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The 2- refined neutrosophic hyperbolic functions with its differential and integrals

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Abstract: This article's goal is to study the 2-refined neutrosophic hyperbolic functions by defining the 2-refined neutrosophic hyperbolic functions, discussing the 2-refined neutrosophic hyperbolic identities, introducing the rules for derivatives and integrals of the 2-refined neutrosophic hyperbolic functions, and presenting the logarithmic forms of the inverse 2-refined neutrosophic hyperbolic functions.

Keywords: 2-refined neutrosophic; hyperbolic fraction; 2-refined neutrosophic derivatives and 2-refined neutrosophic integrals.

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1. Introduction and Preliminaries

As an alternative to the existing logics, Smarandache proposed the neutrosophic Logic to characterize a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, and contradiction. Refined neutrosophic numbers were made available by Smarandache in the following format: $(a, b_1I_1, b_2I_2, \dots, b_nI_n)$ where $a, b_1, b_2, \dots, b_n \in R \text{ or } C$ [1].

The notion of refined neutrosophic algebraic structures was first proposed by Agboola[2]. Furthermore, paper [3] examined the refined neutrosophic rings I, assuming that they divide into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. It then follows logically that: [3]

$$I_1I_1 = {I_1}^2 = I_1 \qquad (1)$$

$$I_2I_2 = {I_2}^2 = I_2 \qquad (2)$$

$$I_1I_2 = I_2I_1 = I_1 \qquad (3)$$

Furthermore, a large number of papers [4-5-6-7-8-12-13-14-15] present research on revised neutrosophic numbers. A study on the refined ah-isometry and its applications in refined neutrosophic surfaces was given by Mehmet Celik and Ahmed Hatip. Smarandache talked about neutrosophic indefinite integral [11]. Additionally, Alhasan gave multiple calculus presentations in which he covered neutrosophic definite and indefinite integrals. Also, he demonstrated the most significant uses of definite integrals in neutrosophic logic [9–10].

This study addressed a variety of topics: the introduction and preliminary information were offered in the first part, while the 2-refined neutrosophic numbers that contain two parts of indeterminacy (I_1, I_2) were discussed in the major discussion part. Conclusion is provided in the final section.

2. Main Discussion

2.1. Neutrosophic hyperbolic functions

Let $f(x, I_1, I_2) = e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2}$, then we can express of the 2- refined neutrosophic function as the following:

$$f(x, I_1, I_2) = e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2}$$
$$= \frac{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} + e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{2}$$
$$+ \frac{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} - e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{2}$$

The odd function is called the 2- refined neutrosophic hyperbolic sine of x and the even function is called the 2- refined neutrosophic hyperbolic cosine of x.

Hence:

$$sinh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$$
$$= \frac{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} - e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{2}$$

$$cosh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$$
$$= \frac{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} + e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{2}$$

For all $x \in (-\infty, +\infty)$, Where $a_1, b_1, c_1, a_2, b_2, c_2$ are real numbers, while $I_1, I_2 =$ indeterminacy.

After that, we can write the definitions that follow:

Definition 1

The 2- refined neutrosophic function

$$cosh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$$
$$= \frac{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} - e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{2}$$

is called 2- refined neutrosophic hyperbolic cosine.

Definition 2

The 2- refined neutrosophic function

$$cosh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$$

$$= \frac{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} + e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{2}$$

is called 2- refined neutrosophic hyperbolic sine.

Definition 3

The 2- refined neutrosophic function

$tanh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$

$$= \frac{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} - e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} + e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}$$

is called 2- refined neutrosophic hyperbolic tangent.

Definition 4

The 2- refined neutrosophic function

$coth\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$

$$= \frac{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} - e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} + e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}$$

is called 2- refined neutrosophic hyperbolic cotangent.

Definition 5

The 2- refined neutrosophic function

$$sech\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) = \frac{1}{cosh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)}$$

$$= \frac{2}{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} + e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}$$

is called 2- refined neutrosophic hyperbolic secant.

Definition 6

The 2- refined neutrosophic function

$$csch\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) = \frac{1}{sinh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)}$$

$$= \frac{2}{e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2} - e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}}$$

is called 2- refined neutrosophic hyperbolic cosecant.

2.2 2- Refined neutrosophic hyperbolic identities

1) $cosh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) + sinh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$

$$= e^{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2}$$

2) $cosh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) - sinh\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)$

$$= e^{-((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2)}$$

3) $cosh^2\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) - sinh^2\big((a_1 + b_1I_1 + c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) = 1$

4) $1-tanh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)=sech^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$

5) $coth^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)-1=csch^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$

6) $sinh\Big(2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\Big)$

$$=2\,sinh\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\,cosh\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$$

7) $cosh\Big(2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\Big)$

$$=cosh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)+sinh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$$

$$or\quad =2\,cosh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)-1$$

$$or\quad =2\,sinh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)+1$$

8) $cosh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$

$$=\frac{1}{2}(cosh\Big(2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\Big)+1)$$

9) $sinh^2\Big(2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\Big)$

$$=\frac{1}{2}(cosh\Big(2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\Big)-1)$$

Note:

We can easily prove of these identities, for example:

Prove (3)

$cosh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)-sinh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$

$$=\left(\frac{e^{\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)}+e^{-\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)}}{2}\right)^2-\left(\frac{e^{\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)}-e^{-\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)}}{2}\right)^2$$

$$=\frac{e^{2\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)}+2+e^{-2\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)}}{4}-\left(\frac{e^{2\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)}-2+e^{-2\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)}}{4}\right)=\frac{4}{4}=1$$

Prove (4)

$$cosh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)-sinh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)=1$$

divide both sides by: $cosh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$, we get

$$1-\frac{sinh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)}{cosh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)}=\frac{1}{cosh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)}$$

Hence:

$$1-tanh^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)=sech^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$$

2.3 Derivatives of the 2- refined neutrosophic hyperbolic functions

1) $$\frac{d}{dx}\big[sinh\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\big]=(a_1+b_1I_1+c_1I_2)\,cosh\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$$

2) $$\frac{d}{dx}\big[cosh\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\big]=(a_1+b_1I_1+c_1I_2)\,sinh\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$$

3) $$\frac{d}{dx}\big[tanh\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\big]=(a_1+b_1I_1+c_1I_2)\;sech^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$$

4) $$\frac{d}{dx}\big[coth\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\big]=-(a_1+b_1I_1+c_1I_2)\;csch^2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$$

5) $$\frac{d}{dx}\big[sech\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\big]=-(a_1+b_1I_1+c_1I_2)\;sech\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\,tanh\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$$

6) $$\frac{d}{dx}\big[csch\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\big]=-(a_1+b_1I_1+c_1I_2)\;csch\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\,coth\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$$

Example 1

1) $$\frac{d}{dx}\big[sinh\big((5+3I_1-I_2)x+2I_1+7I_2\big)\big]=(5+3I_1-I_2)\,cosh\big((5+3I_1-I_2)x+2I_1+7I_2\big)$$

2) $$\frac{d}{dx}\big[tanh\big((-7+I_1+4I_2)x+4-I_1\big)\big]=(-7+I_1+4I_2)\,sech^2\big((-7+I_1+4I_2)x+4-I_1\big)$$

3) $$\frac{d}{dx}\big[csch\big((4+I_1+I_2)x+2+I_1-7I_2\big)\big]$$

$$=(-4-I_1-I_2)\;csch\big((4+I_1+I_2)x+2+I_1-7I_2\big)\,coth\big((4+I_1+I_2)x+2+I_1-7I_2\big)$$

4) $$\frac{d}{dx}\big[cosh(1+9I_1-11I_2)x^4\big]=(4+36I_1-44I_2)x^3\,sinh(1+9I_1-11I_2)x^4$$

5) $$\frac{d}{dx}\Big[sech\sqrt{(1+41I_2)x-3+2I_1-4I_2}\Big]$$

$$=\frac{1+41I_2}{2\sqrt{(1+41I_2)x-3+2I_1-4I_2}}\,sech\sqrt{(1+41I_2)x-3+2I_1-4I_2}\;tanh\sqrt{(1+41I_2)x-3+2I_1-4I_2}$$

6) $\frac{d}{dx}\left[ln\left(tanh\left((1+I_1+I_2)x+3+I_1-2I_2\right)\right)\right] = \frac{(1+I_1+I_2)\,sech^2\left((1+I_1+I_2)x+3+I_1-2I_2\right)}{tanh\left((1+I_1+I_2)x+3+I_1-2I_2\right)}$

2.4 Derivatives of the inverse 2- refined neutrosophic hyperbolic functions

1) $\frac{d}{dx}\left[sinh^{-1}\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)\right]$

$$=\frac{a_1+b_1I_1+c_1I_2}{\sqrt{1+\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)^2}}$$

2) $\frac{d}{dx}\left[cosh^{-1}\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)\right]$

$$=\frac{a_1+b_1I_1+c_1I_2}{\sqrt{\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)^2-1}}\ ;\ x>\frac{1-a_2-b_2I_1-c_2I_2}{a_1+b_1I_1+c_1I_2}$$

3) $\frac{d}{dx}\left[tanh^{-1}\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)\right]$

$$=\frac{a_1+b_1I_1+c_1I_2}{1-\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)^2}\ ;\ |(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2|<1$$

4) $\frac{d}{dx}\left[coth^{-1}\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)\right]$

$$=\frac{a_1+b_1I_1+c_1I_2}{1-\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)^2}\ ;\ |(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2|>1$$

5) $\frac{d}{dx}\left[sech^{-1}\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)\right]$

$$=\frac{-a_1-b_1I_1-c_1I_2}{\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)\sqrt{1-\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)^2}}$$

$$;\frac{-a_2-b_2I_1-c_2I_2}{a_1+b_1I_1+c_1I_2}<x<\frac{1-a_2-b_2I_1-c_2I_2}{a_1+b_1I_1+c_1I_2}$$

6) $\frac{d}{dx}\left[csch^{-1}\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)\right]$

$$=\frac{-a_1-b_1I_1-c_1I_2}{\left|\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)\right|\sqrt{1+\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)^2}}$$

$$;x\neq\frac{-a_2-b_2I_1-c_2I_2}{a_1+b_1I_1+c_1I_2}$$

Example 2

1) $\frac{d}{dx}\left[sinh^{-1}\big((4+I_1+I_2)x+2+I_1-7I_2\big)\right]=\frac{4+I_1+I_2}{\sqrt{1+\big((4+I_1+I_2)x+2+I_1-7I_2\big)^2}}$

2) $\frac{d}{dx}\left[cosh^{-1}\big((2+2I_1+2I_2)x+3+4I_1+5I_2\big)\right]=\frac{2+2I_1+2I_2}{\sqrt{\big((2+2I_1+2I_2)x+3+4I_1+5I_2\big)^2-1}}$

3) $\frac{d}{dx}\left[tanh^{-1}\left(sin\big((7+8I_1+2I_2)x+1+I_1+I_2\big)\right)\right]$

$$=\frac{(7+8I_1+2I_2)\,cos\big((7+8I_1+2I_2)x+1+I_1+I_2\big)}{1-sin^2\big((7+8I_1+2I_2)x+1+I_1+I_2\big)}$$

$$=\frac{(7+8I_1+2I_2)\,cos\big((7+8I_1+2I_2)x+1+I_1+I_2\big)}{cos^2\big((7+8I_1+2I_2)x+1+I_1+I_2\big)}$$

$$=\frac{7+8I_1+2I_2}{cos\big((7+8I_1+2I_2)x+1+I_1+I_2\big)}$$

$$=(7+8I_1+2I_2)sec\big((7+8I_1+2I_2)x+1+I_1+I_2\big)$$

5) $\frac{d}{dx}\left[sech^{-1}\big((-1+5I_1+3I_2)x+4+I_1+3I_2\big)\right]$

$$=\frac{-1+5I_1+3I_2}{\big((-1+5I_1+3I_2)x+4+I_1+3I_2\big)\sqrt{1-\big((-1+5I_1+3I_2)x+4+I_1+3I_2\big)^2}}$$

6) $\frac{d}{dx}\left[csch^{-1}\big((10+12I_1+7I_2)x+1-3I_1+14I_2\big)\right]$

$$=\frac{10+12I_1+7I_2}{|(10+12I_1+7I_2)x+1-3I_1+14I_2|\sqrt{1+\big((10+12I_1+7I_2)x+1-3I_1+14I_2\big)^2}}$$

3. Integrals of the 2- refined neutrosophic hyperbolic functions

Let $a_1\neq 0$, $a_1\neq -c_1, a_1\neq -b_1-c_1$ and a_2,b_2,c_2 are real numbers, while $I_1,I_2=$ indeterminacy, then:

1) $\int cosh\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\,dx$

$$=\left(\frac{1}{a_1}+\left[\frac{-b_1}{(a_1+c_1)(a_1+b_1+c_1)}\right]I_1-\left[\frac{c_1}{a_1(a_1+c_1)}\right]I_2\right)sinh\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)+C$$

2) $\int sinh\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\,dx$

$$= \left(\frac{1}{a_1} + \left[\frac{-b_1}{(a_1+c_1)(a_1+b_1+c_1)}\right] I_1 - \left[\frac{c_1}{a_1(a_1+c_1)}\right] I_2\right) cosh\big((a_1+b_1I_1+c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) + C$$

3) $\int sech^2\big((a+bI)x + c + dI\big)\,dx$

$$= \left(\frac{1}{a_1} + \left[\frac{-b_1}{(a_1+c_1)(a_1+b_1+c_1)}\right] I_1 - \left[\frac{c_1}{a_1(a_1+c_1)}\right] I_2\right) tanh\big((a+bI)x + c + dI\big) + C$$

4) $\int csch^2\big((a_1+b_1I_1+c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)\,dx$

$$= -\left(\frac{1}{a_1} + \left[\frac{-b_1}{(a_1+c_1)(a_1+b_1+c_1)}\right] I_1 - \left[\frac{c_1}{a_1(a_1+c_1)}\right] I_2\right) coth\big((a_1+b_1I_1+c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) + C$$

5) $\int sech\big((a_1+b_1I_1+c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)\, tanh\big((a_1+b_1I_1+c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)\,dx$

$$= -\left(\frac{1}{a_1} + \left[\frac{-b_1}{(a_1+c_1)(a_1+b_1+c_1)}\right] I_1 - \left[\frac{c_1}{a_1(a_1+c_1)}\right] I_2\right) sech\big((a_1+b_1I_1+c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) + C$$

6) $\int csch\big((a_1+b_1I_1+c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)\, coth\big((a_1+b_1I_1+c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big)\,dx$

$$= -\left(\frac{1}{a_1} + \left[\frac{-b_1}{(a_1+c_1)(a_1+b_1+c_1)}\right] I_1 - \left[\frac{c_1}{a_1(a_1+c_1)}\right] I_2\right) csch\big((a_1+b_1I_1+c_1I_2)x + a_2 + b_2I_1 + c_2I_2\big) + C$$

Note 1

Whereas $C = a_0 + b_0I_1 + c_0I_2$ *and* $a_0,\ b_0,\ c_0$ are real numbers.

Example 3

1) $\int cosh^7\big((2+I_1+2I_2)x + 2 - 1I_1 + 15I_2\big)\, sinh\big((2+I_1+2I_2)x + 2 - 1I_1 + 15I_2\big)\,dx$

Solution:

$u = cosh\big((2+I_1+2I_2)x + 2 - 1I_1 + 15I_2\big)$

$\Rightarrow\ du = (2+I_1+2I_2)\, sinh\big((2+I_1+2I_2)x + 2 - 1I_1 + 15I_2\big)dx$

$$\frac{du}{2+I_1+2I_2} = sinh\big((2+I_1+2I_2)x + 2 - 1I_1 + 15I_2\big)\,dx$$

Then:

$$\int cosh^7\big((2+I_1+2I_2)x + 2 - 1I_1 + 15I_2\big)\, sinh\big((2+I_1+2I_2)x + 2 - 1I_1 + 15I_2\big)\,dx$$

$$=\int u^{7}\frac{du}{2+I_1+2I_2}=\frac{1}{2+I_1+2I_2}\,\frac{u^{8}}{8}+C$$

$$=\left(\frac{1}{2}-\frac{1}{20}I_1-\frac{1}{4}I_2\right)\frac{u^{8}}{8}+C=\left(\frac{1}{2}-\frac{1}{20}I_1-\frac{1}{4}I_2\right)cosh^{8}\big((2+I_1+2I_2)x+2-1I_1+15I_2\big)+C$$

2) $\int coth\big((5+2I_1+I_2)x-7+11I_1+9I_2\big)\,dx=\int\frac{cosh\big((5+2I_1+I_2)x-7+11I_1+9I_2\big)}{sinh\big((5+2I_1+I_2)x-7+11I_1+9I_2\big)}dx$

$$u=sinh\big((5+2I_1+I_2)x-7+11I_1+9I_2\big)$$

$$\Rightarrow\quad du=(5+2I_1+I_2)cosh\big((5+2I_1+I_2)x-7+11I_1+9I_2\big)\,dx$$

$$\frac{du}{5+2I_1+I_2}=cosh\big((5+2I_1+I_2)x-7+11I_1+9I_2\big)\,dx$$

Then:

$$\int\frac{cosh\big((5+2I_1+I_2)x-7+11I_1+9I_2\big)}{sinh\big((5+2I_1+I_2)x-7+11I_1+9I_2\big)}dx=\int\frac{1}{u}\,\frac{du}{5+2I_1+I_2}=\frac{1}{5+2I_1+I_2}\int\frac{1}{u}\,du$$

$$=\left(\frac{1}{5}-\frac{1}{24}I_1-\frac{1}{30}I_2\right)ln|u|+C$$

$$=\left(\frac{1}{5}-\frac{1}{24}I_1-\frac{1}{30}I_2\right)ln\left|sinh\big((5+2I_1+I_2)x-7+11I_1+9I_2\big)\right|+C$$

3) $\int sech\big((4+5I_1+6I_2)x+2+I_1+8I_2\big)\,tanh\big((4+5I_1+6I_2)x+2+I_1+8I_2\big)\,dx$
$$=\left(\frac{-1}{4}+\frac{1}{30}I_1+\frac{3}{20}I_2\right)sech\big((4+5I_1+6I_2)x+2+I_1+8I_2\big)+C$$

3.1 Logarithmic forms of inverse 2- refined neutrosophic hyperbolic functions

1) $sinh^{-1}\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$

$$=ln\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2+\sqrt{\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)^{2}+1}\right)$$

2) $cosh^{-1}\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$

$$=ln\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2+\sqrt{\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)^{2}-1}\right)$$

3) $tanh^{-1}\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$

$$=\frac{1}{2}\;ln\left(\frac{1+(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2}{1-(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2}\right)$$

4) $coth^{-1}\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$

$$=\frac{1}{2}\ ln\left(\frac{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2+1}{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2-1}\right)$$

5) $sech^{-1}\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$

$$=ln\left(\frac{1+\sqrt{1-\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)^2}}{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2}\right)$$

6) $csch^{-1}\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$

$$=ln\left(\frac{1}{(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2}+\frac{\sqrt{1+\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)^2}}{|(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2|}\right)$$

Proof (1)

Let: $y=sinh^{-1}\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$, then:

$$(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2=sinhy$$

But:

$$sinhy=\frac{e^y-e^{-y}}{2}$$

$$(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2=\frac{e^y-e^{-y}}{2}$$

$$e^y-e^{-y}=2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)$$

$$e^y-e^{-y}-2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)=0$$

Multiplying by e^y, we get:

$$e^{2y}-2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)e^y-1=0$$

solving by the quadratic formula, we find:

$$\Delta=4\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)^2+4$$

$$=4\left[\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)^2+1\right]$$

$$e^y=\frac{2\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)\pm\sqrt{4\left[\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)^2+1\right]}}{2}$$

$$e^y=(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\pm 2\sqrt{\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)^2+1}$$

Since $e^y>0$, then:

$$e^y=(a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2+2\sqrt{\big((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\big)^2+1}$$

$$y=ln\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2+2\sqrt{\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)^2+1}\right)$$

Hence:

$$sinh^{-1}\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)$$

$$=ln\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2+2\sqrt{\left((a_1+b_1I_1+c_1I_2)x+a_2+b_2I_1+c_2I_2\right)^2+1}\right)$$

Note 2

Similarly, we can prove the rest of the formula.

3.2 Integrals of the inverse 2- refined neutrosophic hyperbolic functions

1) $$\int \frac{1}{\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}dx$$

$$=sinh^{-1}\left(\frac{x}{a_1+b_1I_1+c_1I_2}\right)+C \quad ,or: ln\left(x+\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}\right)+C$$

2) $$\int \frac{1}{\sqrt{x^2-(a_1+b_1I_1+c_1I_2)^2}}dx$$

$$=cosh^{-1}\left(\frac{x}{a_1+b_1I_1+c_1I_2}\right)+C \quad ,or: ln\left(x+\sqrt{x^2-(a_1+b_1I_1+c_1I_2)^2}\right)+C\ ; x>a_1+b_1I_1+c_1I_2$$

3) $$\int \frac{1}{(a_1+b_1I_1+c_1I_2)^2-x^2}dx$$

$$=\begin{cases}\left(\frac{1}{a_1}+\left[\frac{-b_1}{(a_1+c_1)(a_1+b_1+c_1)}\right]I_1-\left[\frac{c_1}{a_1(a_1+c_1)}\right]I_2\right)tanh^{-1}\left(\frac{x}{a_1+b_1I_1+c_1I_2}\right)+C & ; |x|<a_1+b_1I_1+c_1I_2\\ \left(\frac{1}{a_1}+\left[\frac{-b_1}{(a_1+c_1)(a_1+b_1+c_1)}\right]I_1-\left[\frac{c_1}{a_1(a_1+c_1)}\right]I_2\right)coth^{-1}\left(\frac{x}{a_1+b_1I_1+c_1I_2}\right)+C & ; |x|>a_1+b_1I_1+c_1I_2\end{cases}$$

$$or: \quad \frac{1}{2(a_1+b_1I_1+c_1I_2)}ln\left|\frac{a_1+b_1I_1+c_1I_2+x}{a_1+b_1I_1+c_1I_2-x}\right|+C \ ; |x|\neq a_1+b_1I_1+c_1I_2$$

4) $$\int \frac{1}{x\sqrt{(a_1+b_1I_1+c_1I_2)^2-x^2}}dx=\frac{-1}{a_1+b_1I_1+c_1I_2}sech^{-1}\left|\frac{x}{a_1+b_1I_1+c_1I_2}\right|+C$$

$$or: \frac{-1}{a_1+b_1I_1+c_1I_2}ln\left(\frac{a_1+b_1I_1+c_1I_2+\sqrt{(a_1+b_1I_1+c_1I_2)^2-x^2}}{|x|}\right)+C$$

$$; 0<|x|<a_1+b_1I_1+c_1I_2$$

5) $$\int \frac{1}{x\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}dx=\frac{-1}{a_1+b_1I_1+c_1I_2}csch^{-1}\left|\frac{x}{a_1+b_1I_1+c_1I_2}\right|+C$$

$$or: \quad \frac{-1}{a_1+b_1I_1+c_1I_2} ln\left(\frac{a_1+b_1I_1+c_1I_2+\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}{|x|}\right)+C \ ; x \neq 0$$

Note 3

$$\frac{1}{a_1+b_1I_1+c_1I_2}=\frac{1}{a_1}+\left[\frac{-b_1}{(a_1+c_1)(a_1+b_1+c_1)}\right]I_1-\left[\frac{c_1}{a_1(a_1+c_1)}\right]I_2$$

for all the previous rules.

Proof (1)

$$\int \frac{1}{\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}dx = sinh^{-1}\left(\frac{x}{a_1+b_1I_1+c_1I_2}\right)+C$$

$$or: \quad = ln\left(x+\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}\right)+C$$

Method 1

$$\frac{d}{dx}\left[sinh^{-1}\left(\frac{x}{a_1+b_1I_1+c_1I_2}\right)+C\right]=\frac{\frac{1}{a_1+b_1I_1+c_1I_2}}{\sqrt{1+\left(\frac{x}{a_1+b_1I_1+c_1I_2}\right)^2}}=\frac{\frac{1}{a_1+b_1I_1+c_1I_2}}{\sqrt{1+\frac{x^2}{(a_1+b_1I_1+c_1I_2)^2}}}$$

$$=\frac{\frac{1}{a_1+b_1I_1+c_1I_2}}{\sqrt{\frac{(a_1+b_1I_1+c_1I_2)^2+x^2}{(a_1+b_1I_1+c_1I_2)^2}}}=\frac{\frac{1}{a_1+b_1I_1+c_1I_2}}{\frac{1}{a_1+b_1I_1+c_1I_2}\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}$$

$$=\frac{1}{\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}$$

Method 2

$$\frac{d}{dx}\left[ln\left(x+\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}\right)+C\right]=\frac{1+\frac{2x}{2\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}}{x+\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}$$

$$=\frac{\frac{2\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}+2x}{2\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}}{x+\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}=\frac{\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}+x}{\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}\left(x+\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}\right)}$$

$$=\frac{1}{\sqrt{(a_1+b_1I_1+c_1I_2)^2+x^2}}$$

Method 3

Let:

$$y=sinh^{-1}\left(\frac{x}{a_1+b_1I_1+c_1I_2}\right) \quad \Rightarrow \quad \frac{x}{a_1+b_1I_1+c_1I_2}=sinhy$$

by differentiate implicitly with respect to x , we get:

$$\frac{1}{a_1+b_1I_1+c_1I_2}=coshy\ \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{(a_1 + b_1 I_1 + c_1 I_2)coshy} \qquad (*)$$

Since: $cosh^2 y - sinh^2 = 1 \quad \Rightarrow \quad cosh^2 y = 1 + sinh^2 \quad \Rightarrow \quad coshy = \sqrt{1 + sinh^2}$

By substitution in $(*)$, we get:

$$\frac{dy}{dx} = \frac{1}{(a_1 + b_1 I_1 + c_1 I_2)\sqrt{1 + sinh^2}}$$

But: $\frac{x}{a_1 + b_1 I_1 + c_1 I_2} = sinhy$, then:

$$\frac{dy}{dx} = \frac{1}{(a_1 + b_1 I_1 + c_1 I_2)\sqrt{1 + \left(\frac{x}{a_1 + b_1 I_1 + c_1 I_2}\right)^2}} = \frac{1}{(a_1 + b_1 I_1 + c_1 I_2)\sqrt{\frac{(a_1 + b_1 I_1 + c_1 I_2)^2 + x^2}{(a_1 + b_1 I_1 + c_1 I_2)^2}}}$$

$$= \frac{1}{(a_1 + b_1 I_1 + c_1 I_2)\frac{1}{(a_1 + b_1 I_1 + c_1 I_2)}\sqrt{(a_1 + b_1 I_1 + c_1 I_2)^2 + x^2}}$$

$$= \frac{1}{\sqrt{(a_1 + b_1 I_1 + c_1 I_2)^2 + x^2}}$$

Example 4

1) $$\int \frac{1}{\sqrt{(2 + 3I_1 + 5I_2)^2 + x^2}} dx = sinh^{-1}\left(\frac{x}{2 + 3I_1 + 5I_2}\right) + C$$

$$= sinh^{-1}\left(\frac{1}{2} - \frac{3}{70} I_1 - \frac{5}{14} I_2\right) x + C \ , or: \quad ln\left(x + \sqrt{(2 + 3I_1 + 5I_2)^2 + x^2}\right) + C$$

2) $$\int \frac{1}{\sqrt{x^2 - (3 + I_1 + I_2)^2}} dx = \int \frac{1}{\sqrt{x^2 - (3 + I_1 + I_2)^2}} dx$$

$$= cosh^{-1}\left(\frac{x}{3 + I_1 + I_2}\right) + C$$

$$= cosh^{-1}\left(\frac{1}{3} - \frac{1}{20} I_1 - \frac{1}{12} I_2\right) x + C \quad , or: \ ln\left(x + \sqrt{x^2 - (3 + I_1 + I_2)^2}\right) + C$$

4. Conclusions

The studies we presented in the area of neutrosophic derivatives and integrals are expanded upon in this work. Since they make many mathematical processes in our everyday lives easier, derivatives and integrals are crucial concepts in life. This is what led us to introducing the concept the 2- refined neutrosophic hyperbolic functions with its differential and integrals. In addition to proving most of the rules, we presented in this paper in several ways.

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