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Triangular fuzzy quadripartitioned neutrosophic set and its properties

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Abstract: The main objective of the paper is to hybridize the triangular fuzzy number and the quadripartitioned neutrosophic set and develop the triangular fuzzy quadripartitioned neutrosophic set. The triangular fuzzy numbers have great potential to express uncertainty systematically. So, the combination of the triangular fuzzy numbers and quadripartitioned neutrosophic sets is an intelligent mathematical tool that will be a helpful mathematical tool for decision-making. We define some operations on the triangular fuzzy quadripartitioned neutrosophic sets such as union, intersection, and complement. We establish some important theorems on operations like union, intersection in between three triangular fuzzy quadripartitioned neutrosophic sets in an elaborate way. We have defined unity and null triangular fuzzy quadripartitioned neutrosophic sets and established some mathematical operations on them. We establish some fundamental properties of the developed triangular fuzzy quadripartitioned neutrosophic sets.

Keywords: Fuzzy set; neutrosophic set; single-valued neutrosophic set; quadripartitioned neutrosophic set; triangular fuzzy neutrosophic set

1. Introduction

Neutrosophic Set (NS) was first introduced by Smarandache [1] by investigating the properties of Fuzzy Set (FS) [2] and Intuitionistic FS (IFS) [3]) by introducing indeterminacy and falsity as independent membership components. Single- Valued NS (SVNS) was introduced by Wang et al. [4] in 2010 by confining the "truth", "indeterminacy" and "falsity" membership degrees. Several detailed analyses of theories and different applications of NSs and their necessary important extensions were studied by several authors [5-12]. Chatterjee et al. [13] defined Quadripartitioned SVNS (QSVNS) with the introduction of "truth", "falsity", "unknown "and "contradiction" as four independent component membership functions by utilizing four-valued logic [14] and multi-valued neutrosophic refined logic [15]. Interval Quadripartitioned NS (IQNS) was developed by Pramanik [16] by exploring the Interval NS (INS) [17] and QSVNS. Mallick and Pramanik [18] developed the theory of Pentapartitioned NS (PNS) by utilizing multi-valued refined neutrosophic logic [15] where the indeterminacy membership component is split into three independent components, namely "contradiction", "ignorance", and "unknown". Pramanik [19] defined and established the basic properties of Interval PNSs (IPNSs). Triangular Fuzzy Number (TFN) [20] is an important mathematical tool that is very useful for decision-making. Biswas et al. [21] combined the TFN and SVNS and developed the Triangular Fuzzy Neutrosophic Set (TFNS). Sinha & Majumdar (22) investigated the problems of Rural healthcare challenges through quadripartitioned single valued neutrosophic Z-numbers. TFN is not explored in the QSVNS environment. Complications appearing in decision making problems mainly because of uncertain nature of decision makers or attributes can be quiet significantly handled by exploring TFN in QSVNS environment and this approach has not been adapted in the previous studies.

Research gap: No study combining the TFN and QSVNS has been reported in the literature.

Motivation: The research gap motivates us to study by combining the concepts of TFN and QSVNS and develop the theory of Triangular Fuzzy Quadripartitioned Neutrosophic Set (TFQNS).

The TFQNS is a new notion in the field of NS. Since the TFQNS is a hybrid structure, it is capable of expressing uncertainty comprehensively. TFQNS has more advantages for dealing with uncertainty as it can utilize the advantages of TFN and QSVNS. The computational techniques based on TFN or QSVNS alone may not always produce the best results but the hybrid structure TFQNS may yield the best result.

We also investigate some fundamental properties of the newly introduced set.

The paper has four sections given as follows: Section 2 is dedicated to presenting some existing preliminary concepts of NSs. Section 3 represents the concept of TFQNS and some important mathematical operations on TFQNS. Section 4 presents a discussion. Section 5 concludes the chapter by indicating some future scope of research.

2. Preliminaries

Definition 2.1. Let a set Ω be fixed. An NS [1] X over Ω is defined as:

$X = \{\omega, (T_X(\omega), I_X(\omega), F_X(\omega)) : \omega \in \Omega\}$ where $T_X(\omega), I_X(\omega), F_X(\omega) : \Omega \to]^-0,1^+[$,

$^-0 \le T_X(\omega) + I_X(\omega) + F_X(\omega) \le 3^+$. T_X, I_X, F_X represents the truth Membership Function (MF), the contradiction MF, the unknown MF, and the falsity MF respectively.

Definition 2.2. Let a set Ω be fixed. An SVNS X over Ω is defined as:

$X = \{\omega, (T_X(\omega), I_X(\omega), F_X(\omega)) : \omega \in \Omega\}$ where $T_X, I_X, F_X : W \to [0,1]$ and $0 \le T_X(\omega) + I_X(\omega) + F_X(\omega) \le 3$.

T_X, I_X, F_X represents the truth MF, the contradiction MF, the unknown MF, and the falsity MF respectively.

Definition 2.3. Assume that Ω is the finite universe of discourse and $\phi[0,1]$ is the set of all TFNs on [0, 1]. A TFNS [21] X in Ω is represented as: $X = \{\omega, (T_X(\omega), I_X(\omega), F_X(\omega)) : \omega \in \Omega\}$
where $T_X(\omega) : \Omega \to \phi[0,1], I_X(\omega) : \Omega \to \phi[0,1], F_X(\omega) : \Omega \to \phi[0,1]$
The TFN $T_X(\omega) = (T_X^1(\omega), T_X^2(\omega), T_X^3(\omega)), I_X(\omega) = (I_X^1(\omega), I_X^2(\omega), I_X^3(\omega)), F_X(\omega) = (F_X^1(\omega), F_X^2(\omega), F_X^3(\omega))$ respectively presents the truth MF, the contradiction MF, the unknown MF, and the falsity MF and for $\forall w \in \Omega$, $0 \le T_X^3(\omega) + I_X^3(\omega) + F_X^3(\omega) \le 3$

Definition 2.4. A QSVNS [13] X in the universe of discourse Ω (a fixed set) is expressed as:
$X = \{\omega, (T_X(\omega), C_X(\omega), U_X(\omega), F_X(\omega)) : \omega \in \Omega\}$
where $T_X(\omega), C_X(\omega), U_X(\omega), F_X(\omega) : \Omega \to [0,1]$
$T_X(\omega), C_X(\omega), U_X(\omega), F_X(\omega)$ expresses the truth MF, the contradiction MF, the unknown MF, and the falsity MF respectively and with, $0 \le T_X(\omega) + C_X(\omega) + U_X(\omega) + F_X(\omega) \le 4$.

3. The Fundamental Theories of TFQNSs

Definition 3.1. TFQNS
Assume that $\bar{\ddot{\chi}}$ is the finite universe of discourse and $\phi[0,1]$ is the set of all TFNs on [0, 1].

We define a TFQNS $fy\hat{\ddot{\rho}}$ over $\bar{\ddot{\chi}}$ and $fy\hat{\ddot{\rho}}$ is presented as:

$$fy\hat{\ddot{\rho}} = \{\omega, (fy\bar{\ddot{\xi}}^{T}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\mu}}^{C}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\delta}}^{U}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\psi}}^{F}_{fy\hat{\ddot{\rho}}}(\omega)) : \omega \in \bar{\ddot{\chi}}\} \quad \ldots(1)$$

where $fy\bar{\ddot{\xi}}^{T}_{fy\hat{\ddot{\rho}}}(\omega) : \bar{\ddot{\chi}} \to \phi[0,1], fy\bar{\ddot{\mu}}^{C}_{fy\hat{\ddot{\rho}}}(\omega) : \bar{\ddot{\chi}} \to \phi[0,1], fy\bar{\ddot{\delta}}^{U}_{fy\hat{\ddot{\rho}}}(\omega) : \bar{\ddot{\chi}} \to \phi[0,1], fy\bar{\ddot{\psi}}^{F}_{fy\hat{\ddot{\rho}}}(\omega) : \bar{\ddot{\chi}} \to \phi[0,1]$,

$$\begin{aligned} fy\hat{\ddot{\rho}} = \{&(\omega, (fy\bar{\ddot{\xi}}^{T\ (l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\xi}}^{T\ (m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\xi}}^{T\ (u)}_{fy\hat{\ddot{\rho}}}(\omega)), (fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}}(\omega)),\\ &(fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}}(\omega)), (fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}}(\omega)) : \omega \in \bar{\ddot{\chi}}\} \quad \ldots(2) \end{aligned}$$

with $0 \le fy\bar{\ddot{\xi}}^{T\ (l)}_{fy\hat{\ddot{\rho}}}(\omega) \le fy\bar{\ddot{\xi}}^{T\ (m)}_{fy\hat{\ddot{\rho}}}(\omega) \le fy\bar{\ddot{\xi}}^{T\ (u)}_{fy\hat{\ddot{\rho}}}(\omega)) \le 1$

where the TFN,

$fy\bar{\ddot{\xi}}^{T}_{fy\hat{\ddot{\rho}}}(\omega) = (fy\bar{\ddot{\xi}}^{T\ (l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\xi}}^{T\ (m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\xi}}^{T\ (u)}_{fy\hat{\ddot{\rho}}}(\omega)), fy\bar{\ddot{\mu}}^{C}_{fy\hat{\ddot{\rho}}}(\omega) = (fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}}(\omega))$,

$fy\bar{\ddot{\delta}}^{U}_{fy\hat{\ddot{\rho}}} = (fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}}(\omega)), fy\bar{\ddot{\psi}}^{F}_{fy\hat{\ddot{\rho}}} = (fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}}(\omega))$ presents respectively the truth MF, the contradiction MF, the unknown MF, and the falsity MF and

$\forall \hat{\theta} \in \bar{\ddot{\chi}}, 0 \le fy\bar{\ddot{\xi}}^{T\ (u)}_{fy\hat{\ddot{\rho}}}(\omega) + fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}}(\omega) + fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}}(\omega) + fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}}(\omega) \le 4.$

In general, $\forall \omega \in \bar{\ddot{\chi}}, 0 \le fy\bar{\ddot{\xi}}^{T\ (\zeta)}_{fy\hat{\ddot{\rho}}}(\omega) + fy\bar{\ddot{\mu}}^{C(\zeta)}_{fy\hat{\ddot{\rho}}}(\omega) + fy\bar{\ddot{\delta}}^{U(\zeta)}_{fy\hat{\ddot{\rho}}}(\omega) + fy\bar{\ddot{\psi}}^{F(\zeta)}_{fy\hat{\ddot{\rho}}}(\omega) \le 4, \text{for}, \forall \zeta = u, l, m.$

Example 3.1 Consider a real-world problem that is associated with the completion time of a project in the neutrosophic environment.
Assume that there are three executive engineers of a construction company who are present in a meeting called by the managing director of the company to discuss a time frame that should be required for the

completion of a new project. The managing director has raised a question before the three engineers, what should be the time frame for an important construction project, that the company will undertake? The objective of the company is to provide the minimum time of completion for the project with assured quality of work. The company aims to complete the project within the shortest possible time. Three engineers constitute the universe of discourse.

Now, consider about first engineer's ($\bar{\delta}_1$) assessment regarding the expected time of project completion. According to him, the time frame of completion as expected by the company is correct. He is optimistic about the time frame where everything goes smoothly. This constitutes the truth membership function. He is quite confident in completing the project within the desired time and as per him, the time of completion is 6-8 years. On a 0-1 scale, the truth membership degree can be presented as (0.6, 0.6, 0.8) which is a TFN rating. He is quite confident that the project will be completed mostly within 6 years. But at the same time, he has some contradictions about whether the project can be completed in between 4-6 years taking into account potential delays and uncertainties that may come into play. On a 0-1 scale, the contradiction membership function may be expressed as a TFN rating. He is completely unaware that the project can be completed within 2 years. This constitutes the unknown membership function. On a 0-1 scale, the unknown membership degree may be expressed as (0.2, 0.2, 0.2) which is a TFN rating. He never relies upon the fact that the project can never be completed within 3 years. This constitutes the falsity membership function. On a 0-1 scale, the falsity membership function may be expressed as (0.3, 0.3, 0.3) which is a TFN rating. So, his overall rating is expressed as a Triangular Fuzzy Quadripartitioned Neutrosophic Number (TFQNN) as: $\langle(0.6, 0.6, 0.8), (0.4, 0.5, 0.6), (0.2, 0.2, 0.2), (0.3, 0.3, 0.3)\rangle_{\bar{\delta}_1}$.

Similarly, the second engineer's ($\bar{\delta}_2$) assessment regarding the time frame of completion is presented by TFQNN as $\langle(0.8, 0.7, 0.6), (0.5, 0.6, 0.7), (0.3, 0.3, 0.3), (0.2, 0.2, 0.2)\rangle_{\bar{\delta}_2}$. The third engineer ($\bar{\delta}_3$)presents his assessment rating regarding the probable time of completion of the project by a TFQNN rating represented as $\langle(0.7, 0.8, 0.9), (0.6, 0.7, 0.8), (0.4, 0.5, 0.6), (0.3, 0.4, 0.5)\rangle_{\bar{\delta}_3}$. All these three TFQNN ratings are elements of TFQNS represented as:

$$\bar{\Theta}_1 = \langle(0.6, 0.6, 0.8), (0.4, 0.5.0.6), (0.2, 0.2, 0.2), (0.3, 0.3, 0.3)\rangle_{\bar{\delta}_1} + \langle(0.8, 0.7, 0.6), (0.5, 0.6, 0.7), (0.3, 0.3, 0.3), (0.2, 0.2, 0.2)\rangle_{\bar{\delta}_2} + \langle(0.7, 0.8, 0.9), (0.6, 0.7, 0.8), (0.4, 0.5, 0.6), (0.3, 0.4, 0.5)\rangle_{\bar{\delta}_3}.$$

Definition 3.2. We introduce the notion $\hat{0}$ and $\hat{1}$ as follows:

$\hat{0} = \{(0,0,0), (0,0,0), (1,1,1), (1,1,1)\}$ and $\hat{1} = \{(1,1,1), (1,1,1), (0,0,0), (0,0,0)\}$

Definition 3.3. Union of any two TFQNSs $fy\hat{\ddot{\rho}}_1, fy\hat{\ddot{\rho}}_2$ is a TFQNS $fy\hat{\ddot{\rho}}_3$, written as $fy\hat{\ddot{\rho}}_3 = fy\hat{\ddot{\rho}}_1 \cup fy\hat{\ddot{\rho}}_2$,where the ruth MF , the contradiction MF , the unknown MF of unknown, the falsity MF are presented as:

$$\begin{aligned}
&fy\bar{\ddot{\xi}}_{fn\hat{\ddot{\rho}}_3}(\omega) = (\max(fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fn\bar{\ddot{\xi}}^{T(m)}_{fn\hat{\ddot{\rho}}_2}(\omega)),\\
&fy\bar{\ddot{\mu}}_{fy\hat{\ddot{\rho}}_3}(\omega) = (\max(fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\mu}}^{C(n)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\mu}}^{C(n)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\\
&fy\bar{\ddot{\delta}}_{fy\hat{\ddot{\rho}}_3}(\omega) = \min(fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\\
&fy\bar{\ddot{\psi}}_{fy\hat{\ddot{\rho}}_3}(\omega) = \min(fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fn\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \forall \omega \in \bar{\ddot{\chi}}.
\end{aligned}$$

$$fy\hat{\ddot{\rho}}_3 = fy\hat{\ddot{\rho}}_1 \cup fy\hat{\ddot{\rho}}_2$$

$$=\{\ \omega, (\ fy\bar{\ddot{\xi}}_{fy\hat{\ddot{\rho}}_3}(\omega)\ , fy\bar{\ddot{\mu}}_{fy\hat{\ddot{\rho}}_3}(\omega)\ , fy\bar{\ddot{\delta}}_{fy\hat{\ddot{\rho}}_3}(\omega)\ ,)\ fy\bar{\ddot{\psi}}_{fy\hat{\ddot{\rho}}_3}(\omega)\ ) : \omega \in \bar{\ddot{\chi}}\ \}$$

$$\begin{aligned}
= \{\omega, [&(\max(fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_2}(\omega))],\\
&[\max(fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_2}(\omega))],\\
&[\min(fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_2}(\omega))],\\
&[\min(fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_2}(\omega))] : \omega \in \bar{\ddot{\chi}}\}
\end{aligned} \quad \ldots(3)$$

Example 3. 1. Consider two TFQNSs

$$\begin{aligned}
\bar{\Theta}_1 &= \langle(0.6, 0.6, 0.8), (0.4, 0.5.0.6), (0.2, 0.2, 0.2), (0.3, 0.3, 0.3)\rangle_{\bar{\delta}_1}\\
&+\langle(0.8, 0.7, 0.6), (0.5, 0.6, 0.7), (0.3, 0.3, 0.3), (0.2, 0.2, 0.2)\rangle_{\bar{\delta}_2}\\
&+\langle(0.7, 0.8, 0.9), (0.6, 0.7, 0.8), (0.4, 0.5, 0.6), (0.3, 0.4, 0.5)\rangle_{\bar{\delta}_3},
\end{aligned}$$

$$\begin{aligned}
\bar{\Theta}_2 &= \langle(0.4, 0.5, 0.6), (0.3, 0.4, 0.5), (0.4, 0.5, 0.6), (0.6, 0.7, 0.8)\rangle_{\bar{\delta}_1} +\\
&\langle(0.3, 0.4, 0.5), (0.4, 0.5, 0.6), (0.4, 0.5, 0.6), (0.7, 0.8, 0.9)\rangle_{\bar{\delta}_2}\\
&+\langle(0.3, 0.4, 0.5), (0.2, 0.3, 0.4), (0.5, 0.6, 0.7), (0.6, 0.7, 0.8)\rangle_{\bar{\delta}_3}
\end{aligned}$$

Then, $\bar{\Theta}_1 \cup \bar{\Theta}_2 = \langle(0.6, 0.6, 0.8), (0.4, 0.5, 0.6), (0.2, 0.2, 0.2), (0.3, 0.3, 0.3)\rangle_{\bar{\delta}_1} +$

$$\begin{aligned}
&\langle(0.8, 0.7, 0.6), (0.5, 0.6, 0.7), (0.3, 0.3, 0.3), (0.2, 0.2, 0.2)\rangle_{\bar{\delta}_2} +\\
&\langle(0.7, 0.8, 0.9), (0.6, 0.7, 0.8), (0.4, 0.5, 0.6), (0.3, 0.4, 0.5)\rangle_{\bar{\delta}_3}
\end{aligned}$$

Definition 3.4. Intersection of any two TFQNSs $fy\hat{\ddot{\rho}}_1$, $fy\hat{\ddot{\rho}}_2$ is represented as $fy\hat{\ddot{\rho}}_4$ and is expressed as

$$fy\hat{\ddot{\rho}}_4 = fy\hat{\ddot{\rho}}_1 \cap fy\hat{\ddot{\rho}}_2 \quad =\{\ \omega, (\ fy\bar{\ddot{\xi}}_{fy\hat{\ddot{\rho}}_4}(\omega)\ , fy\bar{\ddot{\mu}}_{fy\hat{\ddot{\rho}}_4}(\omega)\ , fy\bar{\ddot{\delta}}_{fy\hat{\ddot{\rho}}_4}(\omega)\ , fy\bar{\ddot{\psi}}_{fy\hat{\ddot{\rho}}_4}(\omega)\ ) : \omega \in \bar{\ddot{\chi}}\ \}\ldots(4)$$

such that

$$\begin{aligned}
&fy\bar{\ddot{\xi}}_{fy\hat{\ddot{\rho}}_4}(\omega) = (\min(fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_2}(\omega))\\
&fy\bar{\ddot{\mu}}_{fy\hat{\ddot{\rho}}_4}(\omega) = (\min(fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\\
&fy\bar{\ddot{\delta}}_{fy\hat{\ddot{\rho}}_4}(\omega) = (\max(fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\\
&fy\bar{\ddot{\psi}}_{fy\hat{\ddot{\rho}}_4}(\hat{\theta}) = (\max(fy\bar{\ddot{\psi}}^{F(1)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\psi}}^{F(1)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\psi}}^{F(2)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \forall \omega \in \bar{\ddot{\chi}}.
\end{aligned}$$

Example 3. 2. Consider two TFQNSs as:

$$\begin{aligned}
\bar{\Theta}_1 &= \langle(0.6, 0.6, 0.8), (0.4, 0.5.0.6), (0.2, 0.2, 0.2), (0.3, 0.3, 0.3)\rangle_{\bar{\delta}_1}\\
&+\langle(0.8, 0.7, 0.6), (0.5, 0.6, 0.7), (0.3, 0.3, 0.3), (0.2, 0.2, 0.2)\rangle_{\bar{\delta}_2}\\
&+\langle(0.7, 0.8, 0.9), (0.6, 0.7, 0.8), (0.4, 0.5, 0.6), (0.3, 0.4, 0.5)\rangle_{\bar{\delta}_3}
\end{aligned}$$

$$\begin{aligned}
\bar{\Theta}_2 &= \langle(0.4, 0.5, 0.6), (0.3, 0.4, 0.5), (0.4, 0.5, 0.6), (0.6, 0.7, 0.8)\rangle_{\bar{\delta}_1} +\\
&\langle(0.3, 0.4, 0.5), (0.4, 0.5, 0.6), (0.4, 0.5, 0.6), (0.7, 0.8, 0.9)\rangle_{\bar{\delta}_2}\\
&+\langle(0.3, 0.4, 0.5), (0.2, 0.3, 0.4), (0.5, 0.6, 0.7), (0.6, 0.7, 0.8)\rangle_{\bar{\delta}_3}
\end{aligned}$$

Then, $\bar{\Theta}_1 \cap \bar{\Theta}_2 = \langle(0.4,0.5,0.6),(0.3,0.4,0.5),(0.4,0.5,0.6),(0.6,0.7,0.8)\rangle_{\bar{\delta}_1} + \langle(0.3,0.4,0.5),(0.4,0.5,0.6),(0.4,0.5,0.6),(0.7,0.8,0.9)\rangle_{\bar{\delta}_2} + \langle(0.3,0.4,0.5),(0.2,0.3,0.4),(0.5,0.6,0.7),(0.6,0.7,0.8)\rangle_{\bar{\delta}_3}$

Definition 3.5. Complement of a TFQNS

The complement of a TFQNS $fy\hat{\ddot{\rho}}$ is expressed as **(** $fy\hat{\ddot{\rho}}$ **)**^{c} and is represented as:

$$(fy\hat{\ddot{\rho}})^c = \{\omega,(fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}}(\omega)),(\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}}(\omega),\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}}(\omega),\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}}(\omega)), ,(fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}}(\omega)),(\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}}(\omega),\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}}(\omega),\bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}}(\omega)) : \omega \in \bar{\ddot{\chi}}\} \quad \ldots(5)$$

Example 3.3 Assume a TFQNS of the form:

$\bar{\Theta}_1 = \langle(0.6,0.6,0.8),(0.4,0.5.0.6),(0.2,0.2,0.2),(0.3,0.3,0.3)\rangle_{\bar{\delta}_1} + \langle(0.8,0.7,0.6),(0.5,0.6,0.7),(0.3,0.3,0.3),(0.2,0.2,0.2)\rangle_{\bar{\delta}_2} + \langle(0.7,0.8,0.9),(0.6,0.7,0.8),(0.4,0.5,0.6),(0.3,0.4,0.5)\rangle_{\bar{\delta}_3}$,

Accordingly, ($\bar{\Theta}_1$)^c

$= \langle(0.3,0.3,0.3),(0.2,0.2,0.2),(0.4,0.5,0.6),(0.6,0.6,0.8)\rangle|_{\bar{\delta}_1} + \langle(0.2,0.2,0.2),(0.3,0.3,0.3),(0.5,0.6,0.7),(0.8,0.7,0.6)\rangle|_{\bar{\delta}_2} + \langle(0.3,0.4,0.5),(0.4,0.5,0.6),(0.6,0.7,0.8),(0.7,0.8,0.9)\rangle|_{\bar{\delta}_3}$

Definition 3.6. Containment

A TFQNS $fy\hat{\ddot{\rho}}_1$ can be defined to be contained in another TFQNS $fy\hat{\ddot{\rho}}_2$ and is denoted by

$fy\hat{\ddot{\rho}}_1 \subseteq fy\hat{\ddot{\rho}}_2$ if and only if,

$$\begin{aligned}
&fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega) \le fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega) \le fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega) \le fy\bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_2}(\omega);\\
&fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega) \le fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega) \le fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega) \le fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_2}(\omega),\\
&fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega) \ge fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega) \ge fy\bar{\ddot{\delta}}^{U(2)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega) \ge fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_2}(\omega),\\
&fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega) \ge fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega) \ge fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\omega) \ge fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_2}(\omega), \forall \omega \in \bar{\ddot{\chi}}
\end{aligned}$$

Theorem 3.1 Assume that $fy\hat{\ddot{\rho}}$ is a TFQNS. Then

a) $fy\hat{\ddot{\rho}} \cup fy\hat{\ddot{\rho}} = fy\hat{\ddot{\rho}}$ b) $fy\hat{\ddot{\rho}} \cap fy\hat{\ddot{\rho}} = fy\hat{\ddot{\rho}}$

Proof : Now $fy\hat{\ddot{\rho}} \cup fy\hat{\ddot{\rho}}$

$$\begin{aligned}
&= \{\omega,[(\max(fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}}(\omega)), \max(fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}}(\omega), \bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}}(\omega)), \max(fy\bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}}(\omega))],\\
&[\max(fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}}(\omega)), \max(fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}}(\omega)), \max(fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}}(\omega))],\\
&,[\min(fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}}(\omega)), \min(fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}}(\omega)), \min(fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}}(\omega))],\\
&[\min(fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}}(\hat{\theta})), \min(fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}}(\omega)), \min(fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}}(\omega))] : \omega \in \bar{\ddot{\chi}}\}
\end{aligned}$$

$$\begin{aligned}
&= \{(\omega,(fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}}(\omega)),(fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}}(\omega)),\\
&(fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}}(\omega)),(fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}}(\omega)) : \omega \in \bar{\ddot{\chi}}\}\\
&= fy\hat{\ddot{\rho}}
\end{aligned}$$

b) $fy\hat{\ddot{\rho}} \cap fy\hat{\ddot{\rho}}$

$$= \{\omega, [(\min(fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}}(\omega)), \min(fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}}(\omega)), \min(fy\bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}}(\omega), \bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}}(\omega))],$$
$$[\min(fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}}(\omega)), \min(fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}}(\omega)), \min(fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}}(\omega))],$$
$$,[\max(fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}}(\omega)), \max(fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}}(\omega)), \max(fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}}(\omega))],$$
$$[\max(fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}}(\omega)), \max(fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}}(\omega)), \max(fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}}(\omega))] : \omega \in \bar{\ddot{\chi}}\}$$

$$= \{(\omega, (fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}}(\omega)), (fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}}(\omega)),$$
$$(fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}}(\omega)), (fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}}(\omega), fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}}(\omega)) : \omega \in \bar{\ddot{\chi}}\}$$
$$= fy\hat{\ddot{\rho}}$$

Theorem 3.2. For any two TFQNSs $fy\hat{\ddot{\rho}}_1$ and $fy\hat{\ddot{\rho}}_2$,

Law of commutation

a) $fy\hat{\ddot{\rho}}_1 \cup fy\hat{\ddot{\rho}}_2 = fy\hat{\ddot{\rho}}_2 \cup fy\hat{\ddot{\rho}}_1$

b) $fy\hat{\ddot{\rho}}_1 \cap fy\hat{\ddot{\rho}}_2 = fy\hat{\ddot{\rho}}_2 \cap fy\hat{\ddot{\rho}}_1$

a) Proof: $fy\hat{\ddot{\rho}}_1 \cup fy\hat{\ddot{\rho}}_2$

$$= \{\omega, [(\max(fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_2}(\omega))],$$
$$[\max(fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_2}(\omega))],$$
$$[\min(fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_2}(\omega))],$$
$$[\min(fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\hat{\theta}), fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_2}(\omega))] : \omega \in \bar{\ddot{\chi}}\}$$

$$= \{\omega, [(\max(fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega)), \max(fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega)), \max(fy\bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega))],$$
$$[\max(fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega)), \max(fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega)), \max(fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega))],$$
$$[\min(fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega)), \min(fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\delta}}^{U(2)}_{fy\hat{\ddot{\rho}}_1}(\omega)),$$
$$\min(fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega))], [\min(fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega)),$$
$$\min(fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega)), \min(fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega)) : \omega \in \bar{\ddot{\chi}}\}$$

$$= fy\hat{\ddot{\rho}}_2 \cup fy\hat{\ddot{\rho}}_1$$

Proof b. $fy\hat{\ddot{\rho}}_1 \cap fy\hat{\ddot{\rho}}_2 = \{\omega, [(\min(fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_2}(\omega))],$

$$[\min(fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_2}(\omega))],$$
$$[\max(fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_2}(\omega))],$$
$$[\max(\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), \bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_2}(\omega))] : \omega \in \bar{\ddot{\chi}}\}$$

$$= \{\omega, [(\min(fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega)), \min(fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega)), \min(fy\bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega))],$$

$$[\min(fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega)), \min(fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega)), \min(fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega))],$$
$$[\max(fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega)), \max(fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega)), \max(fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega))],$$
$$[\max(fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega)), \max(fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega)), \max(fy\bar{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\bar{\ddot{\psi}}^{F(3)}_{fy\hat{\ddot{\rho}}_1}(\omega))] : \omega \in \bar{\ddot{\chi}}\}$$

$$= fy\hat{\ddot{\rho}}_2 \cap fy\hat{\ddot{\rho}}_1$$

Theorem 3.3. For any three TFQNSs $fy\hat{\ddot{\rho}}_1, fy\hat{\ddot{\rho}}_2, fy\hat{\ddot{\rho}}_3$, $fy\hat{\ddot{\rho}}_1 \cup (fy\hat{\ddot{\rho}}_2 \cup fy\hat{\ddot{\rho}}_3) = (fy\hat{\ddot{\rho}}_1 \cup fy\hat{\ddot{\rho}}_2) \cup fy\hat{\ddot{\rho}}_3$

Proof: $fy\hat{\ddot{\rho}}_1 \cup (fy\hat{\ddot{\rho}}_2 \cup fy\hat{\ddot{\rho}}_3)$

$$= \begin{array}{l}\{\omega, (fy\overline{\overline{\ddot{\xi}}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)), (fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)),\\ (fy\overline{\overline{\ddot{\delta}}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega)), (fy\overline{\overline{\ddot{\psi}}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\psi}}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\psi}}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega)) : \omega \in \overline{\overline{\ddot{\chi}}}\}\end{array}$$

$$\begin{array}{l}\{\omega, [\max(fy\overline{\overline{\ddot{\xi}}}^{T(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(l)}_{fy\hat{\ddot{\rho}}_3}(\omega), \max(fy\overline{\overline{\ddot{\xi}}}^{T(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(m)}_{fy\hat{\ddot{\rho}}_3}(\omega), \max(fy\overline{\overline{\ddot{\xi}}}^{T(u)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(u)}_{fy\hat{\ddot{\rho}}_3}(\omega)],\\ \cup\ [\max(fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_3}(\omega), \max(fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_3}(\omega), \max(fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_3}(\omega)],\\ [\min(fy\overline{\overline{\ddot{\delta}}}^{U(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(l)}_{fy\hat{\ddot{\rho}}_3}(\omega), \min(fy\overline{\overline{\ddot{\delta}}}^{U(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(m)}_{fy\hat{\ddot{\rho}}_3}(\omega), \min(fy\overline{\overline{\ddot{\delta}}}^{U(u)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(u)}_{fy\hat{\ddot{\rho}}_3}(\omega)] : \omega \in \overline{\overline{\ddot{\chi}}}\}\end{array}$$

$$\begin{array}{l}= \{\omega, [\max(\ fy\overline{\overline{\ddot{\xi}}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(l)}_{fy\hat{\ddot{\rho}}_3}(\omega)), \max(\ fy\overline{\overline{\ddot{\xi}}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(m)}_{fy\hat{\ddot{\rho}}_3}(\omega)),\\ \max(\ fy\overline{\overline{\ddot{\xi}}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(u)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(u)}_{fy\hat{\ddot{\rho}}_3}(\omega))],\\ [\max(fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_3}(\omega)), \max(fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), \overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_3}(\omega)),\\ \max(fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_3}(\omega))],\\ [\min(fy\overline{\overline{\ddot{\delta}}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(l)}_{fy\hat{\ddot{\rho}}_3}(\omega)), \min(fy\overline{\overline{\ddot{\delta}}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(m)}_{fy\hat{\ddot{\rho}}_3}(\omega)),\\ \min(fy\overline{\overline{\ddot{\delta}}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(u)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(u)}_{fy\hat{\ddot{\rho}}_3}(\omega))],\\ [\min(fy\overline{\overline{\ddot{\psi}}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\psi}}}^{F(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), \overline{\overline{\ddot{\psi}}}^{F(l)}_{fy\hat{\ddot{\rho}}_3}(\omega)), \min(fy\overline{\overline{\ddot{\psi}}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\psi}}}^{F(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\psi}}}^{F(m)}_{fy\hat{\ddot{\rho}}_3}(\omega)),\\ \min(fy\overline{\overline{\ddot{\psi}}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\psi}}}^{F(u)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\psi}}}^{F(u)}_{fy\hat{\ddot{\rho}}_3}(\omega))] : \omega \in \overline{\overline{\ddot{\chi}}}\}\end{array}$$

$$\begin{array}{l}= \{\omega, [\max(fy\overline{\overline{\ddot{\xi}}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\overline{\overline{\ddot{\xi}}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\overline{\overline{\ddot{\xi}}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(u)}_{fy\hat{\ddot{\rho}}_2}(\omega))],\\ [\max(fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\\ \max(fy\overline{\ddot{\mu}}^{C(m)}_{\hat{\ddot{\rho}}_1}(\omega), fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \max(fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_2}(\omega))], [\min(fy\overline{\overline{\ddot{\delta}}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\\ \min(fy\overline{\overline{\ddot{\delta}}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(2)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\\ \min(fy\overline{\overline{\ddot{\delta}}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(u)}_{fy\hat{\ddot{\rho}}_2}(\omega))], [\min(fy\overline{\overline{\ddot{\psi}}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\psi}}}^{F(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)), \min(fy\overline{\overline{\ddot{\psi}}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\psi}}}^{F(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\\ \min(fy\overline{\overline{\ddot{\psi}}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\psi}}}^{F(3)}_{fy\hat{\ddot{\rho}}_2}(\omega))] : \omega \in \overline{\overline{\ddot{\chi}}}\}\\ \cup\{\omega, (fy\overline{\overline{\ddot{\xi}}}^{T(l)}_{fy\hat{\ddot{\rho}}_3}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(m)}_{fy\hat{\ddot{\rho}}_3}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(u)}_{fy\hat{\ddot{\rho}}_3}(\omega)), (fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_3}(\omega), fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_3}(\omega), fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_3}(\omega)), (fy\overline{\overline{\ddot{\delta}}}^{U(l)}_{fy\hat{\ddot{\rho}}_3}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(m)}_{fy\hat{\ddot{\rho}}_3}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(u)}_{fy\hat{\ddot{\rho}}_3}(\omega)),\\ (fy\overline{\overline{\ddot{\psi}}}^{F(l)}_{fy\hat{\ddot{\rho}}_3}(\omega), fy\overline{\overline{\ddot{\psi}}}^{F(m)}_{fy\hat{\ddot{\rho}}_3}(\omega), fy\overline{\overline{\ddot{\psi}}}^{F(u)}_{fy\hat{\ddot{\rho}}_3}(\omega)) : \omega \in \overline{\overline{\ddot{\chi}}}\}\end{array}$$

$$= (fy\hat{\ddot{\rho}}_1 \cup fy\hat{\ddot{\rho}}_2) \cup fy\hat{\ddot{\rho}}_3$$

Theorem 3.4. For any three TFQNSs $fy\hat{\ddot{\rho}}_1, fy\hat{\ddot{\rho}}_2, fy\hat{\ddot{\rho}}_3$, $fy\hat{\ddot{\rho}}_1 \cup (fy\hat{\ddot{\rho}}_2 \cap fy\hat{\ddot{\rho}}_3) = (fy\hat{\ddot{\rho}}_1 \cup fy\hat{\ddot{\rho}}_2) \cap (fy\hat{\ddot{\rho}}_1 \cup fy\hat{\ddot{\rho}}_3)$

Proof: $fy\hat{\ddot{\rho}}_1 \cup (fy\hat{\ddot{\rho}}_2 \cap fy\hat{\ddot{\rho}}_3) =$

$$\begin{array}{l}\{\omega, (fy\overline{\overline{\ddot{\xi}}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)), (fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)),\\ (fy\overline{\overline{\ddot{\delta}}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega)), (fy\overline{\overline{\ddot{\psi}}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\psi}}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega), fy\overline{\overline{\ddot{\psi}}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega)) : \omega \in \overline{\overline{\ddot{\chi}}}\}\end{array}$$

$$\begin{array}{l}\cup\{\omega, [(\min(fy\overline{\overline{\ddot{\xi}}}^{T(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(l)}_{fy\hat{\ddot{\rho}}_3}(\omega)), \min(fy\overline{\overline{\ddot{\xi}}}^{T(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(m)}_{fy\hat{\ddot{\rho}}_3}(\omega)), \min(fy\overline{\overline{\ddot{\xi}}}^{T(u)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\xi}}}^{T(u)}_{fy\hat{\ddot{\rho}}_3}(\omega))],\\ [\min(fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_3}(\omega)), \min(fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_3}(\omega)), \min(fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_3}(\omega))],\\ [\max(fy\overline{\overline{\ddot{\delta}}}^{U(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(l)}_{fy\hat{\ddot{\rho}}_3}(\omega)), \max(fy\overline{\overline{\ddot{\delta}}}^{U(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(m)}_{fy\hat{\ddot{\rho}}_3}(\omega)), \max(fy\overline{\overline{\ddot{\delta}}}^{U(u)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\delta}}}^{U(u)}_{fy\hat{\ddot{\rho}}_3}(\omega))],\\ [\max(fy\overline{\overline{\ddot{\psi}}}^{F(l)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\psi}}}^{F(l)}_{fy\hat{\ddot{\rho}}_3}(\omega)), \max(fy\overline{\overline{\ddot{\psi}}}^{F(m)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\psi}}}^{F(m)}_{fy\hat{\ddot{\rho}}_3}(\omega)), \max(fy\overline{\overline{\ddot{\psi}}}^{F(u)}_{fy\hat{\ddot{\rho}}_2}(\omega), fy\overline{\overline{\ddot{\psi}}}^{F(u)}_{fy\hat{\ddot{\rho}}_3}(\omega))] : \omega \in \overline{\overline{\ddot{\chi}}}\}\end{array}$$

$$= \{\omega,[\max(fy\overline{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),\min(fy\overline{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_2}(\omega),fy\overline{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_3}(\omega))),\max(fy\overline{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),\min(fy\overline{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_2}(\omega),fy\overline{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_3}(\omega))),$$
$$\max(fy\overline{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),\min(fy\overline{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_2}(\omega),fy\overline{\ddot{\xi}}^{(u)}_{fy\hat{\ddot{\rho}}_3}(\omega)))],[\max(fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),\min(fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_2}(\omega),fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_2}(\omega))),$$
$$\max(fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),\min(fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_2}(\omega),fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_2}(\omega))),\max(fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),\min(fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_2}(\omega),fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_2}(\omega)))],$$
$$[\min(fy\overline{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),\max(fy\overline{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_2}(\omega),fy\overline{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_3}(\omega))),$$
$$\min(fy\overline{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),\max(fy\overline{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_2}(\omega),fy\overline{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_3}(\omega))),\min(fy\overline{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),\max(fy\overline{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_2}(\omega),fy\overline{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_3}(\omega)))],$$
$$[\min(fy\overline{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),\max(fy\overline{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_2}(\omega),fy\overline{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_3}(\omega))),$$
$$\min(fy\overline{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),\max(fy\overline{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_2}(\omega),fy\overline{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_3}(\omega))),\min(fy\overline{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),$$
$$\max(fy\overline{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_2}(\omega),fy\overline{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_3}(\omega))):\omega\in\overline{\ddot{\chi}}\}$$

$$= \{\omega,[(\max(fy\overline{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\max(fy\overline{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\max(fy\overline{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_2}(\omega)))],$$
$$[\max(fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\max(fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\max(fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\hat{\theta}),fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_2}(\hat{\theta}))],$$
$$[\min(fy\overline{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\min(fy\overline{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\min(fy\overline{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_2}(\omega))],$$
$$[\min(fy\overline{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\min(fy\overline{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\min(fy\overline{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_2}(\omega))]:\omega\in\overline{\ddot{\chi}}\}$$
$$\cap\{\omega,[\max(fy\overline{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_3}(\omega)),\max(fy\overline{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_3}(\omega)),\max(fy\overline{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_3}(\omega))],$$
$$[\max(fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_3}(\omega)),\max(fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_3}(\omega)),\max(fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_3}(\omega))],$$
$$[\min(fy\overline{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_3}(\omega)),\min(fy\overline{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_3}(\omega)),$$
$$\min(fy\overline{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_3}(\omega))],[\min(fy\overline{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_3}(\omega)),\min(fy\overline{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_3}(\omega)),$$
$$\min(fy\overline{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_3}(\omega))]:\omega\in\overline{\ddot{\chi}}\}$$

$$= (fy\hat{\ddot{\rho}}_1 \cup fy\hat{\ddot{\rho}}_2)\cap(fy\hat{\ddot{\rho}}_1 \cup fy\hat{\ddot{\rho}}_3)$$

Theorem 3.5. For any two TFQNSs $fy\hat{\ddot{\rho}}_1$ and $fy\hat{\ddot{\rho}}_2$, a) $fy\hat{\ddot{\rho}}_1 \cup (fy\hat{\ddot{\rho}}_1 \cap fy\hat{\ddot{\rho}}_2) = fy\hat{\ddot{\rho}}_1$

Proof:

$$fy\hat{\ddot{\rho}}_1 \cup (fy\hat{\ddot{\rho}}_1 \cap fy\hat{\ddot{\rho}}_2) = \{\omega,[\max(fy\overline{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),\min(fy\overline{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\max(fy\overline{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),\min(fy\overline{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)),$$
$$\max(fy\overline{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),\min(fy\overline{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_2}(\omega))],[\max(fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),\min(fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)),$$
$$\max(fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),\min(fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\max(fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),\min(fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_2}(\omega))],$$
$$,[\min(fy\overline{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),\max(fy\overline{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\min(fy\overline{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),$$
$$\max(fy\overline{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\min(fy\overline{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),\max(fy\overline{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_2}(\omega))],$$
$$[\min(fy\overline{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),\max(fy\overline{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\min(fy\overline{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),\max(fy\overline{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)),$$
$$\min(fy\overline{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),\max(fy\overline{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_2}(\omega))]:\omega\in\overline{\ddot{\chi}}\}$$

$$= \{\omega,(fy\overline{\ddot{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)),(fy\overline{\ddot{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)),$$
$$(fy\overline{\ddot{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)),(fy\overline{\ddot{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\overline{\ddot{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)):\omega\in\overline{\ddot{\chi}}\}$$

$$= fy\hat{\ddot{\rho}}_1$$

Similarly, b) $fy\hat{\ddot{\rho}}_1 \cap (fy\hat{\ddot{\rho}}_1 \cup fy\hat{\ddot{\rho}}_2) = fy\hat{\ddot{\rho}}_1$

Proof: $fy\hat{\ddot{\rho}}_1 \cap (fy\hat{\ddot{\rho}}_1 \cup fy\hat{\ddot{\rho}}_2)$

$$
\begin{aligned}
&= \{\omega,[\min(fy\ddot{\bar{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),\max(fy\ddot{\bar{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_2}(\hat{\theta})),\min(fy\ddot{\bar{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),\max(fy\ddot{\bar{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\\
&\min(fy\ddot{\bar{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),\max(fy\ddot{\bar{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_2}(\omega))],[\min(fy\ddot{\bar{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),\max(fy\ddot{\bar{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\\
&\min(fy\ddot{\bar{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),\max(fy\ddot{\bar{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\min(fy\ddot{\bar{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),\max(fy\ddot{\bar{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_2}(\omega)],\\
&[\max(fy\ddot{\bar{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),\min(fy\ddot{\bar{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\\
&\max(fy\ddot{\bar{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),\min(fy\ddot{\bar{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\max(fy\ddot{\bar{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),\min(fy\ddot{\bar{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_2}(\omega)],\\
&[\max(fy\ddot{\bar{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),\min(fy\ddot{\bar{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\max(fy\ddot{\bar{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),\min(fy\ddot{\bar{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_2}(\omega)),\\
&\max(fy\ddot{\bar{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),\min(fy\ddot{\bar{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_2}(\omega)]:\omega\in\ddot{\bar{\chi}}\}\\
&= \{\omega,(fy\ddot{\bar{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)),(fy\ddot{\bar{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)),\\
&(fy\ddot{\bar{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)),(fy\ddot{\bar{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)):\omega\in\ddot{\bar{\chi}}\}
\end{aligned}
$$

$$= fy\hat{\ddot{\rho}}_1$$

Theorem 3.6. For any TFQNS $fy\hat{\ddot{\rho}}_1$, $(fy\hat{\ddot{\rho}}_1{}^{C})^{C} = fy\hat{\ddot{\rho}}_1$

Proof: Now, $\hat{\ddot{\rho}}_1 = \{\omega,(fy\ddot{\bar{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)),(fy\ddot{\bar{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)),(fy\ddot{\bar{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),$

$fy\ddot{\bar{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)),(fy\ddot{\bar{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)):\omega\in\ddot{\bar{\chi}}\}$

So, $(fy\hat{\ddot{\rho}})^{c} = \{\omega,(fy\ddot{\bar{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}}(\omega),fy\ddot{\bar{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}}(\omega),fy\ddot{\bar{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}}(\omega)),(fy\ddot{\bar{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}}(\omega),fy\ddot{\bar{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}}(\omega),fy\ddot{\bar{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}}(\omega)),$

$(fy\ddot{\bar{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}}(\omega),fy\ddot{\bar{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}}(\omega),fy\ddot{\bar{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}}(\omega)),(fy\ddot{\bar{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}}(\omega),fy\ddot{\bar{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}}(\omega),fy\ddot{\bar{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}}(\omega)):\omega\in\ddot{\bar{\chi}}\}$

So, $(fy\hat{\ddot{\rho}}_1{}^{C})^{C} = \{\omega,(fy\ddot{\bar{\xi}}^{T(1)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)),(fy\ddot{\bar{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)),$

$(fy\ddot{\bar{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)),(fy\ddot{\bar{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)):\omega\in\ddot{\bar{\chi}}\}$

$= fy\hat{\ddot{\rho}}_1$

Theorem 3.7. For any TFQNS $fy\hat{\ddot{\rho}}_1$,(a) $fy\hat{\ddot{\rho}}_1 \cap \hat{0} = \hat{0}$, (b) $fy\hat{\ddot{\rho}}_1 \cup \hat{1} = \hat{1}$

Proof. (a) $fy\hat{\ddot{\rho}}_1 \cap \hat{0}$

$$
\begin{aligned}
&= \{\omega,(fy\ddot{\bar{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)),(fy\ddot{\bar{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\mu}}^{C(3)}_{fy\hat{\ddot{\rho}}_1}(\omega)),\\
&(fy\ddot{\bar{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)),(fy\ddot{\bar{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),fy\ddot{\bar{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\omega)):\omega\in\ddot{\bar{\chi}}\}\cap\\
&\{(0,0,0),(0,0,0),(1,1,1),(1,1,1),(1,1,1)\}
\end{aligned}
$$

$$
\begin{aligned}
&= \{\omega,[\min(fy\ddot{\bar{\xi}}^{T(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),0),\min(fy\ddot{\bar{\xi}}^{T(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),0),\min(fy\ddot{\bar{\xi}}^{T(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),0)],[\min(fy\ddot{\bar{\mu}}^{C(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),0),\min(fy\ddot{\bar{\mu}}^{C(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),0),\\
&\min(fy\ddot{\bar{\mu}}^{C(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),0)],[\max(fy\ddot{\bar{\delta}}^{U(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),1),\max(fy\ddot{\bar{\delta}}^{U(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),1),\\
&\max(fy\ddot{\bar{\delta}}^{U(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),1)],[\max(fy\ddot{\bar{\psi}}^{F(l)}_{fy\hat{\ddot{\rho}}_1}(\omega),1),\max(fy\ddot{\bar{\psi}}^{F(m)}_{fy\hat{\ddot{\rho}}_1}(\omega),1),\max(fy\ddot{\bar{\psi}}^{F(u)}_{fy\hat{\ddot{\rho}}_1}(\omega),1)]:\omega\in\ddot{\bar{\chi}}\}\\
&= \{\omega,(0,0,0),(0.0.0),(1,1,1),(1,1,1):\omega\in\ddot{\bar{\chi}}\}\\
&= \hat{0}
\end{aligned}
$$

Proof. (b)

$$fy\hat{\ddot{\rho}}_1 \cup \hat{1} = \{\omega,(fy\ddot{\overline{\xi}}^{T(l)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\xi}}^{T(m)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\xi}}^{T(u)}_{fy\ddot{\rho}_1}(\omega)),(fy\ddot{\overline{\mu}}^{C(l)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\mu}}^{C(m)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\mu}}^{C(u)}_{fy\ddot{\rho}_1}(\omega)),(fy\ddot{\overline{\delta}}^{U(l)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\delta}}^{U(m)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\delta}}^{U(u)}_{fy\ddot{\rho}_1}(\omega)),$$
$$(fy\ddot{\overline{\psi}}^{F(l)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\psi}}^{F(m)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\psi}}^{F(u)}_{fy\ddot{\rho}_1}(\omega)) : \omega \in \ddot{\overline{\chi}}\} \cup$$
$$\{(1,1,1),(1,1,1),(0,0,0),(0,0,0),(0,0,0)\}$$

$$= \{\omega,[\max(fy\ddot{\overline{\xi}}^{T(l)}_{fy\ddot{\rho}_1}(\omega),1),\max(fy\ddot{\overline{\xi}}^{T(m)}_{fy\ddot{\rho}_1}(\omega),1),\min(fy\ddot{\overline{\xi}}^{T(u)}_{fy\ddot{\rho}_1}(\omega),1)],[\max(fy\ddot{\overline{\mu}}^{C(l)}_{fy\ddot{\rho}_1}(\omega),1),\max(fy\ddot{\overline{\mu}}^{C(m)}_{fy\ddot{\rho}_1}(\omega),1),$$
$$\max(fy\ddot{\overline{\mu}}^{C(u)}_{fy\ddot{\rho}_1}(\omega),1)],[\min(fy\ddot{\overline{\delta}}^{U(l)}_{fy\ddot{\rho}_1}(\omega),0),\min(fy\ddot{\overline{\delta}}^{U(m)}_{fy\ddot{\rho}_1}(\omega),0),$$
$$\min(fy\ddot{\overline{\delta}}^{U(u)}_{fy\ddot{\rho}_1}(\omega),0)],[\min(fy\ddot{\overline{\psi}}^{F(l)}_{fy\ddot{\rho}_1}(\omega),0),\min(fy\ddot{\overline{\psi}}^{F(m)}_{fy\ddot{\rho}_1}(\omega),0),\min(fy\ddot{\overline{\psi}}^{F(u)}_{fy\ddot{\rho}_1}(\omega),0)] : \omega \in \ddot{\overline{\chi}}\}$$

$$= \{\omega,(1,1,1),(1,1,1),(0,0,0),(0,0,0),(0,0,0) : \omega \in \ddot{\overline{\chi}}\}$$
$$= \hat{1}$$

Theorem 3.8. For any two TFQNSs $fy\ddot{\overline{\rho}}_1 \;\&\; fy\ddot{\overline{\rho}}_2$, $(fy\ddot{\overline{\rho}}_1 \cap fy\ddot{\overline{\rho}}_2)^C = fy\ddot{\overline{\rho}}_1{}^C \cup fy\ddot{\overline{\rho}}_2{}^C$

Proof. $(fy\hat{\ddot{\rho}}_1 \cap fy\hat{\ddot{\rho}}_2) =$

$$\{\omega,[(\min(fy\ddot{\overline{\xi}}^{T(l)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\xi}}^{T(l)}_{fy\ddot{\rho}_2}(\omega)),\min(fy\ddot{\overline{\xi}}^{T(m)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\xi}}^{T(m)}_{fy\ddot{\rho}_2}(\omega)),\min(fy\ddot{\overline{\xi}}^{T(u)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\xi}}^{T(u)}_{fy\ddot{\rho}_2}(\omega))],$$
$$[\min(fy\ddot{\overline{\mu}}^{C(l)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\mu}}^{C(l)}_{fy\ddot{\rho}_2}(\omega)),\min(fy\ddot{\overline{\mu}}^{C(m)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\mu}}^{C(m)}_{fy\ddot{\rho}_2}(\omega)),\min(fy\ddot{\overline{\mu}}^{C(u)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\mu}}^{C(u)}_{fy\ddot{\rho}_2}(\omega))],$$
$$[\max(fy\ddot{\overline{\delta}}^{U(l)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\delta}}^{U(l)}_{fy\ddot{\rho}_2}(\omega)),\max(fy\ddot{\overline{\delta}}^{U(m)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\delta}}^{U(m)}_{fy\ddot{\rho}_2}(\omega)),$$
$$\max(fy\ddot{\overline{\delta}}^{U(u)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\delta}}^{U(u)}_{fy\ddot{\rho}_2}(\omega))],[\max(fy\ddot{\overline{\psi}}^{F(l)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\psi}}^{F(l)}_{fy\ddot{\rho}_2}(\omega)),\max(fy\ddot{\overline{\psi}}^{F(m)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\psi}}^{F(m)}_{fy\ddot{\rho}_2}(\omega)),$$
$$\max(fy\ddot{\overline{\psi}}^{F(u)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\psi}}^{F(u)}_{fy\ddot{\rho}_2}(\omega))] : \omega \in \ddot{\overline{\chi}}\}$$

So,

$$(fy\hat{\ddot{\rho}}_1 \cap fy\hat{\ddot{\rho}}_2)^C = \{\omega,[\max(fy\ddot{\overline{\psi}}^{F(l)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\psi}}^{F(l)}_{fy\ddot{\rho}_2}(\omega)),\max(fy\ddot{\overline{\psi}}^{F(m)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\psi}}^{F(m)}_{fy\ddot{\rho}_2}(\omega)),\max(fy\ddot{\overline{\psi}}^{F(u)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\psi}}^{F(u)}_{fy\ddot{\rho}_2}(\omega))],$$
$$[\max(fy\ddot{\overline{\delta}}^{U(l)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\delta}}^{U(l)}_{fy\ddot{\rho}_2}(\omega)),\max(fy\ddot{\overline{\delta}}^{U(m)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\delta}}^{U(m)}_{fy\ddot{\rho}_2}(\omega)),\max(fy\ddot{\overline{\delta}}^{U(u)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\delta}}^{U(u)}_{fy\ddot{\rho}_2}(\omega))],$$
$$[\min(fy\ddot{\overline{\mu}}^{C(l)}_{\ddot{\rho}_1}(\omega),fy\ddot{\overline{\mu}}^{C(l)}_{fy\ddot{\rho}_2}(\omega)),\min(fy\ddot{\overline{\mu}}^{C(m)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\mu}}^{C(m)}_{fy\ddot{\rho}_2}(\omega)),$$
$$\min(fy\ddot{\overline{\mu}}^{C(u)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\mu}}^{C(u)}_{fy\ddot{\rho}_2}(\omega))],[\min(fy\ddot{\overline{\xi}}^{T(l)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\xi}}^{T(l)}_{fy\ddot{\rho}_2}(\omega)),\min(fy\ddot{\overline{\xi}}^{T(m)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\xi}}^{T(m)}_{fy\ddot{\rho}_2}(\omega)),$$
$$\min(fy\ddot{\overline{\xi}}^{T(u)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\xi}}^{T(u)}_{fy\ddot{\rho}_2}(\omega))] : \omega \in \ddot{\overline{\chi}}\} \qquad (1)$$

Again,

$fy\hat{\ddot{\rho}}_1{}^C \cup fy\hat{\ddot{\rho}}_2{}^C =$

$$\{\omega,(fy\ddot{\overline{\psi}}^{F(l)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\psi}}^{F(m)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\psi}}^{F(u)}_{fy\ddot{\rho}_1}(\omega)),(fy\ddot{\overline{\delta}}^{U(l)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\delta}}^{U(m)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\delta}}^{U(u)}_{fy\ddot{\rho}_1}(\omega)),$$
$$(fy\ddot{\overline{\mu}}^{C(l)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\mu}}^{C(m)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\mu}}^{C(u)}_{fy\ddot{\rho}_1}(\omega)),(fy\ddot{\overline{\xi}}^{T(l)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\xi}}^{T(m)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\xi}}^{T(u)}_{fy\ddot{\rho}_1}(\omega)) : \omega \in \ddot{\overline{\chi}}\} \cup \{\omega,(fy\ddot{\overline{\psi}}^{F(l)}_{fy\ddot{\rho}_2}(\omega),fy\ddot{\overline{\psi}}^{F(m)}_{fy\ddot{\rho}_2}(\omega),fy\ddot{\overline{\psi}}^{F(u)}_{fy\ddot{\rho}_2}(\omega)),$$
$$(fy\ddot{\overline{\delta}}^{U(l)}_{fy\ddot{\rho}_2}(\omega),fy\ddot{\overline{\delta}}^{U(m)}_{fy\ddot{\rho}_2}(\omega),fy\ddot{\overline{\delta}}^{U(u)}_{fy\ddot{\rho}_2}(\omega)),(fy\ddot{\overline{\mu}}^{C(l)}_{fy\ddot{\rho}_2}(\omega),fy\ddot{\overline{\mu}}^{C(m)}_{fy\ddot{\rho}_2}(\omega),fy\ddot{\overline{\mu}}^{C(u)}_{fy\ddot{\rho}_2}(\omega)),(fy\ddot{\overline{\xi}}^{T(l)}_{fy\ddot{\rho}_2}(\omega),fy\ddot{\overline{\xi}}^{T(m)}_{fy\ddot{\rho}_2}(\omega),fy\ddot{\overline{\xi}}^{T(u)}_{fy\ddot{\rho}_2}(\omega)) : \omega \in \ddot{\overline{\chi}}\}$$

=

$$\{\omega,[\max(fy\ddot{\overline{\psi}}^{F(l)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\psi}}^{F(l)}_{fy\ddot{\rho}_2}(\omega)),\max(fy\ddot{\overline{\psi}}^{F(2)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\psi}}^{F(m)}_{fy\ddot{\rho}_2}(\omega)),\max(fy\ddot{\overline{\psi}}^{F(u)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\psi}}^{F(u)}_{fy\ddot{\rho}_2}(\omega))],$$
$$[\max(fy\ddot{\overline{\delta}}^{U(l)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\delta}}^{U(l)}_{fy\ddot{\rho}_2}(\omega)),\max(fy\ddot{\overline{\delta}}^{U(m)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\delta}}^{U(m)}_{fy\ddot{\rho}_2}(\omega)),\max(fy\ddot{\overline{\delta}}^{U(u)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\delta}}^{U(u)}_{fy\ddot{\rho}_2}(\omega))],$$
$$[\min(fy\ddot{\overline{\mu}}^{C(l)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\mu}}^{C(l)}_{fy\ddot{\rho}_2}(\omega)),\min(fy\ddot{\overline{\mu}}^{C(m)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\mu}}^{C(m)}_{fy\ddot{\rho}_2}(\omega)),$$
$$\min(fy\ddot{\overline{\mu}}^{C(u)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\mu}}^{C(u)}_{fy\ddot{\rho}_2}(\omega))],[\min(fy\ddot{\overline{\xi}}^{T(l)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\xi}}^{T(l)}_{fy\ddot{\rho}_2}(\omega)),\min(fy\ddot{\overline{\xi}}^{T(m)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\xi}}^{T(m)}_{fy\ddot{\rho}_2}(\omega)),$$
$$\min(fy\ddot{\overline{\xi}}^{T(u)}_{fy\ddot{\rho}_1}(\omega),fy\ddot{\overline{\xi}}^{T(u)}_{fy\ddot{\rho}_2}(\omega))] : \omega \in \ddot{\overline{\chi}}\} \qquad (2)$$

Comparing eqn (1) and eqn (2), we have $(fy\ddot{\overline{\rho}}_1 \cap fy\ddot{\overline{\rho}}_2)^C = fy\ddot{\overline{\rho}}_1{}^C \cup fy\ddot{\overline{\rho}}_2{}^C$

4. Discussion

In this paper, the notion of TFQNS is introduced by combining the TFN and the QSVNS to utilize the advantages of TFN and QSVNS. The significance of introducing the hybrid set structure TFQNS is that the computational techniques based on TFN or QSVNS alone may not always produce the best results. But a fusion of them may produce better results. We have presented a real-world example of TFQNS in section 3.1 which is elegant to express uncertainty by utilizing triangular fuzzy number which was not possible using QSVNS alone. So, TFQNS is more advantageous than QSVNS. Limitation of the study: Only a few properties have been established. Many more results are to be explored for TFQNS. In the future, various operators can be defined in TFQNS environment which can be utilized to deal with multi criteria decision making problems. and utilize the theory to deal with practical applications in the areas such as information fusion, information retrieval etc.

5. Conclusions

In this paper, we have introduced a new concept of TFQNS and verified its important properties like unification, intersection, complement etc. We hope that this treatise will enlighten a future scope of development of logical system in information science and some innovative operators can be developed which will be helpful in decision making [23-31], information retrieval system [32-33], cyber security control selection [34], graph theory [35-36], etc.

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