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On Neutrosophic Normed Spaces of I-Convergence DiferenceSequences Defned by Modulus Function

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Abstract. In this paper, we introduce the neutrosophic I-convergent difference sequence spaces $I^{(\mathcal{Y})}_{(\Delta)}(f)$ and $I^{0(\mathcal{Y})}_{(\Delta)}(f)$ defined by modulus function. Also, we define an open ball $B(x,\epsilon,\gamma)(f)$ in neutrosophic norm space defined by modulus function. Furthermore, We construct new topological spaces and look into various topological aspects in neutrosophic I-convergent difference sequence spaces $I^{(\mathcal{Y})}_{(\Delta)}(f)$ and $I^{0(\mathcal{Y})}_{(\Delta)}(f)$ defined by modulus function

Keywords: Difference sequence spaces; Ideal; Filter; I-convergence; Modulus function; Neutrosophic normed spaces.

1. Introduction

Dr. Florentin Smarandache created the idea of neutrosophy in the 1990s as a reaction to these difficulties, providing a broader perspective on uncertainty. Neutrosophy aims to analyse and portray systems, phenomena, and concepts that involve ambiguity, incompleteness, and indeterminacy.

The Neutrosophic Set [1], a mathematical structure that expands on the idea of fuzzy sets [2] and conventional crisp sets, is one of the core ideas of Neutrosophy. Three different elements can occur in a neutrosophic set's membership: truth, falsity, and indeterminacy. Neutosophic sets are used in a variety of real-world contexts, including decision-making, artificial intelligence, medical diagnosis, image processing, and pattern identification. By considering the interplay between truth, falsity, and indeterminacy, neutrosophic sets offer a more robust and flexible approach to modeling real-world uncertainties, making it a valuable tool for addressing complex and contradictory data.

In 2006, [10] F. Samarandache and W.B. Vasantha Kanasamy introduced the concept of neutrosophic algebraic structures. Mahapatra and Bera [7] were the first to introduce the neutrosophic soft linear space. Neutrosophic soft norm linear space, metric, convexity [11], and Cauchy sequence were examined by Bera and Mahapatra [8]. The purpose of the current paper is to change the intuitionistic fuzzy normed space of the structure into neutrosophic normed space. The Cauchy sequence has been studied on neutrosophic normed space in an attempt to investigate some beautiful results in this structure.

Mursaleen [14] introduced and presented the concept of statistical convergence with regard to the intuitionistic fuzzy normed (Saadati and Park [15]). Khan [12] recently defined I-convergence and I-Cauchy sequence in intuitionistic fuzzy normed. Kirişci and Şimşek [4] investigated the statistical convergence in neutrosophic normed space.. Since neutrosophic normed space is a generalisation of intuitionistic fuzzy normed (IFNS), this statistical convergence is an important area for research. This piqued our interest in studying I-convergence in neutrosophic normed space. For further detail on ideal and statistical convergence, see [13, 16–18]. Other important aspects of the neutrosophic norm can be found in [4, 5, 7, 8].

Kizmaz [20] developed the concept of difference sequence spaces by studying the difference sequence spaces $X = l_\infty(\Delta), c(\Delta), c_0(\Delta)$.
Some novel sequence spaces were introduce by means of varius matrix transformation in [19, 21, 22] and [23–25]. As seen below, Kizmaz [20] defines the difference sequence spaces using the difference matrix.

$$X(\Delta) = \{\zeta = \zeta_n : \Delta\zeta \in X\}$$

for $X = c,\ l_\infty,\ c_0$, where $\Delta\zeta_n = \zeta_n - \zeta_{n+1}$ and Δ shows the difference matrix $\Delta = (\Delta_{nm})$ defined by

$$\Delta_{nm} = \begin{cases} (-1)^{n-m}, & \text{if } n \le m \le n+1 \\ 0\ , & \text{if } 0 \le m < n. \end{cases} \tag{1.1}$$

Definition 1.1. [28] A function $f : [0,\infty) \to [0,\infty)$ is called a modulus function if the following conditions are met,
(a) $f(\zeta) = 0 \iff \zeta = 0$,
(b) $f(\zeta_1 + \zeta_2) \le f(\zeta) + f(\zeta_2)$,
(c) f is non-decreasing, and
(d) f is continuous from the right at zero.
Since $|f(\zeta_1) - f(\zeta_2)| \le f(|\zeta_1 - \zeta_2|)$, condition (4) implies that f is continuous on $\mathbb{R}^+ \cup \{0\}$.

Moreover, from condition (2) we have $f(n\zeta) \leq n\ f(\zeta)$, $\forall n \in \mathbb{N}$, and so $f(\zeta) = f(n\zeta(\frac{1}{n}))$. Hence $\frac{1}{n}f(\zeta) \leq f(\frac{\zeta}{n})\ \forall n \in \mathbb{N}$.

It is possible for the modulus function to be either bounded or unbounded. Consider the following example: $f(\zeta) = \frac{\zeta}{1+\zeta}$, then $f(\zeta)$ is bounded. If $f(\zeta) = \zeta d$,$0 < d < 1$, then the modulus function $f(\zeta)$ is unbounded.

In this study, we present the neutrosophic I-convergent difference sequence spaces $I^{(\mathcal{Y})}_{(\Delta)}(f)$ and $I^{0(\mathcal{Y})}_{(\Delta)}(f)$ defined by modulus function and investigate some of its topological properties.

Definition 1.2. [5] A binary operation $\star$ on $[0,1]$ is referred to as CTN if (a) $\star$ is associative, commutative and continuous, (b) $\mu = \mu \star 1$ for any $\mu \in [0,1]$ and (c) for each $\mu_1, \mu_2, \mu_3, \mu_4 \in [0,1]$, if $\mu_3 \geq \mu_1$ and $\mu_4 \geq \mu_2$ then $\mu_3 \star \mu_4 \geq \mu_1 \star \mu_2$.

A binary operation $\circ$ on $[0,1]$ is referred to as CTCN if (a) $\circ$ is associative, commutative and continuous, (b) $\mu = \mu \circ 0$ for any $\mu \in [0,1]$ and (c) for each $\mu_1, \mu_2, \mu_3, \mu_4 \in [0,1]$, if $\mu_3 \geq v_1$ and $\mu_4 \geq \mu_2$ then $\mu_3 \circ \mu_4 \geq \mu_1 \circ \mu_2$.

Definition 1.3. [1] Let $X \neq \phi$ and $\mathcal{Y} \subset X$ Then,

$$\mathcal{Y}_{NS} = \{< \zeta, \mathcal{U}(\zeta), \mathcal{V}(\zeta), \mathcal{W}(\zeta) >: \zeta \in X\},$$

where $\mathcal{U}(\zeta), \mathcal{V}(\zeta), \mathcal{W}(\zeta) : X \to [0,1]$, $\mathcal{U}(\zeta) =$ Truth, $\mathcal{V}(\zeta) =$ Indeterminacy, and $\mathcal{W}(\zeta) =$ Falsehood respectively.

$$0 \leq \mathcal{U}(\zeta) + \mathcal{V}(\zeta) + \mathcal{W}(\zeta) \leq 3.$$

The components of neutrosophic are $\mathcal{U}(\zeta), \mathcal{V}(\zeta)$ and $\mathcal{W}(\zeta)$ independent of each other.

Definition 1.4. [4,27] Assume X is a real vector space, $\star$ and $\diamond$ are CTN and CTCN, respectively, and $\mathcal{Y} = \{< \zeta, \mathcal{U}(\zeta), \mathcal{V}(\zeta), \mathcal{W}(\zeta) >: \zeta \in X\}$ be a neutrosophic set s.t $\mathcal{Y} : X \times (0,\infty) \to [0,1]$. The four-tuple $(X, \mathcal{Y}, \star, \diamond)$ is called a neutosophic normed space (NNS) if the subsequent terms holds; $\forall\ \zeta, y \in X$ and $s, r > 0$

(i) $0 \leq \mathcal{U}(\zeta, s) \leq 1$, $0 \leq \mathcal{V}(\zeta, s) \leq 1$, $0 \leq \mathcal{W}(\zeta, s) \leq 1$, $s \in R^+$,

(ii) $\mathcal{U}(\zeta, s) + \mathcal{V}(\zeta, s) + \mathcal{W}(\zeta, s) \leq 3$, for $s \in R^+$,

(iii) $\mathcal{U}(\zeta, s) = 1$ iff $\zeta = 0$

(iv) $\mathcal{U}(\lambda\zeta, s) = \mathcal{U}(\zeta, \frac{s}{|\lambda|})$,

(v) $\mathcal{U}(\zeta, s) \star \mathcal{U}(y, r) \leq \mathcal{U}(\zeta + y, s + r)$,

vi) $\mathcal{U}(\zeta, \star)$ is continuous non-decreasing function

(vii) $\lim_{s\to\infty} \mathcal{U}(\zeta, s) = 1$

(viii) $\mathcal{V}(\zeta, s) = 0$ iff $\zeta = 0$

(ix) $\mathcal{V}(\lambda\zeta, s) = \mathcal{V}(\zeta, \frac{s}{|\lambda|})$,

(x) $\mathcal{V}(\zeta, s) \diamond \mathcal{V}(y, s) \geq \mathcal{V}(\zeta + y, s + r)$,

(xi) $\mathcal{V}(\zeta, \diamond)$ is continuous non-increasing function,

(xii) $\lim_{s\to\infty} \mathcal{V}(\zeta, s) = 0$,

(xiii) $\mathcal{W}(\zeta, s) = 0$ iff $\zeta = 0$

(xiv) $\mathcal{W}(\lambda\zeta, s) = \mathcal{W}(\zeta, \frac{s}{|\lambda|})$,

(xv) $\mathcal{W}(\zeta, s) \diamond \mathcal{W}(y, s) \geq \mathcal{W}(\zeta + y, s + r)$,

(xvi) $\mathcal{W}(\zeta, \diamond)$ is continuous non-increasing function,

(xvii) $\lim_{s\to\infty} \mathcal{W}(\zeta, s) = 0$,

(xviii) If $s \leq 0$, then $\mathcal{U}(\zeta, s) = 0$, $\mathcal{V}(\zeta, s) = 1$, $\mathcal{W}(\zeta, s) = 1$.

In such case, $\mathcal{Y} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$ is called a neutrosophic normed (NN).

Example 1.1. [27] Suppose $(X, \|\ .\ \|)$ be a normed space, where $||y|| = |y|, \forall y \in \mathbb{R}$. Give the function as $\zeta \circ y = \zeta + y - \zeta y$ and define $\zeta \star y = \min(\zeta, y)$, For $s > ||y||$,

$$\mathcal{U}(\zeta, s) = \frac{s}{s + ||\zeta||},\ \mathcal{V}(\zeta, s) = \frac{\zeta}{s + ||\zeta||},\ \mathcal{W}(\zeta, s) = \frac{||\zeta||}{s} \tag{1.2}$$

$\forall\ \zeta, y \in X$ and $s > 0$.

If we take $s \leq ||\zeta||$, then

$$\mathcal{U}(\zeta, s) = 0, \mathcal{V}(\zeta, s) = 1 \text{ and } \mathcal{W}(\zeta, s) = 1.$$

Hence, $(X, \mathcal{Y}, \circ, \star)$ is neutrosophic norm space s.t $\mathcal{Y} : X \times R^{+} \to [0, 1]$.

Definition 1.5. [5, 29] Let $(X, \mathcal{Y}, \star, \diamond)$ be a NNS. A sequence $x = \{x_n\}$ in X is said to convergent to α_1 with regard to NN-$\mathcal{Y} \iff$ for each $\gamma > 0$, $\epsilon \in (0,1)$, $\exists\ N \in \mathbb{N}$ s.t

$$\mathcal{U}(x_n - \alpha_1, \gamma) > 1 - \epsilon, \mathcal{V}(x_n - \alpha_1, \gamma) < \epsilon, \mathcal{W}(x_n - \alpha_1, \gamma) < \epsilon,\ \forall\ n \in N.$$

i.e, $\gamma > 0$, we have

$$\lim_{n\to\infty} \mathcal{U}(x_n - \alpha_1, \gamma) = 1, \lim_{n\to\infty} \mathcal{V}(x_n - \alpha_1, \gamma) = 0 \text{ and } \lim_{n\to\infty} \mathcal{W}(x_n - \alpha_1, \gamma) = 0.$$

We specify $\mathcal{Y} - \lim x_n = \alpha_1$.

Theorem 1.1. *Let $(X, \mathcal{Y}, \star, \diamond)$ be a NNS. Then, a sequence $x = \{x_n\}$ in X is convergent to $\alpha \in X$ if and only if $\lim_{n\to\infty} \mathcal{U}(x_n - \alpha, \gamma) = 1$, $\lim_{n\to\infty} \mathcal{V}(x_n - \alpha, \gamma) = 0$ and $\lim_{n\to\infty} \mathcal{W}(x_n - \alpha, \gamma) = 0$.*

Definition 1.6. [3, 6, 9] Assemblage of subsets $I \subseteq 2^{\mathbb{N}}$ is known as an ideal in $\mathbb{N}$ if I satisfies these condition;

(1) $\emptyset \in I$

(2) $\mathcal{H}, \mathcal{K} \in I \Rightarrow \mathcal{H} \cup \mathcal{K} \in I$, (additive);

(3) $\mathcal{H} \in I, \mathcal{K} \subseteq \mathcal{H} \Rightarrow \mathcal{K} \in I$. (hereditary);

If $I \neq 2^{\mathbb{N}}$, then $I \subseteq 2^{\mathbb{N}}$ is called nontrivial [13]. A nontrivial ideal $I \subseteq 2^{\mathbb{N}}$ is said to be admissible if I includes every singleton subset of $\mathbb{N}$.

If there isn't a non-trivial ideal $K \neq I$, then I is the maximum non-trivial ideal such that $I \subset K$.

Definition 1.7. [27] Assemblage of subsets $F \subseteq 2^{\mathbb{N}}$ is known as a filter in $\mathbb{N}$ if I satisfies these condition:

(1) $\emptyset \notin F$,

(2) For $\mathcal{H}, \mathcal{K} \in F \implies \mathcal{H} \cap \mathcal{K} \in F$,

(3) If $\mathcal{H} \in F$ and $\mathcal{K} \supset \mathcal{H}$ implies $\mathcal{K} \in F$.

Definition 1.8. [27] Suppose $\{x_n\}$ be a sequence in $(X, \mathcal{Y}, \star, \diamond)$. A sequence $\{x_n\}$ is said to be ideally convergent to α with regard to NN-$\mathcal{Y}$, if, for every $\epsilon > 0$ and $\gamma > 0$

$$P = \{n \in \mathbb{N} : \mathcal{U}(x_n - \alpha, \gamma) \leq 1 - \epsilon \text{ or } \mathcal{V}(x_n - \alpha, \gamma) \geq \epsilon, \mathcal{W}(x_n - \alpha, \gamma) \geq \epsilon\} \in I \tag{1.3}$$

It is denoted by $I_{\mathcal{Y}} - \lim x_n = \alpha$ or $x_n \to \alpha$.

Definition 1.9. [27] Suppose $\{x_n\}$ be a sequence in $(X, \mathcal{Y}, \star, \diamond)$. A sequence $\{x_n\}$ is said to ideally Cauchy sequence with regard to NN-$\mathcal{Y}$, if, for every $\epsilon > 0$ and $\gamma > 0$, $\exists\ k \in \mathbb{N}$ s.t

$$Q = \{n \in \mathbb{N} : \mathcal{U}(x_n - x_k, \gamma) \leq 1 - \epsilon \text{ or } \mathcal{V}(x_n - x_k, \gamma) \geq \epsilon,\ \mathcal{W}(x_n - x_k, \gamma) \geq \epsilon\} \in I.$$

2. Main Results

In this study, we created and examined various topological aspects of neutrosophic ideal convergent difference sequence spaces defined by modulus function, a variant of ideal convergent sequence spaces. Let ω be the space containing all real sequences.

$$I^{0(\mathcal{Y})}_{(\Delta)}(f) = \Big\{x = \{x_n\} \in \omega : \{n \in \mathbb{N} : f\Big(\mathcal{U}(\Delta x_n, \gamma)\Big) \leq 1-\epsilon \text{ or } f\Big(\mathcal{V}(\Delta x_n, \gamma)\Big) \geq \epsilon, f\Big(\mathcal{W}(\Delta x_n, \gamma)\Big) \geq \epsilon\} \in I\Big\} \tag{2.1}$$

$$I^{(\mathcal{Y})}_{(\Delta)}(f) := \Big\{x = \{x_n\} \in \omega : \{n \in \mathbb{N} : \text{for some } \gamma \in \mathbb{R} \text{ , } f\Big(\mathcal{U}(\Delta x_n - \alpha, \gamma)\Big) \leq 1-\epsilon \text{ or } f\Big(\mathcal{V}(\Delta x_n - \alpha, \gamma)\Big) \geq \epsilon, f\Big(\mathcal{W}(\Delta x_n - \alpha, \gamma)\Big) \geq \epsilon\} \in I\Big\} \tag{2.2}$$

We describe an open ball with a radius $\epsilon \in (0,1)$ and a center at x with regard to the neutrosophic $\gamma > 0$ parameter, indicated by $B(x, \epsilon, \gamma)$ as follows:

$$B(x, \epsilon, \gamma) = \Big\{y = \{y_n\} \in I^{(\mathcal{Y})}_{(\Delta)}(f) : \{n \in \mathbb{N} : f\Big(\mathcal{U}(\Delta x_n - \Delta y_n, \gamma)\Big) \leq 1-\epsilon \text{ or } f\Big(\mathcal{V}(\Delta x_n - \Delta y_n, \gamma)\Big) \geq \epsilon, f\Big(\mathcal{W}(\Delta x_n - \Delta y_n, \gamma)\Big) \geq \epsilon, \} \in I\Big\} \tag{2.3}$$

Theorem 2.1. *The inclusion relation* $I^{0(\mathcal{Y})}_{(\Delta)}(f) \subset I^{(\mathcal{Y})}_{(\Delta)}(f)$ *holds.*

The inverse of an inclusion relation is not true. To defend our claim, take a look at the examples below.

Example 2.1. Suppose $(\mathbb{R}, \|.\|)$ be a normed space s.t $\|x\| = \sup_k |x_k|$, and $x_1 * x_2 = \min\{x_1, x_2\}$ and $x_1 \diamond x_2 = \max\{x_1, x_2\}$, $\forall x_1, x_2 \in (0,1)$. For $\beta > ||x||$, now define norms $\mathcal{Y} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$ on $\mathbb{R}^2 \times (0, \infty)$ as follows;

$$\mathcal{U}(x, \beta) = \frac{\beta}{\beta + \|x\|} \text{ , } \mathcal{V}(x, \beta) = \frac{\|x\|}{\beta + \|x\|} \text{ and } \mathcal{W}(x, \beta) = \frac{\|x\|}{\beta}.$$

Then $(\mathbb{R}, \mathcal{Y}, \star, \diamond)$ is a NNS. Consider the sequence (x_k)={1}. It is easy to observe that $(x_k) \in I^{(\mathcal{Y})}_{(\Delta)}(f)$ and $x_k \xrightarrow{I_{(\mathcal{Y})}} 1$, but $x_k \notin I^{0(\mathcal{Y})}_{(\Delta)}(f)$.

Lemma 2.1. Let $x = \{x_n\} \in I^{(\mathcal{Y})}_{(\Delta)}(f)$. Then $\forall\ \epsilon \in (0,1)$ and $\gamma > 0$, the following claims are equivalent ,

(a) $I^{(\mathcal{Y})}_{(\Delta)}(f)$– $\lim(x) = \alpha$,

(b) $\Big\{n \in \mathbb{N} : f\Big(\mathcal{U}(\Delta x_n - \alpha, \gamma)\Big) \leq 1-\epsilon \text{ or } f\Big(\mathcal{V}(\Delta x_n - \alpha, \gamma)\Big) \geq \epsilon, f\Big(\mathcal{W}(\Delta x_n - \alpha, \gamma)\Big) \geq \epsilon\Big\} \in I$,

(c) $\Big\{n \in \mathbb{N} : f\Big(\mathcal{U}(\Delta x_n - \alpha, \gamma)\Big) > 1-\epsilon \text{ and } f\Big(\mathcal{V}(\Delta x_n - \alpha, \gamma)\Big) < \epsilon, f\Big(\mathcal{W}(\Delta x_n - \alpha, \gamma)\Big) < \epsilon\Big\} \in F(I)$

(d) $I-\lim\ f\Big(\mathcal{U}(\Delta x_n-\alpha,\gamma)\Big)=1,\ I-\lim\ f\Big(\mathcal{V}(\Delta x_n-\alpha,\gamma)\Big)=0$ and $I-\lim\ f\Big(\mathcal{W}(\Delta x_n-\alpha,\gamma)\Big)=0.$

Theorem 2.2. *The spaces* $I^{(\mathcal{Y})}_{(\Delta)}(f)$ *and* $I^{0(\mathcal{Y})}_{(\Delta)}(f)$ *are linear spaces.*

Proof. We know that $I^{0(\mathcal{Y})}_{(\Delta)}(f)\subset I^{(\mathcal{Y})}_{(\Delta)}(f)$. Then we show the outcome for $I^{(\mathcal{Y})}_{(\Delta)}(f)$. The proof of linearity of the space $I^{0(\mathcal{Y})}_{(\Delta)}(f)$ follows similarly.
Let $\{x_k\},\{y_k\}\in I^{(\mathcal{Y})}_{(\Delta)}(f)$ and α_1,α_2 be scalars. The proof is trivial for $\alpha_1=0$ and $\alpha_2=0$. Now we take $\alpha_1\neq 0$ and $\alpha_2\neq 0$. For a given $\epsilon>0$, take $r>0$ s.t $(1-\epsilon)*(1-\epsilon)>(1-r)$ and $\epsilon\diamond\epsilon<r$.

$$P_1=\Big\{n\in\mathbb{N}:f\Big(\mathcal{U}(\Delta x_n-\alpha_1,\frac{\gamma}{2|\mu|})\Big)\leq 1-\epsilon \text{ or } f\Big(\mathcal{V}(\Delta x_n-\alpha_1,\frac{\gamma}{2|\mu|})\Big)\geq\epsilon, f\Big(\mathcal{W}(\Delta x_n-\alpha_1,\frac{\gamma}{2|\mu|})\Big)\geq\epsilon\Big\}\in I,$$

$$P_2=\Big\{n\in\mathbb{N}:f\Big(\mathcal{U}(\Delta x_n-\alpha_2,\frac{\gamma}{2|\nu|})\Big)\leq 1-\epsilon \text{ or } f\Big(\mathcal{V}(\Delta x_n-\alpha_2,\frac{\gamma}{2|\nu|})\Big)\geq\epsilon, f\Big(\mathcal{W}(\Delta x_n-\alpha_2,\frac{\gamma}{2|\nu|})\Big)\geq\epsilon\Big\}\in I.$$

Now, we take the complement of P_1 and P_2

$$P_1^c=\Big\{n\in\mathbb{N}:f\Big(\mathcal{U}(\Delta x_n-\alpha_1,\frac{\gamma}{2|\mu|})\Big)>1-\epsilon \text{ and } f\Big(\mathcal{V}(\Delta x_n-\alpha_1,\frac{\gamma}{2|\mu|})\Big)<\epsilon, f\Big(\mathcal{W}(\Delta x_n-\alpha_1,\frac{\gamma}{2|\mu|})\Big)<\epsilon\Big\}\in F(I),$$

$$P_2^c=\Big\{n\in\mathbb{N}:f\Big(\mathcal{U}(\Delta x_n-\alpha_2,\frac{\gamma}{2|\nu|})\Big)>1-\epsilon \text{ and } f\Big(\mathcal{V}(\Delta x_n-\alpha_2,\frac{\gamma}{2|\nu|})\Big)<\epsilon, f\Big(\mathcal{W}(\Delta x_n-\alpha_2,\frac{\gamma}{2|\nu|})\Big)<\epsilon\Big\}\in F(I);$$

Consequently, set $P=\mathcal{P}_1\cup\mathcal{P}_2$ produces $P\in I$. Thus, P^c is a set that is not empty in $\mathcal{F}(I)$. We'll illustrate this for each $\{x_n\},\{y_n\}\in I^{(\mathcal{Y})}_{(\Delta)}(f)$.

$$P^c\subset\Big\{n\in\mathbb{N}:f\Big(\mathcal{U}(\mu\Delta x_n+\nu\Delta y_n)-(\mu\alpha_1+\nu\alpha_2,\gamma)\Big)>1-r \text{ and } f\Big(\mathcal{V}(\mu\Delta x_n+\nu\Delta y_n)-(\mu\alpha_1+\nu\alpha_2,\gamma)\Big)<r, f\Big(\mathcal{W}(\mu\Delta x_n+\nu\Delta y_n)-(\mu\alpha_1+\nu\alpha_2,\gamma)\Big)<r\Big\}$$

Let $i\in\mathcal{P}^c$. In this case,

$$f\Big(\mathcal{U}(\Delta x_i-\alpha_1,\frac{\gamma}{2|\mu|})\Big)>1-\epsilon \text{ and } f\Big(\mathcal{V}(\Delta x_i-\alpha_1,\frac{\gamma}{2|\mu|})\Big)<\epsilon, f\Big(\mathcal{W}(\Delta x_i-\alpha_1,\frac{\gamma}{2|\mu|})\Big)<\epsilon$$

$$f\Big(\mathcal{U}(\Delta y_i-\alpha_2,\frac{\gamma}{2|\nu|})\Big)>1-\epsilon \text{ and } f\Big(\mathcal{V}(\Delta y_i-\alpha_2,\frac{\gamma}{2|\nu|})\Big)<\epsilon, f\Big(\mathcal{W}(\Delta y_i-\alpha_2,\frac{\gamma}{2|\nu|})\Big)<\epsilon$$

Consider

$$\begin{aligned} f\Big(\mathcal{U}(\mu\Delta x_i+\nu\Delta y_i)-(\mu\alpha_1+\nu\alpha_2,\gamma)\Big) &\geq f\Big(\mathcal{U}(\mu\Delta x_i-\mu\alpha_1,\frac{\gamma}{2})\Big)\star f\Big(\mathcal{U}(\nu\Delta y_i-\nu\alpha_2,\frac{\gamma}{2})\Big)\\ &= f\Big(\mathcal{U}(\Delta x_i-\alpha_1,\frac{\gamma}{2|\mu|})\Big)\star f\Big(\mathcal{U}(\Delta y_i-\alpha_2,\frac{\gamma}{2|\nu|})\Big)\\ &> (1-\epsilon)*(1-\epsilon)>1-r \end{aligned}$$

$$\implies f\Big(\mathcal{U}(\mu\Delta x_i+\nu\Delta y_i)-(\mu\alpha_1+\nu\alpha_2,\gamma)\Big)>1-r$$

and

$$\begin{aligned} f\Big(\mathcal{V}(\mu\Delta x_i+\nu\Delta y_i)-(\mu\alpha_1+\nu\alpha_2,\gamma)\Big) &\le f\Big(\mathcal{V}(\mu\Delta x_i-\mu\alpha_1,\frac{\gamma}{2})\Big)\diamond f\Big(\mathcal{V}(\nu\Delta y_i-\nu\alpha_2,\frac{\gamma}{2})\Big)\\ &= f\Big(\mathcal{V}(\Delta x_i-\alpha_1,\frac{\gamma}{2|\mu|})\Big)\diamond f\Big(\mathcal{V}(\Delta y_i-\alpha_2,\frac{\gamma}{2|\nu|})\Big)\\ &< \epsilon\diamond\epsilon<r \end{aligned}$$

$$\implies f\Big(\mathcal{V}(\mu\Delta x_i+\nu\Delta y_i)-(\mu\alpha_1+\nu\alpha_2,\gamma)\Big)<r$$

and

$$\begin{aligned} f\Big(\mathcal{W}(\mu\Delta x_i+\nu\Delta y_i)-(\mu\alpha_1+\nu\alpha_2,\gamma)\Big) &\le f\Big(\mathcal{W}(\mu\Delta x_i-\mu\alpha_1,\frac{\gamma}{2})\Big)\diamond f\Big(\mathcal{W}(\nu\Delta y_i-\nu\alpha_2,\frac{\gamma}{2})\Big)\\ &= f\Big(\mathcal{W}(\Delta x_i-\alpha_1,\frac{\gamma}{2|\mu|})\Big)\diamond f\Big(\mathcal{W}(\Delta y_i-\alpha_2,\frac{\gamma}{2|\nu|})\Big)\\ &< \epsilon\diamond\epsilon<r \end{aligned}$$

$$\implies f\Big(\mathcal{W}(\mu\Delta x_i+\nu\Delta y_i)-(\mu\alpha_1+\nu\alpha_2,\gamma)\Big)<r$$

Thus
$P^c\subset\Big\{n\in\mathbb{N}: f\Big(\mathcal{U}(\mu\Delta x_n+\nu\Delta y_n)-(\mu\alpha_1+\nu\alpha_2,\gamma)\Big)>1-r$ and $f\Big(\mathcal{V}(\mu\Delta x_n+\nu\Delta y_n)-(\mu\alpha_1+\nu\alpha_2,\gamma)\Big)<r, f\Big(\mathcal{W}(\mu\Delta x_n+\nu\Delta y_n)-(\mu\alpha_1+\nu\alpha_2,\gamma)\Big)<r\Big\}$
Since $P^c\in\mathcal{F}(I)$, Thus By the properties of $F(I)$ we have,
$\Big\{n\in\mathbb{N}: f\Big(\mathcal{U}(\mu\Delta x_n+\nu\Delta y_n)-(\mu\alpha_1+\nu\alpha_2,\gamma)\Big)>1-r$ and $f\Big(\mathcal{V}(\mu\Delta x_n+\nu\Delta y_n)-(\mu\alpha_1+\nu\alpha_2,\gamma)\Big)<r, f\Big(\mathcal{W}(\mu\Delta x_n+\nu\Delta y_n)-(\mu\alpha_1+\nu\alpha_2,\gamma)\Big)<r\Big\}\in\mathcal{F}(I)$. Hence $I^{(\mathcal{Y})}_{(\Delta)}(f)$ is a linear space. □

Theorem 2.3. *Every closed ball $B^c(x,\epsilon,\gamma)$ is an open in $I^{(u,v,w)}_{(\Delta)}(f)$, where neutrosophic parameter $\gamma>0$ with centre at x and radius $0<\epsilon<1$.*

Proof. Suppose that $B(x,\gamma,\epsilon)$ is an open ball with a radius of $0<\epsilon<1$ and a neutrosophic parameter $\gamma>0$, with its centre at $x=(x_n)\in I^{(\mathcal{Y})}_{(\Delta)}(f)$.

$$\begin{aligned} B(x,\epsilon,\gamma)(f)=\big\{y\in I^{(\mathcal{Y})}_{(\Delta)}(f): f\Big(\mathcal{U}(\Delta x-\Delta y,\gamma)\Big)\le 1-\epsilon \text{ or } f\Big(\mathcal{V}(\Delta x-\Delta y,\gamma)\Big)\ge\epsilon,\\ f\Big(\mathcal{W}(\Delta x-\Delta y,\gamma)\Big)\ge\epsilon,\in I\big\} \end{aligned}$$

Then

$$\begin{aligned} B^c(x,\epsilon,\gamma)(f)=\big\{y\in I^{(\mathcal{Y})}_{(\Delta)}(f): f\Big(\mathcal{U}(\Delta x-\Delta y,\gamma)\Big)>1-\epsilon \text{ and } f\Big(\mathcal{V}(\Delta x-\Delta y,\gamma)\Big)<\epsilon,\\ f\Big(\mathcal{W}(\Delta x-\Delta y,\gamma)\Big)<\epsilon,\in F(I)\big\} \end{aligned}$$

suppose $y\in\mathcal{B}^c(x,\gamma,\epsilon)$. Then, For

$f\Big(\mathcal{U}(\Delta x - \Delta y, \gamma)\Big) > 1 - \epsilon$ and $f\Big(\mathcal{V}(\Delta x - \Delta y, \gamma)\Big) < \epsilon, f\Big(\mathcal{W}(\Delta x - \Delta y, \gamma)\Big) < \epsilon,$
so there exists $\gamma_0 \in (0, \gamma)$ such that
$f\Big(\mathcal{U}(\Delta x - \Delta y, \gamma_0)\Big) > 1 - \epsilon$ and $f\Big(\mathcal{V}(\Delta x - \Delta y, \gamma_0)\Big) < \epsilon, f\Big(\mathcal{W}(\Delta x - \Delta y, \gamma_0)\Big) < \epsilon.$

Let $\epsilon_0 = f\Big(\mathcal{U}(\Delta x - \Delta y, \gamma_0)\Big)$, we have $\epsilon_0 > 1 - \epsilon$. Then $\exists\, p \in (0,1)$ such that $\epsilon_0 > 1 - p > 1 - \epsilon$. For $\epsilon_0 > 1 - p$, we can have $\epsilon_1, \epsilon_2, \epsilon_3 \in (0,1)$ such that $\epsilon_0 * \epsilon_1 > 1 - p$, $(1 - \epsilon_0) \diamond (1 - \epsilon_2) < p$. and $(1 - \epsilon_0) \diamond (1 - \epsilon_3) < p$.

Let $\epsilon_4 = \max\{\epsilon_1, \epsilon_2, \epsilon_3\}$. Then $(1-p) < \epsilon_0 * \epsilon_1 \le \epsilon_0 * \epsilon_4$ and $(1-\epsilon_0) \diamond (1-\epsilon_4) \le (1-\epsilon_0) \diamond (1-\epsilon_2) < p$.
Consider the closed ball $B^c(y, \gamma - \gamma_0, 1 - \epsilon_4)$ and $B^c(x, \gamma, \epsilon)$.
We prove that $B^c(y, \gamma - \gamma_0, 1 - \epsilon_4) \subset \mathcal{B}^c(x, \gamma, \epsilon)$. Let $z = \{z_n\} \in \mathcal{B}^c(y, \gamma - \gamma_0, 1 - \epsilon_4)$. Then $f\Big(\mathcal{U}(\Delta y - \Delta z, \gamma - \gamma_0)\Big) > \epsilon_4$ and $f\Big(\mathcal{V}(\Delta y - \Delta z, \gamma - \gamma_0)\Big) < 1 - \epsilon_4, f\Big(\mathcal{W}(\Delta y - \Delta z, \gamma - \gamma_0)\Big) < 1 - \epsilon_4$
Therefore

$$\begin{aligned} f\Big(\mathcal{U}(\Delta x - \Delta z, \gamma)\Big) &\ge f\Big(\mathcal{U}(\Delta x - \Delta y, \gamma_0)\Big) \star f\Big(\mathcal{U}(\Delta y - \Delta z, \gamma - \gamma_0)\Big) \\ &\ge \epsilon_0 \star \epsilon_4 \ge \epsilon_0 \star \epsilon_1 \\ &> (1 - p) > \ (1 - \epsilon) \\ f\Big(\mathcal{V}(\Delta x - \Delta z, \gamma)\Big) &\le f\Big(\mathcal{V}(\Delta x - \Delta y, \gamma_0)\Big) \diamond f\Big(\mathcal{V}(\Delta y - \Delta z, \gamma - \gamma_0)\Big) \\ &\le (1 - \epsilon_0) \diamond (1 - \epsilon_4) \le \epsilon_0 \diamond \epsilon_2 \\ &< p < \ \epsilon \end{aligned}$$

and

$$\begin{aligned} f\Big(\mathcal{W}(\Delta x - \Delta z, \gamma)\Big) &\le f\Big(\mathcal{W}(\Delta x - \Delta y, \gamma_0)\Big) \diamond f\Big(\mathcal{W}(\Delta y - \Delta z, \gamma - \gamma_0)\Big) \\ &\le \epsilon_0 \diamond \epsilon_4 \le \epsilon_0 \diamond \epsilon_3 \\ &< p < \ \epsilon \end{aligned}$$

Therefore the set $\Big\{f\Big(\mathcal{U}(\Delta x - \Delta z, \gamma)\Big) > 1 - \epsilon$ and $f\Big(\mathcal{V}(\Delta x - \Delta z, \gamma)\Big) < \epsilon$, $f\Big(\mathcal{W}(\Delta x - \Delta z, \gamma)\Big) < \epsilon\Big\} \in \mathcal{F}(I)$.
$\implies z = (z_n) \in \mathcal{B}^c(x, \gamma, \epsilon)$
$\implies B^c\big(y, \gamma - \gamma_0, 1 - \epsilon_4\big) \subset \mathcal{B}^c\big(x, \gamma, \epsilon\big)$. □

Remark 2.2. It is clear that $I^{(\mathcal{Y})}_{(\Delta)}(f)$ is a neutrosophic normed space with respect to neutrosophic norms $\mathcal{Y} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$. Define

Now define a collection $\tau^{(\mathcal{Y})}_{(\Delta)}(f)$ of a subset of $I^{(\mathcal{Y})}_{(\Delta)}(f)$ as follows:
$\tau^{(\mathcal{Y})}_{(\Delta)}(f) = \big\{P \subset I^{(\mathcal{Y})}_{(\Delta)}(f) :$ for every $x = (x_n) \in P\ \exists\ \gamma > 0$ and $\epsilon \in (0,1)$ s.t $\mathcal{B}^c(x, \gamma, \epsilon) \subset P\big\}$.
Then $\tau^{(\mathcal{Y})}_{(\Delta)}(f)$ is a topolopgy on $I^{(\mathcal{Y})}_{(\Delta)}(f)$

Theorem 2.4. *The topology $\tau^{(\mathcal{Y})}_{(\Delta)}(f)$ on the space $I^{(\mathcal{Y})}_{(\Delta)}(f)$ is first countable.*

Proof. For every $x = \{x_n\} \in I^{(\mathcal{Y})}_{(\Delta)}(f)$, suppose the set $\mathcal{B} = \{B^c(x, \frac{1}{n}, \frac{1}{n})\} : n = 1, 2, 3, 4, ...\}$,which is a local countable basis at $x \in I^{(\mathcal{Y})}_{(\Delta)}(f)$. As a result, the topology $\tau^{(\mathcal{Y})}_{(\Delta)}(f)$ on the space $I^{0(\mathcal{Y})}_{(\Delta)}(f)$ is first countable. □

Theorem 2.5. *The spaces $I^{(\mathcal{Y})}_{(\Delta)}(f)$ and $I^{0(\mathcal{Y})}_{(\Delta)}(f)$ are Hausdorff spaces.*

Proof. We know that $I^{0(\mathcal{Y})}_{(\Delta)}(f) \subset I^{(\mathcal{Y})}_{(\Delta)}(f)$, We will only show the solution for $I^{(\mathcal{Y})}_{(\Delta)}(f)$
Let $x = (x_n), y = (y_n) \in I^{(\mathcal{Y})}_{(\Delta)}(f)$ such that $x \neq y$. Then
$0 < f\Big(\mathcal{U}(\Delta x - \Delta y, \gamma)\Big) < 1$,$0 < f\Big(\mathcal{V}(\Delta x - \Delta y, \gamma)\Big) < 1$ and $0 < f\Big(\mathcal{W}(\Delta x - \Delta y, \gamma)\Big) < 1$

Putting $\epsilon_1 = f\Big(\mathcal{U}(\Delta x - \Delta y, \gamma)\Big)$, $\epsilon_2 = f\Big(\mathcal{V}(\Delta x - \Delta y, \gamma)\Big)$, $\epsilon_3 = f\Big(\mathcal{W}(\Delta x - \Delta y, \gamma)\Big)$
and $\epsilon = \max\{\epsilon_1, 1 - \epsilon_2, 1 - \epsilon_3\}$. Then for each $\epsilon_0 \in (\epsilon, 1)$ there exist $\epsilon_4, \epsilon_5, \epsilon_6 \in (0, 1)$ such that $\epsilon_4 * \epsilon_4 \geq \epsilon_0$, $(1 - \epsilon_5) \diamond (1 - \epsilon_5) \leq (1 - \epsilon_0)$ and $(1 - \epsilon_6) \diamond (1 - \epsilon_6) \leq (1 - \epsilon_0)$.

Once again putting $\epsilon_7 = \max\{\epsilon_4, 1 - \epsilon_5, 1 - \epsilon_6, \}$, think about the closed balls. $B^c(x, 1 - \epsilon_7, \frac{\gamma}{2})$ and $B^c(y, 1 - \epsilon_7, \frac{\gamma}{2})$ respectively centred at x and y.
Then it is obvious that $B^c(x, 1 - \epsilon_7, \frac{\gamma}{2}) \cap \mathcal{B}^c(y, 1 - \epsilon_7, \frac{\gamma}{2}) = \phi$.
If possible let $z = \{z_n\} \in \mathcal{B}^c(x, 1 - \epsilon_7, \frac{\gamma}{2}) \cap \mathcal{B}^c(y, 1 - \epsilon_7, \frac{\gamma}{2})$. Then we have,

$$\begin{aligned}\epsilon_1 &= f\Big(\mathcal{U}(\Delta x - \Delta y, \gamma)\Big) \\ &\geq f\Big(\mathcal{U}(\Delta x - \Delta z, \frac{\gamma}{2})\Big) \star f\Big(\mathcal{U}(\Delta z - \Delta y, \frac{\gamma}{2})\Big) \\ &> \epsilon_7 \star \epsilon_7 \geq \epsilon_4 \star \epsilon_4 \geq \epsilon_0 > \epsilon_1\end{aligned} \tag{2.4}$$

$$\begin{aligned}\epsilon_2 &= f\Big(\mathcal{V}(\Delta x - \Delta y, \gamma)\Big) \\ &\leq f\Big(\mathcal{V}(\Delta x - \Delta z, \frac{\gamma}{2})\Big) \diamond f\Big(\mathcal{V}(\Delta z - \Delta y, \frac{\gamma}{2})\Big) \\ &< (1 - \epsilon_7) \diamond (1 - \epsilon_7) \leq (1 - \epsilon_5) \diamond (1 - \epsilon_5) \\ &\leq (1 - \epsilon_0) < \ \epsilon_2\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\epsilon_3 &= f\Big(\mathcal{W}(\Delta x - \Delta y, \gamma)\Big) \\ &\leq f\Big(\mathcal{W}(\Delta x - \Delta z, \frac{\gamma}{2})\Big) \diamond f\Big(\mathcal{W}(\Delta z - \Delta y, \frac{\gamma}{2})\Big) \\ &< (1 - \epsilon_7) \diamond (1 - \epsilon_7) \leq (1 - \epsilon_6) \diamond (1 - \epsilon_6) \\ &\leq (1 - \epsilon_0) < \ \epsilon_3\end{aligned} \tag{2.6}$$

We have a contradiction from equations (2.4), (2.5) and (2.6). Therefore, $B^c(x, 1-\epsilon_7, \frac{\gamma}{2}) \cap \mathcal{B}^c(y, 1-\epsilon_7, \frac{\gamma}{2}) = \phi$. Hence the space $I^{(\mathcal{Y})}_{(\Delta)}(f)$ is a Hausdorff space. □

Theorem 2.6. *Suppose $\tau^{(\mathcal{Y})}_{(\Delta)}(f)$ be a topology on a neutrosophic norm spaces $I^{(\mathcal{Y})}_{(\Delta)}(f)$, then a sequence $x = \{x_n\} \in I^{(\mathcal{Y})}_{(\Delta)}(f)$ such that (x_n) is Δ-convergent to Δx_0 with regard to NN-$(\mathcal{Y})$,if and only if $f\Big(\mathcal{U}(\Delta x_n - \Delta x_0, \gamma)\Big) \longrightarrow 1$,$f\Big(\mathcal{V}(\Delta x_n - \Delta x_0, \gamma)\Big) \longrightarrow 0$ and $f\Big(\mathcal{W}(\Delta x_n - \Delta x_0, \gamma)\Big) \longrightarrow 0$ as $n \longrightarrow \infty$.*

Proof. Let $B(x_0, \gamma, \epsilon)$ be an open ball with centre $x_0 \in I^{(\mathcal{Y})}_{(\Delta)}(f)$ and radius $\epsilon \in (0,1)$ with $\gamma > 0$, i.e.

$$B(x_0,\epsilon,\gamma)(f) = \Big\{x = \{x_n\} \in I^{(\mathcal{Y})}_{(\Delta)}(f) : \{n \in \mathbb{N} : f\Big(\mathcal{U}(\Delta x_n - \Delta x_0, \gamma)\Big) \leq 1-\epsilon \text{ or } f\Big(\mathcal{V}(\Delta x_n - \Delta x_0, \gamma)\Big) \geq \epsilon, f\Big(\mathcal{W}(\Delta x_0 - \Delta y_n, \gamma)\Big) \geq \epsilon, \} \in I\Big\} \tag{2.7}$$

Consider a sequence $\{x_n\} \in I^{(\mathcal{Y})}_{(\Delta)}(f)$ is Δ-convergent to Δx_0 with respect to neutrosophic norm $(\mathcal{Y})$, then for $\epsilon \in (0,1)$,$\gamma > 0$ $\exists$ $n_0 \in \mathbb{N}$ such that $\{x_n\} \in B^c(x_0, \gamma, \epsilon)$, $\forall\ n \geq n_0$. Thus

$$\Big\{n \in \mathbb{N} : f\Big(\mathcal{U}(\Delta x_n - \Delta x_0, \gamma)\Big) > 1-\epsilon, f\Big(\mathcal{V}(\Delta x_n - \Delta x_0, \gamma)\Big) < \epsilon, f\Big(\mathcal{W}(\Delta x_n - \Delta x_0, \gamma)\Big) < \epsilon\Big\} \in F(I).$$

So

$$1 - f\Big(\mathcal{U}(\Delta x_n - \Delta x_0, \gamma)\Big) > \epsilon, f\Big(\mathcal{V}(\Delta x_n - \Delta x_0, \gamma)\Big) < \epsilon, \text{ and } f\Big(\mathcal{W}(\Delta x_n - \Delta x_0, \gamma)\Big) < \epsilon\ \forall n \geq n_0.$$

$$f\Big(\mathcal{U}(\Delta x_n - \Delta x_0, \gamma)\Big) \to 1, f\Big(\mathcal{V}(\Delta x_n - \Delta x_0, \gamma)\Big) \to 0, \text{ and } f\Big(\mathcal{W}(\Delta x_n - \Delta x_0, \gamma)\Big) \to 0 \text{ as } n \to \infty$$

Conversely, if $\forall\ \gamma > 0$,

$$f\Big(\mathcal{U}(\Delta x_n - \Delta x_0, \gamma)\Big) \to 1, f\Big(\mathcal{V}(\Delta x_n - \Delta x_0, \gamma)\Big) \to 0, \text{ and } f\Big(\mathcal{W}(\Delta x_n - \Delta x_0, \gamma)\Big) \to 0 \text{ as } n \to \infty$$

Then for each $\epsilon \in (0,1)$, $\exists\ n_0 \in \mathbb{N}$ s.t.

$$1 - f\Big(\mathcal{U}(\Delta x_n - \Delta x_0, \gamma)\Big) > \epsilon, f\Big(\mathcal{V}(\Delta x_n - \Delta x_0, \gamma)\Big) < \epsilon, \text{ and } f\Big(\mathcal{W}(\Delta x_n - \Delta x_0, \gamma)\Big) < \epsilon\ \forall n \geq n_0.$$

So,

$$f\Big(\mathcal{U}(\Delta x_n - \Delta x_0, \gamma)\Big) > 1-\epsilon, f\Big(\mathcal{V}(\Delta x_n - \Delta x_0, \gamma)\Big) < \epsilon, f\Big(\mathcal{W}(\Delta x_n - \Delta x_0, \gamma)\Big) < \epsilon\ \forall\ n \geq n_0.$$

Hence $\{x_n\} \in B^c(x_0, \gamma, \epsilon)(f)$, $\forall\ n \geq n_0$. This proves that a sequence (x_n) is Δ-convergent to Δx_0 with regard to the NN-$(\mathcal{Y})$. □

Theorem 2.7. *Let $x = \{x_n\} \in \omega$ be a sequence. If $\exists$ a sequence $y = \{y_n\} \in I^{(\mathcal{Y})}_{(\Delta)}(f)$ such that $f(\Delta(x_n)) = f(\Delta(y_n))$ for almost all n relative to neutrosophic I, then $x \in I^{(\mathcal{Y})}_{(\Delta)}(f)$.*

Proof. Consider $f(\Delta(x_n)) = f(\Delta(y_n))$ for almost all n relative to neutrosophic I. Then $\{n \in \mathbb{N} : f(\Delta(x_n)) \neq f(\Delta(y_n))\} \in I$. This implies. $\{n \in \mathbb{N} : f(\Delta(x_n)) = f(\Delta(y_n))\} \in \mathcal{F}(I)$. Therefore for $n \in \mathcal{F}(I)\ \forall\ \gamma > 0$,

$$f\Big(\mathcal{U}(\Delta x_n - \Delta y_n, \frac{\gamma}{2})\Big) = 1,\ f\Big(\mathcal{V}(\Delta x_n - \Delta y_n, \frac{\gamma}{2})\Big) = 0 \text{ and } f\Big(\mathcal{W}(\Delta x_n - \Delta y_n, \frac{\gamma}{2})\Big) = 0$$

Since $\{y_n\} \in I^{(\mathcal{Y})}_{(\Delta)}(f)$, let (y_n) is Δ-convergent to α. Then for any $\epsilon \in (0, 1)$ and $\gamma > 0$,

$$A_1 = \Big\{n \in \mathbb{N} : f\Big(\mathcal{U}(\Delta y_n - \alpha, \frac{\gamma}{2})\Big) > 1 - \epsilon \text{ and } f\Big(\mathcal{V}(\Delta y_n - \alpha, \frac{\gamma}{2})\Big) < \epsilon, f\Big(\mathcal{W}(\Delta y_n - \alpha, \frac{\gamma}{2}) < \epsilon\Big) \in \mathcal{F}(I).$$

Consider the set,

$$A_2 = \Big\{n \in \mathbb{N} : f\Big(\mathcal{U}(\Delta x_n - \alpha, \frac{\gamma}{2})\Big) > 1 - \epsilon \text{ and } f\Big(\mathcal{V}(\Delta x_n - \alpha, \frac{\gamma}{2})\Big) < \epsilon, f\Big(\mathcal{W}(\Delta x_n - \alpha, \frac{\gamma}{2}) < \epsilon\Big).$$

We show that $A_1 \subset A_2$. So for $n \in A_1$ we have,

$$\begin{aligned} f\Big(\mathcal{U}(\Delta x_n - \alpha, \gamma)\Big) &\geq f\Big(\mathcal{U}(\Delta x_n - \Delta y_n, \frac{\gamma}{2})\Big) \star f\Big(\mathcal{U}(\Delta y_n - \alpha, \frac{\gamma}{2})\Big) \\ &> 1 \star (1 - \epsilon) = \ 1 - \epsilon \end{aligned}$$

$$\begin{aligned} f\Big(\mathcal{V}(\Delta x_n - \alpha, \gamma)\Big) &\leq f\Big(\mathcal{V}(\Delta x_n - \Delta y_n, \frac{\gamma}{2})\Big) \diamond f\Big(\mathcal{V}(\Delta y_n - \alpha, \frac{\gamma}{2})\Big) \\ &< 0 \diamond \epsilon = \ \epsilon \end{aligned}$$

and

$$\begin{aligned} f\Big(\mathcal{W}(\Delta x_n - \alpha, \gamma)\Big) &\leq f\Big(\mathcal{W}(\Delta x_n - \Delta y_n, \frac{\gamma}{2})\Big) \diamond f\Big(\mathcal{W}(\Delta y_n - \alpha, \frac{\gamma}{2})\Big) \\ &< 0 \diamond \epsilon = \ \epsilon \end{aligned}$$

$\implies n \in A_2$ and hence $A_1 \subset A_2$. Since $A_1 \in \mathcal{F}(I)$, therefore $A_2 \in \mathcal{F}(I)$. Hence $x = \{x_n\} \in I^{(\mathcal{Y})}_{(\Delta)}(f)$. □

3. Conclusions

In the current study, using the concept of difference sequence and modulus function, we extend the intriguing idea of I-convergence to the context of neutrosophic norm spaces via difference sequences by modulus function. Also, we have introduce the new notion of I-convergent difference sequence in neutrosophic normed spaces by modulus function and some fundamental properties are examined.

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