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Fermatean Fuzzy α-Homeomorphism in Fermatean Fuzzy Topological Spaces

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Abstract. A relatively new development in fuzzy set theory, Fermatean fuzzy sets (FFSs) were developed for working with higher-level, uncertain data. They provide an even more robust structure for characterizing degree of membership and degree of non-membership, extending on the notions of Pythagorean fuzzy sets (PFSs) and Intuitionistic fuzzy sets (IFSs). There are multiple fields of mathematics and applications that depend extensively on topological generalization of open sets. Topological spaces can be described and analyzed with the aid of homeomorphisms, which provide an empirical way to identify whether two spaces are identical in a topological sense with respect to their essential features. In this study we introduce and investigate the concept of FF α-irresolute, FF α open and clossed mapping, FF α-homeomorphism and, FF α^*-homeomorphism.

Keywords: FF α-open sets; FF α-contiunity: FF α-irresolute; FF α-homeomorphism and, FF α^*-homeomorphism.

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1. Introduction

The way in which imperfection and ambiguity are dealt with in mathematical models and decision-making methods was entirely rewritten by Zadeh L A [15] by adoption of fuzzy sets in 1965, which include AV that lies between 0 and 1, in contrast with classical sets, having binary members that either belong to or do not. This adaptability provides fuzzy sets a stronger basis for dealing with circumstances that exist in real life which are characterized by uncertainty and ambiguities.

In 1983 Atanassov K [4] introduced Intuitionistic fuzzy sets as an extension of traditional fuzzy sets with sum of AV and NAV is less than 1. Pythagorean fuzzy sets introduced by Yager R R [14] extend the flexibility and applicability of fuzzy logic systems, allowing for better modeling of uncertainty in complex scenarios taking sum of squares of AV and NAV is less than 1.

In order to tackle confusion and unpredictability in decision-making processes more effectively, Senapati T [13] introduced FFS in 2020, an extension of Intuitionistic fuzzy sets (IFSs) and

Pythagorean fuzzy sets (PFSs). AVs and $NAVs$ of FFSs reveal their dependence on greater powers with sum of cubes is less than 1. Buyukozkan G [5] have done a sysytametic review on FFSs.

Revathy A [12] applied FF normalised Bonferroni mean operator in MCDM for selection of electric bike and also investigated generalization of FF sets and applied FF PROMETHEE II method for decision-making in [3]. Ibrahim H Z [10] introduced and applied n, m-rung orthopair fuzzy sets in MCDM. Kakati [11] used FF Archimedean Heronian Mean-Based Model for MCDM.

The increase in degree of uncertainty and complexity in many scientific and engineering sectors require FF topology. It provides enhanced and flexible estimation and evaluation methods in topological spaces under ambiguity, that provides beneficial application tools and theoretical foundations for additional mathematical research. Ibrahim H Z [9] introduced the concept of FF topological space and studied FF continuity FF points and study some types of separation axioms. A FFT is further extension in fuzzy topology that makes it possible for even more complicated topological space representations includes the construction of open sets employing FFSs. Farid [6] used FF CODAS approach with topology. α-open sets give a more adaptable design which facilitates the enhancement of vague attributes. Alshami T m [2] has worked on soft α open sets in soft topological spaces. In circumstances where conventional continuity is inappropriately rigid, α-continuity, which is established using α-open sets, can be more flexible than classical continuity. Ajay D [1] coined Pythagorean α-continuity. Granados [8] derived some results in Pythagorean neutrosophic topological spaces. Gonul B N [7] coined Fermatean neutrosophic topological spaces and an application of neutrosophic kano method.

Bijective, continuous mappings with continuous inverses that maintain the topological structure of spaces are known as classical homeomorphisms. However, classical homeomorphisms are not adequate to capture the complexity of fuzzy topological spaces, where uncertainty and fuzziness are inherent. In generalized topology, α-irresolute functions and α-homeomorphisms serve as helpful tools to facilitate the investigation of spaces and functions that struggle to fit into the traditional structure. Their applications vary considerably and include things from the study of dynamic systems and complicated systems to the development of innovative mathematical theories related to approximation and fixed point theorems. An in-depth knowledge of topological characteristics in more extended or unpredictable settings has been rendered possible by these concepts. These concepts are important to the exploration of fuzzy topological spaces, wherein FFSs are utilized to model ambiguity. They tend to be valuable in circumstances in which the exact form of the data is unclear or unreliable, such as decision-making, machine learning, and optimization.

The topological framework of Fermatean fuzzy spaces remains intact under mappings which preserve the fuzzy structure of the spaces by using Fermatean fuzzy α-homeomorphisms, that generalize these classical ideas. This generalization serves as crucial for implementing decisions from classical topology to more complex and uncertain situations. α-homeomorphism is an approach which can be employed in topological studies to examine different characteristics in fuzzy settings and modify classical conclusions. This may result in novel findings in both theoretical and applied mathematics. More generalized forms of continuity such as FF α-irresolute functions, FF α-homeomorphism and other topological characteristics have been introduced and investigated in our proposed study.

The contribution of this article is as follows:

- Preliminaries are given in Section 2.
- FF α-irresolute functions are studied in Section 3.
- In Section 4, FF α-homeomorphism are introduced and their properties are investigated.
- Conclusion and Future work is given in Section 5.

The short form and expansion used in this study are given below.

2. **Preliminaries**

This section conveys some of the essential concepts utilised in this study.

Definition 2.1 ([13]). A set $\mathcal{S} = \{\langle a, \alpha_S(a), \beta_S(a)\rangle : a \in A\}$ in the universe A is called FFs if $0 \leq (\alpha_S(a))^3 + (\beta_S(a))^3 \leq 1$ where $\alpha_S(a) : A \to [0,1]$, $\beta_S(a) : A \to [0,1]$ and $\pi = \sqrt[3]{1 - (\alpha_S(a))^3 - (\beta_S(a))^3}$ are degree of AV, NAV and indeterminacy of a in S and its complement is $\mathcal{S}^c = (\beta_S, \alpha_S)$.

Definition 2.2 ([13]). If $\mathcal{F}_1 = (\mu_{\mathcal{F}_1}, \nu_{\mathcal{F}_1})$ and $\mathcal{F}_2 = (\mu_{\mathcal{F}_2}, \nu_{\mathcal{F}_2})$ are two FFSs then $\mathcal{F}_1 \cup \mathcal{F}_2 = (max\{\mu_{\mathcal{F}_1}, \mu_{\mathcal{F}_2}\}, min\{\nu_{\mathcal{F}_1}, \nu_{\mathcal{F}_2}\})$ and $\mathcal{F}_1 \cap \mathcal{F}_2 = (min\{\mu_{\mathcal{F}_1}, \mu_{\mathcal{F}_2}\}, max\{\nu_{\mathcal{F}_1}, \nu_{\mathcal{F}_2}\})$.

Definition 2.3 ([9]). A FF topological space (FFTS) ia s pair $(\mathcal{S}, \tau)$ if

(1) $1_{\mathcal{S}} \in \tau$

(2) $0_{\mathcal{S}} \in \tau$

(3) for any $\mathcal{F}_1, \mathcal{F}_2 \in \tau$ we have $\mathcal{F}_1 \cup \Gamma_2 \in \tau$

(4) $\mathcal{F}_1 \cap \mathcal{F}_2 \in \tau$ where τ is the family of FFS of non-empty set $\mathcal{S}$.

Acronyms	Expansion
AV	Association value
NAV	Non association value
FS	Fuzzy set
IFS	Intuitionistic fuzzy Set
PFS	Pythagorean fuzzy set
IVFFS	Interval valued Fermatean fuzzy set
FFS	Fermatean fuzzy set
FFTS	Fermatean fuzzy topological space
$FFOS$	Fermatean fuzzy open set
$FF\alpha OS$	Fermatean fuzzy α open set
$FFPOS$	Fermatean fuzzy pre open set
$FFSOS$	Fermatean fuzzy semi open set
$FF\alpha C$	Fermatean fuzzy α continuous function
$FF\alpha I$	Fermatean fuzzy α irresolute function
$FF\alpha O$	Fermatean fuzzy α open mapping
$FF\alpha Cl$	Fermatean fuzzy α closed mapping
$FFPC$	Fermatean fuzzy pre continuous function
$FFSC$	Fermatean fuzzy semi continuous function
$FFS\alpha C$	Fermatean fuzzy strongly α continuous function
FFH	Fermatean fuzzy homeomorphism
$FF\alpha H$	Fermatean fuzzy α homeomorphism
$FF\alpha^*H$	Fermatean fuzzy α^* homeomorphism

The $FFOS$ and $FFCS$ are members of τ and τ^c respectively. FF interior of $\mathcal{F}$ denoted by $int\ \mathcal{F}$ is the union of all FFOSs contained in $\mathcal{F}$ and FF closure of $\mathcal{F}$ denoted by $cl\ \mathcal{F}$ is the intersection of all FFCSs containing $\mathcal{F}$.

Definition 2.4. [[3]] A FFS $\mathcal{F} = (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ of a FFTS $(\mathcal{S}, \tau)$ is

(1) a $FFSOS$ if $\mathcal{F} \subseteq cl\left(int\left(\mathcal{F}\right)\right).$

(2) a $FFPOS$ if $\mathcal{F} \subseteq int\left(cl\left(\mathcal{F}\right)\right).$

(3) a $FF\alpha OS$ if $\mathcal{F} \subseteq int\left(cl\left(int\left(\mathcal{F}\right)\right)\right).$

Their complements are $FFSCS$, $FFPCS$ and $FF\alpha CS$ respectively. The FF α-closure of $\mathcal{F}$, $cl_\alpha\left(\mathcal{F}\right)$ is the intersection of all FF α-closed super sets of $\mathcal{F}$ and the FF α-interior of $\mathcal{F}$, $int_\alpha\left(\mathcal{F}\right)$ is the union of all FF α-open subsets of $\mathcal{F}$.

Definition 2.5. Let (A, τ_A) and (B, τ_B) be FFTS. A function $f : (A, \tau_A) \to (B, \tau_B)$ is a $FFSC$ if the inverse image of each $FFOS$ in (B, τ_B) is a $FFSOS$ in (A, τ_A).

Definition 2.6. Let (A, τ_A) and (B, τ_B) be FFTS. A function $f : (A, \tau_A) \to (B, \tau_B)$ is a $FFPC$ if the inverse image of each $FFOS$ in (B, τ_B) is a $FFPOS$ in $(A, \tau_A$

Definition 2.7. Let (A, τ_A) and (B, τ_B) be FFTS. A function $f : (A, \tau_A) \to (B, \tau_B)$ is a $FF\alpha C$ if the inverse image of each $FFOS$ in (B, τ_B) is a $FF\alpha OS$ in (A, τ_A).

Definition 2.8. A mapping $f : (A, \tau_A) \to (B, \tau_B)$ is called FF open mapping if $f(U)$ is $FFOS$ in (B, τ_B) whenever U is $FFOS$ in (A, τ_A).

Definition 2.9. A mapping $f : (A, \tau_A) \to (B, \tau_B)$ is called FF closed mapping if $f(U)$ is $FFCS$ in (B, τ_B) whenever U is $FFCS$ in (A, τ_A).

Definition 2.10. A FF bijection $f : (A, \tau_A) \to (B, \tau_B)$ is called FFH if f is both FF open and FF closed mapping.

3. Fermatean Fuzzy α-irresolute functions

Definition 3.1. Let (A, τ_A) and (B, τ_B) be FFTS. A function $f : (A, \tau_A) \to (B, \tau_B)$ is a $FF\alpha$-irresolute($FF\alpha I$) if the inverse image of each $FF\alpha OS$ in (B, τ_B) is a $FF\alpha OS$ in (A, τ_A).

Definition 3.2. Let (A, τ_A) and (B, τ_B) be FFTS. A function $f : (A, \tau_A) \to (B, \tau_B)$ is a FF strongly α-continuous ($FFS\alpha C$) function if the inverse image of each $FFSOS$ in (B, τ_B) is a $FF\alpha OS$ in (A, τ_A).

Theorem 3.3. *Every $FF\alpha I$ is a $FFPC$.*

Proof. Let (A, τ_A) and (B, τ_B) be two FFTSs and let $f : (A, \tau_A) \to (B, \tau_B)$ is a $FF\alpha I$. Let us take a $FFOS$, U in (B, τ_B) then U is a $FF\alpha OS$ in (B, τ_B). Since f is $FF\alpha I$, $f^{-1}(U)$ is a $FF\alpha OS$ in (A, τ_A). Since every $FF\alpha OS$ is $FFPOS$, $f^{-1}(U)$ is a $FFPOS$ in (A, τ_A). Hence f is a $FFPC$.

Remark 3.4. The converse of the above theorem need not be true as shown by the following example.

Example 3.5. Let $A = \{a_1, a_2\}$, $\tau_A = \{0_A, 1_A, A_1\}$ where $A_1 = \{(a_1, 0.8, 0.7), (a_2, 0.6, 0.63)\}$ and $B = \{b_1, b_2\}$, $\tau_B = \{0_B, 1_B, B_1\}$ where $B_1 = \{(b_1, 0.45, 0.63), (ab_2, 0.8, 0.7)\}$ be two FFTSs. Define a FF mapping $f : (A, \tau_A) \to (B, \tau_B)$ such that $f(a_1) = b_2$ and $f(a_2) = b_1$. B_1 is a $FFOS$ in (B, τ_B). Since $int\left(cl\left(f^{-1}(B_1)\right)\right) = 1_A$, $f^{-1}(B_1) \subseteq int\left(cl\left(f^{-1}(B_1)\right)\right)$. Hence f is a $FFPC$. But B_1 is a $FF\alpha OS$ in (B, τ_B) and $int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right) = 0_A$ imply $f^{-1}(B_1) \nsubseteq int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right)$. Therefore $f^{-1}(B_1)$ is not a $FF\alpha OS$ in (A, τ_A). Hence f is not $FF\alpha I$.

Theorem 3.6. *Every $FF\alpha I$ is a $FF\alpha C$.*

Proof. Let (A, τ_A) and (B, τ_B) be two FFTSs and let $f : (A, \tau_A) \rightarrow (B, \tau_B)$ is a $FF\alpha I$. Let us take a $FFOS$, U in (B, τ_B) then U is a $FF\alpha OS$ in (B, τ_B). Since f is $FF\alpha I$, $f^{-1}(U)$ is a $FF\alpha OS$ in (A, τ_A). Hence f is a $FF\alpha C$.

Remark 3.7. The converse of the above theorem need not be true as shown by the following example.

Example 3.8. Let $A = \{a_1, a_2, a_3\}$, $\tau_A = \{0_A, 1_A, A_1, A_2, A_1 \cap A_2, A_1 \cup A_2\}$ where $A_1 = \{(a_1, 0.9, 0.6), (a_2, 0.4, 0.7), (a_3, 0.6, 0.5)\}$, $A_2 = \{(a_1, 0.6, 0.9), (a_2, 0.7, 0.4), (a_3, 0.5, 0.6)\}$ and $B = \{b_1, b_2, , b_3\}$, $\tau_B = \{0_B, 1_B, B_1\}$ where $B_1 = \{(b_1, 0.9, 0.6), (b_2, 0.7, 0.4), (b_3, 0.6, 0.5)\}$ be two FFTSs. Define a FF mapping $f : (A, \tau_A) \rightarrow (B, \tau_B)$ such that $f(a_1) = b_1$, $f(a_2) = b_2$ and $f(a_3) = b_3$. B_1 is a $FFOS$ in (B, τ_B). $f^{-1}(B_1) = B_1$ and $int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right) = A_1 \cup A_2$ imply $f^{-1}(B_1) \subseteq int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right)$. Hence f is a $FF\alpha C$. Consider a FFS, $B_2 = \{(b_1, 0.9, 0.5), (b_2, 0.7, 0.3), (b_3, 0.6, 0.5)\}$ in (B, τ_B). Since $B_2 \subseteq int(cl(int(B_2)))$, B_2 is a $FF\alpha OS$ in (B, τ_B). Also $f^{-1}(B_2) = B_2$ and $int\left(cl\left(int\left(f^{-1}(B_2)\right)\right)\right) = A_1 \cup A_2$ imply $f^{-1}(B_2) \nsubseteq int\left(cl\left(int\left(f^{-1}(B_2)\right)\right)\right)$. Therefore $f^{-1}(B_2)$ is not a $FF\alpha OS$ in (A, τ_A). Hence f is not $FF\alpha I$.

Theorem 3.9. *Every $FF\alpha I$ is a $FFSC$.*

Proof. Proof. Let (A, τ_A) and (B, τ_B) be two FFTSs and let $f : (A, \tau_A) \rightarrow (B, \tau_B)$ is a $FF\alpha I$. Let us take a $FFOS$, U in (B, τ_B) then U is a $FF\alpha OS$ in (B, τ_B). Since f is $FF\alpha I$, $f^{-1}(U)$ is a $FF\alpha OS$ in (A, τ_A). Since every $FF\alpha OS$ is $FFSOS$, $f^{-1}(U)$ is a $FFSOS$ in (A, τ_A). Hence f is a $FFSC$.

Remark 3.10. The converse of the above theorem need not be true as shown by the following example.

Example
3.11. Let $A = \{a_1, a_2\}$, $\tau_A = \{0_A, 1_A, A_1\}$ where $A_1 = \{(a_1, 0.6, 0.69), (a_2, 0.23, 0.63)\}$ and $B = \{b_1, b_2\}$, $\tau_B = \{0_B, 1_B, B_1\}$ where $B_1 = \{(b_1, 0.58, 0.63), (b_2, 0.68, 0.67)\}$ be two FFTSs. Define a FF mapping $f : (A, \tau_A) \rightarrow (B, \tau_B)$ such that $f(a_1) = b_2$ and $f(a_2) = b_1$. B_1 is a $FFOS$ in (B, τ_B). Since $cl\left(int\left(f^{-1}(B_1)\right)\right) = 1_A$, $f^{-1}(B_1) \subseteq cl\left(int\left(f^{-1}(B_1)\right)\right)$. Hence f is a $FFSC$. But B_1 is a $FF\alpha OS$ in (B, τ_B) and $int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right) = A_1$ imply $f^{-1}(B_1) \nsubseteq int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right)$. Therefore $f^{-1}(B_1)$ is not a $FF\alpha OS$ in (A, τ_A). Hence f is not $FF\alpha I$.

Theorem 3.12. *Every $FFS\alpha C$ is a $FF\alpha I$.*

Proof. Let (A, τ_A) and (B, τ_B) be two FFTSs and let $f : (A, \tau_A) \rightarrow (B, \tau_B)$ is a $FFS\alpha C$. Consider a $FF\alpha OS$, B_1 in (B, τ_B). Since every $FF\alpha OS$ is $FFSOS$ and f is $FFS\alpha C$, $f^{-1}(B_1)$

is $FF\alpha OS$ in (A, τ_A). Thus the inverse image of $FF\alpha OS$ (B, τ_B) is $FF\alpha OS$ in (A, τ_A). Hence f is $FF\alpha I$.

Remark 3.13. The converse of the above theorem need not be true as shown by the following example.

Example 3.14. Let $A = \{a_1, a_2\}$, $\tau_A = \{0_A, 1_A, A_1\}$ where $A_1 = \{(a_1, 0.63, 0.68), (a_2, 0.73, 0.73)\}$ and $B = \{b_1, b_2\}$, $\tau_B = \{0_B, 1_B, B_1\}$ where $B_1 = \{(b_1, 0.73, 0.73), (b_2, 0.63, 0.68)\}$ be two FFTSs. Define a FF mapping $f : (A, \tau_A) \to (B, \tau_B)$ such that $f(a_1) = b_2$ and $f(a_2) = b_1$. B_1 is a $FFOS$ in (B, τ_B) and so it is $FF\alpha OS$ in (B, τ_B). $f^{-1}(B_1) = \{(a_1, 0.63, 0.68), (a_2, 0.73, 0.73)\}$. Since $int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right) = A_1$, $f^{-1}(B_1) \subseteq int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right)$. Hence $f^{-1}(B_1)$ is a $FF\alpha OS$ in (A, τ_A). Therefore f is a $FF\alpha I$. Consider a FFS, $B_2 = \{(b_1, 0.73, 0.73), (b_2, 0.68, 0.63)\}$ in (B, τ_B). Since $cl(int(B_2)) = B_1^C$, $B_2 \subseteq cl(int(B_2))$. Therefore B_2 is a $FFSOS$ in (B, τ_B). $f^{-1}(B_2) = \{(a_1, 0.68, 0.63), (a_2, 0.73, 0.73)\}$. Since $int\left(cl\left(int\left(f^{-1}(B_2)\right)\right)\right) = A_1$, $f^{-1}(B_2) \nsubseteq int\left(cl\left(int\left(f^{-1}(B_2)\right)\right)\right)$. Therefore $f^{-1}(B_2)$ is a not a $FF\alpha OS$ in (A, τ_A). Hence f is not a $FFS\alpha C$.

Theorem 3.15. *Every $FFS\alpha C$ is a $FF\alpha C$.*

Proof. The proof follows from the theorems 3.12 and 3.6.

Remark 3.16. The converse of the above theorem need not be true as shown by the following example.

Example 3.17. Let $A = \{a_1, a_2\}$, $\tau_A = \{0_A, 1_A, A_1\}$ where $A_1 = \{(a_1, 0.58, 0.69), (a_2, 0.79, 0.79)\}$ and $B = \{b_1, b_2\}$, $\tau_B = \{0_B, 1_B, B_1\}$ where $B_1 = \{(b_1, 0.79, 0.79), (b_2, 0.58, 0.69)\}$ be two FFTSs. Define a FF mapping $f : (A, \tau_A) \to (B, \tau_B)$ such that $f(a_1) = b_2$ and $f(a_2) = b_1$. B_1 is a $FFOS$ in (B, τ_B). $f^{-1}(B_1) = \{(a_1, 0.58, 0.69), (a_2, 0.79, 0.79)\}$. Since $int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right) = A_1$, $f^{-1}(B_1) \subseteq int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right)$. Hence $f^{-1}(B_1)$ is a $FF\alpha OS$ in (A, τ_A). Therefore f is a $FF\alpha C$. Consider a FFS, $B_2 = \{(b_1, 0.79, 0.79), (b_2, 0.69, 0.58)\}$ in (B, τ_B). Since $cl(int(B_2)) = B_1^C$, $B_2 \subseteq cl(int(B_2))$. Therefore B_2 is a $FFSOS$ in (B, τ_B). $f^{-1}(B_2) = \{(a_1, 0.68, 0.63), (a_2, 0.73, 0.73)\}$. Since $int\left(cl\left(int\left(f^{-1}(B_2)\right)\right)\right) = A_1$, $f^{-1}(B_2) \nsubseteq int\left(cl\left(int\left(f^{-1}(B_2)\right)\right)\right)$. Therefore $f^{-1}(B_2)$ is a not a $FF\alpha OS$ in (A, τ_A). Hence f is not a $FFS\alpha C$.

Theorem 3.18. *If $f : (A, \tau_A) \to (B, \tau_B)$ be a mapping from a FFTS A to a FFTS B. Then the following are equivalent*

(1) *f is $FF\alpha I$.*

(2) $f^{-1}(V)$ *is* $FF\alpha CS$ *in* A *for each* $FF\alpha CS$ V *in* B.

(3) $f(cl_{\alpha}U) \subseteq cl_{\alpha}(f(U))$ *for each* FFS U *in* A.

(4) $cl_{\alpha}f^{-1}(V) \subseteq f^{-1}(cl_{\alpha}V)$ *or each* FFS V *in* B.

(5) $f^{-1}(int_{\alpha}V) \subseteq int_{\alpha}\left(f^{-1}V\right)$ *for each* FFS V *in* B.

Proof. (1) $\Longrightarrow$ (2): Consider a $FF\alpha CS$, V in B. Then V^c is $FF\alpha OS$ in B. Since f is $FF\alpha I$, $f^{-1}(V^c) = \left(f^{-1}(V)\right)^c$ is $FF\alpha OS$ in A. Hence $f^{-1}(V)$ is a $FF\alpha CS$ in A. Hence (1) $\Longrightarrow$ (2).
(2) $\Longrightarrow$ (3): Let U be a FFS in A. Then we have $U \subseteq f^{-1}(f(U)) \subseteq f^{-1}(cl_{\alpha}(f(U)))$ $cl_{\alpha}(f(U))$ is $FF\alpha CS$ in B. Then by (2), $f^{-1}(cl_{\alpha}(f(U)))$ is $FF\alpha CS$ in A. $cl_{\alpha}U \subseteq f^{-1}(cl_{\alpha}(f(U)))$ implies $f(cl_{\alpha}(U)) \subseteq f\left(f^{-1}(cl_{\alpha}(f(U)))\right) = cl_{\alpha}(f(U))$. Thus $f(cl_{\alpha}U) \subseteq cl_{\alpha}(f(U))$. Hence (2) $\Longrightarrow$ (3).
(3) $\Longrightarrow$ (4): For a FFs V in B, let $f^{-1} = U$. Then by (3), $f\left(cl_{\alpha}\left(f^{-1}(V)\right)\right) \subseteq cl_{\alpha}\left(f^{-1}(V)\right) \subseteq cl_{\alpha}V$ and $cl_{\alpha}\left(f^{-1}(V)\right) \subseteq f^{-1}\left(f\left(cl_{\alpha}\left(f^{-1}(V)\right)\right)\right) \subseteq cl_{\alpha}V$. Hence $cl_{\alpha}\left(f^{-1}(V)\right) \subseteq f^{-1}(cl_{\alpha}V)$. Hence (3) $\Longrightarrow$ (4).
(4) $\Longrightarrow$ (5): We have $int_{\alpha}(V) = [cl_{\alpha}(V^c)]^c$. Then $f^{-1}(int_{\alpha}(V)) = f^{-1}[cl_{\alpha}(V^c)]^c = [f^{-1}(cl_{\alpha}(V^c))]^c \subseteq [cl_{\alpha}\left(f^{-1}(V^c)\right)]^c = [int_{\alpha}\left(f^{-1}(V)\right)^c]^c \subseteq int_{\alpha}\left(f^{-1}(V)\right)$.
(5) $\Longrightarrow$ (1): Let V be any $FF\alpha OS$ in B. Then $V = int_{\alpha}V$. By (5), $f^{-1}(int_{\alpha}V) = f^{-1}(V) \subseteq int_{\alpha}f^{-1}(V)$. We have $int_{\alpha}f^{-1}(V) \subseteq f^{-1}(V)$. Therefore $int_{\alpha}f^{-1}(V) = f^{-1}(V)$. Thus $f^{-1}(V)$ is a $FF\alpha OS$ in A and so f is $FF\alpha I$. Hence (5) $\Longrightarrow$ (1).

Lemma 3.19. *Let* $f : (A, \tau_A) \to (B, \tau_B)$ *be a mapping and* U_{α} *be a family of* FFS *of* B. *Then*

(1) $f^{-1}\left(\bigcup U_{\alpha}\right) = \bigcup f^{-1}(U)$

(2) $f^{-1}\left(\bigcap U_{\alpha}\right) = \bigcap f^{-1}(U)$

Lemma 3.20. *Let* $f : A_i \to B$ *be a mapping and* U, V *are* $FFSs$ *of* B_1 *and* B_2 *respectively then* $(f_1 \times f_2)^{-1}(U \times V) = f_1^{-1}(U) \times f_2^{-1}(V)$.

Definition 3.21. Let $(A_i, \tau_i)_{i\in\Omega}$ be a family of FFTSs. Then their product is (A, τ_A) where $A = \prod_{i\in\Omega} A_i$ and τ_A is the initial FFT on A generated by the family of FF projection maps $P_i : A \to (A_i, \tau_i)_{i\in\Omega}$.

Lemma 3.22. *Let* $f : (A, \tau_A) \to (B, \tau_B)$ *and* $g : A \to A \times B$ *be two mappings. If* U *and* V *are* $FFSs$ *of* A *and* B *respectively then* $g^{-1}(1_A \times V) = \left(1_A \cap f^{-1}(V)\right)$.

Lemma 3.23. *Let* A *and* B *be FFTSs then* (A, τ_A) *is product related to* (B, τ_B) *if for any* FFS X *in* A, Y *in* B *whenever* $X \nsubseteq U^c, Y \nsubseteq V^c$ *implies* $X \times Y \subseteq U^c \times 1_A \cup 1_B \times V^c$ *then there exists* $A_1 \in \tau_A$, $B_1 \in \tau_B$ *such that* $X \subseteq A_1^c$ *and* $Y \subseteq B_1^c$ *and* $A_1^c \times 1_A \cup 1_B \times B_1^c = A^c \times 1_A \cup 1_B \times B^c$

Lemma 3.24. *Let (A, τ_A) and (B, τ_B) be FFTSs such that (A, τ_A) is product related to (B, τ_B). Then the product $U \times V$ of a $FF\alpha OS$ U in A and a $FF\alpha OS$ V in B is a $FF\alpha OS$ in FF product space $X \times Y$.*

Theorem 3.25. *Let $f : (A, \tau_A) \to (B, \tau_B)$ be a function and assume that (A, τ_A) is product related to (B, τ_B). If the mapping $g : A \to A \times B$ of f is $FF\alpha I$ then so f.*

Proof. Let V be a $FF\alpha OS$ in B. Then by Lemma 3.23, $f^{-1}(V) = 1_A \times f^{-1}(V) = g^{-1}(1_B \times V)$. Now $1_B \times V$ is a $FF\alpha OS$ in $A \times B$. Since g is $FF\alpha I$, $g^{-1}(1_B \times V)$ is $FF\alpha OS$ in A. Thus f is $FF\alpha I$.

Theorem 3.26. *If a function $f : A \to \prod B_i$ is $FF\alpha I$, then $P_i \circ f : A \to B_i$ is $FF\alpha I$, where P_i is the projection of $\prod B_i$ onto B.*

Proof. Let V_i be any $FF\alpha OS$ of B_i. Since P_i is FF continuous and $FFOS$, it is $FF\alpha OS$. Now $P_i : \prod B_i \to B_i$, $P_i^{-1}(V_i)$ is $FF\alpha OS$ of $\prod B_i$. So P_i is $FF\alpha I$. $(P_i \circ f)^{-1}(V_i) = f^{-1}\left(P_i^{-1}(V_i)\right)$, since f is $FF\alpha I$ and $P_i^{-1}(V_i)$ is $FF\alpha OS$, $P_i^{-1}(V_i)$ is $FF\alpha OS$, $f^{-1}(P_i(V_i))$ is $FF\alpha OS$. Thus $P_i \circ f$ is $FF\alpha I$.

Theorem 3.27. *If $f_i : A_i \to B_i$ $i = 1, 2$ are $FF\alpha I$ and A_1 is product related to A_2 then $f_1 \times f_2 : A_1 \times A_2 \to B_1 \times B_2$ is $FF\alpha I$.*

Proof. Let $X = \bigcup(U_i \times V_i)$ where U_i and V_i, $i = 1, 2$ are $FF\alpha OS$ of B_1 and B_2 respectively. Since B_1 is product related to B_2 , by Lemma 3.24, $X = \bigcup(U_i \times V_i)$ is $FF\alpha OS$ of $B_1 \times B_2$. By Lemma 3.19 and Lemma. 3.20 we get $(f_1 \times f_2)^{-1}(X) = (f_1 \times f_2)^{-1}(\bigcup(U_i \times V_i)) = \bigcup\left(f_1^{-1}(U_i) \times f_2^{-1}(V_i)\right)$. Since f_1 and f_2 are $FF\alpha I$, $(f_1 \times f_2)^{-1}(X)$ is a $FF\alpha OS$ in $f_1 \times f_2$ and so $f_1 \times f_2$ is $FF\alpha I$.

Theorem 3.28. *A mapping $f : (A, \tau_A) \to (B, \tau_B)$ is $FF\alpha I$ if and only if for every FF projection, $P_{(\lambda,\mu)}$ in A and $FF\alpha OS$ V in B such that $f\left(P_{(\lambda,\mu)}\right) \in V$, there exists a $FF\alpha OS$ U in A such that $P_{(\lambda,\mu)} \in U$ and $f(U) \subseteq V$.*

Proof. Consider a $FF\alpha I$ f, a FF projection $P_{(\lambda,\mu)}$ in A and a $FF\alpha OS$ V in B such that $f\left(P_{(\lambda,\mu)}\right) \in V$. Then $P_{(\lambda,\mu)} \in f^{-1}(V) = int_\alpha f^{-1}(V)$. Let $U = int_\alpha f^{-1}(V)$. Then U is a $FF\alpha OS$ in A which contains FF projection, $P_{(\lambda,\mu)}$ and $f(U) = int_\alpha f^{-1}(V) \subseteq f\left(f^{-1}(V)\right) = V$. Conversely, let V be a $FF\alpha OS$ in B and $P_{(\lambda,\mu)}$ be a FF projection in A such that $P_{(\lambda,\mu)} \in f^{-1}(V)$. By assumption there exists $FF\alpha OS$ U in A such that $P_{(\lambda,\mu)} \in U$ and $f(U) \subseteq V$. Hence $P_{(\lambda,\mu)} \in U \subseteq f^{-1}(V)$ and $P_{(\lambda,\mu)} \in U = int_\alpha U \subseteq int_\alpha\left(f^{-1}(V)\right)$. Since $P_{(\lambda,\mu)}$ is an arbitrary FF projection $f^{-1}(V)$ is the union of all FF projection containing in $f^{-1}(V)$, thus we get $f^{-1}(V) = int_\alpha\left(f^{-1}(V)\right)$. Hence f is a $FF\alpha I$.

Definition 3.29. Let (A, τ_A) be a FFTS and U be any FFS in A. U is called FF dense set if $cl\,(U) = 1_A$ and no where FF dense set if $int\,(cl\,(U)) = 0_A$.

Theorem 3.30. *If a function $f : (A, \tau_A) \to (B, \tau_B)$ is $FF\alpha I$ then $f^{-1}\,(V)$ is a $FF\alpha CS$ in A for any no where FF dense set V in B.*

Proof. Let V be any no where FF dense set in B. Then $int\,(cl\,(V)) = 0_B$. Now $[int\,(cl\,(V))]^c = 1_B$ implies $cl\,([cl\,(V)]^c) = 1_B$ and so $cl[int\,(V^c)] = 1_B$. Since $int 1_B = 1_B$, $int\,(cl\,(int\,(V^c))) = int 1_B = 1_B$. Hence $V^c \subseteq int\,(cl\,(int\,(V^c))) = 1_B$. Then V^c is $FF\alpha OS$ in B. Since f is $FF\alpha I$, $f^{-1}\,(V^c)$ is $FF\alpha OS$ in A. Hence $f^{-1}\,(V)$ $FF\alpha CS$ in A.

Theorem 3.31. *Consider the two functions $f : (A, \tau_A) \to (B, \tau_B)$ and $g : (B, \tau_B) \to (C, \tau_C)$. Then the following hold.*

(1) *If f is $FF\alpha I$ and g is $FF\alpha I$ then $g \circ f$ is $FF\alpha I$.*

(2) *If f is $FF\alpha I$ and g is $FFS\alpha C$ then $g \circ f$ is $FFS\alpha C$.*

(3) *If f is $FF\alpha I$ and g is $FF\alpha C$ then $g \circ f$ is $FF\alpha I$.*

Proof.

(1) Let V be any $FF\alpha OS$ in (C, τ_C). Since g is $FF\alpha I$, $g^{-1}\,(V)$ is a $FF\alpha OS$ in (B, τ_B). We have $(g \circ f)^{-1}\,(V) = f^{-1}\left(g^{-1}\,(V)\right)$. Since f is $FF\alpha I$, $f^{-1}\left(g^{-1}\,(V)\right)$ is $FF\alpha OS$ in (A, τ_A). Therefore $g \circ f$ is $FF\alpha I$.

(2) Let V be any $FFSOS$ in (C, τ_C). Since g is $FFS\alpha C$, $g^{-1}\,(V)$ is $FF\alpha OS$ in (B, τ_B). Since f is $FF\alpha I$, $f^{-1}\left(g^{-1}\,(V)\right)$ is $FF\alpha OS$ in (A, τ_A). Therefore $g \circ f$ is $FFS\alpha C$.

(3) Let V be any $FFOS$ in (C, τ_C). Since g is $FFS\alpha C$, $g^{-1}\,(V)$ is $FF\alpha OS$ in (B, τ_B). Since f is $FF\alpha I$, $f^{-1}\left(g^{-1}\,(V)\right)$ is $FF\alpha OS$ in (A, τ_A). Therefore $g \circ f$ is $FF\alpha I$.

The Figure 1 illustrate the relation between the various FF mappings.

4. Fermatean fuzzy α-Homeomorphism in FFTS

Definition 4.1. A mapping $f : (A, \tau_A) \to (B, \tau_B)$ is called FF α-open($FF\alpha O$) if $f(U)$ is $FF\alpha OS$ in (B, τ_B) whenever U is $FF\alpha OS$ in (A, τ_A).

Definition 4.2. A mapping $f : (A, \tau_A) \to (B, \tau_B)$ is called FF α-closed($FF\alpha O$) if $f(U)$ is $FF\alpha CS$ in (B, τ_B) whenever U is $FF\alpha CS$ in (A, τ_A).

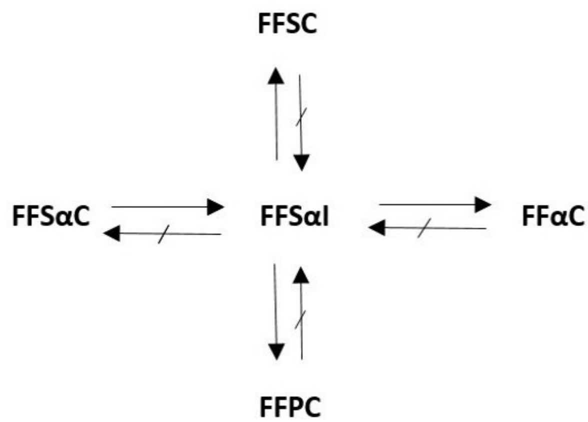

FIGURE 1. Relation between FF mappings

Theorem 4.3. *A mapping $f : (A, \tau_A) \to (B, \tau_B)$ is $FF\alpha O$ if and only if for each $a \in A$ and each $FFOS$ $U \subseteq (A, \tau_A)$ contained a, there exists a $FF\alpha OS$ $W \subseteq (B, \tau_B)$ such that $W \subseteq f(U)$.*

Proof. Obvious.

Theorem 4.4. *A mapping $f : (A, \tau_A) \to (B, \tau_B)$ is $FF\alpha C$ if and only if $cl_\alpha(f(U)) \subseteq f(U)$ for each FFS, $U \subseteq (A, \tau_A)$.*

Proof. Obvious.

Theorem 4.5. *A mapping $f : (A, \tau_A) \to (B, \tau_B)$ be a $FF\alpha O$ and $U \subseteq (A, \tau_A)$ is a $FFCS$ containing $f^{-1}(W)$, then there exists a $FF\alpha CS$ $H \subseteq (B, \tau_B)$ containing W such that $f^{-1}(H) \subseteq U$.*

Proof. Let $H = [f(U^c)]^c$. Since $f^{-1}(W) \subseteq U$ we have $f(U^c) \subseteq W^c$. Since f is $FF\alpha O$, H is $FF\alpha CS$ and $f^{-1}(H) = [f^{-1}(f(U^c))]^c \subseteq (U^c)^c = U$.

Theorem 4.6. *Let $f : (A, \tau_A) \to (B, \tau_B)$ be $FF\alpha c$. If $W \subseteq (B, \tau_B)$ and $U \subseteq (A, \tau_A)$ is a $FFOS$ containing $f^{-1}(W)$, then there exists a $FF\alpha OS$, $H \subseteq (B, \tau_B)$ containing W such that $f^{-1}(H) \subseteq U$.*

Proof. Let $H = [f(U^c)]^c$. Since $f^{-1}(W) \subseteq U$, we have $f(U^c) \subseteq W^c$. Since f is $FF\alpha C$, then H is $FF\alpha OS$ and $f^{-1}(H) = [f^{-1}(f(U^c))]^c \subseteq (U^c)^c = U$.

Corollary 4.7. *Let $f : (A, \tau_A) \to (B, \tau_B)$ be $FF\alpha O$. Then*

(1) $f^{-1}(cl(int(cl(V)))) \subseteq cl\left(f^{-1}(V)\right)$ *for each FFS $V \subseteq B$.*

(2) $f^{-1}(cl(V)) \subseteq cl\left(f^{-1}(V)\right)$ *for each* $FFPOS$ $V \subseteq B$.

Proof.

(1) $cl\left(f^{-1}(V)\right)$ is a $FFCS$ in (A, τ_A) containing $f^{-1}(V)$ for a FFS $V \subseteq B$.By Theorem.4.5, 4.6 there exists a $FF\alpha CS$, $H \subseteq B$, $V \subseteq B$ such that $f^{-1}(H) \subseteq cl\left(f^{-1}(V)\right)$. Thus $f^{-1}(cl(int(cl(V)))) \subseteq f^{-1}(cl(int(cl(H)))) \subseteq f^{-1}(H) \subseteq f^{-1}(V)$.

(2) Similar to (1).

Theorem 4.8. *Let* $f : (A, \tau_A) \to (B, \tau_B)$ *be* $FFPC$ *and* $(FF\alpha O)$ *then the inverse image of each* $FFPOS$ *is* $FFPOS$.

Proof. Let V is a $FFPOS$ in (B, τ_B). So $f^{-1}(V) \subseteq f^{-1}(cl(int(V)) \subseteq int\left(cl\left(f^{-1}(int(cl(V)))\right)\right) \subseteq int\left(cl\left(f^{-1}(cl(V))\right)\right)$. Since f is $FF\alpha O$ by Corollary 4.7 we have $f^{-1}(V) \subseteq int\left(cl\left(f^{-1}(cl(V))\right)\right. \subseteq int\left(cl\left(f^{-1}(cl(V))\right)\right) \subseteq int\left(cl\left(f^{-1}(V)\right)\right) = int\left(cl\left(f^{-1}(V)\right)\right)$. Therefore $f^{-1}(H)$ is $FFPOS$ in (A, τ_A).

Definition 4.9. A FF bijection $f : (A, \tau_A) \to (B, \tau_B)$ is called FF α-homeomorphism $(FF\alpha H)$ if f is both $FF\alpha O$ and $FF\alpha C$.

Example 4.10. Let $A = \{a_1, a_2\}$, $\tau_A = \{0_A, 1_A, A_1\}$ where $A_1 = \{(a_1, 0.6, 0.8), (a_2, 0.9, 0.3)\}$ and $B = \{b_1, b_2\}$, $\tau_B = \{0_B, 1_B, B_1\}$ where $B_1 = \{(b_1, 0.5, 0.85), (b_2, 0.86, 0.45)\}$ be two FFTSs. Define a FF mapping $f : (A, \tau_A) \to (B, \tau_B)$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$. Then f is $FF\alpha H$.

Theorem 4.11. *Every* FFH *is* $FF\alpha H$.

Proof. Let $f : (A, \tau_A) \to (B, \tau_B)$ be FFH. Then f is bijective, FFC and FFO mapping. Let V be a $FFOS$ in (B, τ_B). As f is FFC, $f^{-1}(V)$ is $FFOS$ in (A, τ_A). Since every $FFOS$ is $FF\alpha OS$, $f^{-1}(V)$ is $FF\alpha OS$ in (A, τ_A) which implies f is $FF\alpha C$. Assume U is $FFOS$ in (A, τ_A). Also since f is FFO , $f(U)$ is $FFOS$ in (B, τ_B) which implies f is $FF\alpha O$. Hence f is $FF\alpha H$.

Remark 4.12. Every $FF\alpha H$ need not be FFH as given in the following example.

Example 4.13. Let $A = \{a_1, a_2\}$, $\tau_A = \{0_A, 1_A, A_1\}$ where $A_1 = \{(a_1, 0.6, 0.8), (a_2, 0.9, 0.3)\}$ and $B = \{b_1, b_2\}$, $\tau_B = \{0_B, 1_B, B_1\}$ where $B_1 = \{(b_1, 0.5, 0.85), (b_2, 0.86, 0.45)\}$ be two FFTSs. Define a FF mapping $f : (A, \tau_A) \to (B, \tau_B)$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$. Then f is $FF\alpha H$.

Proposition 4.14. *For a* FF *bijective map* $f : (A, \tau_A) \to (B, \tau_B)$ *the following are equivalent.*

(1) f is $FF\alpha O$.

(2) f is $FF\alpha Cl$.

(3) $f^{-1}:(B,\tau_B)\rightarrow(A,\tau_A)$ is $FF\alpha C$.

Proof. (i) $\implies$ (ii): Let U be $FFCS$ in (A,τ_A). Then U^c is $FFOS$ in (A,τ_A). Since f is $FF\alpha O$, $f\left(U^c\right)$ is $FF\alpha OS$ in (B,τ_B) which implies $[f\left(U\right)]^c$ is $FF\alpha OS$ in (B,τ_B). So $f\left(U\right)$ is $FF\alpha CS$ in (B,τ_B). Hence f is $FF\alpha Cl$.
(ii) $\implies$ (iii): let U be $FFCS$ in (A,τ_A). Since f is $FF\alpha Cl$, $f\left(U\right)$ is $FF\alpha CS$ in (B,τ_B). Since f is FF bijective $f\left(U\right)=\left(f^{-1}\right)^{-1}\left(U\right)$, f^{-1} $FF\alpha Cl$.
(iiI) $\implies$ (i): Let U be $FFOS$ in (A,τ_A) and hence $FF\alpha OS$ in (A,τ_A). $\left(f^{-1}\right)^{-1}\left(U\right)$ is $FF\alpha OS$ in (B,τ_B). Therefore $f\left(U\right)$ is $FF\alpha OS$ in (B,τ_B). Hence f is $FF\alpha O$.

Theorem 4.15. *$f:(A,\tau_A)\rightarrow(B,\tau_B)$ be Ff bijective and $FF\alpha C$ then the following are equivalent*

(1) *f is $FF\alpha O$.*

(2) *f is $FF\alpha H$.*

(3) *f is $FF\alpha Cl$.*

Proof. (i) $\implies$ (ii): Since f is FF bijective, $FF\alpha C$ and $FF\alpha O$, by definition f is $FF\alpha H$.
(ii) $\implies$ (iii): let U be $FFCS$ in (A,τ_A). Then U^c is $FFOS$ in (A,τ_A). By hypothesis, $f\left(U^c\right)=[f\left(U\right)]^c$, is $FF\alpha OS$ in (B,τ_B). Therefore $f\left(V\right)$ is $FF\alpha CS$ in (B,τ_B). Thus f is $FF\alpha Cl$.
(iii) $\implies$ (i): let U be $FFOS$ in (A,τ_A). Then U^c is $FFCS$ in (A,τ_A). By hypothesis, $f\left(U^c\right)=[f\left(U\right)]^c$, is $FFCS$ in (B,τ_B). Therefore $f\left(U\right)$ is $FF\alpha OS$ in (B,τ_B). Thus f is $FF\alpha O$.

Definition 4.16. A FF bijection $f:(A,\tau_A)\rightarrow(B,\tau_B)$ is said to be $FF\alpha^*H$ homeomorphism $(FF\alpha^*H)$ if f is both f^{-1} are $FF\alpha I$.

Theorem 4.17. *Let $f:(A,\tau_A)\rightarrow(B,\tau_B)$ and $g:(B,\tau_B)\rightarrow(C,\tau_C)$ are $FF\alpha^*H$ then $g\circ f$ is a $FF\alpha^*H$.*

Proof. Let U be a $FF\alpha OS$ in (C,τ_C). Since g is $FF\alpha I$, $g^{-1}\left(U\right)$ is $FF\alpha OS$ in (B,τ_B). Since f is $FF\alpha I$, $f^{-1}\left(g^{-1}\left(U\right)\right)=\left(g\circ f\right)^{-1}\left(U\right)$ is $FF\alpha OS$ in (A,τ_A). Therefore $g\circ f$ is $FF\alpha I$. Let V be $FF\alpha OS$ in (A,τ_A). $\left(g\circ f\right)\left(V\right)=g[f\left(V\right)]$ is $FF\alpha OS$ in (C,τ_C). Thus $\left(g\circ f\right)\left(V\right)$ is $FF\alpha OS$ in (C,τ_C). Therefore $\left(g\circ f\right)^{-1}$ is $FF\alpha I$. Also $\left(g\circ f\right)$ is a FF bijection. Hence $\left(g\circ f\right)$ is $FF\alpha^*H$.

Theorem 4.18. *Every $FF\alpha^*H$ is $FF\alpha H$.*

Proof. Let $f:(A,\tau_A)\rightarrow(B,\tau_B)$ be a $FF\alpha^*H$. Then f and f^{-1} are $FF\alpha I$. Let U is $FFOS$ in (B,τ_B). Then U is $FF\alpha OS$ in (B,τ_B). Since f is $FF\alpha I$, $f^{-1}(U)$ is $FF\alpha OS$ in (A,τ_A). Thus U is $FF\alpha OS$ in (B,τ_B) implies $f^{-1}(U)$ is $FF\alpha CS$ in (A,τ_A). Therefore f is $FF\alpha C$. Since f^{-1} is $FF\alpha I$, $\left(f^{-1}\right)^{-1}(U)=f(U)$ is $FF\alpha OS$ in (B,τ_B). Thus U is $FF\alpha OS$ implies $f^{-1}(U)$ is $FF\alpha OS$. Therefore f is $FF\alpha O$. Hence f is $FF\alpha H$.

5. **Conclusion and Future work**

In this study we have introduced generalised FF continuity such as $FF\alpha I$,FF α open and clossed mapping, $FF\alpha H$ and $FF\alpha^*H$. Topological equivalence can be interpreted more generally with the help of α-homeomorphisms, whereby spaces may be considered identical under more kinds of instances. When studying spaces with complicated or unpredictable topologies, such fractals or particular functions of spaces, this is particularly insightful. The FF compactness, FF connectedness and FF separation axioms will be studied in future. These ideas are currently being developed as part of current studies, and there is a demand for additional research into their philosophical foundations and practical significance, especially in sectors where uncertainty in data is an important issue.

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