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Exploring Topological characteristics of Neutrosophic Banach Spaces

Pandiselvi. M[1], and Jeyaraman. M[2]

[1]PG and Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai, Affiliated to Alagappa University, Karaikudi, Tamil Nadu, India. E-mail: mpandiselvi2612@gmail.com
[2]PG and Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai, Affiliated to Alagappa University, Karaikudi, Tamil Nadu, India. E-mail: jeya.math@gmail.com

Abstract: The open mapping theorem, the closed graph theorem, and other topological characteristics of neurosophic Banach spaces are examined in this paper. Moreover, in neutrosophic normed spaces, the closedness characteristics of the sum of two linear operators has been studied.

Keywords: Neutrosophic normed space, Neutrosophic Banach space, Closed linear operator, First category, Second category.

1. Introduction

Zadeh [23], laid the foundation for fuzzy mathematics in 1965. This concept takes standard set theory to a higher level of analysis. After then, the idea received a number of proven improvements, and the reasoning has been used in a variety of scientific and engineering fields, including the study of approximation theory [1], linear systems [6] [17] and matrix theory. Several authors investigated at the theory with its topological aspects from their own angle and came to some important basic results that seem important when examining the idea in connection to different other generalized spaces.

In 1992, Felbin [9] presented a new idea of fuzzy norms on linear spaces. Xiao and Zhu [22] expanded the concept of fuzzy norm by studying the topological properties of fuzzy normed linear spaces. Another fuzzy norm was established by Bag and Samanta [4]. Bag and Samanta [5] developed weak fuzzy boundedness, weak fuzzy continuity, strong fuzzy boundedness, fuzzy continuity, sequential fuzzy continuity, and the fuzzy norm of linear operators with respects to an associated fuzzy norm. Atanassov [2] developed the idea of an intuitionistic fuzzy set in 1984. He did this by designating a new kind of membership function that indicates how much an item does not belong in a particular set. Park [15] defined the notion of Intuitionistic Fuzzy Metric Space with the help of continuous t-norms and continuous. Amazing work was done on intuitionistic fuzzy topological spaces by sadati and park [18]. Many authors have since published their own works in the literature (see [12] [14] [16]). Among them are those who have made multiple important contributions to convergence theory and proposed convergent sequence spaces within the intuitionistic fuzzy normed space framework. Some important topological findings in fuzzy Banach spaces were studied in 2005 by Saadati and Vaezpour [19]. In 1998, Smarandache[21] developed the ideas of neutrosophic logic and Neutrosophic Set. Kirisci and Simsek [13] founded the concept of Neutrosophic Metric Spaces which addresses membership, non-membership and neutralness. The aim of this study is to investigate topological characteristics of neutrosophic Banach space, building on the results of [19] earlier studies published in [19]. Furthermore, the information provides some unresearched results.

2. Preliminaries

Definition 2.1[10]: A binary operation $*:\ [0,1]\times[0,1]\to[0,1]$ is called a continuous t-norm if following conditions hold:

(i) $\mathfrak{z} * \tilde{\varsigma} = \tilde{\varsigma} * \mathfrak{z}$ for all $\mathfrak{z}, \tilde{\varsigma} \in [0,1]$;
(ii) $*$ is continuous;
(iii) $\mathfrak{z} * 1 = \mathfrak{z}$, for all $\mathfrak{z} \in [0, 1]$;
(iv) $*$ is associative;
If $\mathfrak{z} \le \tilde{\varsigma}$ and $\mathfrak{d} \le \vartheta$, with $\mathfrak{z}, \tilde{\varsigma}, \mathfrak{d}, \vartheta \in [0,1]$, then $\mathfrak{z} * \mathfrak{d} \le \tilde{\varsigma} * \vartheta$.

Definition 2.2[10]: A binary operation $\odot$: $[0,1]\times[0,1]\to[0,1]$ is called a continuous t-conorm if it holds the followings assertions:

(i) $\mathfrak{z} \odot \tilde{\varsigma} = \tilde{\varsigma} \odot \mathfrak{z}$ for all $\mathfrak{z}, \tilde{\varsigma} \in [0,1]$;
(ii) $\odot$ is continuous;
(iii) $\mathfrak{z} \odot 0 = 0$;
(iv) $\odot$ is associative;
(v) If $\mathfrak{z} \le \tilde{\varsigma}$ and $\mathfrak{d} \le \vartheta$, with $\mathfrak{z}, \tilde{\varsigma}, \mathfrak{d}, \vartheta \in [0,1]$, then $\mathfrak{z} \odot \mathfrak{d} \le \tilde{\varsigma} \odot \vartheta$.

Definition 2.3: The 6-tuple $(\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$ is said to be a Neutrosophic Normed Linear Space [*NNLS*], if $\widetilde{\mathcal{N}}$ is a vector space over a field $\mathbb{R}$, $\star$ is a continuous t-norm, $\diamond$ is a continous t-conorm, and $\eta, \upsilon, \varsigma$ are functions from $\widetilde{\mathcal{N}} \times \mathbb{R} \to [0,1]$ meets the following conditions for every $\mathfrak{e}, \mathfrak{w} \in \widetilde{\mathcal{N}}$ and $\sigma, \mathfrak{k} \in \mathbb{R}$

(n1) $0 \le \eta(\mathfrak{e}, \mathfrak{k}) \le 1;\ 0 \le \upsilon(\mathfrak{e}, \mathfrak{k}) \le 1;\ 0 \le \varsigma(\mathfrak{e}, \mathfrak{k}) \le 1$;
(n2) $\eta(\mathfrak{e}, \mathfrak{k}) + \upsilon(\mathfrak{e}, \mathfrak{k}) + \rho(\mathfrak{e}, \mathfrak{k}) \le 3$;
(n3) $\eta(\mathfrak{e}, \mathfrak{k}) > 0$;
(n4) $\eta(\mathfrak{e}, \mathfrak{k}) = 1 \Leftrightarrow \mathfrak{v} = 0$;
(n5) $\eta(\sigma\mathfrak{e}, \mathfrak{k}) = \eta\left(\mathfrak{e}, \frac{\mathfrak{k}}{|\sigma|}\right)$ for $\sigma \ne 0$;
(n6) $\eta(\mathfrak{e}, \sigma) \star \eta(\mathfrak{w}, \mathfrak{k}) \le \eta(\mathfrak{e} + \mathfrak{w},\ \sigma + \mathfrak{k})$;
(n7) $\eta(\mathfrak{e}, \mathfrak{k}) : (0, \infty) \to [0,1]$ is continuous;
(n8) $\lim_{\mathfrak{k}\to\infty} \eta(\mathfrak{e}, \mathfrak{k}) = 1$ and $\lim_{\mathfrak{k}\to 0} \eta(\mathfrak{e}, \mathfrak{k}) = 0$;
(n9) $\upsilon(\mathfrak{e}, \mathfrak{k}) < 1$;
(n10) $\upsilon(\mathfrak{e}, \mathfrak{k}) = 0 \Leftrightarrow \mathfrak{v} = 0$;
(n11) $\upsilon(\sigma\mathfrak{e}, \mathfrak{k}) = \upsilon\left(\mathfrak{e}, \frac{\mathfrak{k}}{|\sigma|}\right)$ for $\sigma \ne 0$;
(n12) $\upsilon(\mathfrak{e}, \sigma) \diamond \upsilon(\mathfrak{w}, \mathfrak{k}) \ge \upsilon(\mathfrak{e} + \mathfrak{w},\ \sigma + \mathfrak{k})$;
(n13) $\upsilon(\mathfrak{e}, \tau) : (0, \infty) \to [0,1]$ is continuous;
(n14) $\lim_{\mathfrak{k}\to\infty} \upsilon(\mathfrak{e}, \mathfrak{k}) = 0$ and $\lim_{\mathfrak{k}\to 0} \upsilon(\mathfrak{e}, \mathfrak{k}) = 1$;
(n15) $\varsigma(\mathfrak{e}, \mathfrak{k}) < 1$;
(n16) $\varsigma(\mathfrak{e}, \mathfrak{k}) = 0 \Leftrightarrow \mathfrak{e} = 0$;
(n17) $\varsigma(\sigma\mathfrak{e}, \mathfrak{k}) = \varsigma\left(\mathfrak{e}, \frac{\mathfrak{k}}{|\sigma|}\right)$ for $\sigma \ne 0$;
(n18) $\varsigma(\mathfrak{e}, \sigma) \diamond \varsigma(\mathfrak{w}, \mathfrak{k}) \ge \varsigma(\mathfrak{e} + \mathfrak{w}, \sigma + \mathfrak{k})$;
(n19) $\varsigma(\mathfrak{e}, \mathfrak{k}) : (0, \infty) \to [0,1]$ is continuous;
(n20) $\lim_{\mathfrak{k}\to\infty} \varsigma(\mathfrak{e}, \mathfrak{k}) = 0$ and $\lim_{\mathfrak{k}\to 0} \varsigma(\mathfrak{e}, \mathfrak{k}) = 1$.

Remark 2.4 [15]:

(i) For every pair $0 < \mathfrak{z}_1, \mathfrak{z}_2 < 1$ we can find $0 < \tilde{\varsigma}_1, \tilde{\varsigma}_2 < 1$ such that $\mathfrak{z}_1 \le \mathfrak{z}_2 * \tilde{\varsigma}_2$ and $\mathfrak{z}_2 \ge \tilde{\varsigma}_1 \diamond \mathfrak{z}_1$.
(ii) For every $0 < \mu < 1$ we can find $0 < \tilde{\varsigma}_1, \tilde{\varsigma}_2 < 1$ such that $\mu \le \mathfrak{z}_1 * \mathfrak{z}_2$ and $\mathfrak{z}_1 \diamond \mathfrak{z}_2 \ge \mu$.

Definition 2.5: In neutrosophic normed linear space, the open ball $\mathcal{B}_{\mathfrak{v}}(\varepsilon, \mathfrak{k})$ centred at $\mathfrak{e}$ is defined as $\mathcal{B}_{\mathfrak{e}}(\mathfrak{r}, \mathfrak{k}) = \{ \mathfrak{e} \in \widetilde{\mathcal{N}} : 1 - \eta(\mathfrak{e} - \mathfrak{w}, \mathfrak{k}) < \mathfrak{r}, \upsilon(\mathfrak{e} - \mathfrak{w}, \mathfrak{k}) > \varepsilon$ and $\varsigma(\mathfrak{e} - \mathfrak{w}, \mathfrak{k}) > \mathfrak{r} \}$ where $0 < \mathfrak{r} < 1$ and $\mathfrak{k} > 0$. Similarly, the closed ball centred at $\mathfrak{e}$ is defined as

$\mathcal{B}_{\mathfrak{e}}[\mathfrak{r}, \mathfrak{k}] = \{ \mathfrak{e} \in \widetilde{\mathcal{N}} : 1 - \eta(\mathfrak{e} - \mathfrak{w}, \mathfrak{k}) \leq \mathfrak{r}, \upsilon(\mathfrak{e} - \mathfrak{w}, \mathfrak{k}) \geq \varepsilon$ and $\varsigma(\mathfrak{e} - \mathfrak{w}, \mathfrak{k}) \geq \mathfrak{r} \}$ where $0 < \mathfrak{r} < 1$ and $\mathfrak{k} > 0$.

Remark 2.6 [15]: Every open ball is an open set in neutrosophic normed linear space.

Lemma 2.7: Let $(\eta, \upsilon, \varsigma)$ be neutrosophic norm on $(\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$ then

(i) $\eta(\mathfrak{e}, \mathfrak{k})$ is a non decreasing, $\upsilon(\mathfrak{e}, \mathfrak{k})$ is non a non increasing and $\varsigma(\mathfrak{e}, \mathfrak{k})$ is decreasing but not strictly with respect to $\mathfrak{k}$ for each $\mathfrak{e} \in \widetilde{\mathcal{N}}$.

(ii) $\eta(\mathfrak{e} - \mathfrak{w}, \mathfrak{k}) = \eta(\mathfrak{w} - \mathfrak{e}, \mathfrak{k})$, $\upsilon(\mathfrak{e} - \mathfrak{w}, \mathfrak{k}) = \upsilon(\mathfrak{w} - \mathfrak{e}, \mathfrak{k})$ and $\varsigma(\mathfrak{e} - \mathfrak{w}, \mathfrak{k}) = \varsigma(\mathfrak{w} - \mathfrak{e}, \mathfrak{k})$.

Proof: Let $\mathfrak{k}_1 < \mathfrak{k}_2$, and $\theta = \mathfrak{k}_2 - \mathfrak{k}_1$ or $\mathfrak{k}_2 = \theta + \mathfrak{k}_1$, then

(i) $\eta(\mathfrak{e}, \mathfrak{k}_1) = \eta(\mathfrak{e}, \mathfrak{k}_1) \star 1 = \eta(\mathfrak{e}, \mathfrak{k}_1) \star \eta(0, \theta) \leq \eta(\mathfrak{e} + 0, \mathfrak{k}_1 + \theta) = \eta(\mathfrak{e}, \mathfrak{k}_2)$ .... (1)

$\Rightarrow \eta(\mathfrak{e}, \mathfrak{k}_1) \leq \eta(\mathfrak{e}, \mathfrak{k}_2)$.

$\upsilon(\mathfrak{e}, \mathfrak{k}_1) = \upsilon(\mathfrak{e}, \mathfrak{k}_1) \diamond 0 = \upsilon(\mathfrak{e}, \mathfrak{k}_1) \star \upsilon(0, \theta) \geq \upsilon(\mathfrak{e} + 0, \mathfrak{k}_1 + \theta) = \upsilon(\mathfrak{e}, \mathfrak{k}_2)$ .... (2)

$\Rightarrow \upsilon(\mathfrak{e}, \mathfrak{k}_1) \geq \upsilon(\mathfrak{e}, \mathfrak{k}_2)$ and

$\varsigma(\mathfrak{e}, \mathfrak{k}_1) = \varsigma(\mathfrak{e}, \mathfrak{k}_1) \diamond 0 = \varsigma(\mathfrak{e}, \mathfrak{k}_1) \star \varsigma(0, \theta) \geq \varsigma(\mathfrak{e} + 0, \mathfrak{k}_1 + \theta) = \varsigma(\mathfrak{e}, \mathfrak{k}_2)$ .... (3)

$\Rightarrow \varsigma(\mathfrak{e}, \mathfrak{k}_1) \geq \varsigma(\mathfrak{e}, \mathfrak{k}_2)$.

(ii) $\eta(\mathfrak{e} - \mathfrak{w}, \mathfrak{k}) = \eta(-(\mathfrak{w} - \mathfrak{e}), \mathfrak{k}) = \eta\left(\mathfrak{w} - \mathfrak{e}, \frac{\mathfrak{k}}{|-1|}\right) = \eta(\mathfrak{w} - \mathfrak{e}, \mathfrak{k})$ .... (4)

$\upsilon(\mathfrak{e} - \mathfrak{w}, \mathfrak{k}) = \upsilon(-(\mathfrak{w} - \mathfrak{e}), \mathfrak{k}) = \upsilon\left(\mathfrak{w} - \mathfrak{e}, \frac{\mathfrak{k}}{|-1|}\right) = \upsilon(\mathfrak{w} - \mathfrak{e}, \mathfrak{k})$ .... (5)

$\varsigma(\mathfrak{e} - \mathfrak{w}, \mathfrak{k}) = \varsigma(-(\mathfrak{w} - \mathfrak{e}), \mathfrak{k}) = \varsigma\left(\mathfrak{w} - \mathfrak{e}, \frac{\mathfrak{k}}{|-1|}\right) = \varsigma(\mathfrak{w} - \mathfrak{e}, \mathfrak{k})$. .... (6)

Definition 2.8 A point $\mathfrak{e} \in (\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$ is said to be an interior point if there exist an open ball centred at $\mathfrak{e}$ is contained in $(\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$.

Definition 2.9 Let $\mathcal{I} \subseteq (\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$ then the interior of the set $\mathcal{I}$ of all the interior point of $\mathcal{I}$ with regard to neutrosophic norm $(\eta, \upsilon, \varsigma)$.

Definition 2.10 A set $\mathcal{I} \subseteq (\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$ is said to be nowhere dense in $(\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$ if the closure of $\mathcal{I}$ has no interior point.

Definition 2.11 A neutrosophic normed space $(\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$ is called first category if $\widetilde{\mathcal{N}} = \cup_i^{\infty} \mathcal{I}_i$, for each i, $\mathcal{I}_i$ is nowhere dense in $(\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$.

A neutrosophic normed space which is not first category is said to be second category.

Definition 2.12 $(\mathfrak{e}_n)$ is said to be a Cauchy sequence in $(\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$ if for all $0 < \varepsilon < 1$ there exist $\mathfrak{m} \in \mathbb{N}$ such that $1 - \eta(\mathfrak{e}_i - \mathfrak{e}_j, \mathfrak{k}) \leq \varepsilon, \upsilon(\mathfrak{e}_i - \mathfrak{e}_j, \mathfrak{k}) < \varepsilon$ and $\varsigma(\mathfrak{e}_i - \mathfrak{e}_j, \mathfrak{k}) < \varepsilon$ for all $i, j \geq \mathfrak{m}$ and $\mathfrak{k} > 0$.

Definition 2.13 A neutrosophic normed linear space $(\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$ is said to be complete, if every Cauchy sequence $(\mathfrak{e}_n)$ in $(\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$ converges in $(\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$.

Definition 2.14 Let $(\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$ and $(\widetilde{\mathcal{M}}, \eta, \upsilon, \varsigma, \star, \diamond)$ are neutrosophic normed linear space. The linear operator $\Psi : \mathfrak{C} \to \widetilde{\mathcal{M}}$, where $\mathfrak{C} \subseteq \widetilde{\mathcal{N}}$ is closed $\Leftrightarrow$ it satisfies the following condition that $\Psi(\mathfrak{e}_n) \to \Psi(\mathfrak{e})$ whenever $\mathfrak{e}_n \to \mathfrak{e}$ and $\mathfrak{e} \in \mathfrak{C}$ for all n.

3. Main Results

Theorem 3.1 A complete neutrosophic normed linear space $(\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$ is of second category space.

Proof: Suppose the statement is not true. i.e $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ is not of second category and hence $\widetilde{\mathcal{N}} = \bigcup_{i=1}^{\infty}(\mathfrak{F}_i,\eta,\upsilon,\varsigma,\star,\diamond)$ for each i, $(\mathfrak{F}_i,\eta,\upsilon,\varsigma,\star,\diamond)$ is nowhere dense in $\widetilde{\mathcal{N}} = (\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$.

Now for $i=1$, $\mathfrak{F}_1$ is nowhere dense in $\widetilde{\mathcal{N}}$. So the closure of $\mathfrak{F}_1$ is not an open set, this implies $\overline{\mathfrak{F}_1}^c$ contains an interior point, let $\mathfrak{e}_1 \in \overline{\mathfrak{F}_1}^c$ such that for $\mathfrak{k} > 0$ and every $0<\mathfrak{r}_1 < \frac{1}{2}$ there exists a ball centred at $\mathfrak{e}_1$, $\mathcal{B}_1 = \mathcal{B}_{\mathfrak{e}_1}(\mathfrak{r},\mathfrak{k}) = \{\xi \in \mathfrak{N} : 1-\eta(\mathfrak{e}_1-\xi,\mathfrak{k}) < \varepsilon, \upsilon(\mathfrak{e}_1-\xi,\mathfrak{k}) < \varepsilon \text{ and } \varsigma(\mathfrak{e}_1-\xi,\mathfrak{k}) < \varepsilon\} \subseteq \overline{\mathfrak{F}_1}^c$. Again $\overline{\mathfrak{F}_2}^c$ is not open in $\widetilde{\mathcal{N}}$ therefore $\overline{\mathfrak{F}_2}^c \cap \mathcal{B}_{\mathfrak{e}_1}\left(\frac{\mathfrak{r}_1}{2},\mathfrak{k}\right) = \emptyset$; where $\mathfrak{r}_1 < \frac{\mathfrak{r}}{2}$, on the other hand, $\overline{\mathfrak{F}_2}^c$ intersect the ball $\mathcal{B}_{\mathfrak{e}_1}\left(\frac{\mathfrak{r}_1}{2},\mathfrak{k}\right)$. Now let $\overline{\mathfrak{F}_2}^c \cap \mathcal{B}_{\mathfrak{e}_1}(\mathfrak{r}_2,\mathfrak{k})$ contains a ball $\mathcal{B}_2 = \mathcal{B}_{\mathfrak{e}_2}\left(\frac{\mathfrak{r}_1}{2},\mathfrak{k}\right)$ where $\mathfrak{r}_2 < \frac{\mathfrak{r}_1}{2}$, continuing this process of forming the ball $\mathcal{B}_n = \mathcal{B}_{\mathfrak{e}_n}(\mathfrak{r}_n,\mathfrak{k})$, we shall have $\mathcal{B}_{n+1} \subseteq \mathcal{B}_n$

Where $\mathfrak{r}_{n+1} < \frac{\mathfrak{r}_n}{2}$ and $\mathfrak{r}_n < \frac{1}{2^n}$. We get $(\mathfrak{e}_n)$ of the centres of the balls $\mathcal{B}_n$. Now we show that $(\mathfrak{e}_n)$ is a Cauchy sequence. Let $n_{\mathfrak{r}} \in \mathbb{N}$ and let $n > m > n_{\mathfrak{r}} \Rightarrow \mathcal{B}_n \subseteq \mathcal{B}_m$, take $\varpi \in \mathcal{B}_m$, then

$$1-\eta\left(\mathfrak{e}_n - \varpi, \frac{\mathfrak{k}}{2}\right) < \frac{1}{2^n},\ \upsilon\left(\mathfrak{e}_n - \varpi, \frac{\mathfrak{k}}{2}\right) < \frac{1}{2^n} \text{ and } \varsigma\left(\mathfrak{e}_n - \varpi, \frac{\mathfrak{k}}{2}\right) < \frac{1}{2^n} \tag{3.1.1}$$

$$1-\eta\left(\mathfrak{e}_m - \varpi, \frac{\mathfrak{k}}{2}\right) < \frac{1}{2^m},\ \upsilon\left(\mathfrak{e}_m - \varpi, \frac{\mathfrak{k}}{2}\right) < \frac{1}{2^m} \text{ and } \varsigma\left(\mathfrak{e}_m - \varpi, \frac{\mathfrak{k}}{2}\right) < \frac{1}{2^m}. \tag{3.1.2}$$

Now,
$$\eta(\mathfrak{e}_n - \mathfrak{e}_m,\mathfrak{k}) = \eta(\mathfrak{e}_n - \varpi + \varpi - \mathfrak{e}_m,\mathfrak{k}) \geq \eta\left(\mathfrak{e}_n - \varpi,\frac{\mathfrak{k}}{2}\right) \star \eta\left(\varpi - \mathfrak{e}_m,\frac{\mathfrak{k}}{2}\right) > \left(1-\frac{1}{2^n}\right) \star \left(1-\frac{1}{2^m}\right) > 1-\mathfrak{r}' \tag{3.1.3}$$

$$\upsilon(\mathfrak{e}_n - \mathfrak{e}_m,\mathfrak{k}) = \upsilon(\mathfrak{e}_n - \varpi + \varpi - \mathfrak{e}_m,\mathfrak{k}) \leq \upsilon\left(\mathfrak{e}_n - \varpi,\frac{\mathfrak{k}}{2}\right) \diamond \upsilon\left(\varpi - \mathfrak{e}_m,\frac{\mathfrak{k}}{2}\right) < \frac{1}{2^n} \diamond \frac{1}{2^m} < \mathfrak{r}' \tag{3.1.4}$$

$$\varsigma(\mathfrak{e}_n - \mathfrak{e}_m,\mathfrak{k}) = \varsigma(\mathfrak{e}_n - \varpi + \varpi - \mathfrak{e}_m,\mathfrak{k}) \leq \varsigma\left(\mathfrak{e}_n - \varpi,\frac{\mathfrak{k}}{2}\right) \diamond \varsigma\left(\varpi - \mathfrak{e}_m,\frac{\mathfrak{k}}{2}\right) < \frac{1}{2^n} \diamond \frac{1}{2^m} < \mathfrak{r}'. \tag{3.1.5}$$

Since for every n and m we can $0<\mathfrak{r}' < 1$ such that $\left(1-\frac{1}{2^n}\right) \star \left(1-\frac{1}{2^m}\right) > 1-\mathfrak{r}'$ and $\frac{1}{2^n} \diamond \frac{1}{2^m} < \mathfrak{r}'$.

Thus from equations (3.1.3), (3.1.4) and (3.1.5), we conclude that $(\mathfrak{e}_n)$ is a Cauchy sequence with respect to neutrosophic norm $(\eta,\upsilon,\varsigma)$, let $(\mathfrak{e}_n)$ converges at $\mathfrak{e} \in \widetilde{\mathcal{N}}$. $\mathfrak{e}$ lies in some $\overline{\mathfrak{F}_\iota}$, because $\widetilde{\mathcal{N}}$ is complete, $\mathfrak{e} \in \overline{\mathfrak{F}_t}$ for a particular t, therefore $\overline{\mathfrak{F}_t}$ contains some open ball $\mathcal{B}_{\mathfrak{e}}(\mathfrak{e},\mathfrak{k})$ which contradict that $\overline{\mathfrak{F}_t}$ is nowhere dense in $\widetilde{\mathcal{N}}$. Thus theorem is concluded.

Theorem: 3.2: Let $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ and $(\widetilde{\mathcal{M}},\eta_1,\upsilon_1,\varsigma_1,\star,\diamond)$ be Neutrosophic Banach spaces and Ψ be a continuous linear oprator from $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ onto $(\widetilde{\mathcal{M}},\eta_1,\upsilon_1,\varsigma_1,\star,\diamond)$. Then Ψ is an open mapping.

Proof.

Step- I. Let $\mathfrak{P}$ be a ball centred at 0 in $\widetilde{\mathcal{N}} = (\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$, we shall show that $0 \in int(\Psi(\mathfrak{P}))$, let $\mathfrak{F}$ is a neighbourhood of 0 such that $\mathfrak{F} + \mathfrak{F} \subset \mathfrak{P}$, now $\widetilde{\mathcal{M}} = \Psi(\widetilde{\mathcal{N}})$, since Ψ is a surjective mapping, then by Theorem (3.1) we obtain that if $\widetilde{\mathcal{M}} = \bigcup_{n\geq 1}\Psi(\mathfrak{F}_n)$ then there exist $p_0 \in \mathbb{N}$ such that $int(\Psi(\overline{\mathfrak{F}_0}))$ is empty, therefore $0 = \Psi(0) \in \text{int}\left(\Psi(\overline{\mathfrak{Z}})\right) - \text{int}\left(\Psi(\overline{\mathfrak{Z}})\right) \subseteq \Psi(\overline{\mathfrak{Z}}) - \Psi(\overline{\mathfrak{Z}}) = \Psi(\mathfrak{Z}) - \Psi(\mathfrak{z}) = \Psi(\mathfrak{Z} - \mathfrak{z}) \subset \Psi(\overline{\mathfrak{Z}})$. This shows that $\Psi-$image of the neighbourhood of 0 belongs to $\widetilde{\mathcal{N}}$ contains a neighbourhood of 0 in $\widetilde{\mathcal{M}}$. ρ_n

Step II. Let $0 \in \mathfrak{P}$ and $\mathfrak{P}$ is open then $\mathfrak{P}$ contains a ball $\mathcal{B}_0(\delta,\mathfrak{k}_0)$ for some $0<\delta<1$ and $\mathfrak{k}_0 > 0$, a sequence $(\mathfrak{e}_n)$ can be find, where $0 < \mathfrak{e}_n < 1$, such that $\mathfrak{e}_n \to 0$ as $n \to \infty$ and by remark (2.4) we have

$$\lim_n[(1-\mathfrak{e}_1) \star (1-\mathfrak{e}_2) \ldots (1-\mathfrak{e}_n)] > 1-\delta. \tag{3.2.1}$$

However, if we construct a sequence of neighborhoods $\mathcal{B}_0(\mathfrak{e}_n,\tau_n) = \mathcal{B}_n$ (say), where $\tau_n = \frac{\mathfrak{k}_0}{2^n}$ then by step-I, $\Psi(\mathcal{B}_n)$ contains a neighbourhood $\mathfrak{W}_n = \mathcal{B}_0(\lambda_n,\rho_n) \subset \Psi(\mathcal{B}_n)$, where $0 < \lambda_n < 1$ and $\mathfrak{k}_0 > 0$, we have $\tau_n \to 0$ as $n \to \infty$ this lead us to choose λ_n and ρ_n such that $\lambda_n,\rho_n \to 0$ as $n \to \infty$. We get $\mathfrak{W}_1 \subset int(\Psi(\mathfrak{P}))$, take $\mathcal{k} \in \mathfrak{W}_1$ then $\mathcal{k} \in \Psi(\mathcal{B}_1)$, we now form a ball centred at $\mathcal{k}$, $\mathcal{B}_{\mathcal{k}}(\lambda_2,\rho_2)$ such that $\mathcal{B}_{\mathcal{k}}(\lambda_2,\rho_2) \cap \Psi(\mathcal{B}_1) \neq \emptyset$, there there exist $\mathfrak{d}_1 \in \mathcal{B}_1$ such that $\Psi(\mathfrak{d}_1) \in \Psi(\mathcal{B}_1)$.

$$\eta_1(\mathcal{k} - \Psi(\mathfrak{d}_1)),\rho_2) > 1-\lambda_2,\ \upsilon_1(\mathcal{k} - \Psi(\mathfrak{d}_1)),\rho_2) < \lambda_2 \text{ and } \varsigma_1(\mathcal{k} - \Psi(\mathfrak{d}_1)),\rho_2) < \lambda_2 \tag{3.2.2}$$

$\Rightarrow\ \mathcal{k} - \Psi(\mathfrak{d}_1) \in \mathfrak{W}_2$ and $\mathfrak{W}_2 \subset \Psi(\mathcal{B}_2)$ then there exists $\mathfrak{d}_2 \in \mathcal{B}_2$ such that $\Psi(\mathfrak{d}_2) \in \Psi(\mathcal{B}_2)$ and $\eta_1(\mathcal{k} - \Psi(\mathfrak{d}_1) - \Psi(\mathfrak{d}_2)), \rho_3) > 1 - \lambda_3$, $\upsilon_1(\mathcal{k} - \Psi(\mathfrak{d}_1) - \Psi(\mathfrak{d}_2)), \rho_3) < \lambda_3$ and $\varsigma_1(\mathcal{k} - \Psi(\mathfrak{d}_1) - \Psi(\mathfrak{d}_2)), \rho_3) < \lambda_3$. (3.2.3)

$\Rightarrow \mathcal{k} - \Psi(\mathfrak{d}_1) - \Psi(\mathfrak{d}_2) \in \mathfrak{W}_3$, keeping up this procedure, we get a sequence $(\mathfrak{d}_n)$ such that $\mathfrak{d}_n \in \mathcal{B}_n$ and $\eta_1(\mathcal{k} - \sum_{i=}^{n-1} \Psi(\mathfrak{d}_i), \rho_n)$ >1-λ_n, $\upsilon_1(\mathcal{k} - \sum_{i=}^{n-1} \Psi(\mathfrak{d}_i), \rho_n)$<$\lambda_n$ and $\varsigma_1(\mathcal{k} - \sum_{i=}^{n-1} \Psi(\mathfrak{d}_i), \rho_n)$< λ_n. (3.2.4)

Now we demonstrate that $(\mathfrak{G}_n)$ to be a Cauchy sequence, where $\mathfrak{G}_n = \sum_{i=1}^{n} \mathfrak{d}_i$. When $n \to \infty$, $\tau_n \to 0$, this implies that there exist some $n_0 \in \mathbb{N}$ such that o< $\tau_n < \mathfrak{k}'$ for all $n \geq n_0$ where $\mathfrak{k}' = \min\{\mathfrak{k}_1, \mathfrak{k}_2, \dots \mathfrak{k}_i\}$. Let $\mathfrak{p} > \mathfrak{q} > n_0$ and then

$\eta(\mathfrak{G}_n - \mathfrak{G}_m, \mathfrak{k}) = \eta(\sum_{j=\mathfrak{q}+1}^{\mathfrak{q}+i} \mathfrak{d}_j, \mathfrak{k}) \geq \eta(\mathfrak{d}_{\mathfrak{q}+1}, \mathfrak{k}_1) \star \eta(\mathfrak{d}_{\mathfrak{q}+2}, \mathfrak{k}_2) \star \dots \star \eta(\mathfrak{d}_{\mathfrak{q}+i}, \mathfrak{k}_i) > 1 - \mathrm{e}$ (3.2.5)

$\upsilon(\mathfrak{G}_n - \mathfrak{G}_m, \mathfrak{k}) = \upsilon(\sum_{j=\mathfrak{q}+1}^{\mathfrak{q}+i} \mathfrak{d}_j, \mathfrak{k}) \leq \upsilon(\mathfrak{d}_{\mathfrak{q}+1}, \mathfrak{k}_1) \diamond \upsilon(\mathfrak{d}_{\mathfrak{q}+2}, \mathfrak{k}_2) \diamond \dots \diamond \upsilon(\mathfrak{d}_{\mathfrak{q}+i}, \mathfrak{k}_i) < \mathrm{e}$ (3.2.6)

$\varsigma(\mathfrak{G}_n - \mathfrak{G}_m, \mathfrak{k}) = \varsigma(\sum_{j=\mathfrak{q}+1}^{\mathfrak{q}+i} \mathfrak{d}_j, \mathfrak{k}) \leq \varsigma(\mathfrak{d}_{\mathfrak{q}+1}, \mathfrak{k}_1) \diamond \varsigma(\mathfrak{d}_{\mathfrak{q}+2}, \mathfrak{k}_2) \diamond \dots \diamond \varsigma(\mathfrak{d}_{\mathfrak{q}+i}, \mathfrak{k}_i) < \mathrm{e}$. (3.2.7)

Since, we had (see remark (2.4) and lemma (2.7))

$$\begin{aligned}\eta(\mathfrak{d}_{\mathfrak{q}+1}, \mathfrak{k}_1) \star \eta(\mathfrak{d}_{\mathfrak{q}+2}, \mathfrak{k}_2) \star \dots \star \eta(\mathfrak{d}_{\mathfrak{q}+i}, \mathfrak{k}_i) &\geq \eta(\mathfrak{d}_{\mathfrak{q}+1}, \mathfrak{k}') \star \eta(\mathfrak{d}_{\mathfrak{q}+2}, \mathfrak{k}') \star \dots \star \eta(\mathfrak{d}_{\mathfrak{q}+i}, \mathfrak{k}')\\ &\geq \eta(\mathfrak{d}_{\mathfrak{q}+1}, \tau_{\mathfrak{q}+1}) \star \eta(\mathfrak{d}_{\mathfrak{q}+2}, \tau_{\mathfrak{q}+2}) \star \dots \star \eta(\mathfrak{d}_{\mathfrak{q}+i}, \tau_{\mathfrak{q}+i})\\ &> (1 - \mathrm{e}_{\mathfrak{q}+1}) \star (1 - \mathrm{e}_{\mathfrak{q}+1}) \star \dots \star (1 - \mathrm{e}_{\mathfrak{q}+i}) > 1 - \mathrm{e} \qquad (3.2.8)\end{aligned}$$

$$\begin{aligned}\upsilon(\mathfrak{d}_{\mathfrak{q}+1}, \mathfrak{k}_1) \diamond \upsilon(\mathfrak{d}_{\mathfrak{q}+2}, \mathfrak{k}_2) \diamond \dots \diamond \upsilon(\mathfrak{d}_{\mathfrak{q}+i}, \mathfrak{k}_i) &\leq \upsilon(\mathfrak{d}_{\mathfrak{q}+1}, \mathfrak{k}') \diamond \upsilon(\mathfrak{d}_{\mathfrak{q}+2}, \mathfrak{k}') \diamond \dots \diamond \upsilon(\mathfrak{d}_{\mathfrak{q}+i}, \mathfrak{k}')\\ &\leq \upsilon(\mathfrak{d}_{\mathfrak{q}+1}, \tau_{\mathfrak{q}+1}) \diamond \upsilon(\mathfrak{d}_{\mathfrak{q}+2}, \tau_{\mathfrak{q}+2}) \diamond \dots \diamond \upsilon(\mathfrak{d}_{\mathfrak{q}+i}, \tau_{\mathfrak{q}+i})\\ &< \mathrm{e}_{\mathfrak{q}+1} \diamond \mathrm{e}_{\mathfrak{q}+1} \diamond \dots \diamond \mathrm{e}_{\mathfrak{q}+i} < \mathrm{e} \qquad (3.2.9)\end{aligned}$$

$$\begin{aligned}\varsigma(\mathfrak{d}_{\mathfrak{q}+1}, \mathfrak{k}_1) \diamond \varsigma(\mathfrak{d}_{\mathfrak{q}+2}, \mathfrak{k}_2) \diamond \dots \diamond \varsigma(\mathfrak{d}_{\mathfrak{q}+i}, \mathfrak{k}_i) &\leq \varsigma(\mathfrak{d}_{\mathfrak{q}+1}, \mathfrak{k}') \diamond \varsigma(\mathfrak{d}_{\mathfrak{q}+2}, \mathfrak{k}') \diamond \dots \diamond \varsigma(\eta, \mathfrak{k}')\\ &\leq \varsigma(\mathfrak{d}_{\mathfrak{q}+1}, \tau_{\mathfrak{q}+1}) \diamond \varsigma(\mathfrak{d}_{\mathfrak{q}+2}, \tau_{\mathfrak{q}+2}) \diamond \dots \diamond \varsigma(\mathfrak{d}_{\mathfrak{q}+i}, \tau_{\mathfrak{q}+i})\\ &< \mathrm{e}_{\mathfrak{q}+1} \diamond \mathrm{e}_{\mathfrak{q}+1} \diamond \dots \diamond \mathrm{e}_{\mathfrak{q}+i} < \mathrm{e}. \qquad (3.2.10)\end{aligned}$$

Thus, from equations (3.2.5), (3.2.6) and (3.2.7) we obtain that

$\eta(\mathfrak{G}_n - \mathfrak{G}_m, \mathfrak{k}) \to 1, \upsilon(\mathfrak{G}_n - \mathfrak{G}_m, \mathfrak{k}) \to 0$ and $\varsigma(\mathfrak{G}_n - \mathfrak{G}_m, \mathfrak{k}) \to 0$ for every $\mathfrak{k} > 0$, hence $(\mathfrak{G}_n)$ is a Cauchy sequence. Let $(\mathfrak{G}_n)$ converges to $\mathfrak{d} \in \widetilde{\mathcal{N}}$, since $\widetilde{\mathcal{N}}$ is Banach space, which implies $\mathfrak{d} = \sum_{j\geq 1} \mathfrak{d}_j$, now $\rho_n \to 0$, so for a fix $\rho > 0$ we can find n_0 such that $\rho > \rho_n$ for $n > n_0$, it follows

$\eta_1(\mathcal{k} - \Psi(\mathfrak{G}_{n-1})), \rho) > \eta_1(\mathcal{k} - \Psi(\mathfrak{G}_{n-1})), \rho_n) > 1 - \lambda_n$ (3.2.11)

$\upsilon_1(\mathcal{k} - \Psi(\mathfrak{G}_{n-1})), \rho) < \upsilon_1(\mathcal{k} - \Psi(\mathfrak{G}_{n-1})), \rho_n) < \lambda_n$ (3.2.12)

$\varsigma_1(\mathcal{k} - \Psi(\mathfrak{G}_{n-1})), \rho) < \varsigma_1(\mathcal{k} - \Psi(\mathfrak{G}_{n-1})), \rho_n) < \lambda_n$. (3.2.13)

$\Longrightarrow\ \mathcal{k} = \Psi(\mathfrak{G}_{n-1})$ as $n \to \infty$ or $\mathcal{k} = \Psi(\sum_{i=}^{n-1} \mathfrak{d}_i)$ and $n \to \infty$, but it is known that $\mathfrak{d} = \lim_{n\to\infty} \sum_{i=}^{n-1} \mathfrak{d}_i$ was in $\mathcal{B}_0(\delta, \mathfrak{k}_0)$. Because $\eta(\mathfrak{d}, \mathfrak{k}_0) \geq \lim_{n\to\infty} \eta(\sum_{i=}^{n} \mathfrak{d}_i, \mathfrak{k}_0) \geq \lim_{n\to\infty} [\eta(\mathfrak{d}_1, \tau_1) \star \eta(\mathfrak{d}_2, \tau_2) \star \dots \star \eta(\mathfrak{d}_n, \tau_n)$

$$> \lim_{n\to\infty} (1 - \mathrm{e}_1) \star (1 - \mathrm{e}_2) \star \dots \star (1 - \mathrm{e}_n) > 1 - \delta \qquad (3.2.14)$$

$$\upsilon(\mathfrak{d}, \mathfrak{k}_0) \leq \lim_{n\to\infty} \upsilon(\sum_{i=}^{n} \mathfrak{d}_i, \mathfrak{k}_0) \leq \lim_{n\to\infty} [\upsilon(\mathfrak{d}_1, \tau_1) \diamond \upsilon(\mathfrak{d}_2, \tau_2) \diamond \dots \diamond \upsilon(\mathfrak{d}_n, \tau_n)$$

$$< \lim_{n\to\infty} \mathrm{e}_1 \diamond \mathrm{e}_2 \diamond \dots \diamond \mathrm{e}_n < \delta \qquad (3.2.15)$$

$\in$

$$\varsigma(\mathfrak{d}, \mathfrak{k}_0) \leq \lim_{n\to\infty} \varsigma(\sum_{i=}^{n} \mathfrak{d}_i, \mathfrak{k}_0) \leq \lim_{n\to\infty} [\varsigma(\mathfrak{d}_1, \tau_1) \diamond \varsigma(\mathfrak{d}_2, \tau_2) \diamond \dots \diamond \varsigma(\mathfrak{d}_n, \tau_n)$$

$$< \lim_{n\to\infty} \mathrm{e}_1 \diamond \mathrm{e}_2 \diamond \dots \diamond \mathrm{e}_n < \delta. \qquad (3.2.16)$$

Since, $\mathfrak{k}_0 = \sum_{n=1}^{\infty} \tau_n = \sum_{n=1}^{\infty} \frac{\mathfrak{k}_0}{2^n} = \mathfrak{k}_0 \sum_{n=1}^{\infty} \frac{1}{2^n}$. Thus it is shown that $\Psi -$ image of $\mathcal{B}_0(\delta, \mathfrak{k}_0)$ contains a neighbourhood of $0 = \Psi(0) \in \widetilde{\mathcal{M}}$.

Step-III We will demonstrate that $\Psi(\mathcal{F})$ is open in $\widetilde{\mathcal{M}}$ if $\mathcal{F}$ is open in $\widetilde{\mathcal{N}}$. Let $\mathcal{k} = \Psi(\mathfrak{d})$ this implies $\mathfrak{d} \in \mathcal{F}$, since $\mathcal{F}$ is open, therefore $\mathcal{F}$ contains $\mathcal{B}_{\mathfrak{d}}(\mathfrak{r}, \mathfrak{k})$, a neighbourhood of $\mathfrak{d}$. We have already proved that if $\mathfrak{d}_0 \in \mathcal{B}_0(\delta, \mathfrak{k}_0) \subset \mathfrak{P}$ then $\Psi(\mathfrak{d}_0) \in \mathfrak{W}_1 \subseteq int\Psi(\mathfrak{P})$. Hence if $\mathfrak{d} \in \mathcal{F}$and $\mathcal{B}_{\mathfrak{d}}(\mathfrak{r}, \mathfrak{k}) \subset \mathcal{F}$ then $\Psi(\mathfrak{d}) \in int(\Psi(\mathcal{F}))$. Consequently $\Psi(\mathcal{F})$ is open.

Example 3.3: Let $\widetilde{\mathcal{N}} = \widetilde{\mathcal{M}} = \mathbb{R}$, and $(\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$ $(\widetilde{\mathcal{M}}, \eta_1, \upsilon_1, \varsigma_1, \star, \diamond)$ be Neutrosophic Banach spaces. Also $\eta, \upsilon, \varsigma$ are defined by $\eta(\mathfrak{d}, \mathfrak{k}) = \frac{|\mathfrak{d}|}{|\mathfrak{d}|+\mathfrak{k}}$, $\upsilon(\mathfrak{d}, \mathfrak{k}) = \frac{|\mathfrak{k}|}{|\mathfrak{d}|+\mathfrak{k}}$ and $\varsigma(\mathfrak{d}, \mathfrak{k}) = \frac{|\mathfrak{d}|}{\mathfrak{k}}$.

Similarly for $\widetilde{\mathcal{M}}$ the norms are by $\eta_1(\mathfrak{d},\mathfrak{k}) = \frac{|\mathfrak{d}|}{|\mathfrak{d}|+\mathfrak{k}}$, $\upsilon_1(\mathfrak{d},\mathfrak{k}) = \frac{|\mathfrak{k}|}{|\mathfrak{d}|+\mathfrak{k}}$ and $\varsigma_1(\mathfrak{d},\mathfrak{k}) = \frac{|\mathfrak{d}|}{\mathfrak{k}}$.

The continuous linear operator Ψ from$(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ onto $(\widetilde{\mathcal{M}},\eta_1,\upsilon_1,\varsigma_1,\star,\diamond)$ defined by $\Psi(\mathfrak{d}) = \frac{\mathfrak{d}}{2}$. Then Ψ is an open mapping

Proof.

Let $\mathfrak{P}$ be an open set in $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$, the image of $\mathfrak{P}$ under Ψ is $\Psi(\mathfrak{P}) = \{\frac{\mathfrak{d}}{2}/\mathfrak{d} \in \mathfrak{P}\}$.

To Prove Ψ is an open map, it is enough to show that $\Psi(\mathfrak{P})$ is open.

Let $p \in \Psi(\mathfrak{P})$. Therefore, there exists $\mathfrak{d} \in \mathfrak{P}$ such that $p = \Psi(\mathfrak{d}) = \frac{\mathfrak{d}}{2}$.

Since $\mathfrak{d} \in \mathfrak{P}$ and $\mathfrak{P}$ is open, therefore there exist $0< \mathfrak{r} < 1$ such that open ball centered at $\mathfrak{d}$, with respect to the norm η,υ,ς , $\mathcal{B}_{\eta,\upsilon,\varsigma}(\mathfrak{d},\mathfrak{r})$ (say) is contained in $\mathfrak{P}$.

$\mathcal{B}_{\eta,\upsilon,\varsigma}(\mathfrak{d},\mathfrak{r}) \subset \mathfrak{P}$.

Since, $\Psi(\mathcal{B}_{\eta}(\mathfrak{d},\mathfrak{r})) = \mathcal{B}_{\eta_1,\upsilon_1,\varsigma_1}\left(\frac{\mathfrak{d}}{2},\frac{\mathfrak{r}}{2}\right)$ and $\mathcal{B}_{\eta}(\mathfrak{d},\mathfrak{r}) \subset \mathfrak{P}$

$\Longrightarrow \Psi(\mathcal{B}_{\eta,\upsilon,\varsigma}(\mathfrak{d},\mathfrak{r})) \subset \Psi(\mathfrak{P})$

$\Longrightarrow \mathcal{B}_{\eta_1,\upsilon_1,\varsigma_1}\left(\frac{\mathfrak{d}}{2},\frac{\mathfrak{r}}{2}\right) \subset \Psi(\mathfrak{P})$

$\Longrightarrow \mathcal{B}_{\eta_1,\upsilon_1,\varsigma_1}\left(p,\frac{\mathfrak{r}}{2}\right) \subset \Psi(\mathfrak{P})$.

Therefore there exists a open ball centered at p is contained in $\Psi(\mathfrak{P})$.

Since p belongs to $\Psi(\mathfrak{P})$ is arbitrary, there every point of $\Psi(\mathfrak{P})$ in an interior point.

Hence $\Psi(\mathfrak{P})$ is open. Therefore Ψ is an open mapping.

Theorem 3.4: Let $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ and $(\widetilde{\mathcal{M}},\eta_1,\upsilon_1,\varsigma_1,\star,\diamond)$ be Neutrosophic Banach space and Ψ be a linear oprator from $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ to $(\widetilde{\mathcal{M}},\eta_1,\upsilon_1,\varsigma_1,\star,\diamond)$. Moreover if for every sequence $(\mathfrak{d}_n)$ of the elements of $\widetilde{\mathcal{N}}$ converges to $\mathfrak{d} \in \widetilde{\mathcal{N}}$, the sequence $\Psi(\mathfrak{d}_n)$ converges to $\mathcal{k} \in \widetilde{\mathcal{M}}$ with the property $\mathcal{k} = \Psi(\mathfrak{d})$ then Ψ is continuous

Proof: We firstly define Neutrosophic norm (γ,ϑ,β) on $\widetilde{\mathcal{N}} \times \widetilde{\mathcal{M}}$ defined as

$\gamma(((\mathfrak{d},\mathcal{k}),\mathfrak{k}) = \gamma(\mathfrak{d},\mathfrak{k}) \star \gamma_1(\mathcal{k},\mathfrak{k})$

$\vartheta(((\mathfrak{d},\mathcal{k}),\mathfrak{k}) = \vartheta(\mathfrak{d},\mathfrak{k}) \diamond \vartheta_1(\mathcal{k},\mathfrak{k})$ and

$\beta(((\mathfrak{d},\mathcal{k}),\mathfrak{k}) = \beta(\mathfrak{d},\mathfrak{k}) \diamond \beta(\mathcal{k},\mathfrak{k})$. Let $(\mathfrak{d}_n)$ be a Cauchy sequence in $\widetilde{\mathcal{N}}$, then we prove that $\Psi(\mathfrak{d}_n)$ will be Cauchy in $\widetilde{\mathcal{M}}$, for this, it is enough to show that $\widetilde{\mathcal{N}} \times \widetilde{\mathcal{M}}$ is complete with respect to Neutrosophic norms (γ,ϑ,β). Let $(\mathfrak{d}_n,\mathcal{k}_n) \in \widetilde{\mathcal{N}} \times \widetilde{\mathcal{M}}$ be a Cauchy, then for all $0 < \mathfrak{r} < 1$ and $\mathfrak{k} > 0$ there exists a $n_0 \in \mathbb{N}$ such that $\gamma((\mathfrak{d}_n,\mathcal{k}_n) - (\mathfrak{d}_m,\mathcal{k}_m),\mathfrak{k}) > 1 - \mathfrak{r}$, $\vartheta((\mathfrak{d}_n,\mathcal{k}_n) - (\mathfrak{d}_m,\mathcal{k}_m),\mathfrak{k}) < \mathfrak{r}$ and $\beta((\mathfrak{d}_n,\mathcal{k}_n) - (\mathfrak{d}_m,\mathcal{k}_m),\mathfrak{k}) < \mathfrak{r}$ for all $n,m > n_0$.

$$\begin{aligned}\gamma((\mathfrak{d}_n,\mathcal{k}_n) - (\mathfrak{d}_m,\mathcal{k}_m),\mathfrak{k}) &= \gamma((\mathfrak{d}_n - \mathfrak{d}_m) - (\mathcal{k}_n - \mathcal{k}_m),\mathfrak{k}) \\ &= \gamma((\mathfrak{d}_n - \mathfrak{d}_m),\mathfrak{k}) \star \gamma((\mathcal{k}_n - \mathcal{k}_m),\mathfrak{k}) > 1 - \mathfrak{r}, \qquad (3.4.1)\end{aligned}$$

$\Rightarrow \gamma((\mathfrak{d}_n - \mathfrak{d}_m),\mathfrak{k}) > 1 - \mathfrak{r}_1$ and $\gamma((\mathcal{k}_n - \mathcal{k}_m),\mathfrak{k}) > 1 - \mathfrak{r}_2$.

$$\begin{aligned}\vartheta((\mathfrak{d}_n,\mathcal{k}_n) - (\mathfrak{d}_m,\mathcal{k}_m),\mathfrak{k}) &= \vartheta((\mathfrak{d}_n - \mathfrak{d}_m) - (\mathcal{k}_n - \mathcal{k}_m),\mathfrak{k}) \\ &= \vartheta((\mathfrak{d}_n - \mathfrak{d}_m),\mathfrak{k}) \diamond \vartheta((\mathcal{k}_n - \mathcal{k}_m),\mathfrak{k}) < \mathfrak{r} \qquad (3.4.2)\end{aligned}$$

$\Rightarrow \vartheta((\mathfrak{d}_n - \mathfrak{d}_m),\mathfrak{k}) < \mathfrak{r}_1$ and $\vartheta((\mathcal{k}_n - \mathcal{k}_m),\mathfrak{k}) < \mathfrak{r}_2$.

$$\begin{aligned}\beta((\mathfrak{d}_n,\mathcal{k}_n) - (\mathfrak{d}_m,\mathcal{k}_m),\mathfrak{k}) &= \beta((\mathfrak{d}_n - \mathfrak{d}_m) - (\mathcal{k}_n - \mathcal{k}_m),\mathfrak{k}) \\ &= \beta((\mathfrak{d}_n - \mathfrak{d}_m),\mathfrak{k}) \diamond \beta((\mathcal{k}_n - \mathcal{k}_m),\mathfrak{k}) < \mathfrak{r}. \qquad (3.4.3)\end{aligned}$$

$\Rightarrow \beta((\mathfrak{d}_n - \mathfrak{d}_m),\mathfrak{k}) < \mathfrak{r}_1$ and $\beta((\mathcal{k}_n - \mathcal{k}_m),\mathfrak{k}) < \mathfrak{r}_2$ for every $0 < \mathfrak{r} < 1$, we can find $0 < \mathfrak{r}_1 < 1$ and $0 < \mathfrak{r}_2 < 1$, such that $(1 - \mathfrak{r}_1) \star (1 - \mathfrak{r}_2) > 1 - \mathfrak{r}$ and $\mathfrak{r}_1 \diamond \mathfrak{r}_2 < \mathfrak{r}$. Consequently, for equations (3.3.1) (3.32) and (3.3.3) we get that $(\mathfrak{d}_n)$ converges at $\mathfrak{d} \in \widetilde{\mathcal{N}}$ and $(\mathcal{k}_n)$ converges at $\mathcal{k} \in \widetilde{\mathcal{M}}$, this implies that $(\mathfrak{d}_n,\mathcal{k}_n)$ converges at $(\mathfrak{d},\mathcal{k}) \in \widetilde{\mathcal{N}} \times \widetilde{\mathcal{M}}$ which establish the property that the sequence $\Psi(\mathfrak{d}_n)$ is Cauchy in $\widetilde{\mathcal{M}}$ whenever $(\mathfrak{d}_n)$ is a Cauchy sequence in $\widetilde{\mathcal{N}}$. Hence Ψ is continuous.

Remark 3.5 [15] Every open ball is an open set in neutrosophic normed space.

Example 3.6: Let $\widetilde{\mathcal{N}} = \widetilde{\mathcal{M}} = \mathbb{R}$, and $(\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$ $(\widetilde{\mathcal{M}}, \eta, \upsilon, \varsigma, \star, \diamond)$ be Neutrosophic Banach spaces. Also $\eta, \upsilon, \varsigma$ are defined by $\eta(\mathfrak{d}, \mathfrak{k}) = \frac{|\mathfrak{d}|}{|\mathfrak{d}|+\mathfrak{k}}$, $\upsilon(\mathfrak{d}, \mathfrak{k}) = \frac{|\mathfrak{k}|}{|\mathfrak{d}|+\mathfrak{k}}$ and $\varsigma(\mathfrak{d}, \mathfrak{k}) = \frac{|\mathfrak{d}|}{\mathfrak{k}}$ The continuous linear oprator Ψ from $(\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$ onto $(\widetilde{\mathcal{M}}, \eta_1, \upsilon_1, \varsigma_1, \star, \diamond)$ defined by $\Psi(\mathfrak{d}) = \frac{\mathfrak{d}}{8}$. If $(\mathfrak{d}_n)$ convergesat $\mathfrak{d} \in \widetilde{\mathcal{N}}$ imples the sequence $\Psi(\mathfrak{d}_n)$ converges to $\mathscr{k} \in \widetilde{\mathcal{M}}$ with the property $\mathscr{k} = \Psi(\mathfrak{d})$. then Ψ is continuous.

Theorem 3.7: Let $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{M}}$ are two Neutrosophic normed linear spaces. If $\Psi_1, \Psi_2 : \widetilde{\mathcal{N}} \longrightarrow \widetilde{\mathcal{M}}$ are two linear operators, Ψ_1 is closed and Ψ_2 is bounded then $\Psi_1 + \Psi_2$ is closed with respect to neutrosophic norm $(\eta, \upsilon, \varsigma)$.

Proof: Let $(\mathfrak{d}_n)$ be a sequence in $\widetilde{\mathcal{N}}$ such that $\mathfrak{d}_n \longrightarrow \mathfrak{d}$ with respect to neutrosophic norm $(\eta, \upsilon, \varsigma)$, i.e. for every $0 < \delta < 1$ and $\mathfrak{k} > 0$ there exists $n_0 \in \mathbb{N}$ such that fir all $n \geq n_0$ we have $\eta(\mathfrak{d}_n) - \mathfrak{d}, \mathfrak{k}) > 1 - \delta, \upsilon(\mathfrak{d}_n - \mathfrak{d}, \mathfrak{k}) < \delta$ and $\varsigma(\mathfrak{d}_n - \mathfrak{d}, \mathfrak{k}) < \delta$. (3.7.1)

Now, by hypothesis $\Psi_1(\mathfrak{d}_n) \longrightarrow \Psi(\mathfrak{d})$ with respect to neutrosophic norm $(\eta, \upsilon, \varsigma)$ and $\mathfrak{d} \in \widetilde{\mathcal{N}}$.

Therefore, for every $0 < \lambda < 1$ and $\mathfrak{k} > 0$ there exists $n_1 \in \mathbb{N}$ such that for all $n > n_1$

$\eta(\Psi_1(\mathfrak{d}_n) - \Psi_1(\mathfrak{d}), \mathfrak{k}) > 1 - \lambda, \upsilon(\Psi_1(\mathfrak{d}_n) - \Psi_1(\mathfrak{d}), \mathfrak{k}) < \lambda$ and $\varsigma(\Psi_1(\mathfrak{d}_n) - \Psi_1(\mathfrak{d}), \mathfrak{k}) < \lambda$. (3.7.2)

Futhermore, Ψ_2 is bounded therefore there exists $\mathrm{K} > 0$, such that $\|\Psi_2\| \leq K$. Now we prove that $(\Psi_1 + \Psi_2)(\mathfrak{d}_n) \longrightarrow (\Psi_1 + \Psi_2)(\mathfrak{d})$ with respect to neutrosophic norm $(\eta, \upsilon, \varsigma)$ and $\mathfrak{d} \in \widetilde{\mathcal{N}}$. Let $n_{\mathfrak{r}} = \max\{n_0, n_1\}$ and for every $0 < \lambda, \delta < 1$ there exists $0 < \mathfrak{r} < 1$ such that

$(1-\lambda) \star (1 - \delta) > 1 - \mathfrak{r}$ and $\lambda \diamond \delta < \mathfrak{r}$, then for every $n \geq n_{\mathfrak{r}}$ we get

$$\begin{aligned}
\eta((\Psi_1 + \Psi_2)(\mathfrak{d}_n) - (\Psi_1 + \Psi_2)(\mathfrak{d}), \mathfrak{k}) &= \eta((\Psi_1(\mathfrak{d}_n) - \Psi_1(\mathfrak{d}) + \Psi_2(\mathfrak{d}_n) - \Psi_2(\mathfrak{d}), \mathfrak{k}) \\
&\geq \eta((\Psi_1(\mathfrak{d}_n) - \Psi_1(\mathfrak{d}), \tfrac{\mathfrak{k}}{2}) \star \eta((\Psi_2(\mathfrak{d}_n) - \Psi_2(\mathfrak{d}), \tfrac{\mathfrak{k}}{2}) \\
&\geq (1 - \lambda) \star \eta(\mathfrak{d}_n - \mathfrak{d}, \tfrac{\mathfrak{k}}{2\|\Psi_2\|}) \geq (1 - \lambda) \star \eta(\mathfrak{d}_n - \mathfrak{d}, \mathfrak{k}) \\
&> (1 - \lambda) \star (1 - \delta) > 1 - \mathfrak{r}, \qquad (3.7.3)
\end{aligned}$$

$$\begin{aligned}
\upsilon((\Psi_1 + \Psi_2)(\mathfrak{d}_n) - (\Psi_1 + \Psi_2)(\mathfrak{d}), \mathfrak{k}) &= \upsilon((\Psi_1(\mathfrak{d}_n) - \Psi_1(\mathfrak{d}) + \Psi_2(\mathfrak{d}_n) - \Psi_2(\mathfrak{d}), \mathfrak{k}) \\
&\leq \upsilon((\Psi_1(\mathfrak{d}_n) - \Psi_1(\mathfrak{d}), \tfrac{\mathfrak{k}}{2}) \diamond \upsilon((\Psi_2(\mathfrak{d}_n) - \Psi_2(\mathfrak{d}), \tfrac{\mathfrak{k}}{2}) \\
&\leq (1 - \lambda) \diamond \upsilon(\mathfrak{d}_n - \mathfrak{d}, \tfrac{\mathfrak{k}}{2\|\Psi_2\|}) \leq (1 - \lambda) \diamond \upsilon(\mathfrak{d}_n - \mathfrak{d}, \mathfrak{k}) \\
&< \lambda \diamond \delta < \mathfrak{r} \qquad (3.7.4)
\end{aligned}$$

$$\begin{aligned}
\varsigma((\Psi_1 + \Psi_2)(\mathfrak{d}_n) - (\Psi_1 + \Psi_2)(\mathfrak{d}), \mathfrak{k}) &= \varsigma((\Psi_1(\mathfrak{d}_n) - \Psi_1(\mathfrak{d}) + \Psi_2(\mathfrak{d}_n) - \Psi_2(\mathfrak{d}), \mathfrak{k}) \\
&\leq \varsigma((\Psi_1(\mathfrak{d}_n) - \Psi_1(\mathfrak{d}), \tfrac{\mathfrak{k}}{2}) \diamond \varsigma((\Psi_2(\mathfrak{d}_n) - \Psi_2(\mathfrak{d}), \tfrac{\mathfrak{k}}{2}) \\
&\leq (1 - \lambda) \diamond \varsigma(\mathfrak{d}_n - \mathfrak{d}, \tfrac{\mathfrak{k}}{2\|\Psi_2\|}) \leq (1 - \lambda) \diamond \varsigma(\mathfrak{d}_n - \mathfrak{d}, \mathfrak{k}) \\
&< \lambda \diamond \delta < \mathfrak{r}. \qquad (3.7.5)
\end{aligned}$$

We use $\tau = \frac{\mathfrak{k}}{2K}$ in the above equations. Now equations (3.7.3), (3.7.4) and (3.7.5) simultaneously conclude that $(\Psi_1 + \Psi_2)(\mathfrak{d}_n) \longrightarrow (\Psi_1 + \Psi_2)(\mathfrak{d})$ with respect to neutrosophic norm $(\eta, \upsilon, \varsigma)$. Also it should to noted that Ψ_1 is closed, then we obtain $\mathfrak{d} \in \widetilde{\mathcal{N}}$, by the definition (2.12).

Conclusion

In this paper, we have developed open mapping and closed graph theorem in neutrosophic Banach space and we have presented some suitable examples that support our main results. We hope that the result proved in this paper will form new connection for those who are working in the in neutrosophic Banach space and this work opens a new path for researchers in the concerned field.

References

1. G. A. Anastassiou, Fuzzy approximation by fuzzy convolution type operators, Comput. Math. Appl 48 (9) (2004) 1369–1386.
2. K. T. Atanassov, Intuitionistic fuzzy sets, in: Intuitionistic fuzzy sets, Springer, 1999, pp. 1–137. 165.

3. C. Alaca, D. Turkoglu, C. Yildiz, Fixed points in intuitionistic fuzzy metric spaces, Chaos Solitons Fractals 29 (5) (2006) 1073–1078.
4. T. Bag and S.K. Samanta, Finite dimensional fuzzy normed linear spaces, J. Fuzzy Math. 11 (2003), no. 3, 687–705.
5. T. Bag and S.K. Samanta, Fuzzy bounded linear operators, Fuzzy Sets Syst. 151 (2005), 513–547.
6. J. Buckley, Y. Qu, Solving systems of linear fuzzy equations, Fuzzy Sets Syst. 43 (1) (1991) 33–43.
7. Z. Deng, Fuzzy pseudo-metric spaces, J. Math. Anal. Appl. 86 (1) (1982) 170 74–95.
8. M. A. Erceg, Metric spaces in fuzzy set theory, J. Math. Anal. Appl. 69 (1) (1979) 205–230.
9. C. Felbin, Finite dimensional fuzzy normed linear space, Fuzzy Sets Syst. 48 (1992), 239–248.
10. A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets Syst. 64 (3) (1994) 395–399.
11. M. Jeyaraman , A.N. Mangayarkkarasi, V. Jeyanthi, R. Pandiselvi, Hyer-Ulam-Rassias stability for functional equation in neutrosophic normed space, 18(2022) 127-27-143.
12. S. Karakus, K. Demirci, O. Duman, Statistical convergence on intuitionistic fuzzy normed spaces, Chaos Solitons Fractals 35 (4) (2008) 763–769.
13. M. Kirişci, N. Simsek, Neutrosophic metric spaces, Math. Sci., 14 (2020), 241–248.
14. M. Mursaleen, S. Mohiuddine, Statistical convergence of double sequences in intuitionistic fuzzy normed spaces, Chaos Solitons Fractals 41 (5) (2009) 2414–2421.
15. J. H. Park, Intuitionistic fuzzy metric spaces, Chaos Solitons Fractals 22 (5) (2004) 1039–1046. 185.
16. S. Pandit, A. Ahmad, A study on statistical convergence of triple sequences in intuitionistic fuzzy normed space, Sahand Commun. Math. Anal. 19 (3) (2022) 1–12.
17. K. Peeva, Fuzzy linear systems, Fuzzy Sets Syst. 49 (3) (1992) 339–355.
18. R. Saadati, J. H. Park, On the intuitionistic fuzzy topological spaces, Chaos Solitons Fractals 27 (2) (2006) 331–344.
19. R. Saadati, S. Vaezpour, Some results on fuzzy Banach spaces, J. Appl. 195 Math. Comput. 17 (1) (2005) 475–484.
20. Shailendra Pandit, Ayaz Ahmad, Ayha Esai, A Study on topological properties of intuitionistic fuzzy Banach spaces, Research Square (2023).
21. F.Smarandache, Neutrosophy: Neutrosophic probability, set and logic, Rehoboth: American Research Press, 1998.
22. J. Xiao and X. Zhu, On linearly topological structure and property of fuzzy normed linear space, Fuzzy Sets Systems 125(2002), 153-161.
23. L. Zadeh, Fuzzy sets, Inform. Control 8 (3) (1965) 338–353.

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