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The Role of Lacunary Statistical Convergence for Double sequences in Neutrosophic Normed Spaces

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Abstract: This paper introduces and explores the concept of lacunary statistical convergence of double sequence within the framework neutrosophic normed spaces. Neutrosophic normed spaces extend classical normed spaces by incorporating neutrosophic numbers, which account for the inherent uncertainty, indeterminacy, and vagueness present in real - world data. The study begins by defining lacunary statistical convergence for double sequences in this extended context and proceeds to establish fundamental theorems and properties related to this new notion. In addition, we present a new idea in this context: statistical completeness. We demonstrate that, while neutrosophic normed space is statistically complete, it is not complete.

Keywords: Neutrosophic Normed Spaces; Lacunary Statistical Convergence and Cauchyness; Statistical Completeness.

1. Introduction

Fuzzy theory has been a hot topic of study in a number of scientific domains in the last few years. Many studies have been published on this theory since Zadeh originally put it forth in 1965. Saadati and Park introduced the idea of intuitionistic fuzzy normed space initially. Smarandache introduced the concept of neutrosophic sets as an extension of the intuitionistic fuzzy set. The requirement can be met when the component sum is equal to one by using neutronosophic set operators. While indeterminacy is treated by neutrosophic operators on the same plane as truth-membership and falsehood-nonmembership, intuitionistic fuzzy operators disregard indeterminacy and may produce different results.

Using the idea of density of positive natural numbers, Fast and Steinhaus separately created statistical convergence in 1951. Mursaleen and Edely have defined and studied double sequence statistical convergence. Karakus et al.'s recent study examined statistical convergence in intuitionistic fuzzy normed space. Rough statistical convergence and statistical $\triangle^m$ convergence were recently established in neutrosophic normed spaces by Jeyaraman and Jenifer.

Lacunary statistical convergence was first proposed by Fridy and Orhan. The lacunary statistical Cauchy and convergence for double sequences in neutrosophic normed space will be examined in this paper. The results presented in this paper contribute to the growing field of neutrosophic mathematics and provide a deeper understanding of convergence behavior in spaces characterized by uncertainty and indeterminacy.

2. Preliminaries

Some of the fundamental notions and definitions that are needed in the following sections are presented in this section.

Let $\mathfrak{J}$ represent a subset of the natural number set $\mathbb{N}$. Next, we define the asymptotic density of $\mathfrak{J}$, represented by $\delta(\mathfrak{J})$, as follows: $\delta(\mathfrak{J}) = \lim_{n} \frac{1}{n} |\{k \leq n : k \in \mathfrak{J}\}|$. The cardinality of the contained set is indicated by the vertical bars. The sequence of numbers $\mathfrak{x} = (\mathfrak{x}_k)$ is statistically convergent to ℓ if, for any $\epsilon > 0$, the set $\mathfrak{J}(\epsilon) = \{k \leq n : |\mathfrak{x}_k - \ell| > \epsilon\}$ has asymptotic density zero, that is, $\lim_{n} \frac{1}{n} |\{k \leq n : |\mathfrak{x}_k - \ell| \geq \epsilon\}| = 0$.In this instance, $\mathfrak{st} - \lim \mathfrak{x} = \ell$ is written. It should be noted that while the converse may not always be true, any convergent sequence approaches the same limit statistically.

If, for any $\epsilon > 0$, the set $\{(j,k), j \leq m \ and \ k \leq n : |\mathfrak{x}_{jk} - \ell| \geq \epsilon\}$ has double natural density zero, then the real double sequence $\mathfrak{x} = (\mathfrak{x}_{jk})$ is statistically convergent to the number ℓ. We indicate the set of all statistically convergent double sequences by $\aleph_2$ in this instance, and the set of all limited statistically convergent double sequences by$\aleph_2^{\infty}$. In this example, we write $\mathfrak{st}_2 - \lim \mathfrak{x} = \ell$.

Definition 2.1 Let $(\mathfrak{X}, \tau, \varphi, \omega *, \diamond, \circledast)$ be an $\mathfrak{NNS}$. Here, $\mathfrak{X}$ is a vector space, $*$ is a continuous t-norm, $\diamond$ and $\circledast$ are continuous t-conorm, and τ, φ and ω are fuzzy sets on $\mathfrak{X} \times (0, \infty)$ satisfy the following conditions. For every $\mathfrak{x}, \mathfrak{y} \in \mathfrak{X}$ and $\zeta, \lambda > 0$,

(i) $\tau(\mathfrak{x}, \lambda) + \nu(\mathfrak{x}, \lambda) + \omega(\mathfrak{x}, \lambda) \leq 1$;
(ii) $\tau(\mathfrak{x}, \lambda) > 0$;
(iii) $\tau(\mathfrak{x}, \lambda) = 1$ iff $\mathfrak{x} = 0$;
(iv) $\tau(\alpha\mathfrak{x}, \lambda) = \tau\left(\mathfrak{x}, \frac{\lambda}{|\alpha|}\right)$ for each $\alpha \neq 0$;
(v) $\tau(\mathfrak{x}, \lambda) * \tau(\mathfrak{y}, \zeta) \leq \tau(\mathfrak{x} + \mathfrak{y},)$;
(vi) $\tau(\mathfrak{x}, \cdot) : (0, \infty) \to [0,1]$ is continuous;
(vii) $\lim_{\lambda \to \infty} \tau(\mathfrak{x}, \lambda) = 1$ and $\lim_{\lambda \to 0} \tau(\mathfrak{x}, \lambda) = 0$;
(viii) $\nu(\mathfrak{x}, \lambda) < 1$;
(ix) $\nu(\mathfrak{x}, \lambda) = 0$ iff $\mathfrak{x} = 0$;
(x) $\nu(\alpha\mathfrak{x}, \lambda) = \nu\left(\mathfrak{x}, \frac{\lambda}{|\alpha|}\right)$ for each $\alpha \neq 0$;
(xi) $\nu(\mathfrak{x}, \lambda) \diamond \nu(\mathfrak{y}, \zeta) \geq \nu(\mathfrak{x} + \mathfrak{y}, \lambda + \zeta)$;
(xii) $\nu(\mathfrak{x}, \cdot) : (0, \infty) \to [0,1]$ is continuous;
(xiii) $\lim_{\lambda \to \infty} \nu(\mathfrak{x}, \lambda) = 0$ and $\lim_{\lambda \to 0} \nu(\mathfrak{x}, \lambda) = 1$;
(xiv) $\omega(\mathfrak{x}, \lambda) < 1$;
(xv) $\omega(\mathfrak{x}, \lambda) = 0$ iff $\mathfrak{x} = 0$;
(xvi) $\omega(\alpha\mathfrak{x}, \lambda) = \omega\left(\mathfrak{x}, \frac{\lambda}{|\alpha|}\right)$ for each $\alpha \neq 0$;
(xvii) $\omega(\mathfrak{x}, \lambda) \circledast \omega(\mathfrak{y}, \zeta) \geq \omega(\mathfrak{x} + \mathfrak{y}, \lambda + \zeta)$;
(xviii) $\omega(\mathfrak{x}, \cdot) : (0, \infty) \to [0,1]$ is continuous;
(ixx) $\lim_{\lambda \to \infty} \omega(\mathfrak{x}, \lambda) = 0$ and $\lim_{\lambda \to 0} \omega(\mathfrak{x}, \lambda) = 1$.
In this case, (μ, ν, ω) is called an $\mathfrak{NNS}$.

Definition 2.2. Let a $\mathfrak{NNS}$ be $(\mathfrak{X}, \mu, \nu, \omega, *, \diamond, \circledast)$. According to the $\mathfrak{NN}$ (μ, ν, ω), $\mathfrak{x} = (\mathfrak{x}_k)$ is said to be convergent to $\ell \in \mathfrak{X}$ if, $\forall \ \epsilon > 0$ and $\lambda > 0$, $\exists \ k_0 \in \mathbb{N} : \mu(\mathfrak{x}_k - \ell, \lambda) > 1 - \epsilon$, $\nu(\mathfrak{x}_k - \ell, \lambda) < \epsilon$ and $\omega(\mathfrak{x}_k - \ell, \lambda) < \epsilon \ \forall \ k \geq k_0$. In this instance, we write $\mathfrak{x}_k \xrightarrow{(\mu,\nu,\omega)} \ell$ as $k \to \infty$ or $(\mu, \nu, \omega) - \lim \mathfrak{x} = \ell$.

Definition 2.3. Let a $\mathfrak{NNS}$ be $(\mathfrak{X}, \mu, \nu, \omega, *, \diamond, \circledast)$. Then, for every $\epsilon > 0$ and $\lambda > 0$, $\exists\, k_0 \in \mathbb{N}$ such that $\mu(\mathfrak{x}_k - \mathfrak{x}_\ell, \lambda) > 1-\epsilon$, $\nu(\mathfrak{x}_k - \mathfrak{x}_\ell, \lambda) < \epsilon$ and $\omega(\mathfrak{x}_k - \mathfrak{x}_\ell, \lambda) < \epsilon$ $\forall$ $k, \ell \geq k_0$. This indicates that $\mathfrak{x} = (\mathfrak{x}_k)$ is a Cauchy sequence with respect to the $\mathfrak{NN}$ (μ, ν, ω).

Remark 2.4 [13]. The real normed linear space $(\mathfrak{X}, \|\cdot\|)$ has the following properties: $\mu(\mathfrak{x}, \lambda) := \frac{\lambda}{\lambda + \|\mathfrak{x}\|}$, $\nu(\mathfrak{x}, \lambda) := \frac{\|\mathfrak{x}\|}{\lambda + \|\mathfrak{x}\|}$ and $\omega(\mathfrak{x}, \lambda) := \frac{\|\mathfrak{x}\|}{\lambda}$for all $\mathfrak{x} \in \mathfrak{X}$ and $\lambda > 0$.Subsequently, $\mathfrak{x}_n \xrightarrow{\|\cdot\|} \mathfrak{x}$ iff $\mathfrak{x}_n \xrightarrow{(\mu,\nu,\omega)} \mathfrak{x}$.

3. Lacunary Statistical Convergence ($\mathfrak{LSt}\mathfrak{C}$) of double sequences in $\mathfrak{NNS}$

The idea of $\mathfrak{LSt}\mathfrak{C}$ sequences in $\mathfrak{NNS}$ is examined in this section. First, let's define what we mean by θ-density:

Definition 3.1 A $\mathfrak{LS}$ is an ascending integer sequence $\theta = (\mathfrak{J}_\mathfrak{r})$ such that $\mathfrak{h}_\mathfrak{r} := \mathfrak{J}_\mathfrak{r} - \mathfrak{J}_{\mathfrak{r}-1} \to \infty$ as $\mathfrak{r} \to \infty$ and $\mathfrak{J}_0 = 0$ are considered.
In this study, the intervals identified by θ will be represented as $I_\mathfrak{r} := (\mathfrak{J}_{\mathfrak{r}-1}, \mathfrak{J}_\mathfrak{r}]$, and the $\mathfrak{J}_\mathfrak{r}/\mathfrak{J}_{\mathfrak{r}-1}$ ratio will be shortened to $\mathfrak{Q}_\mathfrak{r}$. Allow N to $\subseteq \mathbb{N}$. Assuming the limit exists, the θ -density of $\mathfrak{J}$ is given by the number $\delta_\theta(N) = \lim_{\mathfrak{r}} \frac{1}{\mathfrak{h}_\mathfrak{r}} |\{\mathfrak{J} \in I_\mathfrak{r} : \mathfrak{J} \in N\}|$.

Definition 3.2 Consider the θ. If, for each $\epsilon > 0$, the set $\mathfrak{J}(\epsilon)$ has θ-density zero, where $\mathfrak{J}(\epsilon) := \{k \in I_\mathfrak{r} : |\mathfrak{x}_k - \ell| \geq \epsilon\}$, then a sequence $\mathfrak{x} = (\mathfrak{x}_k)$ is said to be $\aleph_\theta$-convergent to the number ℓ. $\aleph_\theta - \lim \mathfrak{x} = \ell$ or $\mathfrak{x}_k \to \ell(\aleph_\theta)$ is written in this instance.

Now we define the $\aleph_\theta$-convergence of double sequences with respect to $\mathfrak{NNS}$.

Definition 3.3 Let θ be a $\mathfrak{LS}$ and $(\mathfrak{X}, \mu, \nu, \omega, *, \diamond, \circledast)$ be a $\mathfrak{NNS}$. Then, $\forall \epsilon > 0$and $\lambda > 0$,
$\delta_\theta\big(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk} - \ell, \lambda) \leq 1-\epsilon \text{ or } \nu(\mathfrak{x}_{jk} - \ell, \lambda) \geq \epsilon, \omega(\mathfrak{x}_{jk} - \ell, \lambda) \geq \epsilon\}\big) = 0$
or equivalently
$\delta_\theta\,(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk} - \ell, \lambda) > 1-\epsilon,\ \nu(\mathfrak{x}_{jk} - \ell, \lambda) < \epsilon \text{ and } \omega(\mathfrak{x}_{jk} - \ell, \lambda) < \epsilon\}) = 1.$
Here, we write $\aleph_\theta^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = \ell$ or $\mathfrak{x}_{jk} \xrightarrow{(\mu,\nu,\omega)} \ell(\aleph_\theta)$, where ℓ is referred to as $\aleph_\theta^{(\mu,\nu,\omega)} - \lim \mathfrak{x}$, and We signify the collection of all $\aleph_\theta$-convergent sequences with regard to the $\mathfrak{NN}(\mu, \nu, \omega)$ by $\aleph_\theta^{(\mu,\nu,\omega)}$.

Lemma 3.4 Consider a $\mathfrak{NNS}$ $(\mathfrak{X}, \mu, \nu, \omega, *, \diamond, \circledast)$. Let θ be a $\mathfrak{LS}$. Then, $\forall\, \epsilon > 0$ and $\lambda > 0$, the statements that follow are comparable:
$\aleph_\theta^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = \ell$.
$\delta_\theta\big(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk} - \ell, \lambda) \leq 1-\epsilon\}\big) = \delta_\theta\big(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu(\mathfrak{x}_{jk} - \ell, \lambda) \geq \epsilon \text{ and } \omega(\mathfrak{x}_{jk} - \ell, \lambda) \geq \epsilon\}\big) = 0.$
$\delta_\theta\big(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk} - \ell, \lambda) > 1-\epsilon, \nu(\mathfrak{x}_{jk} - \ell, \lambda) < \epsilon \text{ and } \omega(\mathfrak{x}_{jk} - \ell, \lambda) < \epsilon\}\big) = 1.$
$\delta_\theta\big(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - \ell, \lambda) > 1-\epsilon\}\big) = \delta_\theta\big(\{k \in : \nu(x_{jk} - \ell, \lambda) < \epsilon\}\big) = \delta_\theta\big(\{k \in : \omega(x_{jk} - \ell, \lambda) < \epsilon\}\big) = 1.$
$\aleph_\theta - \lim \mu(\mathfrak{x}_{jk} - \ell, \lambda) = 1$, $\aleph_\theta - \lim \nu(\mathfrak{x}_{jk} - \ell, \lambda) = 0$ and $\aleph_\theta - \lim\ \omega(\mathfrak{x}_{jk} - \ell, \lambda) = 0$.

Theorem 3.5 Let θ be a $\mathfrak{LS}$ and $(\mathfrak{X}, \mu, \nu, \omega, *, \diamond, \circledast)$ be a $\mathfrak{NNS}$. $\aleph_\theta^{(\mu,\nu,\omega)}$-limit is unique if $\mathfrak{x} = (\mathfrak{x}_{jk})$ is $\mathfrak{LSt}\mathfrak{C}$ with regard to the $\mathfrak{NN}(\mu, \nu, \omega)$.

Proof. Assume that $\aleph_\theta^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = \ell_1$ and $\aleph_\theta^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = \ell_2$. Consider $\epsilon > 0$ and choose $> 0 : (1 - \mathfrak{d}) * (1 - \mathfrak{d}) > 1-\epsilon$, $\mathfrak{d} \diamond \mathfrak{d} < \epsilon$ and $\mathfrak{d} \circledast \mathfrak{d} < \epsilon$. Next, define the following sets as follows for every $\lambda > 0$:

$\mathfrak{I}_{\mu,1}(\mathfrak{d},\lambda) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk} - \ell_1, \lambda) \leq 1 - \mathfrak{d}\}$,
$\mathfrak{I}_{\mu,2}(\mathfrak{d},\lambda) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk} - \ell_2, \lambda) \leq 1 - \mathfrak{d}\}$,
$\mathfrak{I}_{\nu,1}(\mathfrak{d},\lambda) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu(\mathfrak{x}_{jk} - \ell_1, \lambda) \geq \mathfrak{d}\}$,
$\mathfrak{I}_{\nu,2}(\mathfrak{d},\lambda) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu(\mathfrak{x}_{jk} - \ell_2, \lambda) \geq \mathfrak{d}\}$.
$\mathfrak{I}_{\omega,1}(\mathfrak{d},\lambda) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \omega(\mathfrak{x}_{jk} - \ell_1, \lambda) \geq \mathfrak{d}\}$,
$\mathfrak{I}_{\omega,2}(\mathfrak{d},\lambda) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \omega(\mathfrak{x}_{jk} - \ell_2, \lambda) \geq \mathfrak{d}\}$

We have to use Lemma 3.1 since $\aleph_\theta^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = \ell_1$.

$\delta_\theta\left(\mathfrak{I}_{\mu,1}(\epsilon,\lambda)\right) = \delta_\theta\left(\mathfrak{I}_{\nu,1}(\epsilon,\lambda)\right) = \delta_\theta\left(\mathfrak{I}_{\omega,1}(\epsilon,\lambda)\right) = 0$ for all $\lambda > 0$.

Additionally, using $\aleph_{\mu,\nu,\omega}^\theta - \lim \mathfrak{x} = \ell_2$, we get

$\delta_\theta\left(\mathfrak{I}_{\mu,2}(\epsilon,\lambda)\right) = \delta_\theta\left(\mathfrak{I}_{\nu,2}(\epsilon,\lambda)\right) = \delta_\theta\left(\mathfrak{I}_{\omega,2}(\epsilon,\lambda)\right) = 0$ for all $\lambda > 0$.

Let's now

$\mathfrak{I}_{\mu,\nu,\omega}(\epsilon,\lambda) = \left(\mathfrak{I}_{\mu,1}(\epsilon,\lambda) \cup \mathfrak{I}_{\mu,2}(\epsilon,\lambda)\right) \cap \left(\mathfrak{I}_{\nu,1}(\epsilon,\lambda) \cup \mathfrak{I}_{\nu,2}(\epsilon,\lambda)\right) \cap \left(\mathfrak{I}_{\omega,1}(\epsilon,\lambda) \cup \mathfrak{I}_{\omega,2}(\epsilon,\lambda)\right)$.

Next, note that $\delta_\theta\left(\mathfrak{I}_{\mu,\nu,\omega}(\epsilon,\lambda)\right) = 0$ which suggests

$\delta_\theta\left(\mathbb{N} \setminus \mathfrak{I}_{\mu,\nu,\omega}(\epsilon,\lambda)\right) = 1$. If $k \in \mathbb{N} \setminus \mathfrak{I}_{\mu,\nu,\omega}(\epsilon,\lambda)$, hence, there are three scenarios that could occur.

(a) $k \in \mathbb{N} \setminus \left(\mathfrak{I}_{\mu,1}(\epsilon,\lambda) \cup \mathfrak{I}_{\mu,2}(\epsilon,\lambda)\right)$ and

(b) $k \in N \setminus \left(\mathfrak{I}_{\nu,1}(\epsilon,\lambda) \cup \mathfrak{I}_{\nu,2}(\epsilon,\lambda)\right)$.

(c) $k \in N \setminus \left(\mathfrak{I}_{\omega,1}(\epsilon,\lambda) \cup \mathfrak{I}_{\omega,2}(\epsilon,\lambda)\right)$.

We first consider that $k \in N \setminus \left(\mathfrak{I}_{\mu,1}(\epsilon,\lambda) \cup \mathfrak{I}_{\mu,2}(\epsilon,\lambda)\right)$. Then we have

$\mu(\ell_1 - \ell_2, \lambda) \geq \mu\left(\mathfrak{x}_k - \ell_1, \frac{\lambda}{2}\right) * \mu\left(\mathfrak{x}_k - \ell_2, \frac{\lambda}{2}\right) > (1 - \mathfrak{d}) * (1 - \mathfrak{d})$.

$\mu(\ell_1 - \ell_2, \lambda) > 1 - \epsilon$ since $(1 - \mathfrak{d}) * (1 - \mathfrak{d}) > 1 - \epsilon$.

Since $\epsilon > 0$ was random, we obtain $\mu(\ell_1 - \ell_2, \lambda) = 1$ for any $\lambda > 0$, which results in $\ell_1 = \ell_2$.

As an alternative, we can write

$\nu(\ell_1 - \ell_2, \lambda) \leq \nu\left(\mathfrak{x}_k - \ell_1, \frac{\lambda}{2}\right) \diamond \nu\left(\mathfrak{x}_k - \ell_2, \frac{\lambda}{2}\right) < \mathfrak{d} \diamond \mathfrak{d}$ if $k \in N \setminus \left(\mathfrak{I}_{\nu,1}(\epsilon,\lambda) \cup \mathfrak{I}_{\nu,2}(\epsilon,\lambda)\right)$.

Using the knowledge that $\mathfrak{d} \diamond \mathfrak{d} < \epsilon$, we can now observe that $\nu(\ell_1 - \ell_2, \lambda) < \epsilon$.
Thus, for any $\lambda > 0$, $\nu(\ell_1 - \ell_2, \lambda) = 0$, suggesting that $\ell_1 = \ell_2$.

Also, if $k \in N \setminus \left(\mathfrak{I}_{\omega,1}(\epsilon,\lambda) \cup \mathfrak{I}_{\omega,2}(\epsilon,\lambda)\right)$, after which we could write

$\omega(\ell_1 - \ell_2, \lambda) \leq \omega\left(\mathfrak{x}_k - \ell_1, \frac{\lambda}{2}\right) \circledast \omega\left(\mathfrak{x}_k - \ell_2, \frac{\lambda}{2}\right) < s \circledast s$.

We can observe that $\omega(\ell_1 - \ell_2, \lambda) < \epsilon$ by using the information that $\mathfrak{d} \circledast \mathfrak{d} < \epsilon$.
Thus, for any $\lambda > 0$, $\omega(\ell_1 - \ell_2, \lambda) = 0$ implies $\ell_1 = \ell_2$.

Hence, we deduce that $\aleph_\theta^{(\mu,\nu,\omega)}$-limit is unique in each case.

Hereby, the theorem's proof is concluded.

Theorem 3.6 Let θ be any $\mathfrak{LS}$ and $(\mathfrak{X}, \mu, \nu, \omega, *, \diamond, \circledast)$ be a $\mathfrak{NNS}$. If $(\mu,\nu,\omega) - \lim \mathfrak{x} = \ell$, then $\aleph_\theta^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = \ell$.

Proof. Let $(\mu,\nu,\omega) - \lim \mathfrak{x} = \ell$. Then for every $\epsilon > 0$ and $\lambda > 0$, there is a number $k_0 \in \mathbb{N}$ such that $\mu(\mathfrak{x}_k - \ell, \lambda) > 1 - \epsilon$ and $\nu(\mathfrak{x}_k - \ell, \lambda) < \epsilon$ and $\omega(\mathfrak{x}_k - \ell, \lambda) < \epsilon$ for all $k \geq k_0$.

Hence the set $\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk} - \ell, \lambda) \leq 1 - \epsilon \text{ or } \nu(\mathfrak{x}_{jk} - \ell, \lambda) \geq \epsilon, \omega(\mathfrak{x}_{jk} - \ell, \lambda) \geq \epsilon\}$ possesses certain number of terms. Given that each finite subset of $\mathbb{N}$ has a density of zero,

$\delta_\theta(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk} - \ell, \lambda) \leq 1 - \epsilon \text{ or } \nu(\mathfrak{x}_{jk} - \ell, \lambda) \geq \epsilon, \omega(\mathfrak{x}_{jk} - \ell, \lambda) \geq \epsilon\}) = 0$,

that is, $\aleph_\theta^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = \ell$. This concludes the theorem's proof.

Example 3.7 Let $(\mathfrak{X}, \|\cdot\|)$ denote the space of all real numbers with the usual norm, and let $a * b = ab$, $a \diamond b = \min\{a + b, 1\}$ and $a \circledast b = \min\{a + b, 1\}$ for all $a, b \in [0,1]$. For all $x \in \mathbb{R}$ and $\lambda > 0$, consider $\mu(\mathfrak{x}, \lambda) := \frac{\lambda}{\lambda + \|\mathfrak{x}\|}$, $\nu(\mathfrak{x}, \lambda) := \frac{\|\mathfrak{x}\|}{\lambda + \|\mathfrak{x}\|}$ and $\omega(\mathfrak{x}, \lambda) := \frac{\|\mathfrak{x}\|}{\lambda}$. Then $(\mathfrak{X}, \mu, \nu, \omega, *, \diamond, \circledast)$ be a $\mathfrak{NNS}$.

Now we define a sequence $\mathfrak{x} = (\mathfrak{x}_{jk})$ by

$$\mathfrak{x}_{jk} = \begin{cases} (j,k); \text{for } j_r - [\mathfrak{h}_\mathfrak{r}] + 1 \le j \le j_\mathfrak{r}, k_r - [\mathfrak{h}_\mathfrak{r}] + 1 \le k \le k_\mathfrak{r}, \mathfrak{r} \in \mathbb{N} \\ 0; \text{otherwise.} \end{cases}$$

Let for $\epsilon > 0,\ \lambda > 0$.

$\mathfrak{I}_\mathfrak{r}(\epsilon,\lambda) = \left\{(j,k) \in \mathbb{N} \times \mathbb{N}: \frac{\lambda}{\lambda + \|\mathfrak{x}_{jk}\|} \le 1 - \epsilon \text{ or} \frac{\|\mathfrak{x}_{jk}\|}{\lambda + \|\mathfrak{x}_{jk}\|} \ge \epsilon, \frac{\|\mathfrak{x}_{jk}\|}{\lambda} \ge \epsilon \right\},$

$= \left\{(j,k) \in \mathbb{N} \times \mathbb{N}: \|\mathfrak{x}_{jk}\| \ge \frac{\epsilon\lambda}{1-\epsilon} > 0 \right\},$

$= \{(j,k) \in \mathbb{N} \times \mathbb{N}: \|\mathfrak{x}_{jk}\| = (j,k) \},$

$= \{(j,k) \in \mathbb{N} \times \mathbb{N}: j_r - \left[\sqrt{\mathfrak{h}_\mathfrak{r}}\right] + 1 \le j \le j_\mathfrak{r}, k_r - \left[\sqrt{\mathfrak{h}_\mathfrak{r}}\right] + 1 \le k \le k_\mathfrak{r}, \mathfrak{r} \in \mathbb{N}\},$

and so, we get

$$\frac{1}{\mathfrak{h}_\mathfrak{r}}|\mathfrak{I}_\mathfrak{r}(\epsilon,\lambda)| \le \frac{1}{\mathfrak{h}_\mathfrak{r}}\left|\{(j,k) \in \mathbb{N} \times \mathbb{N}: j_r - \left[\sqrt{\mathfrak{h}_\mathfrak{r}}\right] + 1 \le j \le j_\mathfrak{r}, k_r - \left[\sqrt{\mathfrak{h}_\mathfrak{r}}\right] + 1 \le k \le k_\mathfrak{r}, \mathfrak{r} \in \mathbb{N}\}\right| \le \frac{\sqrt{\mathfrak{h}_\mathfrak{r}}}{\mathfrak{h}_\mathfrak{r}},$$

which implies that $\lim_\mathfrak{r} \frac{1}{\mathfrak{h}_\mathfrak{r}}|\mathfrak{I}_\mathfrak{r}(\epsilon,\lambda)| = 0$. Hence

$\delta_\theta(\mathfrak{I}_\mathfrak{r}(\epsilon,\lambda)) = \lim \frac{\sqrt{\mathfrak{h}_\mathfrak{r}}}{\mathfrak{h}_\mathfrak{r}} = 0$ as $\mathfrak{r} \to \infty$ implies that $\mathfrak{x}_{jk} \to 0(\aleph_\theta)$.

On the other hand $\mathfrak{x}_{jk} \nrightarrow 0$, since

$$\mu(\mathfrak{x}_{jk},\lambda) = \frac{\lambda}{\lambda + \|\mathfrak{x}_{jk}\|} = \begin{cases} \frac{\lambda}{\lambda + \|jk\|}; \text{for } j_r - [\mathfrak{h}_\mathfrak{r}] + 1 \le j \le j_\mathfrak{r}, k_r - [\mathfrak{h}_\mathfrak{r}] + 1 \le k \le k_\mathfrak{r}, \mathfrak{r} \in \mathbb{N} \\ 1; \text{otherwise.} \end{cases} \le 1, \text{ and}$$

$$\nu(\mathfrak{x}_{jk},\lambda) = \frac{\|\mathfrak{x}_{jk}\|}{\lambda + \|\mathfrak{x}_{jk}\|} = \begin{cases} \frac{\|jk\|}{\lambda + \|jk\|}; \text{for } j_r - [\mathfrak{h}_\mathfrak{r}] + 1 \le j \le j_\mathfrak{r}, k_r - [\mathfrak{h}_\mathfrak{r}] + 1 \le k \le k_\mathfrak{r}, \mathfrak{r} \in \mathbb{N} \\ 0 \quad ; \text{otherwise.} \end{cases} \ge 0, \text{ also}$$

$$\omega(\mathfrak{x}_{jk},\lambda) = \frac{\|\mathfrak{x}_{jk}\|}{\lambda + \|\mathfrak{x}_{jk}\|} = \begin{cases} \frac{\|jk\|}{\lambda + \|jk\|}; \text{for } j_r - [\mathfrak{h}_\mathfrak{r}] + 1 \le j \le j_\mathfrak{r}, k_r - [\mathfrak{h}_\mathfrak{r}] + 1 \le k \le k_\mathfrak{r}, \mathfrak{r} \in \mathbb{N} \\ 0 \quad ; \text{otherwise.} \end{cases} \ge 0.$$

This completes the proof.

4. Lacunary statistically($\mathfrak{LS}t$) Cauchy double sequences in $\mathfrak{NNS}$

This section introduces a new notion of statistical completeness and defines lacunary statistically Cauchy double sequences with regard to a $\mathfrak{NNS}$.

Definition 4.1 Let θ be a $\mathfrak{LS}$ and $(\mathfrak{X}, \mu, \nu, \omega, *, \diamond, \circledast)$ be a $\mathfrak{NNS}$. Then, $\forall\ \epsilon > 0$ and $\lambda > 0$, $\exists\ n = n(\epsilon)$ and $m = m(\epsilon)$ such that

$\delta_\theta(\{(j,k) \in \mathbb{N} \times \mathbb{N}: \mu(\mathfrak{x}_{jk} - \mathfrak{x}_{mn},\lambda) \le 1 - \epsilon \ or\ \nu(\mathfrak{x}_{jk} - \mathfrak{x}_{mn},\lambda), \omega(\mathfrak{x}_{jk} - \mathfrak{x}_{mn},\lambda) \ge \epsilon\}) = 0.$

This indicates that the sequence $\mathfrak{x} = (\mathfrak{x}_{jk})$ is $\mathfrak{LS}t-$ Cauchy (or $\aleph_\theta$-Cauchy) with regard to the (μ,ν,ω).

Theorem 4.2 Consider a $\mathfrak{NNS}$ $(\mathfrak{X}, \mu, \nu, \omega, *, \diamond, \circledast)$ with any $\mathfrak{LS}$ θ. If a sequence $\mathfrak{x} = (\mathfrak{x}_{jk})$ is $\aleph_\theta$-Cauchy with regard to the (μ,ν,ω), then it is $\aleph_\theta$-convergent.

Proof. Let $\mathfrak{x} = (\mathfrak{x}_k)$ be $\aleph_\theta$-convergent to ℓ with respect to the $\mathfrak{NN}(\mu,\nu,\omega)$, i.e., $\aleph_\theta^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = \ell$. Then

$\delta_\theta\left(\left\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu\left(\mathfrak{x}_{jk} - \ell, \frac{\lambda}{2}\right) \le 1 - \epsilon \ or\ \nu\left(\mathfrak{x}_{jk} - \ell, \frac{\lambda}{2}\right) \ge \epsilon, \omega\left(\mathfrak{x}_{jk} - \ell, \frac{\lambda}{2}\right) \ge \epsilon\right\}\right) = 0.$

Specifically, for $k = N$

$\delta_\theta\left(\left\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu\left(\mathfrak{x}_{mn} - \ell, \frac{\lambda}{2}\right) \le 1 - \epsilon \ or\ \nu\left(\mathfrak{x}_{mn} - \ell, \frac{\lambda}{2}\right) \ge \epsilon, \omega\left(\mathfrak{x}_{mn} - \ell, \frac{\lambda}{2}\right) \ge \epsilon\right\}\right) = 0.$

Since

$\mu(\mathfrak{x}_{jk} - \mathfrak{x}_{mn},\lambda) = \mu\left(\mathfrak{x}_{jk} - \ell - \mathfrak{x}_{mn} + \ell, \frac{\lambda}{2} + \frac{\lambda}{2}\right) \ge \mu\left(\mathfrak{x}_{jk} - \ell, \frac{\lambda}{2}\right) * \mu\left(\mathfrak{x}_{mn} - \ell, \frac{\lambda}{2}\right)$

and since

$\nu(\mathfrak{x}_{jk}-\mathfrak{x}_{mn},\lambda) \leq \nu\left(\mathfrak{x}_{jk}-\ell,\frac{\lambda}{2}\right) \diamond \nu\left(\mathfrak{x}_{mn}-\ell,\frac{\lambda}{2}\right)$, $\omega(\mathfrak{x}_{jk}-\mathfrak{x}_{mn},\lambda) \leq \omega\left(\mathfrak{x}_{jk}-\ell,\frac{\lambda}{2}\right) \circledast \omega\left(\mathfrak{x}_{mn}-\ell,\frac{\lambda}{2}\right)$,

we have

$\delta_\theta(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk}-\mathfrak{x}_{mn},\lambda) \leq 1-\epsilon \ or \ \nu(\mathfrak{x}_{jk}-\mathfrak{x}_{mn},\lambda) \geq \epsilon, \omega(\mathfrak{x}_{jk}-\mathfrak{x}_{mn},\lambda) \geq \epsilon\}) = 0,$

that is, with regard to the $\mathfrak{NN}$ (μ,ν,ω), $\mathfrak{x}$ is $\aleph_\theta$-Cauchy.

In contrast, let $\mathfrak{x} = (\mathfrak{x}_{jk})$ be $\aleph_\theta$-Cauchy, but with respect to the $\mathfrak{S}$ (μ,ν,ω), it is not $\aleph_\theta$-convergent.

Consequently, N exists such that $\delta_\theta(\mathfrak{A}(\epsilon,\lambda)) = 0,$

(3)

$\delta_\theta(\mathfrak{B}(\epsilon,\lambda)) = 0, \quad i.e. \ \delta_\theta(\mathfrak{B}^C(\epsilon,\lambda)) = 1;$

(4)

where

$\mathfrak{A}(\epsilon,\lambda) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk}-\mathfrak{x}_{mn},\lambda) \leq 1-\epsilon \text{ or } \nu(\mathfrak{x}_{jk}-\mathfrak{x}_{mn},\lambda) \geq \epsilon, \omega(\mathfrak{x}_{jk}-\mathfrak{x}_{mn},\lambda) \geq \epsilon\},$

$\mathfrak{B}(\epsilon,\lambda) = \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu\left(\mathfrak{x}_{jk}-\ell,\frac{\lambda}{2}\right) > \frac{1-\epsilon}{2}, \ \nu\left(\mathfrak{x}_{jk}-\ell,\frac{\lambda}{2}\right) < \frac{\epsilon}{2} \text{ and } \omega\left(\mathfrak{x}_{jk}-\ell,\frac{\lambda}{2}\right) < \frac{\epsilon}{2}\right\}.$

Since

$\mu(\mathfrak{x}_{jk}-\mathfrak{x}_{mn},\lambda) \geq 2\mu\left(\mathfrak{x}_{jk}-\ell,\frac{\lambda}{2}\right) > 1-\epsilon,$

$\nu(\mathfrak{x}_{jk}-\mathfrak{x}_{mn},\lambda) \leq 2\nu\left(\mathfrak{x}_{jk}-\ell,\frac{\lambda}{2}\right) < \epsilon$ and

$\omega(\mathfrak{x}_{jk}-\mathfrak{x}_{mn},\lambda) \leq 2\omega\left(\mathfrak{x}_{jk}-\ell,\frac{\lambda}{2}\right) < \epsilon$

if $\left(\mathfrak{x}_{jk}-\ell,\frac{\lambda}{2}\right) > \frac{1-\epsilon}{2}$, $\nu\left(\mathfrak{x}_{jk}-\ell,\frac{\lambda}{2}\right) < \frac{\epsilon}{2}$ and $\omega\left(\mathfrak{x}_{jk}-\ell,\frac{\lambda}{2}\right) < \frac{\epsilon}{2}$.

Therefore,

$\delta_\theta(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk}-\mathfrak{x}_{mn},\lambda) > 1-\epsilon, \ \nu(\mathfrak{x}_{jk}-\mathfrak{x}_{mn},\lambda) < \epsilon \text{ and } \omega(\mathfrak{x}_{jk}-\mathfrak{x}_{mn},\lambda) < \epsilon\}) = 0,$

since $\mathfrak{x}$ was $\aleph_\theta$-Cauchy with respect to $\mathfrak{NN}$ (μ,ν,ω), $\delta_\theta(\mathfrak{A}(\epsilon,\lambda)) = 1$, which defies (3).

Hence, with regard to (μ,ν,ω), $\mathfrak{x}$ must be $\aleph_\theta$-convergent.

Definition 4.3 If all of the Cauchy sequences in $(\mathfrak{X},\tau,\varphi,\omega,*,\diamond,\circledast)$, then the $\mathfrak{NNS}$ $(\mathfrak{X},\tau,\varphi,\omega,*,\diamond,\circledast)$ is considered complete.

Definition 4.4 If every $\aleph_\theta$- Cauchy sequence in relation to $\mathfrak{NN}$ (τ,φ,ω), is $\aleph_\theta$- convergent in relation to $\mathfrak{NN}$ (τ,φ,ω), then a $\mathfrak{NNS}$ $(\mathfrak{X},\tau,\varphi,\omega,*,\diamond,\circledast)$ is statistically complete ($\aleph_\theta$- complete).

Theorem 4.5 Let any $\mathfrak{L}\mathcal{S}$ be represented by θ. In that case, any $\mathfrak{NNS}$ $(\mathfrak{X},\tau,\varphi,\omega,*,\diamond,\circledast)$ is $\aleph_\theta$-complete, but not necessarily complete.

Proof. Given a $\mathfrak{NN}$ (τ,φ,ω), let $\mathfrak{x} = (\mathfrak{x}_{jk})$ be $\aleph_\theta$- Cauchy but not $\aleph_\theta$- convergent.

Assuming $\epsilon > 0$ and $\lambda > 0$, select $\mathfrak{d} > 0$: $(1-\epsilon) * (1-\epsilon) > 1-\mathfrak{d}$, $\epsilon \diamond \epsilon < \mathfrak{d}$ and $\epsilon \circledast \epsilon < \mathfrak{d}$.

Now,

$\tau(\mathfrak{x}_{jk}-\mathfrak{x}_{mn},\lambda) \geq \tau\left(\mathfrak{x}_{jk}-\ell,\frac{\lambda}{2}\right) * \tau\left(\mathfrak{x}_{mn}-\ell,\frac{\lambda}{2}\right) > (1-\epsilon) * (1-\epsilon) > 1-\mathfrak{d},$

$\varphi(\mathfrak{x}_{jk}-\mathfrak{x}_{mn},\lambda) \leq \varphi\left(\mathfrak{x}_{jk}-\ell,\frac{\lambda}{2}\right) * \varphi\left(\mathfrak{x}_{mn}-\ell,\frac{\lambda}{2}\right) < \epsilon \diamond \epsilon < \mathfrak{d}$ and

$\omega(\mathfrak{x}_{jk}-\mathfrak{x}_{mn},\lambda) \leq \omega\left(\mathfrak{x}_{jk}-\ell,\frac{\lambda}{2}\right) * \omega\left(\mathfrak{x}_{mn}-\ell,\frac{\lambda}{2}\right) < \epsilon \circledast \epsilon < \mathfrak{d}$, as $\mathfrak{x}$ is not $\aleph_\theta$- convergent.

As a result, $\delta_\theta(\mathfrak{H}^C(\epsilon,\lambda)) = 0,$ where

$\mathfrak{H}(\epsilon,\lambda) = \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : \varphi_{\mathfrak{x}_{jk}-\mathfrak{x}_{mn}}(\epsilon) \leq 1-\mathfrak{r}, \omega_{\mathfrak{x}_{jk}-\mathfrak{x}_{mn}}(\epsilon) \leq 1-\mathfrak{r}\right\}.$

Consequently, $\delta_\theta(\mathfrak{H}(\epsilon,\lambda)) = 1$, which is a contradiction, since $\mathfrak{x}$ was $\aleph_\theta$- Cauchy with regard to $\mathfrak{N}\mathfrak{N}$ (τ,φ,ω). Thus, $\mathfrak{x}$ needs to be $\aleph_\theta$- convergent in relation to $\mathfrak{N}\mathfrak{N}$ (τ,φ,ω). As a result, each $\mathfrak{N}\mathfrak{N}\mathfrak{S}$ is $\aleph_\theta$- complete.

We can observe from the following example that a $\mathfrak{N}\mathfrak{N}\mathfrak{S}$ is not complete in general.

Example 4.6 For $\mathfrak{X} = (0,1]$, let $\tau(\mathfrak{x},\lambda) := \frac{\lambda}{\lambda+\|\mathfrak{x}\|}$, $\varphi(\mathfrak{x},\lambda) := \frac{\|\mathfrak{x}\|}{\lambda+\|\mathfrak{x}\|}$ and $\omega(\mathfrak{x},\lambda) := \frac{\|\mathfrak{x}\|}{\lambda}$. When the sequence $\left(\frac{1}{n}\right)$ is Cauchy sequence with respect to $\mathfrak{N}\mathfrak{N}$ (τ,φ,ω) but not convergent with respect to $\mathfrak{N}$ (τ,φ,ω), then $(\mathfrak{X},\tau,\varphi,\omega,min,max,max)$ is $\mathfrak{N}\mathfrak{N}\mathfrak{S}$ but not complete.

This concludes the theorem's proof.

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