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The Study of Neutrosophic Algebraic Structures and itsApplication in Medical Science

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Abstract. Neutrosophic exact sequence and its construction is presented as algebraic structure in this article. A mathematical construction known as a neutrosophic exact sequence expands the implications of an exact sequence from classical algebra to the domain of neutrosophic sets. This work introduces a full description of neutrosophic exact sequence of R-modules as a structural preserving tools and discusses the fundamental features. Additionally we examine the role of beta level sets in neutrosophic sets and investigate the neutrosophic split exact sequence.

Keywords: Neutrosophic set, Neutrosophic submodule, Neutrosophic exact sequence, Neutrosophic homomorphism, neutrosophic split exact sequence

1. Introduction

A mathematical paradigm known as the neutrosophic set, put forth by Florentine Smarandache [20,22,24], addresses the ambiguity and indeterminacy of contradicting information. Hierarchical component membership, which generalises the contemporary ideas of fuzzy sets [26] and intuitionistic fuzzy sets [2], can be used to manage the prescribed set. It classifies each element of a set using three distinct types of membership grades: Truth, Indeterminacy, and Falsity. A few experts have examined the algebraic structure associated with uncertainty in pure mathematics. Abstract algebra was among the first few disciplines to use the concept of a neutrosophic set for research. Neutrosophic algebraic structures and their application to advanced neutrosophical models were first introduced by W. B. Vasantha Kandasamy and Florentin Smarandache [11]. Vidan Cetkin [7,14] combined the neutrosophic set theory and algebraic structures for the creation of neutrosophic subgroups and neutrosophic submodules.

Homomorphism is a structural persistence map between two algebraic structures, that serves a purpose akin to or equivalent to continuous functions and inflexible geometry motions. Exact sequences serves as a comprehensive representation of module homomorphisms, including their images and kernels. In the field of mathematical sciences, there is a rapid growth toward the consolidation of the neutrosophic set hypothesis with algebraic structures. Compared to traditional crisp set-based structures, algebraic neutrosophic set-based structures have greater expressive capacity in all dimension. The algebraic structure module [1] also sums up the idea of abelian group which is a module over $\mathbb{Z}$ and the idea of vector space over a field is generalised by the module concept over a ring.. A sequence of morphisms between modules that are exact is one in which the image of one morphism is the kernel of the next. In the area of abstract algebra, this study is an exploratory evaluation using the neutrosophic set and the exact module sequence. Exact sequences are beneficial for unifying module homomorphisms with their images and kernels introduced by Hurewicz [9] and Kelley [12]. A.K. Sahni's [17] study on fuzzy exact sequences over semirings is one of the generalisations in exact sequences. We introduced the notion of exact sequence of R-submodules in neutrosophic set, the most generalized form of the three valued logic in this study. The structure of the proposed study is as follows:Section 1 provides a brief summary of related studies conducted in the past, and Section 2 gives a general overview of the core findings that will be needed to understand the sessions to follow. Section 3 explains the exact sequence of neutrosophic modules along with direct sum of neutrosophic submodules and its algebraic properties followed by neutrosophic split exact sequences. The final session conclusion outlines the significance, extent, and possible follow-up research for the current study.

2. Materials and Methods

(1) Construction of neutrosophic exact sequence
(2) Derivation of the relation connecting neutrosophic exact sequence and direct sum of neutrosophic submodules
(3) Study of neutrosophic split exact sequence of an R-module
(4) Analyze the concept of restriction map, β level sets of neutrosophic submodules with the neutrosophic exact sequence.

3. Preliminaries

We provide some of the fundamental concepts and conclusions in this session, which are necessary for a better comprehension of the sessions that follow.

Definition 3.1. [8, 10, 16, 23] A pair of module homomorphisms $M \xrightarrow{f} N \xrightarrow{g} P$ is said to be exact at N if $Im\ f = ker\ g$. A sequence of module homomorphisms

$$....... \xrightarrow{f_{i-1}} M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} ....$$

is exact provided that $Im\ f_i = ker\ f_{i+1}\ \forall i$.

Definition 3.2. *[18, 19, 21, 25] Any element ω in the universal set Σ has the form $\Omega = \{(\omega, \varphi_\Omega(\omega), \chi_\Omega(\omega), \psi_\Omega(\omega)) : \omega \in \Sigma\}$ where $\varphi_\Omega, \chi_\Omega, \psi_\Omega : \Sigma \to [0,1]$, then Ω is referred as a single valued nuetrosophic set on Σ. The percentage of truth, indeterminacy, and non-membership value are represented by the three components, $\varphi_\Omega, \chi_\Omega$ and ψ_Ω.*

Definition 3.3. *[3, 4, 18] Let Ω and Υ be two neutrosophic sets on Σ. Then Ω is contained in Υ, denoted as $\Omega \subseteq \Upsilon$ if and only if $\Omega(\omega) \leqslant \Upsilon(\omega)\ \forall \omega \in \Sigma$, this means that*

$$\varphi_\Omega(\omega) \leq \varphi_\Upsilon(\omega), \chi_\Omega(\omega) \leq \chi_\Upsilon(\omega), \psi_\Omega(\omega) \geq \psi_\Upsilon(\omega)$$

Remark 3.1. *[21]*

(1) *U^Σ denotes the set of all neutrosophic subsets of Σ or neutrosophic power set of Σ.*

Definition 3.4. *[4, 18] For any neutrosophic subset $\Omega = \{(\omega, \varphi_\Omega(\omega), \chi_\Omega(\omega), \psi_\Omega(\omega)) : \omega \in \Sigma\}$ of Σ, the support Ω^* of the neutrosophic set Ω can be defined as*

$$\Omega^* = \{\omega \in \Sigma, \varphi_\Omega(\omega) > 0, \chi_\Omega(\omega) > 0, \psi_\Omega(\omega) < 1\}.$$

Definition 3.5. *[3, 6, 18] For any $\omega \in \Sigma$, the neutrosophic point $\hat{N}_{\{\omega\}}$ is defined as*

$$\hat{N}_{\{\omega\}}(s) = \{s, \varphi_{\hat{N}_{\{\omega\}}}(s), \chi_{\hat{N}_{\{\omega\}}}(s), \psi_{\hat{N}_{\{\omega\}}}(s) : s \in \Sigma\}$$

where

$$\hat{N}_{\{\omega\}}(s) = \begin{cases} (1,1,0) & \omega = s \\ (0,0,1) & \omega \neq s \end{cases}$$

Remark 3.2. *Let Σ be a non empty set. The neutrosophic point $\hat{N}_{\{0\}}$ in Σ is $\hat{N}_{\{0\}}(\omega) = \{\omega, \varphi_{\hat{N}_{\{0\}}}(\omega), \chi_{\hat{N}_{\{0\}}}(\omega), \psi_{\hat{N}_{\{0\}}}(\omega) : \omega \in \Sigma\}$ where*

$$\hat{N}_{\{0\}}(\omega) = \begin{cases} (1,1,0) & \omega = 0 \\ (0,0,1) & \omega \neq 0 \end{cases}$$

Definition 3.6. *[3, 4, 13, 18] Let Σ and Π be two non empty sets and $g : \Sigma \to \Pi$ be a mapping. Let Ω and Υ be neutrosophic subsets of Σ and Π respectively. Then the image of Ω under the map g is denoted by $g(\Omega)$ and is defined as $g(\Omega) = \{\upsilon, \varphi_{g(\Omega)}(\upsilon), \chi_{g(\Omega)}(\upsilon), \psi_{g(\Omega)}(\upsilon) : \upsilon \in \Pi\}$ where*

$$\varphi_{g(\Omega)}(\upsilon) = \begin{cases} \vee\varphi_\Omega(\omega) : \omega \in g^{-1}(\upsilon)) & if\ g^{-1}(\upsilon) \neq \emptyset \\ 0 & otherwise \end{cases}$$

$$\chi_{g(\Omega)}(\upsilon) = \begin{cases} \vee\chi_\Omega(\omega) : \omega \in g^{-1}(\upsilon)) & if\ g^{-1}(\upsilon) \neq \emptyset \\ 0 & otherwise \end{cases}$$

$$\psi_{g(\Omega)}(\upsilon) = \begin{cases} \wedge\psi_\Omega(\omega) : \omega \in g^{-1}(\upsilon)) & if\ g^{-1}(\upsilon) \neq \emptyset \\ 1 & otherwise \end{cases}$$

Furthermore, the inverse of g, denoted by $g^{-1} : \Pi \to \Omega$ *is defined by* $g^{-1}(\Upsilon) = \{\omega, \varphi_{g^{-1}(\Upsilon)}(\omega), \chi_{g^{-1}(\Upsilon)}(\omega), \psi_{g^{-1}(\Upsilon)}(\omega) : g(\omega) \in \Upsilon\}$ *where* $\varphi_{g^{-1}(\Upsilon)}(\omega) = \varphi_\Upsilon(g(\omega)),\ \chi_{g^{-1}(\Upsilon)}(\omega) = \chi_\Upsilon(g(\omega)),\ \psi_{g^{-1}(\Upsilon)}(\omega) = \psi_\Upsilon(g(\omega))$

Definition 3.7. [7] Let M be an R module. Let $\Omega \in U^M$. Then a neutrosophic subset $\Omega = \{\omega, \varphi_P(\omega), \chi_P(\omega),$
$\psi_\Omega(\omega) : \omega \in M\}$ in M satisfies the following conditions, It is referred to as a neutrosophic submodule of M

(1) $\varphi_\Omega(0) = 1, \chi_\Omega(0) = 1, \psi_\Omega(0) = 0$

(2) $\varphi_\Omega(\omega + \upsilon) \geq \varphi_\Omega(\omega) \wedge \varphi_\Omega(\upsilon)$
$\chi_\Omega(\omega + \upsilon) \geq \chi_\Omega(\omega) \wedge \chi_\Omega(\upsilon)$
$\psi_\Omega(\omega + \upsilon) \leq \psi_\Omega(\omega) \vee \psi_\Omega(\upsilon),\ \forall \omega, \upsilon \in M$

(3) $\varphi_\Omega(r\omega) \geq \varphi_\Omega(\omega),\ \chi_\Omega(r\omega) \geq \chi_\Omega(\omega),\ \psi_\Omega(r\omega) \leq \psi_\Omega(\omega), \forall\ \omega \in M, \forall\ r \in R$.

Remark 3.3. *$U(M)$ is the space of all neutrosophic submodules of R-module M.*

Definition 3.8. *[4, 5, 18] A homomorphism Γ of M into N is called a weak neutrosophic homomorphism of Ω onto Υ if $\Gamma(\Omega) \subseteq \Upsilon$. If Γ is a **weak neutrosophic homomorphism** of Ω onto Υ, then Ω is weakly homomorphic to Υ and we write $\Omega \sim \Upsilon$. A homomorphism Γ of M into N is called a **neutrosophic homomorphism** of Ω onto Υ if $\Gamma(\Omega) = \Upsilon$ and we represent it as $\Omega \approx \Upsilon$.*

Definition 3.9. *[3, 15] Let Ω, Υ and $\Psi \in U(M)$, then Ω is said to be the direct sum of neutrosophic submodules Υ and Ψ, we write $\Omega = \Upsilon \oplus \Psi$, if*

(1) $\Omega = \Upsilon + \Psi$

(2) $\Upsilon \cap \Psi = \hat{N}_{\{0\}}$

Definition 3.10. *[18] Let $\Omega \in U^\Sigma$. If for all $\beta \in [0, 1]$, the β-level sets of of Ω, can be denoted and defines as $\Omega_\beta = \{\omega \in \Sigma : \varphi_\Omega(\omega) \geq \beta, \chi_\Omega(\omega) \geq \beta, \psi_\Omega(\omega) \leq \beta\}$ and the strict β level sets of Ω can be denoted and defined as $\Omega^*_\beta = \{\omega \in \Sigma : \varphi_\Omega(\omega) > \beta, \chi_\Omega(\omega) > \beta, \psi_\Omega(\omega) < \beta\}$.*

14. **Neutrosophic Exact Sequences**

This section discusses the notion of exact sequence in the field of neutrosophic algebra and its algebraic properties.

Definition 4.1. *Let M_i be an arbitrary family of R-modules and $\Omega_i \in U(M_i)$. Suppose that* $\ldots\ldots \xrightarrow{g_{i-1}} M_{i-1} \xrightarrow{g_i} M_i \xrightarrow{g_{i+1}} M_{i+1} \xrightarrow{g_{i+2}} \ldots\ldots$ *is an exact sequence of R-modules. Then the sequence* $\ldots\ldots \xrightarrow{g_{i-1}} \Omega_{i-1} \xrightarrow{g_i} \Omega_i \xrightarrow{g_{i+1}} \Omega_{i+1} \xrightarrow{g_{i+2}} \ldots\ldots$ *of neutrosophic R-modules is said to be neutrosophic exact sequence if* $\forall\ i$

(1) $g_{i+1}(\Omega_i) \subseteq \Omega_{i+1}$

(2) $(g_i(\Omega_{i-1}))^* = ker(g_{i+1})$ *or*

$$\begin{cases} g_i(\Omega_{i-1})(\omega) > 0 & if\ \omega \in ker\ g_{i+1} \\ g_i(\Omega_{i-1})(\omega) = 0 & if\ \omega \notin ker\ g_{i+1} \end{cases}$$

Remark 4.1. *For convenience we have denoted the $\hat{N}_{\{0\}} \in U(M)$ by 0.*

Theorem 4.1. *Let $\Omega, \Upsilon \in U(M)$ be such that $\Omega \oplus \Upsilon$ is the direct sum of neutrosophic R-modules of M so that $\Omega^* \oplus \Upsilon^*$ is a direct sum of R submodules of M. Then the sequence*

$$0 \to \Omega \xrightarrow{I} \Omega \oplus \Upsilon \xrightarrow{\pi} \Upsilon \longrightarrow 0$$

is exact sequence considering $\Omega \in U(\Omega^*)$ and $\Upsilon \in U(\Upsilon^*)$.

Proof. Consider the exact sequence $0 \to \Omega^* \xrightarrow{I} \Omega^* \oplus \Upsilon^* \xrightarrow{\pi} \Upsilon^* \longrightarrow 0$ where I and π are injection and projection mappings respectively.
we have to prove that the sequence $0 \to \Omega \xrightarrow{I} \Omega \oplus \Upsilon \xrightarrow{\pi} \Upsilon \longrightarrow 0$ is an exact sequence. Let $\omega \in \Omega^* + \Upsilon^*$. Then $I(\Omega)(\omega) = \{\omega, \varphi_{I(\Omega)}(\omega), \chi_{I(\Omega)}(\omega), \psi_{I(\Omega)}(\omega)\}$ where

$$\varphi_{I(\Omega)}(\omega) = \begin{cases} \vee(\varphi_\Omega(\eta) : \eta \in \Omega^*, I(\eta) = \omega) & I^{-1}(\omega) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \varphi_\Omega(\omega) & if\ \omega \in \Omega^* \\ 0 & if\ \omega \notin \Omega^* \end{cases}$$

$$\chi_{I(\Omega)}(\omega) = \begin{cases} \vee(\chi_\Omega(\eta) : \eta \in \Omega^*, I(\eta) = \omega) & I^{-1}(\omega) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \chi_\Omega(\omega) & if\ \omega \in \Omega^* \\ 0 & if\ \omega \notin \Omega^* \end{cases}$$

$$\Psi_{I(\Omega)}(\omega) = \begin{cases} \wedge(\Psi_\Omega(\eta) : \eta \in \Omega^*, I(\eta) = \omega) & I^{-1}(y) \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \Psi_{\Omega}(\omega) & if\ \omega \in \Omega^* \\ 1 & if\ \omega \notin \Omega^* \end{cases}$$

Also $\xi = \Omega + \Upsilon$ *and* $\xi(\omega) = \{\omega, \varphi_{\xi}(\omega), \chi_{\xi}(\omega), f_{\xi}(\omega) : \omega \in \Omega^* + \Upsilon^*\}$ where

$$\begin{aligned} \varphi_{\xi}(\omega) &= \vee\{\varphi_{\Omega}(\upsilon) \wedge \varphi_{\Upsilon}(\rho) : \upsilon, \rho \in M, \omega = \upsilon + \rho\} \\ &= \varphi_{\Omega}(\omega) \text{ if } \omega \in \Omega^* \end{aligned}$$

$$\begin{aligned} \chi_{\xi}(\omega) &= \vee\{\chi_{\Omega}(\upsilon) \wedge \chi_{\Upsilon}(\rho) : \upsilon, \rho \in M, \omega = \upsilon + \rho\} \\ &= \chi_{\Omega}(\omega)\ if\ \omega \in \Omega^* \end{aligned}$$

$$\begin{aligned} \Psi_{\xi}(\omega) &= \wedge\{\Psi_{\Omega}(\upsilon) \vee \Psi_{\Upsilon}(\rho) : \upsilon, \rho \in M, \omega = \upsilon + \rho\} \\ &= \Psi_{\Omega}(\omega)\ if\ \omega \in \Omega^* \end{aligned}$$

(Note that $\Omega \oplus \Upsilon$ is a direct sum and $\omega = \upsilon + \rho$ where $\omega \in P^*$, then the only option is $\omega = \omega + 0$ *or* $\omega = \upsilon + \rho;\ \upsilon, \rho \in \Omega^*$. But in the second case $t_{\Upsilon}(\rho) = i_{\Upsilon}(\rho) = f_{\Upsilon}(\rho) = 0.)$ From the above two derivations, $I(\Omega) \subseteq \Omega + \Upsilon$.

Now we consider, $\omega \in \Upsilon^*$,

$\pi(\Omega + \Upsilon)(\omega) = \{\omega, t_{\pi(\Omega+\Upsilon)}(\omega), i_{\pi(\Omega+\Upsilon)}(\omega), f_{\pi(\Omega+\Upsilon)}(\omega))\}$ where

$$\begin{aligned} \varphi_{\pi(\Omega+\Upsilon)}(\omega) &= \vee\{\varphi_{\Omega+\Upsilon}(\eta) : \eta \in \Omega^* + \Upsilon^*,\ \pi(\eta) = \omega\} \\ &= \vee\{\varphi_{\Omega+\Upsilon}(r + \omega) : r \in \Omega^*\} \\ &\quad (since\ \pi : \Omega^* \oplus \Upsilon^* \to \Upsilon^* \\ &\quad is\ the\ projection\ map) \\ &= \vee\{\varphi_{\Omega}(r) \wedge \varphi_{\Upsilon}(\omega) : r \in \Omega^*\} \\ &\quad (Property\ of\ direct\ Sum) \\ &= \varphi_{\Upsilon}(\omega)\ [\varphi_{\Omega}(r) = 1\ with\ r = 0] \end{aligned}$$

Similarly, $\chi_{\pi(\Omega+\Upsilon)}(\omega) = \chi_{\Upsilon}(\omega),\ \Psi_{\pi(\Omega+\Upsilon)}(\omega) = \Psi_{\Upsilon}(\omega)$. So $\pi(\Omega + \Upsilon) = \Upsilon$

Now $I(\Omega)(\omega) = \{\omega, \varphi_{\Omega}(\omega), \chi_{\Omega}(\omega), \Psi_{\Omega}(\omega)) : \omega \in \Omega^*\}$ and 0 otherwise. $\Rightarrow (I(\Omega))^* = ker\ \pi$

$\therefore 0 \to \Omega \xrightarrow{I} \Omega \oplus \Upsilon \xrightarrow{\pi} \Upsilon \longrightarrow 0$ is an exact sequence of neutrosophic R-modules. Hence the proof. □

Definition 4.2. *Let* Ω, Υ *and* $\Omega \oplus \Upsilon \in U(M)$. *Then the exact sequence* $0 \to \Omega \xrightarrow{I} \Omega \oplus \Upsilon \xrightarrow{\pi} \Upsilon \longrightarrow 0$ *of an* R*-modules is called a neutrosophic split exact sequence of* R*-modules.*

Theorem 4.2. *Let* $M \xrightarrow{g} N \xrightarrow{h} S$ *be a sequence of an* R*-modules exact at* N. *Let* $\Omega \in U(M),\ \Upsilon \in U(N)$ *and* $\xi \in U(S)$. *Then the sequence of neutrosophic submodules* $\Omega \xrightarrow{g} \Upsilon \xrightarrow{h}$

ξ is exact at Υ only if $\Omega^* \xrightarrow{g^{`}} \Upsilon^* \xrightarrow{h^{`}} \xi^*$ is exact at Υ^*, where $g^{`}$ and $h^{`}$are restrictions of g and h to Ω^* and Υ^* respectively.

Proof. Suppose the sequence of neutrosophic submodules $\Omega \xrightarrow{g} \Upsilon \xrightarrow{h} \xi$ is exact at Υ. Then by definition $g(\Omega) \subseteq \Upsilon,\ h(\Upsilon) \subseteq \xi\ and\ (g(\Omega))^* = ker\ h$ Now consider the sequence $\Omega^* \xrightarrow{g^{`}} \Upsilon^* \xrightarrow{h^{`}} \xi^*$ and we claim that this sequence is exact at Υ^*. Now consider

$$\begin{aligned}
\omega \in (g(\Omega))^* \quad &\Leftrightarrow \quad \varphi_{g(\Omega)}(\omega) > 0,\ \chi_{g(\Omega)}(\omega) > 0,\ \psi_{g(\Omega)}(\omega) < 1 \\
&\Leftrightarrow \quad \vee\{\varphi_\Omega(\eta) : \eta \in M, g(\eta) = \omega\} > 0, \\
&\qquad \vee\{\chi_\Omega(\eta) : \eta \in M, g(\eta) = \omega\} > 0\ and \\
&\qquad \wedge\{\psi_\Omega(\eta) : \eta \in M, g(\eta) = \omega\} < 1 \\
&\Leftrightarrow \quad \exists\ \eta \in M\ such\ that \\
&\qquad \{\omega = g(\eta) : \varphi_\Omega(\eta) > 0, \chi_\Omega(\eta) > 0,\ \psi_\Omega(\eta) < 1\} \\
&\Leftrightarrow \quad \omega \in g(\Omega^*)
\end{aligned}$$

Thus we get $(g(\Omega))^* = g(\Omega^*)$. Similarly we can prove $(h(\Upsilon))^* = h(\Upsilon^*)$.Therefore $g^{`}(\Omega^*) = g(\Omega^*) = (g(\Omega))^* \subseteq \Upsilon^*$ Now $(g(\Omega))^* = ker\ h \Rightarrow (g^{`}(\Omega))^* = ker\ h^{`}$.
Thus the sequence $\Omega^* \xrightarrow{g^{`}} \Upsilon^* \xrightarrow{h^{`}} \xi^*$ is exact at Υ^*. □

Remark 4.2. *The converse of the above theorem need not be true as we see in the following example.*

Example 4.1. Let M be an R-module, N and L are submodules of M such that $N \oplus L$ is a direct sum. Define $\Omega \in U(N),\ \Upsilon \in U(L)\ and\ \xi \in U(N \oplus L)$ as follows

$$\varphi_\Omega(\omega) = \begin{cases} 1 & \omega = 0 \\ 0.8 & \omega \in N - \{0\} \end{cases};$$

$$\chi_\Omega(\omega) = \begin{cases} 1 & \omega = 0 \\ 0.8 & \omega \in N - \{0\} \end{cases};$$

$$\psi_\Omega(\omega) = \begin{cases} 0 & \omega = 0 \\ 0.1 & \omega \in N - \{0\} \end{cases}$$

$$\varphi_\Upsilon(\omega) = \begin{cases} 1 & \omega = 0 \\ 0.5 & \omega \in L - \{0\} \end{cases};$$

$$\chi_{\Upsilon}(\omega) = \begin{cases} 1 & \omega = 0 \\ 0.5 & \omega \in L - \{0\} \end{cases};$$

$$\psi_{\Upsilon}(\omega) = \begin{cases} 0 & \omega = 0 \\ 0.3 & \omega \in L - \{0\} \end{cases}$$

$$\varphi_{\xi}(\omega) = \begin{cases} 1 & \omega = 0 \\ 0.3 & \omega \in N \oplus L - \{0\} \end{cases};$$

$$\chi_{\xi}(\omega) = \begin{cases} 1 & \omega = 0 \\ 0.3 & \omega \in N \oplus L - \{0\} \end{cases}$$

$$\psi_{\xi}(\omega) = \begin{cases} 0 & \omega = 0 \\ 0.5 & \omega \in N \oplus L - \{0\} \end{cases}$$

Then $\Omega^* = N,\ \chi^* = L \ and\ \xi^* = N \oplus L$. So $N^* \xrightarrow{I} N \oplus L \xrightarrow{\pi} L^*$ is exact at $N \oplus L$
$\Rightarrow\ \Omega^* \xrightarrow{I} \xi^* \xrightarrow{\pi} \chi*$ is exact at ξ^*.
Now, $\varphi_{I(\Omega)}(\omega) = \vee\{\varphi_{\Omega}(z) : z \in N, I(z) = \omega\} = \varphi_{\Omega}(\omega),\ \chi_{I(\Omega)}(\omega) = \vee\{\chi_{\Omega}(z) : z \in N, I(z) = \omega\} = \chi_{\Omega}(\omega)$ and $\psi_{I(\Omega)}(\omega) = \wedge\{\psi_{\Omega}(z) : z \in N, I(z) = \omega\} = \psi_{\Omega}(\omega)$
$\Rightarrow\ i(\Omega) = \Omega\ but\ \Omega \nsubseteq \chi$. Therefore the sequence $\Omega \xrightarrow{I} \xi \xrightarrow{\pi} \chi$ is not exact.

Theorem 4.3. *Let $M \xrightarrow{g} N \xrightarrow{h} S$ be a sequence of an R-modules exact at N and let $\Omega \in U(M),\ \Upsilon \in U(N)$ and $\xi \in U(S)$ be such that $\Omega \xrightarrow{g} \Upsilon \xrightarrow{h} T$ is a sequence of neutrosophic submodules exact at χ. Then $g(\Omega^*_\beta) \subseteq ker\ h\ \forall\ \beta \in [0,1]$.*

Proof. Consider the strict β level subsets $\Omega^*_\beta, \Upsilon^*_\beta, \xi^*_\beta$ of $\Omega, \Upsilon\ and\ \xi$ respectively. Then
$\omega \in g(\Omega^*_\beta) \Rightarrow \exists\ \theta\ such\ that\ \omega = g(\theta)$

$\Rightarrow \varphi\,\Omega(\theta) > \beta,\ \chi_\Omega(\theta) > \beta\ and\ \psi_\Omega(\beta) < \beta$

$\Rightarrow\ \vee\{\varphi_\Omega(\theta) : \omega = g(\theta)\} > \beta$

$\Rightarrow\ \vee\{\chi_\Omega(\theta) : \omega = g(\theta)\} > \beta$

$\Rightarrow\ \wedge\{\psi_\Omega(\theta) : \omega = g(\theta)\} < \beta$

$\Rightarrow\ It\ follows\ that\ g(\Omega)(\omega) > \beta$

$\Rightarrow\ \omega \in (g(\Omega))^*_\beta$

$\Rightarrow\ g(\Omega^*_\beta) \subseteq (g(\Omega))^*_\beta$

Similarly it can be prove that $h(\Upsilon^*_\beta) \subseteq (h(\Upsilon))^*_\beta$.

Now $\omega \in g(\Omega^*_\beta) \subseteq (g(\Omega))^*_\beta \Rightarrow g(\Omega)(\omega) > \beta \Rightarrow g(\Omega)(\omega) > 0$

$\Rightarrow\ \omega \in ker\ h$ [Since by definition 4.1, $g(\Omega)(\omega) > 0\ if\ and\ only\ if\ \omega \in ker\ h$]

Hence $g(\Omega^*_\beta) \subseteq ker\ h\ \forall\ \beta \in [0, 1]$. □

5. Conclusion

This paper reviews a comprehensive range of traditional and more recent findings around how classical algebra and neutrosophic sets interact. Neutrosophic exact sequences constitute an important development in the mathematical modelling of uncertainty, enhancing exact sequence theoretical foundations and real-world applications. Direct sum in neutrosophic domain is used to examine the fundamental properties of the exact sequence of neutrosophic submodules, and examples have been constructed to strengthen the study. Additionally, this work analyzes how kernels and restriction maps influence the exact sequence of neutrosophic submodules. The main objective of this study is to expand, generalize, and offer a new perspective on the theory of exact sequences within neutrosophic algebra. The current study is beneficial for shortest path problems based on neutrosophic weighted automata and neutrosophic algorithms that utilize nuetrosophic algebraic structures. On the basis of this approach, future research has produced category theory and proper exact sequences in the neutrosophic domain.

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