac{\gamma - w_{\tilde{A}}}{1 - w_{\tilde{A}}}\right)^{\frac{1}{p_{F_2}}}, c_2 - (c_2 - \bar{c}_2)\left(\frac{\gamma - w_{\tilde{A}}}{1 - w_{\tilde{A}}}\right)^{\frac{1}{q_{F_2}}}\Bigg] \tag{3.15}$$

3.2.1. *Addition of two GNLTNN*

This section presents addition of two NLTNN using (p,q,α,β,γ)-cut. For simplicity, we consider $p_{T_i} = q_{T_i} = n_1$, $p_{I_i} = q_{I_i} = n_2$, $q_{F_i} = q_{F_i} = n_3$, $i = 1, 2$. Let

$$C_{(p,q,\alpha,\beta,\gamma)} = A_{(p_1,q_1,\alpha,\beta,\gamma)} + B_{(p_2,q_2,\alpha,\beta,\gamma)} \tag{3.16}$$

Using Equations (3.14) and (3.15), we get

$$C_{(p,q,\alpha,\beta,\gamma)} = \Bigg[\underline{\zeta}_1\left(1 - \frac{\alpha}{u_{\tilde{A}}}\right)^{\frac{1}{n_1}}, \bar{\zeta}_1\left(1 - \frac{\alpha}{u_{\tilde{A}}}\right)^{\frac{1}{n_1}}; \underline{\xi}_1\left(\frac{\beta - v_{\tilde{A}}}{1 - v_{\tilde{A}}}\right)^{\frac{1}{n_2}}, \bar{\zeta}_1\left(\frac{\beta - v_{\tilde{A}}}{1 - v_{A}}\right)^{\frac{1}{n_2}}; \underline{\xi}_1\left(\frac{\gamma - w_{\tilde{A}}}{1 - w_{\tilde{A}}}\right)^{\frac{1}{n_3}}, \bar{\zeta}_1\left(\frac{\gamma - w_{\tilde{A}}}{1 - w_{\tilde{A}}}\right)^{\frac{1}{n_3}}\Bigg]. \tag{3.17}$$

The membership function of the above Equation (3.17) is as follows.

$$T_C(x) = \begin{cases} u_{\tilde{A}}\left\{1 - \quad \dfrac{x - c_1 - c_2}{\underline{\xi}_1}\right)^{n_1}\right\}, & \text{for } \underline{c}_1 + \underline{c}_2 \leqslant x < c_1 + c_2 \\ u_{\tilde{A}}, \quad \text{for } x = c_1 + c_2 \\ u_{\tilde{A}}\left\{1 - \left(\dfrac{x - c_1 + c_2}{\bar{\xi}_1}\right)^{n_1}\right\}, & \text{for } c_1 + c_2 < x \leqslant \bar{c}_1 + \bar{c}_2 \\ 0, \quad \text{otherwise} \end{cases} \tag{3.18}$$

$$I_C(x) = \begin{cases} v_{\tilde{A}} + (1 - v_{\tilde{A}})\left(\dfrac{c_1 + c_2 - x}{\underline{c}_1 + \underline{c}_2}\right)^{n_2}, & \text{for } \underline{c}_1 + \underline{c}_2 \leqslant x < c_1 + c_2 \\ 0, \quad \text{for } x = c_1 + c_2 \\ v_{\tilde{A}} + (1 - v_{\tilde{A}})\left(\dfrac{c_1 + c_2 - x}{\bar{\xi}_1}\right)^{n_2}, & \text{for } c_1 + c_2 < x \leqslant \bar{c}_1 + \bar{c}_2 \\ 1, \quad \text{otherwise} \end{cases} \tag{3.19}$$

$$F_C(x) = \begin{cases} w_{\tilde{A}} + (1 - w_{\tilde{A}}) \left(\dfrac{c_1 + c_2 - x}{\underline{c}_1 + \underline{c}_2} \right)^{n_3}, & \text{for } \underline{c}_1 + \underline{c}_2 \leqslant x < c_1 + c_2 \\ 0, & \text{for } x = c_1 + c_2 \\ w_{\tilde{A}} + (1 - w_{\tilde{A}}) \left(\dfrac{c_1 + c_2 - x}{\bar{\xi}_1} \right)^{n_3}, & \text{for } c_1 + c_2 < x \leqslant \bar{c}_1 + \bar{c}_2 \\ 1, & \text{otherwise} \end{cases} \tag{3.20}$$

where $\underline{\zeta}_1 = c_1 + c_2 - (c_1 + c_2 - \underline{c}_1 - \underline{c}_2)$, $\bar{\zeta}_1 = c_1 + c_2 - (c_1 + c_2 - \bar{c}_1 - \bar{c}_2)$, $\underline{\xi}_1 = c_1 + c_2 - (\underline{c}_1 + \underline{c}_2)$ and $\bar{\xi}_1 = c_1 + c_2 - (\bar{c}_1 + \bar{c}_2)$ Addition of two GNLTNN is illustrated in Example 1.

Example 1: If $\tilde{A} = (3, 5, 7; 1, 0.5, 0.75)$, $\tilde{B} = (4, 6, 8; 1, 0.5, 0.75)$ and $n_1 = n_2 = n_3 = 3$ then using Equation (3.17), we obtain

$$C_{(3,3,\alpha,\beta,\gamma)} = \Bigg[11 - 4(1-\alpha)^{\frac{1}{3}}, 11 + 4(1-\alpha)^{\frac{1}{3}}; 11 - 7\left(\frac{\beta - 0.5}{0.5}\right)^{\frac{1}{3}} 11 + 4\left(\frac{\beta - 0.5}{0.5}\right)^{\frac{1}{3}};$$
$$11 - 7\left(\frac{\gamma - 0.75}{0.25}\right)^{\frac{1}{3}}, 11 + 4\left(\frac{\gamma - 0.75}{0.25}\right)^{\frac{1}{3}} \Bigg].$$

The graph of above equation is presented in Figure 4.

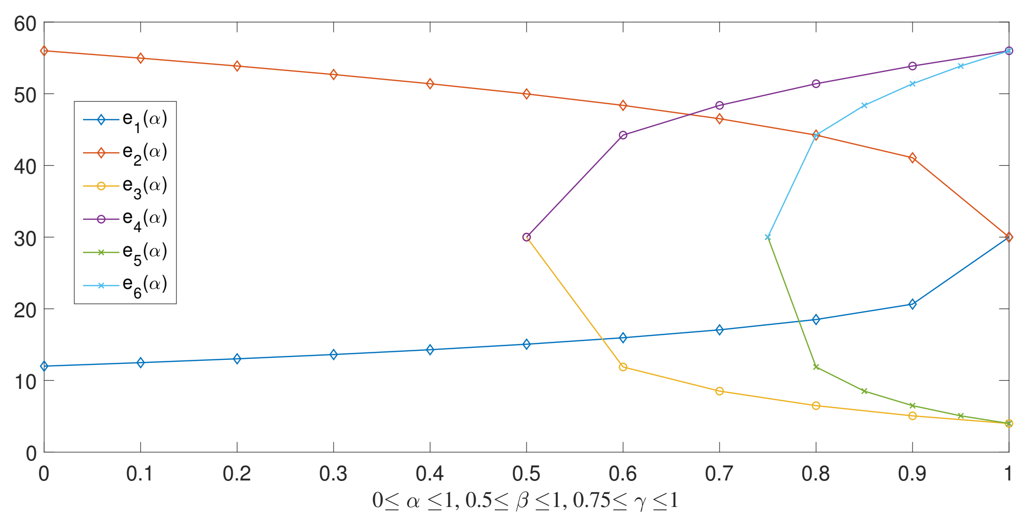

FIGURE 4. Addition of two GNLTNN

3.2.2. *Subtraction of two GNLTNN*

This section presents subtraction of two NLTNN using (p,q,α,β,γ)-cut. Let

$$C_{(p,q,\alpha,\beta,\gamma)} = A_{(p_1,q_1,\alpha,\beta,\gamma)} - B_{(p_2,q_2,\alpha,\beta,\gamma)}$$

Using Equations (3.14) and (3.15), we get

$$C_{(p,q,\alpha,\beta,\gamma)} = \Bigg[\underline{\zeta}_2\left(1-\frac{\alpha}{u_{\tilde{A}}}\right)^{\frac{1}{n_1}}, \bar{\zeta}_2\left(1-\frac{\alpha}{u_{\tilde{A}}}\right)^{\frac{1}{n_1}}; \underline{\xi}_2\left(\frac{\beta-v_{\tilde{A}}}{1-v_{\tilde{A}}}\right)^{\frac{1}{n_2}}, \bar{\zeta}_2\left(\frac{\beta-v_{\tilde{A}}}{1-v_A}\right)^{\frac{1}{n_2}};$$
$$\underline{\xi}_2\left(\frac{\gamma-w_{\tilde{A}}}{1-w_{\tilde{A}}}\right)^{\frac{1}{n_3}}, \bar{\zeta}_2\left(\frac{\gamma-w_{\tilde{A}}}{1-w_{\tilde{A}}}\right)^{\frac{1}{n_3}}\Bigg]. \tag{3.21}$$

The membership function of the above Equation is as follows.

$$T_C(x) = \begin{cases} u_{\tilde{A}}\left\{1-\left(\dfrac{x-c_1+c_2}{\underline{\xi}_2}\right)^{n_1}\right\}, & \text{for } \underline{c}_1-\underline{c}_2 \leqslant x < c_1-c_2 \\ u_{\tilde{A}}, & \text{for } x = c_1-c_2 \\ u_{\tilde{A}}\left\{1-\left(\dfrac{x-c_1+c_2}{\bar{\xi}_2}\right)^{n_1}\right\}, & \text{for } c_1-c_2 < x \leqslant \bar{c}_1-\bar{c}_2 \\ 0, & \text{otherwise} \end{cases} \tag{3.22}$$

$$I_C(x) = \begin{cases} v_{\tilde{A}} + (1-v_{\tilde{A}})\left(\dfrac{c_1-c_2-x}{\underline{c}_1-\underline{c}_2}\right)^{n_2}, & \text{for } \underline{c}_1-\underline{c}_2 \leqslant x < c_1-c_2 \\ 0, & \text{for } x = c_1-c_2 \\ v_{\tilde{A}} + (1-v_{\tilde{A}})\left(\dfrac{c_1-c_2-x}{\bar{\xi}_2}\right)^{n_2}, & \text{for } c_1-c_2 < x \leqslant \bar{c}_1-\bar{c}_2 \\ 1, & \text{otherwise} \end{cases} \tag{3.23}$$

$$F_C(x) = \begin{cases} w_{\tilde{A}} + (1-w_{\tilde{A}})\left(\dfrac{c_1-c_2-x}{\underline{c}_1-\underline{c}_2}\right)^{n_3}, & \text{for } \underline{c}_1-\underline{c}_2 \leqslant x < c_1-c_2 \\ 0, & \text{for } x = c_1-c_2 \\ w_{\tilde{A}} + (1-w_{\tilde{A}})\left(\dfrac{c_1-c_2-x}{\bar{\xi}_2}\right)^{n_3}, & \text{for } c_1-c_2 < x \leqslant \bar{c}_1-\bar{c}_2 \\ 1, & \text{otherwise} \end{cases} \tag{3.24}$$

where $\underline{\zeta}_2 = c_1-c_2-(c_1-c_2-\underline{c}_1+\underline{c}_2)$, $\bar{\zeta}_2 = c_1-c_2-(c_1-c_2-\bar{c}_1+\bar{c}_2)$, $\underline{\xi}_2 = c_1-c_2-(\underline{c}_1-\underline{c}_2)$ and $\bar{\xi}_2 = c_1-c_2-(\bar{c}_1-\bar{c}_2)$.

Subtraction of two GNLTNN is presented in Example 2.

Example 2: If $\tilde{A} = (1,2,3;1,0.5,0.75)$, $\tilde{B} = (4,6,8;1,0.5,0.75)$ and $n_1 = n_2 = n_3 = 3$ then using Equation (3.21), we get

$$C_{(3,3,\alpha,\beta,\gamma)} = \Bigg[-4+(1-\alpha)^{\frac{1}{3}}, -4-(1-\alpha)^{\frac{1}{3}}; -4+3\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}, -4-\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}};$$
$$-4+3\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}, -4-\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}\Bigg].$$

The graph of above equation is presented in Figure 5.

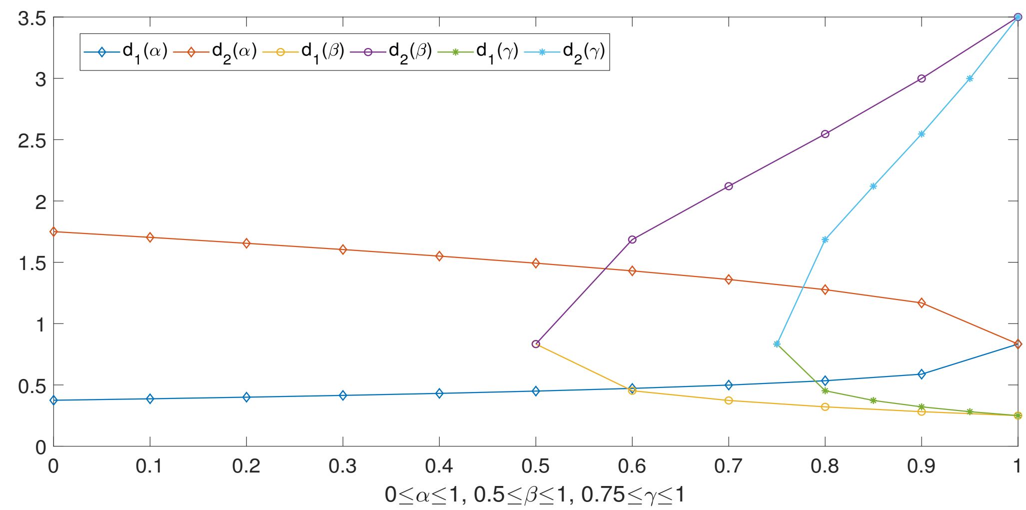

FIGURE 5. Subtraction of two NLTNN

3.2.3. *Multiplication and division of two GNLTNN*

This section presents the multiplication and division of two NLTNN using interval rule base method. Consider $[a_1, a_2]$ and $[a_3, a_4]$, where $a_i \neq 0, i = 1,2,3,4$. Now $[a_1, a_2] . [a_3, a_4] = [\min\{a_1a_3, a_1a_4, a_2a_3, a_2a_4\}, \max\{a_1a_3, a_1a_4, a_2a_3, a_2a_4\}]$ and $\frac{[a_1,a_2]}{[a_3,a_4]} = \left[\min\left\{\frac{a_1}{a_3}, \frac{a_1}{a_4}, \frac{a_2}{a_3}, \frac{a_2}{a_4}\right\}, \max\left\{\frac{a_1}{a_3}, \frac{a_1}{a_4}, \frac{a_2}{a_3}, \frac{a_2}{a_4}\right\}\right]$. Using this technique, we can obtain the product of two NLTNN as follows.

For multiplication,

$$C_{\alpha,\beta,\gamma} = A_{\alpha,\beta,\gamma} * B_{\alpha,\beta,\gamma} = [e_1(\alpha), e_2(\alpha); e_1(\beta), e_2(\beta); e_1(\gamma), e_2(\gamma)], \tag{3.25}$$

where

$$e_i\,(\mathrm{j}) = \begin{cases} \min[A_1(j)\cdot B_1(j), A_1(j)\cdot B_2(\alpha), A_2(j)\cdot B_1(j), A_2(j)\cdot B_2(j)], \text{ for } i = 1, j = \alpha, \beta, \gamma \\ \max[A_1(j)\cdot B_1(j), A_1(j)\cdot B_2(j), A_2(j)\cdot B_1(j), A_2(j)\cdot B_2(j)], \text{ for } i = 2, j = \alpha, \beta, \gamma \end{cases}$$

For division,

$$C_{\alpha,\beta,\gamma} = \frac{A_{\alpha,\beta,\gamma}}{B_{\alpha,\beta,\gamma}} = [d_1(\alpha), d_2(\alpha); d_1(\beta), d_2(\beta); d_1(\gamma), d_2(\gamma)], \tag{3.26}$$

where

$$d_i(j) = \begin{cases} \min\left[\frac{A_1(j)}{B_1(j)}, \frac{A_1(j)}{B_2(j)}, \frac{A_2(j)}{B_1(j)}, \frac{A_2(j)}{B_2(j)}\right], \text{ for } i = 1, j = \alpha, \beta, \gamma \\ \max\left[\frac{A_1(j)}{B_1(j)}, \frac{A_1(j)}{B_2(j)}, \frac{A_2(j)}{B_1(j)}, \frac{A_2(j)}{B_2(j)}\right], \text{ for } i = 2, j = \alpha, \beta, \gamma \end{cases}$$

Example 3 If $\tilde{A} = (3,5,7;1,0.5,0.75)$, $\tilde{B} = (4,6,8;1,0.5,0.75)$ and $n_1 = n_2 = n_3 = 3$ then

$$A_{(3,3,\alpha,\beta,\gamma)} = \left[5-2(1-\alpha)^{\frac{1}{3}}, 5+2(1-\alpha)^{\frac{1}{3}}, 5-3\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}, 5+2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}},\right.$$
$$\left. 5-3\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}, 5+2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}\right].$$

$$B_{(3,3,\alpha,\beta,\gamma)} = \left[6-2(1-\alpha)^{\frac{1}{3}}, 6+2(1-\alpha)^{\frac{1}{3}}, 6-4\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}, 6+2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}},\right.$$
$$\left. 6-4\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}, 6+2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}\right].$$

$$e_i(\alpha) = \begin{cases} \min\, k_1(\alpha) & \text{if } i=1 \\ \max\, k_1(\alpha) & \text{if } i=2 \end{cases}$$
$$e_i(\beta) = \begin{cases} \min\, k_1(\beta) & \text{if } i=1 \\ \max\, k_1(\beta) & \text{if } i=2 \end{cases}$$
$$e_i(\gamma) = \begin{cases} \min\, k_1(\gamma) & \text{if } i=1 \\ \max\, k_1(\gamma) & \text{if } i=2 \end{cases}$$

where

$$k_1(\alpha) = \left\{5-2(1-\alpha)^{\frac{1}{3}} \cdot 6-2(1-\alpha)^{\frac{1}{3}}, 5+2(1-\alpha)^{\frac{1}{3}} \cdot 6-2(1-\alpha)^{\frac{1}{3}}\right.,$$
$$\left. 5-2(1-\alpha)^{\frac{1}{3}} \cdot 6+2(1-\alpha)^{\frac{1}{3}}, 5+2(1-\alpha)^{\frac{1}{3}} \cdot 6+2(1-\alpha)^{\frac{1}{3}}\right\},$$

$$k(\beta) = \left\{5-3\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}} \cdot 6-4\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}, 5+2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}} \cdot 6-4\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}},\right.$$
$$\left. 5-3\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}} \cdot 6+2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}, 5+2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}} \cdot 6+2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}\right\},$$

$$k(\gamma) = \left\{5-3\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}} \cdot 6-4\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}, 5+2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}} \cdot 6-4\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}},\right.$$
$$\left. 5-3\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}} \cdot 6+2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}, 5+2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}} \cdot 6+2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}\right\},$$

By varying the values of α, β and γ, it is observed that

$$e_1(\alpha) = \left(5-2(1-\alpha)^{\frac{1}{3}}\right) \cdot \left(6-2(1-\alpha)^{\frac{1}{3}}\right),$$
$$e_2(\alpha) = \left(5+2(1-\alpha)^{\frac{1}{3}}\right) \cdot \left(6+2(1-\alpha)^{\frac{1}{3}}\right),$$
$$e_1(\beta) = \left\{5-3\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}\right\} \cdot \left\{6-4\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}\right\},$$

$$e_2(\beta)=\left\{5+2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}\right\}\cdot\left\{6+2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}\right\},$$

$$e_1(\gamma)=\left\{5-3\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}\right\}\cdot\left\{6-4\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}\right\},$$

$$e_2(\gamma)=\left\{5+2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}\right\}\cdot\left\{6+2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}\right\}.$$

By interval rule base method, the $(\alpha.\ \beta,\ \gamma)$-cut of $\tilde{e}$ is given by

$$[\tilde{e}]_{(p,q,\alpha,\beta,\gamma)}=\Bigg[\left(5-2(1-\alpha)^{\frac{1}{3}}\right)\left(6-2(1-\alpha)^{\frac{1}{3}}\right),\left(5+2(1-\alpha)^{\frac{1}{3}}\right)\left(6+2(1-\alpha)^{\frac{1}{3}}\right);$$

$$\left\{5-3\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}\right\}\left\{6-4\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}\right\},\left\{5+2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}\right\}\left\{6+2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}\right\};$$

$$\left\{5-3\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}\right\}\left\{6-4\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}\right\},\left\{5+2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}\right\}\left\{6+2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}\right\}\Bigg].$$

The graph of above result is presented in Figure 6. Similarly,

$$d_i\left(\alpha\right)=\begin{cases}\min\ k_2\left(\alpha\right)\ \text{ if } i=1\\ \max\ k_2\left(\alpha\right)\ \text{ if } i=2\end{cases}$$

$$d_i\left(\beta\right)=\begin{cases}\min\ k_2\left(\beta\right)\ \text{ if } i=1\\ \max\ k_2\left(\beta\right)\ \text{ if } i=2\end{cases}$$

$$d_i\left(\gamma\right)=\begin{cases}\min\ k_2\left(\gamma\right)\ \text{ if } i=1\\ \max\ k_2\left(\gamma\right)\ \text{ if } i=2\end{cases}$$

where

$$k_2\left(\alpha\right)=\left\{\frac{5-2(1-\alpha)^{\frac{1}{3}}}{6-2(1-\alpha)^{\frac{1}{3}}},\frac{5+2(1-\alpha)^{\frac{1}{3}}}{6-2(1-\alpha)^{\frac{1}{3}}},\frac{5-2(1-\alpha)^{\frac{1}{3}}}{6+2(1-\alpha)^{\frac{1}{3}}},\frac{5+2(1-\alpha)^{\frac{1}{3}}}{6+2(1-\alpha)^{\frac{1}{3}}}\right\},$$

$$k_2(\beta)=\left\{\frac{5-3\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}}{6-4\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}},\frac{5+2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}}{6-4\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}},\frac{5-3\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}}{6+2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}},\frac{5+2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}}{6+2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}}\right\},$$

$$k_3(\gamma)=\left\{\frac{5-3\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}}{6-4\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}},\frac{5+2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}}{6-4\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}},\frac{5-3\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}}{6+2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}},\frac{5+2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}}{6+2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}}\right\}.$$

It is observed that

$$d_1\left(\alpha\right)=\frac{5+2(1-\alpha)^{\frac{1}{3}}}{6-2(1-\alpha)^{\frac{1}{3}}},\ d_2\left(\alpha\right)=\frac{5-2(1-\alpha)^{\frac{1}{3}}}{6+2(1-\alpha)^{\frac{1}{3}}},$$

$$d_1(\beta)=\frac{5+2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}}{6-4\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}},d_2(\beta)=\frac{5-3\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}}{6+2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}},$$

$$d_1(\gamma)=\frac{5+2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}}{6-4\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}},d_2(\gamma)=\frac{5-3\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}}{6+2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}},$$

By interval rule base method, the $(\alpha.\ \beta,\ \gamma)$-cut of $\tilde{d}$ is given by

$$[\tilde{d}]_{(3,3,\alpha,\beta,\gamma)}=\left\{\frac{5+2(1-\alpha)^{\frac{1}{3}}}{6-2(1-\alpha)^{\frac{1}{3}}},\frac{5-2(1-\alpha)^{\frac{1}{3}}}{6+2(1-\alpha)^{\frac{1}{3}}},\frac{5+2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}}{6-4\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}},\frac{5-3\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}}{6+2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{3}}}\right.$$
$$\left.\frac{5+2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}}{6-4\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}},\frac{5-3\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}}{6+2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{3}}}\right\}.$$

The graph of above result is presented in Figure 7.

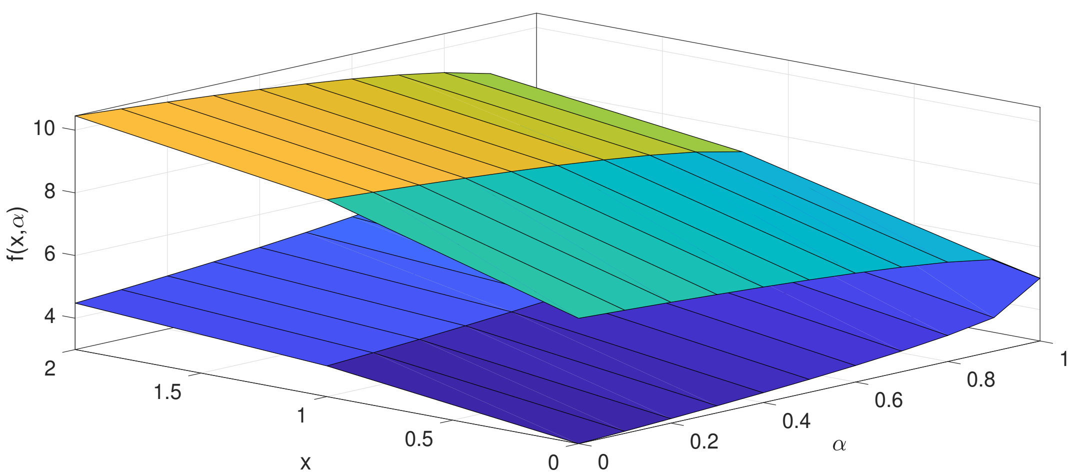

FIGURE 6. Multiplication of two NLTNN

3.3. *Neutrosophic valued function*

Let $\underline{H}(\alpha)$, $\overline{H}(\alpha)$, $\underline{H}(\beta)$, $\overline{H}(\beta)$, $\underline{H}(\gamma)$ and $\overline{H}(\gamma)$ are the continuous functions on the interval I. The set $\tilde{H}$ can be obtained by membership function as follows:

$$T_{\tilde{H}}\left(l\left(\alpha\right)\right)=\begin{cases}\alpha, & l\left(\alpha\right)=\underline{H}\left(\alpha\right),0\leqslant\alpha\leqslant1\\ \alpha, & l\left(\alpha\right)=\bar{H}\left(\alpha\right),0\leqslant\alpha\leqslant1\\ 0, & \text{otherwise}\end{cases}\tag{3.27}$$

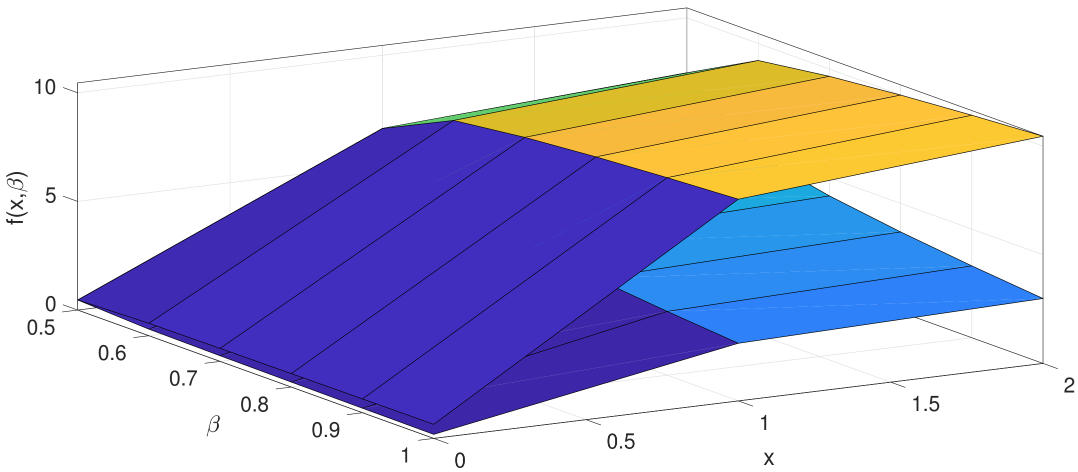

FIGURE 7. Division of two NLTNN

$$T_{\tilde{H}}\left(m\left(\beta\right)\right)=\begin{cases}\beta, & m\left(\beta\right)=\underline{H}\left(\beta\right), 0\leqslant\beta\leqslant 1\\ \beta, & m\left(\beta\right)=\bar{H}\left(\beta\right), 0\leqslant\beta\leqslant 1\\ 1, & \text{otherwise}\end{cases} \tag{3.28}$$

$$T_{\tilde{H}}\left(n\left(\gamma\right)\right)=\begin{cases}\gamma, & n\left(\gamma\right)=\underline{H}\left(\gamma\right), 0\leqslant\gamma\leqslant 1\\ \gamma, & n\left(\gamma\right)=\bar{H}\left(\gamma\right), 0\leqslant\gamma\leqslant 1\\ 1, & \text{otherwise}\end{cases} \tag{3.29}$$

The (α, β, γ)-cuts of neutrosophic valued function $\tilde{H}(t)$ is as follows:

$$[\tilde{H}(t)]_(\alpha,\beta,\gamma)=[\underline{H}(\alpha),\overline{H}(\alpha);\underline{H}(\beta),\overline{H}(\beta);\underline{H}(\gamma)\overline{H}(\gamma)]. \tag{3.30}$$

Example 4: Consider a neutrosophic valued function $\tilde{F}(t) = at^2 + t + 1$, where a is a neutrosophic coefficient. Let $\tilde{H}(t)_(p, q) = (2, 4, 6, 1, 0.5, 0.75)$, $p_i = q_i = 4$ where $i = T, I, F$. Using equations $(3.5) - (3.7)$, we obtain

$$\tilde{F}\left(t\right)_{(4,4,\alpha,\beta,\gamma)}=\Bigg[\left\{4-2(1-\alpha)^{\frac{1}{4}}\right\}t^2+t+1,\left\{4+2(1-\alpha)^{\frac{1}{4}}\right\}t^2+t+1;$$
$$\left\{4-2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{4}}\right\}t^2+t+1,\left\{4+2\left(\frac{\beta-0.5}{0.5}\right)^{\frac{1}{4}}\right\}t^2+t+1;$$
$$\left\{4-2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{4}}\right\}t^2+t+1,\left\{4+2\left(\frac{\gamma-0.75}{0.25}\right)^{\frac{1}{4}}\right\}t^2+t+1\Bigg].$$

The solution of the above equation is presented in Tables $2-4$

α	$t=0$		$t=1$		$t=2$		$t=3$	
	$\underline{F}(t,\alpha)$	$\bar{F}(t,\alpha)$	$\underline{F}(t,\alpha)$	$\bar{F}(t,\alpha)$	$\underline{F}(t,\alpha)$	$\bar{F}(t,\alpha)$	$\underline{F}(t,\alpha)$	$\bar{F}(t,\alpha)$
0	4.0000	8.0000	11.0000	27.0000	22.0000	58.0000	37.0000	101.0000
0.1	4.0520	7.9480	11.2080	26.7920	22.4679	57.5321	37.8319	100.1681
0.2	4.1085	7.8915	11.4341	26.5659	22.9767	57.0233	38.7363	99.2637
0.3	4.1706	7.8294	11.6825	26.3175	23.5356	56.4644	39.7299	98.2701
0.4	4.2398	7.7602	11.9591	26.0409	24.1580	55.8420	40.8364	97.1636
0.5	4.3182	7.6818	12.2728	25.7272	24.8639	55.1361	42.0913	95.9087
0.6	4.4095	7.5905	12.6378	25.3622	25.6851	54.3149	43.5513	94.4487
0.7	4.5198	7.4802	13.0793	24.9207	26.6785	53.3215	45.3174	92.6826
0.8	4.6625	7.3375	13.6501	24.3499	27.9627	52.0373	47.6003	90.3997
0.9	4.8753	7.1247	14.5013	23.4987	29.8779	50.1221	51.0051	86.9949
1	6.0000	6.0000	19.0000	19.0000	40.0000	40.0000	69.0000	69.0000

TABLE 2. Solution of the equation for various values of α and t..

β	$t=0$		$t=1$		$t=2$		$t=3$	
	$\underline{F}(t,\beta)$	$\bar{F}(t,\beta)$	$\underline{F}(t,\beta)$	$\bar{F}(t,\beta)$	$\underline{F}(t,\beta)$	$\bar{F}(t,\beta)$	$\underline{F}(t,\beta)$	$\bar{F}(t,\beta)$
0.5000	6.0000	6.0000	19.0000	19.0000	40.0000	40.0000	69.0000	69.0000
0.6000	4.6625	7.3375	13.6501	24.3499	27.9627	52.0373	47.6003	90.3997
0.7000	4.4095	7.5905	12.6378	25.3622	25.6851	54.3149	43.5513	94.4487
0.8000	4.2398	7.7602	11.9591	26.0409	24.1580	55.8420	40.8364	97.1636
0.9000	4.1085	7.8915	11.4341	26.5659	22.9767	57.0233	38.7363	99.2637
1.0000	4.0000	8.0000	11.0000	27.0000	22.0000	58.0000	37.0000	101.0000

TABLE 3. Solution of the equation for various values of β and t..

4. Values and ambiguities of GNLTNN

In this section, we aim to explore the values and ambiguities associated with GNLTNN. Let $\tilde{A}_{(p.q)}$ be GNLTNN. The values of GNLTNN for the $\alpha-$ cut , $\beta-$ cut and $\gamma-$ cut is denoted as $V_T(\tilde{A})$, $V_I(\tilde{A})$ and $V_F(\tilde{A})$ respectively. These values are defined as follows

$$V_T(\tilde{A}) = \int_0^{u_{\tilde{A}}} \left\{ T_{\tilde{A}}^L(\alpha, p_T) + T_{\tilde{A}}^R(\alpha, q_T) \right\} \phi_1(\alpha)\, d\alpha, \phi_1(\alpha) \in [0,1], \tag{4.1}$$

$$V_I(\tilde{A}) = \int_{v_{\tilde{A}}}^1 \left\{ I_{\tilde{A}}^L(\beta, p_I) + I_{\tilde{A}}^R(\beta, q_I) \right\} \phi_2(\beta)\, d\beta, \phi_2(\beta) \in [0,1], \tag{4.2}$$

γ	$t=0$		$t=1$		$t=2$		$t=3$	
	$\underline{F}(t,\gamma)$	$\bar{F}(t,\gamma)$	$\underline{F}(t,\gamma)$	$\bar{F}(t,\gamma)$	$\underline{F}(t,\gamma)$	$\bar{F}(t,\gamma)$	$\underline{F}(t,\gamma)$	$\bar{F}(t,\gamma)$
0.75	6.0000	6.0000	19.0000	19.0000	40.0000	40.0000	69.0000	69.0000
0.80	4.6625	7.3375	13.6501	24.3499	27.9627	52.0373	47.6003	90.3997
0.85	4.4095	7.5905	12.6378	25.3622	25.6851	54.3149	43.5513	94.4487
0.90	4.2398	7.7602	11.9591	26.0409	24.1580	55.8420	40.8364	97.1636
0.95	4.1085	7.8915	11.4341	26.5659	22.9767	57.0233	38.7363	99.2637
1.00	4.0000	8.0000	11.0000	27.0000	22.0000	58.0000	37.0000	101.0000

TABLE 4. Solution of the equation for various values of γ and t

$$V_F(\tilde{A}) = \int_{w_{\tilde{A}}}^{1} \left\{ F^L_{\tilde{A}}(\gamma, p_F) + F^R_{\tilde{A}}(\gamma, q_F) \right\} \phi_3(\gamma)\, d\gamma, \phi_3(\gamma) \in [0,1], \tag{4.3}$$

where

- $\phi_1(\alpha)$ is monotonic and non-decreasing in the interval $\left[0, u_{\tilde{A}}\right]$ and $\phi_1(0) = 0$
- $\phi_2(\beta)$ is monotonic and non-increasing in the interval $\left[v_{\tilde{A}}, 1\right]$ and $\phi_2(1) = 0$
- $\phi_3(\gamma)$ is monotonic and non-increasing in the interval $\left[w_{\tilde{A}}, 1\right]$ and $\phi_3(1) = 0$

Without loss of generality, we assume $\phi_1(\alpha) = \alpha$, $\phi_2(\beta) = 1 - \beta$ and $\phi_3(\gamma) = 1 - \gamma$ Using Equations (3.5) – (3.7) in Equations (4.1) – (4.3) respectively, we get the following results

$$\begin{aligned} V_T(\tilde{A}) &= \int_0^{u_{\tilde{A}}} \left\{ 2c - \left(1 - \frac{\alpha}{u_{\tilde{A}}}\right)^{\frac{1}{n_1}} (2c - \underline{c} - \bar{c}) \right\} \alpha d\alpha \\ &= c u_{\tilde{A}} - \frac{(2c - \underline{c} - \bar{c}) \left(u_{\tilde{A}}\right)^2}{\left(\frac{1}{n_1} + 1\right)\left(\frac{1}{n_1} + 2\right)}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} V_I(\tilde{A}) &= \int_{v_{\tilde{A}}}^{1} \left\{ 2c - \frac{\left(\underline{c} + c - \bar{c}\right)\left(\beta - v_{\tilde{A}}\right)^{\frac{1}{n_2}}}{\left(1 - v_{\tilde{A}}\right)^{\frac{1}{n_2}}} \right\} (1 - \beta)\, d\beta \\ &= c\left(v_{\tilde{A}} - 1\right)^2 - \frac{{n_2}^2 \left({v_{\tilde{A}}}^2 - 2v_{\tilde{A}} + 1\right)\left(\underline{c} + c - \bar{c}\right)}{\left(2{n_2}^2 + 3n_2 + 1\right)}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} V_F(\tilde{A}) &= \int_{w_{\tilde{A}}}^{1} \left\{ 2c - \frac{\left(\underline{c} + c - \bar{c}\right)\left(\gamma - w_{\tilde{A}}\right)^{\frac{1}{n_3}}}{\left(1 - w_{\tilde{A}}\right)^{\frac{1}{n_3}}} \right\} (1 - \gamma)\, d\gamma \\ &= c\left(w_{\tilde{A}} - 1\right)^2 - \frac{{n_3}^2 \left({w_{\tilde{A}}}^2 - 2w_{\tilde{A}} + 1\right)\left(\underline{c} + c - \bar{c}\right)}{\left(2{n_3}^2 + 3n_3 + 1\right)}. \end{aligned} \tag{4.6}$$

Let $G_T(\tilde{A})$, $G_I(\tilde{A})$ and $G_F(\tilde{A})$ denotes the ambiguities of the GNLTNS for α, β and γ cut sets respectively, then

$$G_T(\tilde{A}) = \int_0^{u_{\tilde{A}}} \left\{ T^R_{\tilde{A}}(\alpha, q_T) - T^L_{\tilde{A}}(\alpha, p_T) \right\} \phi_1(\alpha)\, d\alpha, \phi_1(\alpha) \in [0,1] \tag{4.7}$$

$$G_I(\tilde{A}) = \int_{v_{\tilde{A}}}^{1} \left\{ I^R_{\tilde{A}}(\beta, q_F) - I^L_{\tilde{A}}(\beta, p_F) \right\} \phi_2(\beta)\, d\beta, \phi_2(\beta) \in [0,1] \tag{4.8}$$

$$G_F(\tilde{A}) = \int_{w_{\tilde{A}}}^{1} \left\{ F^R_{\tilde{A}}(\gamma, q_F) - F^L_{\tilde{A}}(\gamma, p_F) \right\} \phi_3(\gamma)\, d\gamma, \phi_3(\gamma) \in [0,1]\,. \tag{4.9}$$

Similarly, using Equations (3.5) – (3.7) in Equations (4.7) – (4.9), we get

$$G_T(\tilde{A}) = \frac{(\bar{c} - \underline{c})\left(u_{\tilde{A}}\right)^2}{\left(\frac{1}{n_1} + 1\right)\left(\frac{1}{n_1} + 2\right)} \tag{4.10}$$

$$G_I(\tilde{A}) = \frac{{n_2}^2\left(\underline{c} + \bar{c} - c\right)\left({v_{\tilde{A}}}^2 - 2v_{\tilde{A}} + 1\right)}{2{n_2}^2 + 3n_2 + 1} \tag{4.11}$$

$$G_F(\tilde{A}) = \frac{{n_3}^2\left(\underline{c} + \bar{c} - c\right)\left({w_{\tilde{A}}}^2 - 2w_{\tilde{A}} + 1\right)}{2{n_3}^2 + 3n_3 + 1} \tag{4.12}$$

These values provide insights into the degree of truth, indeterminacy, and falsity associated with a GNLTNN within specific intervals defined by the cut sets. By analyzing these values, we can understand the distribution of uncertainty and ambiguity inherent in the GNLTNN.

5. **Linear Fredholm Integral Equation in Neutrosophic Environment**

This section delves into the realm of neutrosophic integral equations, a significant branch within integral equations. Such equations play a pivotal role in theoretical analysis, offering a robust framework for deriving analytical solutions, exploring the existence and uniqueness of solutions, and examining qualitative properties of various physical systems.

5.1. *Neutrosophic integral equation (NIE)*

We consider Fredholm integral equation of second kind

$$f(x) = g(x) + \lambda \int_a^b k(s,t)\, u(t)\, dt$$

The above Equation is said be NIE if

(1) $g(x)$ is neutrosophic valued function,
(2) $k(s,t)$ is neutrosophic valued function,
(3) Both $g(x)$ and $k(s,t)$ are neutrosophic valued functions.

5.2. *Condition for existence for solution of NIE*

Let the solution NIE be $\tilde{f}(x)$ and its (α, β, γ)- cuts be

$$\left[\tilde{f}(x)\right]_{(\alpha,\beta,\gamma)} = \left[\underline{f}(x,\alpha), \bar{f}(x,\alpha); \underline{f}(x,\beta), \bar{f}(x,\beta); \underline{f}(x,\gamma), \bar{f}(x,\gamma)\right].$$

The solution is said to be strong solution if

(1) $\frac{\partial \underline{f}(x,\alpha)}{\partial \alpha} > 0, \frac{\partial \bar{f}(x,\alpha)}{\partial \alpha} < 0 \quad \forall \alpha \in \left[0, u_{\tilde{A}}\right], \underline{f}(x,1) \leqslant \bar{f}(x,1)$
(2) $\frac{\partial \underline{f}(x,\beta)}{\partial \beta} < 0, \frac{\partial \bar{f}(x,\beta)}{\partial \beta} > 0 \quad \forall \beta \in \left[v_{\tilde{A}}, 1\right], \underline{f}(x,0) \leqslant \bar{f}(x,0)$
(3) $\frac{\partial \underline{u}(x,\gamma)}{\partial \gamma} < 0, \frac{\partial \bar{f}(x,\gamma)}{\partial \gamma} > 0 \quad \forall \gamma \in \left[w_{\tilde{A}}, 1\right], \underline{f}(x,0) \leqslant \bar{f}(x,0)$

The rest of the cases are weak solutions.

5.3. *Evaluation of NIE*

Consider the integral equation $f(x) = \bar{g}(x) + \lambda \int_a^b \bar{k}(s,t)\, f(t)\, dt$. Here we consider both $\bar{g}(x)$ and $\bar{k}(s,t)$ are neutrosophic valued function. Taking Laplace transform,, we get

$$L[f(x)) = L[\bar{g}(x)] + \lambda L[\bar{k}(x,t)]L[f(t)] \tag{5.1}$$

Case(i) When $k(x,t)$ is a positive function

$$\underline{f}(x,i) = \mathscr{L}^{-1}\left[\frac{\mathscr{L}\left\{\underline{g}(s,i)\right\}}{1-\lambda\mathscr{L}\left\{\underline{k}(s,i)\right\}}\right], i = \alpha, \beta, \gamma \tag{5.2}$$

$$\bar{f}(x,i) = \mathscr{L}^{-1}\left[\frac{\mathscr{L}\left\{\bar{g}(s,i)\right\}}{1-\lambda\mathscr{L}\left\{\bar{k}(s,i)\right\}}\right], i = \alpha, \beta, \gamma \tag{5.3}$$

Case(ii) When $k(x,t)$ is a negative function say $k(x,t) = -h(x,t)$ then

$$L[f(x)) = L[g(x)] + \lambda L[-h(x,t)]L[f(t)] \tag{5.4}$$

$$\underline{f}(x,i) = \mathscr{L}^{-1}\left[\frac{\lambda\mathscr{L}\left\{\underline{h}(x,t,i)\right\}\mathscr{L}\left\{\underline{g}(s,i)\right\}}{1-\lambda^2\mathscr{L}\left\{\underline{h}(x,t,i)\right\}\mathscr{L}\left\{\bar{h}(x,t,i)\right\}}\right], i = \alpha, \beta, \gamma \tag{5.5}$$

$$\bar{f}(x,i) = \mathscr{L}^{-1}\left[\frac{\lambda L\left\{\bar{h}(x,t,i)\right\}\mathscr{L}\left\{\bar{g}(s,i)\right\}}{1-\lambda^2\mathscr{L}\left\{\underline{h}(x,t,i)\right\}\mathscr{L}\left\{\bar{h}(x,t,i)\right\}}\right], i = \alpha, \beta, \gamma \tag{5.6}$$

Example 5: Consider a scenario of heat transfer in a one-dimensional rod. Let $f(x)$ represent the temperature distribution along the rod at position x, $g(x)$ represent the initial temperature distribution and $k(x,t)$ represent the thermal conductivity between two points x and t. The equation $f(x) = g(x) + \lambda \int_a^b k(x,t)\, f(t)\, dt$ describes how the temperature distribution evolves over time due to the heat flow between different points along the rod. let initial temperature distribution $g(x) = (3, 5, 7; 1, 0.5, 0.75)e^{-x}$, proportionality constant $\lambda = 1$, length of the rod a = 0,b = x, thermal conductivity $k(x,t) = \cos(x-t)$. Using Equations (4.1) − (4.3), we can get the solution for the integral equation as follows

$$\underline{f}(x,\alpha) = \left(5 - 2(1-\alpha)^{\frac{1}{3}}\right) m(x), \bar{f}(x,\alpha) = \left(5 + 2(1-\alpha)^{\frac{1}{3}}\right) m(x),$$

$$\underline{f}(x,\beta) = \left\{5 - 3\left(\frac{\beta - 0.5}{0.5}\right)^{\frac{1}{3}}\right\} m(x), \bar{f}(x,\beta) = \left\{5 + 2\left(\frac{\beta - 0.5}{0.5}\right)^{\frac{1}{3}}\right\} m(x),$$

$$\underline{f}(x,\gamma) = \left\{5 - 3\left(\frac{\gamma - 0.75}{0.25}\right)^{\frac{1}{3}}\right\} m(x), \bar{f}(x,\gamma) = \left\{5 + 2\left(\frac{\gamma - 0.75}{0.25}\right)^{\frac{1}{3}}\right\} m(x),$$

where

$$m(x) = \Bigg[\frac{e^{\frac{x}{2}}}{3} \quad \sqrt{3}\sin\frac{\sqrt{3}x}{2} + \cos\frac{\sqrt{3}x}{2}\Bigg) + \frac{2e^{-x}}{3}\Bigg].$$

Graphical representation of the above result is presented in Figures 8 − 10.

6. Result discussion

The conclusions drawn from Table 5 − 7 offer valuable insights into the behavior of temperature distribution $f(x)$ in one-dimensional heat transfer modeling using NLTNN. From Table 5, it is observed that as α increases from 0 to 1, the values of $\underline{f}(x,\alpha)$ increase gradually, whereas the values of $\bar{f}(x,\alpha)$ decrease. From Table 6, it is observed that as β rises from 0.5 to 1, $\underline{f}(x,\beta)$ decreases gradually, whereas the values of $\bar{f}(x,\beta)$ increase. Table 7 behaves the same as Table 6. As γ increases from 0.75 to 1, $\underline{f}(x,\gamma)$ decreases gradually, whereas the values of $\bar{f}(x,\gamma)$ increase. The temperature distribution $\underline{f}(x)$ and $\bar{f}(x)$ are influenced by the parameter values, exhibiting distinct patterns based on the values of α, β, and γ.

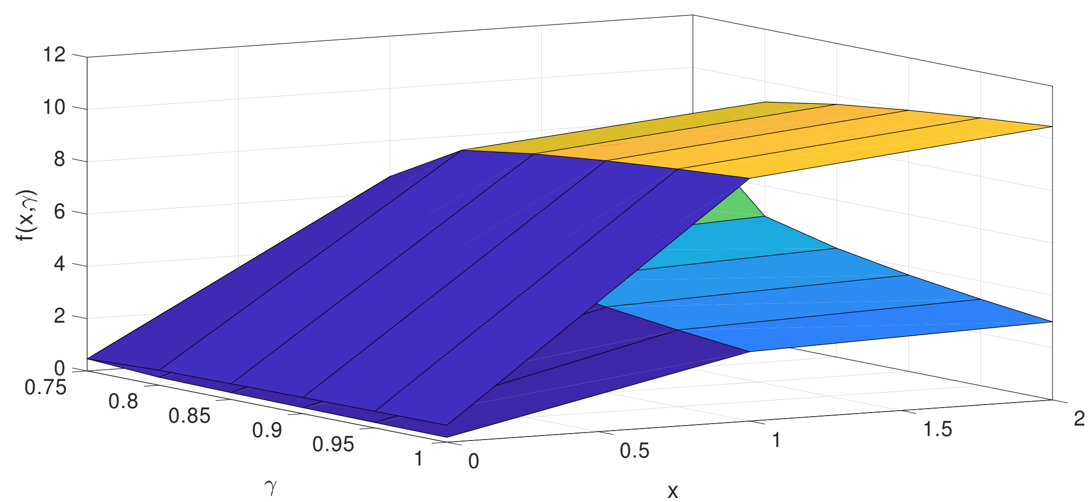

FIGURE 8. Solution of integral equation $f(x,\alpha)$

α	$x=0$		$x=1$		$x=2$	
	$\underline{f}(x,\alpha)$	$\bar{f}(x,\alpha)$	$\underline{f}(x,\alpha)$	$\bar{f}(x,\alpha)$	$\underline{f}(x,\alpha)$	$\bar{f}(x,\alpha)$
0.0	3.0000	7.0000	3.9792	9.2849	4.4814	10.4565
0.1	3.0690	6.9310	4.0708	9.1933	4.5845	10.3534
0.2	3.1434	6.8566	4.1694	9.0947	4.6955	10.2423
0.3	3.2242	6.7758	4.2766	8.9875	4.8162	10.1216
0.4	3.3131	6.6869	4.3946	8.8695	4.9491	9.9887
0.5	3.4126	6.5874	4.5265	8.7376	5.0977	9.8402
0.6	3.5264	6.4736	4.6774	8.5867	5.2677	9.6702
0.7	3.6611	6.3389	4.8562	8.4079	5.4689	9.4689
0.8	3.8304	6.1696	5.0807	8.1834	5.7218	9.2161
0.9	4.0717	5.9283	5.4007	7.8634	6.0822	8.8556
1.0	5.0000	5.0000	6.6321	6.6321	7.4689	7.4689

TABLE 5. Solution of the integral equation for various values of α and x.

β	$x=0$		$x=1$		$x=2$	
	$\underline{f}(x,\beta)$	$\bar{f}(x,\beta)$	$\underline{f}(x,\beta)$	$\bar{f}(x,\beta)$	$\underline{f}(x,\beta)$	$\bar{f}(x,\beta)$
0.5	5.0000	5.0000	6.6321	6.6321	7.4689	7.4689
0.6	3.2456	6.1696	4.3050	8.1834	4.8482	9.2161
0.7	2.7896	6.4736	3.7001	8.5867	4.1670	9.6702
0.8	2.4697	6.6869	3.2758	8.8695	3.6892	9.9887
0.9	2.2150	6.8566	2.9381	9.0947	3.3088	10.2423
1.0	2.0000	7.0000	2.6528	9.2849	2.9876	10.4565

TABLE 6. Solution of the integral equation for various values of β and x.

γ	$x=0$		$x=1$		$x=2$	
	$\underline{f}(x,\gamma)$	$\bar{f}(x,\gamma)$	$\underline{f}(x,\gamma)$	$\bar{f}(x,\gamma)$	$\underline{f}(x,\gamma)$	$\bar{f}(x,\gamma)$
0.75	5.0000	5.0000	6.6321	6.6321	7.4689	7.4689
0.8	3.2456	6.1696	4.3050	8.1834	4.8482	9.2161
0.85	2.7896	6.4736	3.7001	8.5867	4.1670	9.6702
0.9	2.4697	6.6869	3.2758	8.8695	3.6892	9.9887
0.9.5	2.2150	6.8566	2.9381	9.0947	3.3088	10.2423
1.0	2.0000	7.0000	2.6528	9.2849	2.9876	10.4565

TABLE 7. Solution of the integral equation for various values of γ and x.

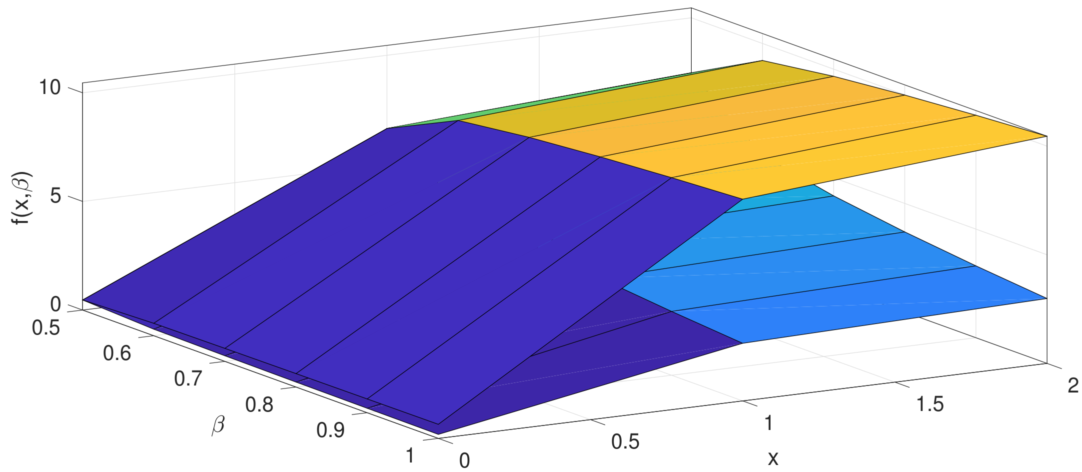

FIGURE 9. Solution of integral equation $f(x, \beta)$

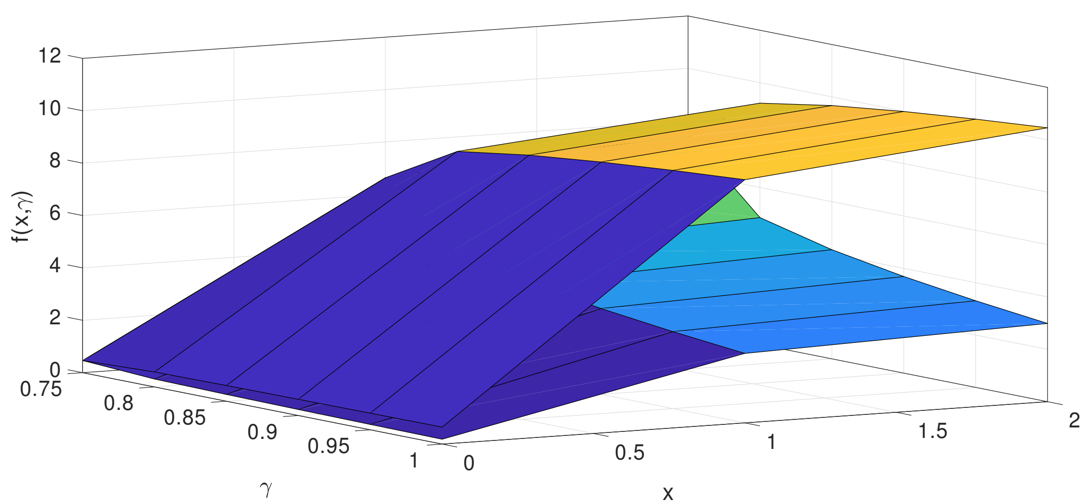

FIGURE 10. Solution of integral equation $f(x, \gamma)$

7. **Conclusion and future work**

In conclusion, this paper presents a refined version of GNLNN, which mitigates certain constraints observed in prior literature. We outline the fundamental properties of GNLNN, particularly emphasizing the notions of (α, β, γ). Moreover, we illustrate the application of this numerical framework in integral equations set within a neutrosophic context. This research paves the way for further exploration, offering avenues to tackle diverse categories of both differential and integral equations. Such endeavors hold promise for broadening the scope of mathematical modeling and analysis, thereby enhancing their utility and significance.

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