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A Study on n-th Derivative of Neutrosophic Function and Neutrosophic Differential Equation

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Abstract. Neutrosophic sets play an important role in addressing uncertainty, vagueness, and indeterminacy in problem-solving. In this research article, we deal with the concepts of the derivative of neutrosophic real functions. This article aims to understand the n-th order derivative of neutrosophic real functions. In addition, the n-th order neutrosophic differential equation is defined, and at the same time, we have established both the existence and uniqueness of the solution to the n-th order neutrosophic differential equation.

Keywords: neutrosophic set; neutrosophic real number; derivative of neutrosophic function; n-th order neutrosophic differential equation

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List of Abbreviation

Complete Metric Space : CMS

Initial Value Problem : IVP

Lattice Problem : LP

Neutrosophic Complex Number : NCN

Neutrosophic Function : NF

Neutrosophic Mereo Continuity : NMC

Neutrosophic Mereo Derivative : NMD

Neutrosophic Mereo Integral : NMI

Neutrosophic Mereo Limit : NML

Neutrosophic Real and Complex Number : NRCN

Neutrosophic Real Number : NRN

Neutrosophic Real Numbers : NRNs

Pentagonal Neutrosophic Number : PNN

Shortest Path Problem : SPP

Single Valued Neutrosophic Polygroups : SVNPs

Triangular Neutrosophic Number : TNN

1. Introduction

The American scientist and philosopher F. Smarandache introduced neutrosophic logic, and this logic is a generalization of fuzzy logic invented by Zadeh in 1965 [1]. Neutrosophic sets were introduced into the literature by Smarandache, supporting incomplete, undefined, and inconsistent information. In neutrosophic sets, the vagueness is explicitly measured by a new parameter I. True membership (T), ambiguity of membership (I), and false membership (F) are three independent parameters that allow us to characterize a Neutrosophic number. Smarandache proposed neutrosophic logic to represent a mathematical model of uncertainty, inaccuracy, ambiguity, imprecision, vagueness, unknown, incompleteness, inconsistency, redundancy, and contradiction, the concept of neutrosophy being a new branch of philosophy introduced by Smarandache [2–15]. He introduced the meaning of the standard form of an NRN and the circumstances for the division of two NRNs, characterized the standard form of an NCN, and found the root $n \geq 2$ of an NRCN [2–5]. While studying the concept of neutrosophic probability and neutrosophic statistics, Professor Smarandache [2–9] entered the concept of provisional calculus, where he first introduced the concepts of the NML, NMC, NMD, and NMI. Al-Tahan [11] presented the results on SVNPs. Edalatpanah [12] proposed a new simple algorithm for solving linear neutrosophic programming, in which the variables and the RHS represent the TNNs. Chakraborty [13,14] applied the PNN to the LP and the SPP. Mondal et al. [16] described the application of the neutrosophic differential equation on Mine safety via a single-valued neutrosophic number. Sumathi et al. [17] discussed the differential equation in a neutrosophic environment and the solution of a second-order linear differential equation with trapezoidal neutrosophic numbers as boundary conditions. Acharya et al. [18] used the differential equations in a neutrosophic environment under Hukuhara differentiability to model the amount of glucose distribution and absorption rates in blood. Lathamaheswari et al. [19] solved the neutrosophic differential equation by using bipolar trapezoidal neutrosophic number and applied this concept in predicting bacterial reproduction over separate bodies. Parikh et al. [20] describe the solution of a first-order linear non-homogeneous fuzzy differential equation with initial conditions in a neutrosophic environment. He also introduced the neutrosophic analytical method and the fourth-order Runge-Kutta numerical method by using triangular neutrosophic numbers. Recently Alhasan [21–23] introduced the differential and integral of neutrosophic real functions. Salamah et al. [24] used the concepts of continuity, differentiability, and integrability from real analysis to study the derivative and integration of a neutrosophic real function with one variable depending on the geometry isometry(AH-Isometry), also they studied the neutrosophic differential equation by using One-Dimensional geometry AH-Isometry, where they discuss the methods of finding the solution of neutrosophic identical linear differential equation and neutrosophic non-homogeneous linear differential equation [25].

In this article, the main contributions are:

- An n-th order derivative of neutrosophic functions.
- An n-th order neutrosophic differential equation.
- Existence and uniqueness to the solution of n-th order neutrosophic differential equation.

1.1. **Motivation**

In our literature review, we have seen that few works have been done on neutrosophic differential equations. However, there are almost no work has been done on nth-order differential equations for neutrosophic numbers of the form $\eta_1 + \eta_2 I$, where $\eta_1, \eta_2 \in R$, and I represent literal indeterminacy (I- neutrosophic structure). Therefore, there is a lot of scope and opportunity to work in this area. So, to proceed in this direction, we must first define the nth-order derivative of a neutrosophic functions by using the first-order neutrosophic derivative, which has been previously done. Also, we must define the first-order neutrosophic differential equation to discuss the n-th order differential equation and the existence of its solution.

In our reality, many things cannot be precisely defined, and they contain an indeterminacy part. As a result, we can't find an accurate solution to such a problem by using a differential equation of classical real numbers, and this motivates us to think about similar types of development and modification in a neutrosophic environment.

This article is structured into six sections. The first section offers an introduction that provides a scientific overview of neutrosophists. It also includes information about the contributions and motivation of this article. The second section explains some basic concepts such as NRNs, the derivative of neutrosophic real functions, its rule, and the properties of the derivative. The third section introduces the n-th order neutrosophic derivative and provides examples. In the fourth section, we study the simple first and second-order neutrosophic differential equations with examples by using the concepts from classical differential equations. We also study the existence and uniqueness of a solution to the n-th order neutrosophic differential equation. In the fifth section, we mention a few applications of neutrosophic differential equations, which we plan to study in the future. Lastly, in the sixth section, we provide a conclusion to this article.

2. Preliminaries

2.1. NRN [3]

Let η be a NRN, then it takes the standard form $\eta = \eta_1 + \eta_2 I$, where $\eta_1, \eta_2 \in R$, and I represent literal indeterminacy, such that $I.0 = 0$ and $I^n = I \quad \forall n \in Z^+$.

2.2. Division of two NRNs [3]

If $\eta = \eta_1 + \eta_2 I, \quad \gamma = \gamma_1 + \gamma_2 I$ are two NRNs

Then, $\dfrac{\eta_1 + \eta_2 I}{\gamma_1 + \gamma_2 I} = \dfrac{\eta_1}{\gamma_1} + \dfrac{\gamma_1\eta_2 - \eta_1\gamma_2}{\gamma_1(\gamma_1 + \gamma_2)}.I$, provided, $\gamma_1 \neq 0$ and $\gamma_1 \neq -\gamma_2$.

2.3. The neutrosophic derivative [20]

Let $g : D_g \subseteq R \to R_g \cup \{I\}$, if

$$\lim_{h+h_0 I \to 0+0I} \frac{g(s+h+h_0 I) - g(s,I)}{h+h_0 I}$$

exist, then we say that the function $g(s,I)$ is differentiable w.r.t. s and is given by

$$g'(s,I) = \lim_{h+h_0 I \to 0+0I} \frac{g(s+h+h_0 I) - g(s,I)}{h+h_0 I}$$

Where $h + h_0 I$ is the number of small indetermined changes in s, and $h, h_0 \in R$, while I is Indeterminacy.

2.3.1. Some rules of neutrosophic derivatives [20]

By the above definition given in 2.3, we can prove the following formulas:

(1) $\dfrac{d}{ds}(\gamma_1 + \gamma_2 I) = 0 + 0I$; where $\gamma_1, \gamma_2 \in R$.

(2) $\dfrac{d}{ds}[(\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I)] = \eta_1 + \eta_2 I$; where $\eta_1, \eta_2 \gamma_1, \gamma_2 \in R$.

(3) $\dfrac{d}{ds}[(\eta_1 + \eta_2 I)s^n] = n(\eta_1 + \eta_2 I)s^{n-1}$; $n \in R$.

(4) $\dfrac{d}{ds}[e^{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)}] = (\eta_1 + \eta_2 I)e^{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)}$

(5) $\dfrac{d}{ds}(\eta_1 + \eta_2 I)^s = (\eta_1 + \eta_2 I)^s \ln(\eta_1 + \eta_2 I)$; where $\gamma_1 > 0, \gamma_2 > 0$ and $I \geq 0$ or $\gamma > 0, d < 0$ and $I \leq 0$

(6) $\dfrac{d}{ds}\left[\log_{\eta_1+\eta_2 I} s\right] = \dfrac{1}{s\ln(\eta_1 + \eta_2 I)}$; where $\gamma_1 > 0, \gamma_2 > 0$ and $I \geq 0$ or $\gamma_1 > 0, \gamma_2 < 0$ and $I \leq 0$

(7) $\frac{d}{ds}[\ln((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))] = \frac{\eta_1+\eta_2 I}{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)}$

(8) $\frac{d}{ds}\left[\sqrt{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)}\right] = \frac{\eta_1+\eta_2 I}{2\sqrt{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)}}$

(9) $\frac{d}{ds}[\sin\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}] = (\eta_1+\eta_2 I)\cos\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}$

(10) $\frac{d}{ds}[\cos\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}] = -(\eta_1+\eta_2 I)\sin\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}$

(11) $\frac{d}{ds}[\tan\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}] = (\eta_1+\eta_2 I)\sec^2\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}$

(12) $\frac{d}{ds}[\cot\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}] = -(\eta_1+\eta_2 I)\csc^2\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}$

(13) $\frac{d}{ds}[\sec\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}] = (\eta_1+\eta_2 I)\sec\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}\tan\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}$

(14) $\frac{d}{ds}[\csc\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}] = -(\eta_1+\eta_2 I)\csc\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}\cot\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}$

2.4. Properties of neutrosophic derivative [20]

(1) $\frac{d}{ds}[f(s,I)\pm g(s,I)] = \frac{d}{ds}[f(s,I)] \pm \frac{d}{ds}[g(s,I)]$; where $f(s,I)$ and $g(s,I)$ are any two differentiable NF.

(2) $\frac{d}{ds}[(\gamma_1+\gamma_2 I)f(s,I)] = (\gamma_1+\gamma_2 I)\frac{d}{ds}[f(s,I)]$; where $\gamma_1, \gamma_2 \in R$.

(3) $\frac{d}{ds}[f(s,I).g(s,I)] = f(s,I)\frac{d}{ds}[g(s,I)] + [g(s,I)]\frac{d}{ds}[f(s,I)]$; where $f(s,I)$ and $g(s,I)$ are any two differentiable NF.

(4) $\frac{d}{ds}\left[\frac{f(s,I)}{g(s,I)}\right] = \frac{f(s,I)\frac{d}{ds}[g(s,I)] - [g(s,I)]\frac{d}{ds}[f(s,I)]}{[g(s,I)]^2}$; where $f(s,I)$ and $g(s,I)$ are any two differentiable NF.

(5) If $y = f(t,I)$ and $t = g(s,I)$ are two NF, then

$$\frac{dy}{ds} = \frac{dy}{dt}.\frac{dt}{ds} \Rightarrow \frac{dy}{ds} = f'(t, I).g'(s, I)$$

This is the derivative of composite NFs, which is also known as Chain rule.

Main Work

3. N-th order derivative of a NF

Let $f : D_f \subseteq R \to R_f \cup \{I\}$ and $y = f(s, I)$ be any NF. Then $\frac{dy}{ds} = f'(s, I)$ is called the first-order derivative of the NF w.r.t. s. Again if we differentiate the first-order derivative w.r.t. s, that is $\frac{d}{ds}\Big(\frac{dy}{ds}\Big) = f''(s, I)$ or $\frac{d^2y}{ds^2} = f''(s, I)$, then it is said to be second-order derivative of the NF. Similarly the derivative $\frac{d}{ds}\Big(\frac{d^2y}{ds^2}\Big) = \frac{d^3y}{ds^3}$ is called the third-order derivative of the NF and so on.

Continuing this process up to n-times, we get the nth-order derivative of the NF. That is the derivative $\frac{d^ny}{ds^n} = f^n(s, I)$ is said to be the nth order derivative of the NF.

If $y = f(s, I)$, then it's derivatives are denoted by

$$\begin{array}{c} y_1 \quad y_2 \quad y_3 \quad \cdots \quad y_n \\ \text{or } y' \quad y'' \quad y''' \quad \ldots \quad y^{(n)} \\ \text{or } \dot{y} \quad \ddot{y} \quad \dddot{y} \quad \ldots \quad y^{(n)} \\ \text{or } f'(s, I) \quad f''(s, I) \quad f'''(s, I) \quad \ldots \quad f^{(n)}(s, I) \\ \text{or } \frac{dy}{ds} \quad \frac{d^2y}{ds^2} \quad \frac{d^3y}{ds^3} \quad \ldots \quad \frac{d^ny}{ds^n} \qquad \text{and so on.} \end{array}$$

3.1. The n-th derivative of some special functions

(1) $f(s, I) = (\eta_1 + \eta_2 I)s^n$, where n is any positive integer.

Solution: We have, $f(s, I) = (\eta_1 + \eta_2 I)s^n$, then

$f'(s, I) = n.(\eta_1 + \eta_2 I).s^{n-1}$

$f''(s, I) = n.(n - 1).(\eta_1 + \eta_2 I).s^{n-2}$

$f'''(s, I) = n.(n - 1).(n - 2).(\eta_1 + \eta_2 I).s^{n-3}$

Proceeding in this way upto n-times, we get

$f^n(s, I) = n.(n - 1).(n - 2).(n - (n - 1)).(\eta_1 + \eta_2 I).s^{n-n}$

or $f^n(s, I) = n.(n - 1).(n - 2).1.(\eta_1 + \eta_2 I).1$

or $f^n(s, I) = n!.(\eta_1 + \eta_2 I)$.

(2) $f(s, I) = ((\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I))^m$, where m is any integer.

Solution: We have, $f(s, I) = ((\eta_1 + \eta_2 I)s + (\gamma_1 + \gamma_2 I))^m$, then

$f'(s,I) = m.(\eta_1+\eta_2 I).((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))^{m-1}$

$f''(s,I) = m.(m-1).(\eta_1+\eta_2 I)^2.((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))^{m-2}$

$f'''(s,I) = m.(m-1).(m-2).(\eta_1+\eta_2 I)^3.((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))^{m-3}$

Proceeding in this way upto n-times, we get

$f^n(s,I) = m.(m-1).(m-2).(m-(n-1)).(\eta_1+\eta_2 I)^n.((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))^{m-n}$

or $f^n(s,I) = \dfrac{m.(m-1)...(m-n+1).(m-n)...3.2.1}{(m-n)...3.2.1}.(\eta_1+\eta_2 I)^n.((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))^{m-n}$

or $f^n(s,I) = \dfrac{m!}{(m-n)!}.(\eta_1+\eta_2 I)^n.((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))^{m-n}$

(3) $f(s,I) = e^{(\eta_1+\eta_2 I)s}$

Solution: We have, $f(s,I) = e^{(\eta_1+\eta_2 I)s}$, then

$f'(s,I) = (\eta_1+\eta_2 I)e^{(\eta_1+\eta_2 I)s}$

$f''(s,I) = (\eta_1+\eta_2 I)^2 e^{(\eta_1+\eta_2 I)s}$

$f'''(s,I) = (\eta_1+\eta_2 I)^3 e^{(\eta_1+\eta_2 I)s}$

Proceeding in this way upto n-times, we get

$f^n(s,I) = (\eta_1+\eta_2 I)^n e^{(\eta_1+\eta_2 I)s}$

Note: If $f(s,I) = e^{((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))}$, then $f^n(s,I) = (\eta_1+\eta_2 I)^n e^{((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))s}$

(4) $f(s,I) = \log((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))$

Solution: We have, $f(s,I) = \log((\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I))$, then

$f'(s,I) = \dfrac{(\eta_1+\eta_2 I)}{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)} = (\eta_1+\eta_2 I)\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}^{-1}$

$f''(s,I) = (-1)(\eta_1+\eta_2 I)^2\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}^{-2}$

$f'''(s,I) = (-1)(-2)(\eta_1+\eta_2 I)^3\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}^{-3}$

or $f'''(s,I) = \dfrac{(-1)^2 2!(\eta_1+\eta_2 I)^3}{\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}^3}$

$f^{iv}(s,I) = (-1)(-2)(-3)(\eta_1+\eta_2 I)^4\{(\eta_1+\eta_2 I)s+(c+dI)\}^{-4}$

or $f^{iv}(s,I) = \dfrac{(-1)^3 3!(\eta_1+\eta_2 I)^4}{\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}^4}$

Proceeding in this way upto n-times, we get

$f^n(s,I) = \dfrac{(-1)^{n-1}(n-1)!(\eta_1+\eta_2 I)^n}{\{(\eta_1+\eta_2 I)s+(\gamma_1+\gamma_2 I)\}^n}$

(5) $f(s,I) = \sin\left((\eta_1+\eta_2 I)s + (\gamma_1+\gamma_2 I)\right)$

Solution: We have, $f(s,I) = \sin\left((\eta_1+\eta_2 I)s + (\gamma_1+\gamma_2 I)\right)$, then

$f'(s,I) = (\eta_1+\eta_2 I)\cos\left((\eta_1+\eta_2 I)s + (\gamma_1+\gamma_2 I)\right)$

or $f'(s,I) = (\eta_1+\eta_2 I)\sin\left(\frac{\pi}{2} + (\eta_1+\eta_2 I)s + (\gamma_1+\gamma_2 I)\right)$

$f''(s,I) = (\eta_1+\eta_2 I)^2\cos\left(\frac{\pi}{2} + (\eta_1+\eta_2 I)s + (\gamma_1+\gamma_2 I)\right)$

or $f''(s,I) = (\eta_1+\eta_2 I)^2\sin\left(\frac{2\pi}{2} + (\eta_1+\eta_2 I)s + (\gamma_1+\gamma_2 I)\right)$

$f'''(s,I) = (\eta_1+\eta_2 I)^3\cos\left(\frac{2\pi}{2} + (\eta_1+\eta_2 I)s + (\gamma_1+\gamma_2 I)\right)$

or $f'''(s,I) = (\eta_1+\eta_2 I)^3\sin\left(\frac{3\pi}{2} + (\eta_1+\eta_2 I)s + (\gamma_1+\gamma_2 I)\right)$

Proceeding in this way upto n-times, we get

$f^n(s,I) = (\eta_1+\eta_2 I)^n\sin\left(\frac{n\pi}{2} + (\eta_1+\eta_2 I)s + (\gamma_1+\gamma_2 I)\right)$

(6) $f(s,I) = \cos\left((\eta_1+\eta_2 I)s + (\gamma_1+\gamma_2 I)\right)$

Solution: We have, $f(s,I) = \cos\left((\eta_1+\eta_2 I)s + (\gamma_1+\gamma_2 I)\right)$, then

$f'(s,I) = (\eta_1+\eta_2 I)[-\sin\left((\eta_1+\eta_2 I)s + (\gamma_1+\gamma_2 I)\right)]$

or $f'(s,I) = (\eta_1+\eta_2 I)\cos\left(\frac{\pi}{2} + (\eta_1+\eta_2 I)s + (\gamma_1+\gamma_2 I)\right)$

$f''(s,I) = (\eta_1+\eta_2 I)^2\Big[-\sin\left(\frac{\pi}{2} + (\eta_1+\eta_2 I)s + (\gamma_1+\gamma_2 I)\right)\Big]$

or $f''(s,I) = (\eta_1+\eta_2 I)^2\cos\left(\frac{2\pi}{2} + (\eta_1+\eta_2 I)s + (\gamma_1+\gamma_2 I)\right)$

$f'''(s,I) = (\eta_1+\eta_2 I)^3\Big[-\sin\left(\frac{2\pi}{2} + (\eta_1+\eta_2 I)s + (\gamma_1+\gamma_2 I)\right)\Big]$

or $f'''(s,I) = (\eta_1+\eta_2 I)^3\cos\left(\frac{3\pi}{2} + (\eta_1+\eta_2 I)s + (\gamma_1+\gamma_2 I)\right)$

Proceeding in this way upto n-times, we get

$f^n(s,I) = (\eta_1+\eta_2 I)^n\cos\left(\frac{n\pi}{2} + (\eta_1+\eta_2 I)s + (\gamma_1+\gamma_2 I)\right)$

(7) $f(s,I) = e^{(\eta_1+\eta_2 I)s}\sin\{(\gamma_1+\gamma_2 I)s\}$

Solution: We have, $f(s,I) = e^{(\eta_1+\eta_2 I)s}\sin\{(\gamma_1+\gamma_2 I)s\}$, then

$f'(s,I) = e^{(\eta_1+\eta_2 I)s}(\eta_1+\eta_2 I)\sin\{(\gamma_1+\gamma_2 I)s\} + e^{(\eta_1+\eta_2 I)s}\cos\{(\gamma_1+\gamma_2 I)s\}(\gamma_1+\gamma_2 I)$

$\Rightarrow f'(s,I) = e^{(\eta_1+\eta_2 I)s}[(a\eta_1+\eta_2 I)\sin\{(\gamma_1+\gamma_2 I)s\} + (\gamma_1+\gamma_2 I)\cos\{(\gamma_1+\gamma_2 I)s\}]$

Let $\eta_1+\eta_2 I = r\cos\Theta$, $\quad \gamma_1+\gamma_2 I = r\sin\Theta$, where $r = r_1 + r_2 I$ and $\Theta = \theta + I$, then

$r = \left[(\eta_1+\eta_2 I)^2 + (\gamma_1+\gamma_2 I)^2\right]^{\frac{1}{2}}$

$\Rightarrow r = \left[(\eta_1{}^2 + \gamma_1{}^2) + \{\eta_2{}^2 + \gamma_2{}^2 + 2(\eta_1\eta_2 + \gamma_1\gamma_2)\}I\right]^{\frac{1}{2}}$

$\Rightarrow r = \left(u_1 + u_2 I\right)^{\frac{1}{2}}$, where $u_1 = (\eta_1{}^2 + \gamma_1{}^2)$ and $u_2 = \eta_2{}^2 + \gamma_2{}^2 + 2(\eta_1\eta_2 + \gamma_1\gamma_2)$

and $\Theta = tan^{-1}\dfrac{\gamma_1 + \gamma_2 I}{\eta_1 + \eta_2 I}$

$\Rightarrow \Theta = tan^{-1}\dfrac{\gamma_1}{\eta_1} + \dfrac{\eta_1\gamma_2 - \eta_2\gamma_1}{\eta_1(\eta_1 + \eta_2)}.I$

Thus, we get

$f'(s, I) = e^{(\eta_1+\eta_2 I)s}[r\cos\Theta \sin\{(\gamma_1 + \gamma_2 I)s\} + r\sin\Theta\cos\{(\gamma_1 + \gamma_2 I)s\}]$

$\Rightarrow f'(s, I) = re^{(\eta_1+\eta_2 I)s}\sin[(\gamma_1 + \gamma_2 I)s + \Theta]$

Again differentiating the above equation with respect to s, we get,

$f''(s, I) = r\Big[e^{(\eta_1+\eta_2 I)s}(\eta_1 + \eta_2 I)\sin\{(\gamma_1 + \gamma_2 I)s + \Theta\} + e^{(\eta_1+\eta_2 I)s}(\gamma_1 + \gamma_2 I)\cos\{(\gamma_1 + \gamma_2 I)s + \Theta\}\Big]$

$\Rightarrow f''(s, I) = re^{(\eta_1+\eta_2 I)s}\Big[(\eta_1+\eta_2 I)\sin\{(\gamma_1+\gamma_2 I)s+\Theta\}+(\gamma_1+\gamma_2 I)\cos\{(\gamma_1+\gamma_2 I)s+\Theta\}\Big]$

$\Rightarrow f''(s, I) = re^{(\eta_1+\eta_2 I)s}\Big[r\cos\Theta\sin\{(\gamma_1 + \gamma_2 I)s + \Theta\} + r\sin\Theta\cos\{(\gamma_1 + \gamma_2 I)s + \Theta\}\Big]$

$\Rightarrow f''(s, I) = r^2 e^{(\eta_1+\eta_2 I)s}\sin\Big[(\gamma_1 + \gamma_2 I)s + 2\Theta\Big]$

Similarly,

$f'''(s, I) = r^3 e^{(\eta_1+\eta_2 I)s}\sin\Big[(\gamma_1 + \gamma_2 I)s + 3\Theta\Big]$

Proceeding in this way upto n-times, we get

$f^n(s, I) = r^n e^{(\eta_1+\eta_2 I)s}\sin\Big[(\gamma_1 + \gamma_2 I)s + n\Theta\Big]$

$f^n(s, I) = \Big((\eta_1 + \eta_2 I)^2 + (\gamma_1 + \gamma_2 I)^2\Big)^{\frac{n}{2}}.e^{(\eta_1+\eta_2 I)s}\sin\Big[(\gamma_1 + \gamma_2 I)s + n\tan^{-1}\dfrac{\gamma_1 + \gamma_2 I}{\eta_1 + \eta_2 I}\Big]$

or, $f^n(s, I) = \left(u_1 + u_2 I\right)^{\frac{n}{2}}.e^{(\eta_1+\eta_2 I)s}\sin\Big[(\gamma_1 + \gamma_2 I)s + n\tan^{-1}\dfrac{\gamma_1}{\eta_1} + \dfrac{\eta_1\gamma_2 - \eta_2\gamma_1}{\eta_1(\eta_1 + \eta_2)}.I\Big]$

(8) If $f(s, I) = e^{(\eta_1+\eta_2 I)s}\cos\{(\gamma_1 + \gamma_2 I)s\}$, then

$f^n(s, I) = \Big((\eta_1 + \eta_2 I)^2 + (\gamma_1 + \gamma_2 I)^2\Big)^{\frac{n}{2}} e^{(\eta_1+\eta_2 I)s}\cos\Big[(\gamma_1 + \gamma_2 I)s + n\tan^{-1}\dfrac{\gamma_1 + \gamma_2 I}{\eta_1 + \eta_2 I}\Big]$

or, $f^n(s, I) = \left(u_1 + u_2 I\right)^{\frac{n}{2}}.e^{(\eta_1+\eta_2 I)s}\cos\Big[(\gamma_1 + \gamma_2 I)s + n\tan^{-1}\dfrac{\gamma_1}{\eta_1} + \dfrac{\eta_1\gamma_2 - \eta_2\gamma_1}{\eta_1(\eta_1 + \eta_2)}.I\Big]$

Where, $u_1 = (\eta_1{}^2 + \gamma_1{}^2)$ and $u_2 = \eta_2{}^2 + \gamma_2{}^2 + 2(\eta_1\eta_2 + \gamma_1\gamma_2)$

(9) If $f(s, I) = e^{(\eta_1+\eta_2 I)s}\sin\{(\gamma_1 + \gamma_2 I)s + (\zeta_1 + \zeta_2 I)\}$, then

$f^n(s, I) = \Big((\eta_1 + \eta_2 I)^2 + (\gamma_1 + \gamma_2 I)^2\Big)^{\frac{n}{2}} e^{(\eta_1+\eta_2 I)s}\sin\Big[(\gamma_1 + \gamma_2 I)s + (\zeta_1 + \zeta_2 I) +$

$n\tan^{-1}\frac{\gamma_1+\gamma_2 I}{\eta_1+\eta_2 I}\Big]$

or, $f^n(s,I) = \big(u_1 + u_2 I\big)^{\frac{n}{2}}.e^{(\eta_1+\eta_2 I)s}\sin\Big[(\gamma_1+\gamma_2 I)s + (\zeta_1+\zeta_2 I) + n\tan^{-1}\Big(\frac{\gamma_1}{\eta_1} + \frac{\eta_1\gamma_2-\eta_2\gamma_1}{\eta_1(\eta_1+\eta_2)}.I\Big)\Big]$

Where, $u_1 = (\eta_1{}^2+\gamma_1{}^2)$ and $u_2 = \eta_2{}^2+\gamma_2{}^2+2(\eta_1\eta_2+\gamma_1\gamma_2)$

(10) If $f(s,I) = e^{(\eta_1+\eta_2 I)s}\cos\{(\gamma_1+\gamma_2 I)s + (\zeta_1+\zeta_2 I)\}$, then

$f^n(s,I) = \Big((\eta_1+\eta_2 I)^2 + (\gamma_1+\gamma_2 I)^2\Big)^{\frac{n}{2}} e^{(\eta_1+\eta_2 I)s}\cos\Big[(\gamma_1+\gamma_2 I)s + (\zeta_1+\zeta_2 I) + n\tan^{-1}\frac{\gamma_1+\gamma_2 I}{\eta_1+\eta_2 I}\Big]$

or, $f^n(s,I) = \big(u_1 + u_2 I\big)^{\frac{n}{2}}.e^{(\eta_1+\eta_2 I)s}\cos\Big[(\gamma_1+\gamma_2 I)s + (\zeta_1+\zeta_2 I) + n\tan^{-1}\Big(\frac{\gamma_1}{\eta_1} + \frac{\eta_1\gamma_2-\eta_2\gamma_1}{\eta_1(\eta_1+\eta_2)}.I\Big)\Big]$

Where, $u_1 = (\eta_1{}^2+\gamma_1{}^2)$ and $u_2 = \eta_2{}^2+\gamma_2{}^2+2(\eta_1\eta_2+\gamma_1\gamma_2)$

4. N-th order Neutrosophic differential equation

Definition 4.1 The equation which includes the first-order neutrosophic derivative as its highest derivative is called the first-order neutrosophic differential equation and is represented by

$$y'(s,I) = g(s, y(s,I))$$

In other way, an equation of the form $F(s, y(s,I), y'(s,I)) = 0$ is called a first-order neutrosophic differential equation and the function $f(s,I)$ that satisfies $F(s, y(s,I), y'(s,I)) = 0$ for every value of s is called its solution.

Similarly, the equation that includes the second-order neutrosophic derivative as its highest derivative is called second-order neutrosophic differential equation and is represented by

$$y''(s,I) = g(s, y(s,I), y'(s,I))$$
$$\text{or} \qquad F(s, y(s,I), y'(s,I), y''(s,I)) = 0$$

In the same way, we can define the n-th order neutrosophic differential equation by

$$y^n(s,I) = g(s, y(s,I), y'(s,I), y''(s,I), ..., y^{n-1}(s,I))$$
$$\text{or} \qquad F(s, y(s,I), y'(s,I), y''(s,I), ..., y^n(s,I)) = 0$$

Where $y(s,I)$ is a NF of s and I. Here, y^j denotes the j^{th} derivative of y w.r.t. s, i.e. $y^j = \frac{d^j y}{ds^j}$, j=0,1,2,...,n.

If $y(s_0, I) = l_1, \quad y'(s_0, I) = l_2, \quad y''(s_0, I) = l_3, \quad ..., \quad y^{n-1}(s_0, I) = l_n$ are the given initial values, where $l_j \quad ; \quad j = 1, 2, ..., n$ are the NRNs and are of the form $l_j = l_{j1} + l_{j2}I$, then the Cauchy problem of n-th order is given by

$$\begin{aligned} &y^n(s, I) = g(s, y(s, I), y'(s, I), y''(s, I), ..., y^{n-1}(s, I)); \\ &y(s_0, I) = l_1, \quad y'(s_0, I) = l_2, \quad y''(s_0, I) = l_3, ..., y^{n-1}(s_0, I) = l_n. \end{aligned} \tag{1}$$

In this section, we studied the existence and uniqueness of solutions to the n-th order neutrosophic differential equation under the conditions of contraction mapping. For this, we define the metric in definition 4.2.

Example 4.1.1 $y' = \sqrt{7}s = (2.6457.....)s$ is a 1st-order classical differential equation, but since $\sqrt{7}$ has infinitely many decimals, we cannot work with this exact number in our real life. Hence, we need to approximate this number, and we may write it as

$$y'(s, I) = (2 + I)s; \text{ where } I = (0.6, 0.7)$$

Which becomes first-order neutrosophic differential equation and we can find its solution by integrating both sides.

So, integrating both sides, we get

$$\begin{aligned} y(s, I) &= (2 + I)\frac{s^2}{2} + C \\ \Rightarrow y(s, I) &= \Big(\frac{2 + I}{2}\Big)s^2 + C \\ \Rightarrow y(s, I) &= \Big(\frac{2}{2} + \frac{2 \times 1 - 2 \times 0}{2(2 + 0)}.I\Big)s^2 + C \\ \Rightarrow y(s, I) &= \Big(1 + \frac{1}{2}.I\Big)s^2 + C \\ \Rightarrow y(s, I) &= (1 + 0.5I)s^2 + C \end{aligned}$$

Where C is indeterminate real constant.

Since $I = (0.6, 0.7)$, and let $C = 10 = 10 + 0.I$, then, the solution of the given neutrosophic differential equation $y'(s, I) = (2 + I)s$ is bounded by the lines marked by red and blue colors in the figure 1.

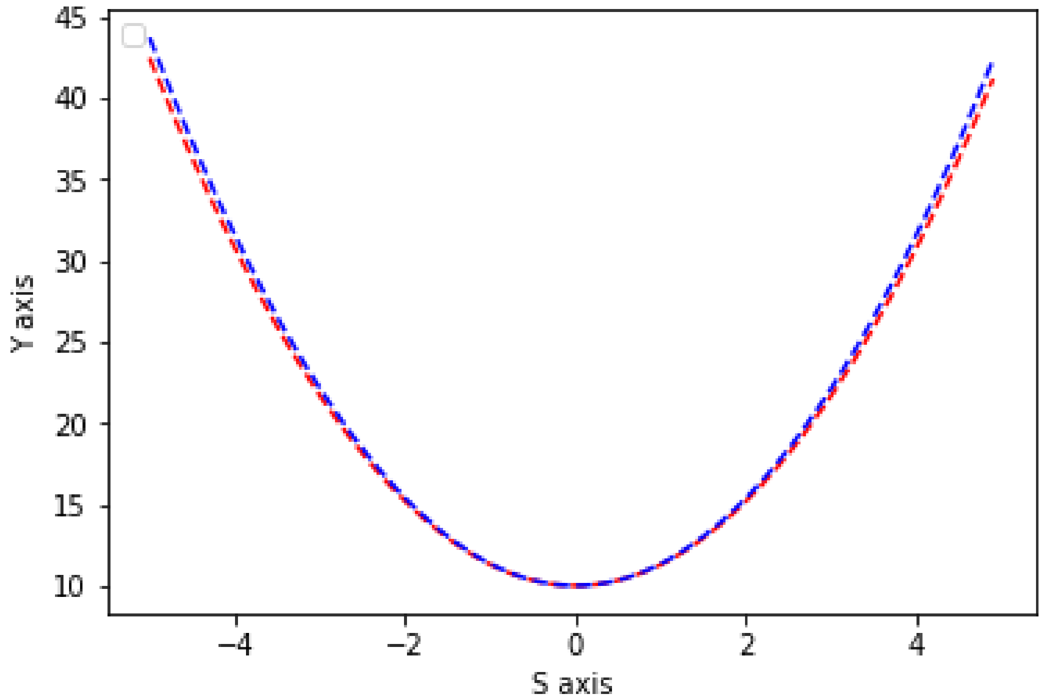

FIGURE 1. Graph of $y(s, I) = (1 + 0.5I)s^2 + C$

Example 4.1.2 Let us consider the second-order neutrosophic differential equation $y''(s, I) = 24Is$, which is directly integrable, so its solution can be found by integrating two times.

$$y'(s, I) = 24\frac{s^2}{2}I + A$$

$$\Rightarrow y'(s, I) = 12Is^2 + A$$

$$\Rightarrow y(s, I) = 4Is^3 + As + B$$

where A and B are indeterminate real constants.

Since, there is indeterminacy in the solution of the given differential equations, so, the graphical representation can only be shown approximately through dot lines, as illustrated in the figure 2 below.

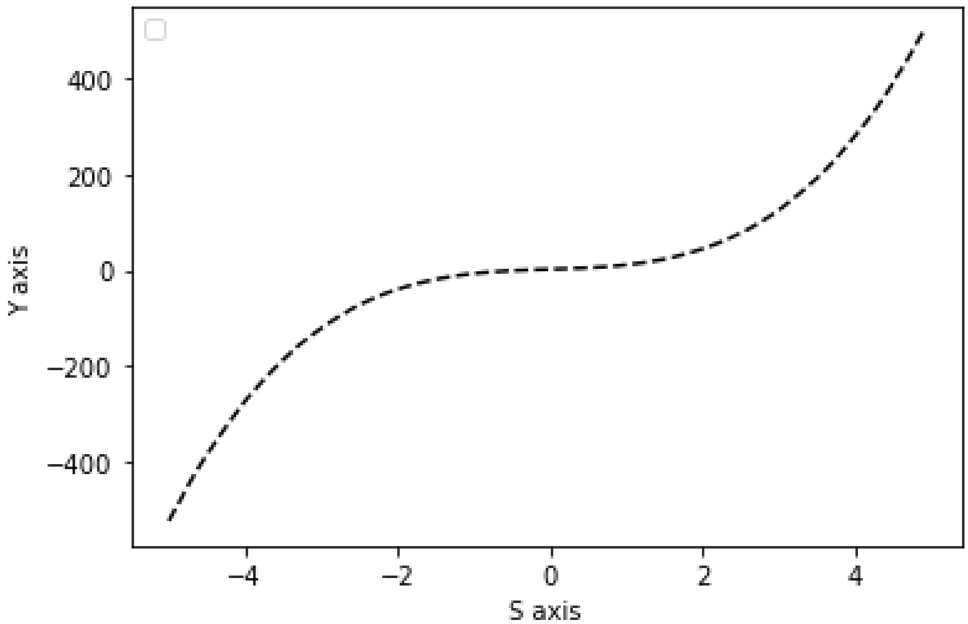

FIGURE 2. Graph of $y(s, I) = 4Is^3 + As + B$

Definition 4.2 The metric d can be defined on R_N^n (Set of all ordered n tuples of NRN) by

$$d(x_1 + x_2I, y_1 + y_2I) = \max_{1\leq k\leq n} |(x_{j1} + x_{j2}I) - (y_{j1} + y_{j2}I)|;$$
$$x_1 + x_2I, y_1 + y_2I \in R_N^n \quad ; \quad x_1, x_2, y_1, y_2 \in R$$

Let us denote the space of continuous function $x = x_1 + x_2I : J = [s_0, S] \to R_N^n$ by $C(J, R_N^n)$ and it is a CMS with the distance

$$D(x_1 + x_2I, y_1 + y_2I) = \sup_{s\in J} \left[d\{(x_1 + x_2I)(s), (y_1 + y_2I)(s)\}e^{-\rho s}\right]$$

Where $\rho \in R$ is fixed. For $x_1 + x_2I, y_1 + y_2I \in C^n(J, R_N^n)$, we consider the distance

$D_n(x_1 + x_2I, y_1 + y_2I) = D\big[x_1 + x_2I, y_1 + y_2I\big] + D\big[(x_1 + x_2I)', (y_1 + y_2I)'\big] + ... + D\big[(x_1 + x_2I)^n, (y_1 + y_2I)^n\big]$

or $D_n(x_1 + x_2I, y_1 + y_2I) = \displaystyle\sum_{j=0}^{n} D\big[(x_1 + x_2I)^j, (y_1 + y_2I)^j\big]$

Lemma 4.1 $(C^n(J, R_N^n), D_n)$ is a CMS.

Proof: Let $\{x_a\}_{a=1}^{\infty} = \{x_{a1} + x_{a2}I\}_{a=1}^{\infty} \subset C^n(J, R_N^n)$ be any cauchy sequence in $(C^n(J, R_N^n), D_n)$. Then

$D_n(x_{a1} + x_{a2}I, x_{b1} + x_{b2}I) = D(x_{a1} + x_{a2}I, x_{b1} + x_{b2}I) + D\big[(x_{a1} + x_{a2}I)', (x_{b1} + x_{b2}I)'\big] + ... + D\big[(x_{a1} + x_{a2}I)^n, (x_{b1} + x_{b2}I)^n\big] \to 0$ as $a, b \to +\infty$

This implies

$D(x_{a1}+x_{a2}I, x_{b1}+x_{b2}I) \to 0, D\big[(x_{a1}+x_{a2}I)', (x_{b1}+x_{b2}I)'\big] \to 0, ..., D\big[(x_{a1}+x_{a2}I)^n, (x_{b1}+x_{b2}I)^n\big] \to 0$

This shows that $\{x_{a1}+x_{a2}I\}_{a=1}^{\infty}, \{(x_{a1}+x_{a2}I)'\}_{a=1}^{\infty}, ..., \{(x_{a1}+x_{a2}I)^n\}_{a=1}^{\infty}$ are the cauchy sequence in the CMS $(C(J, R_N^n), D)$. So, $\exists \quad x_{01}+x_{02}I, x_{11}+x_{12}I, x_{21}+x_{22}I, ..., x_{n1}+x_{n2}I \in C(J, R_N^n)$ such that $\{x_{a1}+x_{a2}I\} \to x_{01}+x_{02}I, \{(x_{a1}+x_{a2}I)'\} \to x_{11}+x_{12}I, ..., \{(x_{a1}+x_{a2}I)^n\} \to x_{n1}+x_{n2}I$ as $a \to +\infty$.

We have to show that $x_{01}+x_{02}I \in C^n(J, R_N^n)$ and $(x_{01}+x_{02}I)' = x_{11}+x_{12}I, (x_{01}+x_{02}I)'' = x_{21}+x_{22}I, ..., (x_{01}+x_{02}I)^n = x_{n1}+x_{n2}I$.

In this case, we have

$D_n(x_{a1}+x_{a2}I, x_{01}+x_{02}I) = D(x_{a1}+x_{a2}I, x_{01}+x_{02}I) + D\big[(x_{a1}+x_{a2}I)', (x_{01}+x_{02}I)'\big] + ... + D\big[(x_{a1}+x_{a2}I)^n, (x_{01}+x_{02}I)^n\big]$.

$\Rightarrow D_n(x_{a1}+x_{a2}I, x_{01}+x_{02}I) = D(x_{a1}+x_{a2}I, x_{01}+x_{02}I) + D\big[(x_{a1}+x_{a2}I)', x_{11}+x_{12}I\big] + ... + D\big[(x_{a1}+x_{a2}I)^n, x_{n1}+x_{n2}I\big] \to 0$ as $a \to +\infty$.

This shows that $\{x_{a1}+x_{a2}I\} \to x_{01}+x_{02}I \in C^n(J, R_N^n)$ and hence $(C^n(J, R_N^n), D_n)$ is a CMS.

Theorem 4.1. *Let $f : [s_0, S] \times R_N^n \times R_N^n \times ... \times R_N^n \to R_N^n$ is continuous, then a mapping $y : [s_0, S] \to R_N^n$ is a solution of the IVP (1) iff $y \in C^n(J, R_N^n)$ satisfy the following integral equation for all $u_1 \in [s_0, S]$:*

$$y(u_1, I) = l_1 + l_2(u_1 - s_0) + (l_3 \int_{s_0}^{u_1} (u_2 - s_0)du_2 + l_4 \int_{s_0}^{u_1}\int_{s_0}^{u_2} (u_3 - s_0)du_3du_2 + ... + l_n \int_{s_0}^{u_1} ... \int_{s_0}^{u_{n-2}} (u_{n-1} - s_0)du_{n-1}...du_2 + \int_{s_0}^{u_1}\int_{s_0}^{u_2} ... \int_{s_0}^{u_n} f((s, I), y(s, I), ..., y^{n-1}(s, I))dsdu_n...du_2) \tag{2}$$

Proof: *We have*

$$y^n(s, I) = g(s, y(s, I), y'(s, I), y''(s, I), ..., y^{n-1}(s, I))$$

For $u_n \in [s_0, S]$, we have

$y^n(u_n, I) = g(u_n, y(u_n, I), y'(u_n, I), ...y^{n-1}(u_n, I))$

$\Rightarrow \int_{s_0}^{u_n} y^n(u_n, I) = \int_{s_0}^{u_n} g(u_n, y(u_n, I), y'(u_n, I), ...y^{n-1}(u_n, I))$

$\Rightarrow \Big[y^{n-1}(s, I)\Big]_{s_0}^{u_n} = \int_{s_0}^{u_n} g(s, y(s, I), y'(s, I), ...y^{n-1}(s, I))ds$

$\Rightarrow y^{n-1}(u_n, I) = \int_{s_0}^{u_n} g(s, y(s, I), y'(s, I), ...y^{n-1}(s, I))ds + y^{n-1}(s_0, I)$

$\Rightarrow y^{n-1}(u_n, I) = l_n + \int_{s_0}^{u_n} g(s, y(s, I), y'(s, I), ...y^{n-1}(s, I))ds$

Integrating above equality from s_0 to $u_{n-1} \in [s_0, S]$ w.r.t. u_n, we get

$y^{n-2}(u_{n-1},I) = l_{n-1} + l_n(u_{n-1} - s_0) + \int_{s_0}^{u_{n-1}} \int_{s_0}^{u_n} g(s, y(s,I), y'(s,I), ...y^{n-1}(s,I))dsdu_n$

Proceeding in this way for $u_2 \in [s_0, S]$, *we get*

$y'(u_2,I) = l_2 + l_3(u_2 - s_0) + l_4 \int_{s_0}^{u_2} (u_3 - s_0)du_3 + ... + l_n \int_{s_0}^{u_2} ... \int_{s_0}^{u_{n-2}} (u_{n-1} - s_0)du_{n-1}...du_2 + \int_{s_0}^{u_2} ... \int_{s_0}^{u_n} f(s, y(s,I), ..., y^{n-1}(s,I))dsdu_n...du_2$

Now, integrating the above equality from s_0 *to* $u_1 \in [s_0, S]$ *w.r.t.* u_2, *we get*

$y(u_1,I) = l_1 + l_2(u_1 - s_0) + l_3 \int_{s_0}^{u_1} (u_2 - s_0)du_2 + l_4 \int_{s_0}^{u_1} \int_{s_0}^{u_2} (u_3 - s_0)du_3du_2 + ... + l_n \int_{s_0}^{u_1} ... \int_{s_0}^{u_{n-2}} (u_{n-1} - s_0)du_{n-1}...du_2 + \int_{s_0}^{u_1} \int_{s_0}^{u_2} ... \int_{s_0}^{u_n} f(s, y(s,I), ..., y^{n-1}(s,I))dsdu_n...du_2))$

Definition 4.3 *[17] Let us introduce the following function defined on* $[s_0, S]$ *by*

$$\Phi_{\rho,0(u_n)} = \int_{s_0}^{u_n} e^{\rho s} ds$$

$$\Phi_{\rho,1(u_{n-i})} = \int_{s_0}^{u_{n-1}} \int_{s_0}^{u_n} e^{\rho s} dsdu_n$$

Similarly, for $2 \leq i \leq n-1$

$$\Phi_{\rho,i(u_{n-i})} = \int_{s_0}^{u_{n-j}} \int_{s_0}^{u_{n-j+1}} ... \int_{s_0}^{u_n} e^{\rho s} dsdu_n...du_{n-i+1}$$

Theorem 4.2. *Let* $f : [s_0, S] \times R_N^n \times R_N^n \times ... \times R_N^n \to R_N^n$ *be continuous and suppose that* $\exists \quad G_1, G_2, ..., G_n > 0$ *such that*

$$d(f(s, x_1, x_2, ..., x_n), f(s, y_1, y_2, ..., y_n)) \leq \sum_{j=1}^{n} G_j d_\infty(x_j, y_j) \tag{3}$$

$\forall s \in [s_0, S], \quad x_j, y_j \in R_N^n, \quad j = 1, 2, ..., n.$ *Then the IVP* (1) *has a unique solution.*

Proof: *Let us consider the CMS* $\big(C^n(J, R_N^n), D_n\big)$ *and* $J = [s_0, S]$

Let us define the operator $B_{n-1} : C^{n-1}(I, R_N^n) \to C^{n-1}(I, R_N^n) \quad$ *by* $\quad x \to B_{n-1}x$

Let $B_{n-1}x(u_1) = l_1 + l_2(u_1 - s_0) + l_3 \int_{s_0}^{u_1} (u_2 - s_0)du_2 + l_4 \int_{s_0}^{u_1} \int_{s_0}^{u_2} (u_3 - s_0)du_3du_2 + ... + l_n \int_{s_0}^{u_1} ... \int_{s_0}^{u_{n-2}} (u_{n-1} - s_0)du_{n-1}...du_2 + \int_{s_0}^{u_1} \int_{s_0}^{u_2} ... \int_{s_0}^{u_n} f(s, x(s,I), ..., x^{n-1}(s,I))dsdu_n... du_2))$

To prove the theorem, it is sufficient to show that B_{n-1} *is a contraction mapping for* $\rho > 0$.

Now, we have

$D_{n-1}(B_{n-1}x, B_{n-1}y) = \sum_{j=0}^{n-1} D\big((B_{n-1}x)^j, (B_{n-1}y)^j\big)$

$= \sup_{u_1 \in J} \Big[d\{(l_1 + l_2(u_1 - s_0) + l_3 \int_{s_0}^{u_1} (u_2 - s_0)du_2 + l_4 \int_{s_0}^{u_1} \int_{s_0}^{u_2} (u_3 - s_0)du_3du_2 + ... + l_n \int_{s_0}^{u_1}$

$$...\int_{s_0}^{u_{n-2}}(u_{n-1}-s_0)du_{n-1}...du_2+\int_{s_0}^{u_1}\int_{s_0}^{u_2}...\int_{s_0}^{u_n}f(s,x(s,I),...,x^{n-1}(s,I))dsdu_n...du_2),$$
$$(l_1+l_2(u_1-s_0)+l_3\int_{s_0}^{u_1}(u_2-s_0)du_2+l_4\int_{s_0}^{u_1}\int_{s_0}^{u_2}(u_3-s_0)du_3du_2+...+l_n\int_{s_0}^{u_1}...\int_{s_0}^{u_{n-2}}$$
$$(u_{n-1}-s_0)du_{n-1}...du_2+\int_{s_0}^{u_1}\int_{s_0}^{u_2}...\int_{s_0}^{u_n}f(s,y(s,I),...,y^{n-1}(s,I))dsdu_n...du_2)\}e^{-\rho u_1}\Big]$$
$$+\sup_{u_2\in J}\Big[d\{(l_2+l_3(u_2-s_0)du_2+l_4\int_{s_0}^{u_2}du_3+...+l_n\int_{s_0}^{u_2}...\int_{s_0}^{u_{n-2}}(u_{n-1}-s_0)du_{n-1}...du_3+$$
$$\int_{s_0}^{u_2}...\int_{s_0}^{u_n}f(s,x(s,I),...,x^{n-1}(s,I))dsdu_n...du_3),(l_2+l_3(u_2-s_0)du_2+l_4\int_{s_0}^{u_2}du_3+...$$
$$+l_n\int_{s_0}^{u_2}...\int_{s_0}^{u_{n-2}}(u_{n-1}-s_0)du_{n-1}...du_3+\int_{s_0}^{u_2}...\int_{s_0}^{u_n}f(s,y(s,I),...,y^{n-1}(s,I))dsdu_n...du_3)$$
$$\}e^{-\rho u_2}\Big]+...+\sup_{u_n\in J}\Big[d\{(l_n+\int_{s_0}^{u_n}f(s,x(s,I),...,x^{n-1}(s,I))ds),(l_n+\int_{s_0}^{u_n}f(s,y(s,I),...,$$
$$y^{n-1}(s,I))ds)\}e^{-\rho u_n}\Big]$$

$$=\sup_{u_1\in J}\Big[\Big\{\max_{1\le k\le n}|(l_1+l_2(u_1-s_0)+l_3\int_{s_0}^{u_1}(u_2-s_0)du_2+l_4\int_{s_0}^{u_1}\int_{s_0}^{u_2}(u_3-s_0)du_3du_2+...+$$
$$l_n\int_{s_0}^{u_1}...\int_{s_0}^{u_{n-2}}(u_{n-1}-s_0)du_{n-1}...du_2+\int_{s_0}^{u_1}\int_{s_0}^{u_2}...\int_{s_0}^{u_n}f(s,x(s,I),...,x^{n-1}(s,I))dsdu_n...$$
$$du_2)|-|(l_1+l_2(u_1-s_0)+l_3\int_{s_0}^{u_1}(u_2-s_0)du_2+l_4\int_{s_0}^{u_1}\int_{s_0}^{u_2}(u_3-s_0)du_3du_2+...+l_n\int_{s_0}^{u_1}$$
$$...\int_{s_0}^{u_{n-2}}(u_{n-1}-s_0)du_{n-1}...du_2+\int_{s_0}^{u_1}\int_{s_0}^{u_2}...\int_{s_0}^{u_n}f(s,y(s,I),...,y^{n-1}(s,I))dsdu_n...du_2)|\Big\}$$
$$e^{-\rho u_1}\Big]+\sup_{u_2\in J}\Big[\Big\{\max_{1\le k\le n}|(l_2+l_3(u_2-s_0)+l_4\int_{s_0}^{u_2}(u_3-s_0)du_3+...+l_n\int_{s_0}^{u_2}...\int_{s_0}^{u_{n-2}}du_{n-1}...du_3+$$
$$\int_{s_0}^{u_2}...\int_{s_0}^{u_n}f(s,x(s,I),...,x^{n-1}(s,I))dsdu_n...du_3)|-|(l_2+l_3(u_2-s_0)+l_4\int_{s_0}^{u_2}(u_3-s_0)du_3+...+$$
$$l_n\int_{s_0}^{u_2}...\int_{s_0}^{u_{n-2}}du_{n-1}...du_3+\int_{s_0}^{u_2}...\int_{s_0}^{u_n}f(s,y(s,I),...,y^{n-1}(s,I))dsdu_n...du_3)|\Big\}e^{-\rho u_2}\Big]+...+$$
$$\sup_{u_n\in J}\Big[\Big\{\max_{1\le k\le n}|l_n+\int_{s_0}^{u_n}f(s,x(s,I),...,x^{n-1}(s,I))ds|-|l_n+\int_{s_0}^{u_n}f(s,y(s,I),...,y^{n-1}(s,I))$$
$$ds|\Big\}e^{-\rho u_n}\Big]$$

$$=\sup_{u_1\in J}\Big[\Big\{\max_{1\le k\le n}|\int_{s_0}^{u_1}\int_{s_0}^{u_2}...\int_{s_0}^{u_n}\{f(s,x(s,I),...,x^{n-1}(s,I))-f(s,y(s,I),...,y^{n-1}(s,I))\}ds$$
$$du_n...du_2|\Big\}e^{-\rho u_1}\Big]+\sup_{u_2\in J}\Big[\Big\{\max_{1\le k\le n}|\int_{s_0}^{u_2}...\int_{s_0}^{u_n}\{f(s,x(s,I),...,x^{n-1}(s,I))-f(s,y(s,I),...,$$
$$y^{n-1}(s,I))\}dsdu_n...du_3|\Big\}e^{-\rho u_2}\Big]+...+\sup_{u_n\in J}\Big[\Big\{\max_{1\le k\le n}|\int_{s_0}^{u_n}\{f(s,x(s,I),...,x^{n-1}(s,I))-$$
$$f(s,y(s,I),...,y^{n-1}(s,I))\}ds|\Big\}e^{-\rho u_n}\Big]$$

$$=\sup_{u_1\in J}\Big[\int_{s_0}^{u_1}\int_{s_0}^{u_2}...\int_{s_0}^{u_n}dsdu_n...du_2\Big\{\max_{1\le k\le n}|(f(s,x(s,I),...,x^{n-1}(s,I))-f(s,y(s,I),...,$$
$$y^{n-1}(s,I)))|\Big\}e^{-\rho u_1}\Big]+\sup_{u_2\in J}\Big[\int_{s_0}^{u_2}...\int_{s_0}^{u_n}dsdu_n...du_3\Big\{\max_{1\le k\le n}|(f(s,x(s,I),...,x^{n-1}(s,I))-$$
$$f(s,y(s,I),...,y^{n-1}(s,I)))|\Big\}e^{-\rho u_2}\Big]+...+\sup_{u_n\in J}\Big[\int_{s_0}^{u_n}ds\Big\{\max_{1\le k\le n}|(f(s,x(s,I),...,x^{n-1}(s,I))-$$

$f(s, y(s, I), ..., y^{n-1}(s, I)))|\Big\}e^{-\rho u_n}\Big]$

$=$

$\sup_{u_1\in J}\Big[e^{-\rho u_1}\int_{s_0}^{u_1}...\int_{s_0}^{u_n} dsdu_n...du_2 d\Big\{f(s, x(s, I), ..., x^{n-1}(s, I)), f(s, y(s, I), ..., y^{n-1}(s, I))\Big\}\Big] +$
$\sup_{u_2\in J}\Big[e^{-\rho u_2}\int_{s_0}^{u_2}...\int_{s_0}^{u_n} dsdu_n...du_3 d\Big\{f(s, x(s, I), ..., x^{n-1}(s, I)), f(s, y(s, I), ..., y^{n-1}(s, I))\Big\}\Big] +$
$... + \sup_{u_n\in J}\Big[e^{-\rho u_n}\int_{s_0}^{u_n} dsd\Big\{f(s, x(s, I), ..., x^{n-1}(s, I)), f(s, y(s, I), ..., y^{n-1}(s, I))\Big\}\Big]$

$\leq \sup_{u_1\in J}\Big[e^{-\rho u_1}\int_{s_0}^{u_1}...\int_{s_0}^{u_n}\sum_{j=0}^{n-1} G_{j+1} d(x^j(s, I), y^j(s, I))dsdu_n...du_2\Big] + \sup_{u_2\in J}\Big[e^{-\rho u_2}\int_{s_0}^{u_2}...\int_{s_0}^{u_{n-1}}$
$\int_{s_0}^{u_n}\sum_{j=0}^{n-1} G_{j+1} d(x^j(s, I), y^j(s, I))dsdu_n...du_3\Big] + ... + \sup_{u_n\in J}\Big[e^{-\rho u_n}\int_{s_0}^{u_n}\sum_{j=0}^{n-1} G_{j+1}$
$d(x^j(s, I), y^j(s, I))ds\Big]$

$\leq \max\{G_1, G_2, ..., G_n\} D_{n-1}(x, y)\sum_{i=0}^{n-1}\sup_{u_{n-i}\in J}\{e^{-\rho u_{n-i}}\Phi_{i(u_{n-i})}\}$

For every continuous function F on $[a, b]$, we have

$$\int_a^y dy_n\int_a^{y_n} dy_{n-1}...\int_a^{y_2} F(y_1)dy_1 = \frac{1}{(n-1)!}\int_a^y (y-t)^{n-1}F(t)dt$$

Also, we have

$$\Phi_{\rho,i(u_{n-i})} = \int_{s_0}^{u_{n-i}}\int_{s_0}^{u_{n-i+1}}...\int_{s_0}^{u_n} e^{\rho s}dsdu_n...du_{n-i+1}$$
$$\Rightarrow \Phi_{\rho,i(u_{n-i})} = \frac{1}{i!}\int_{s_0}^{u_{n-i}}(u_{n-i}-s)^i e^{\rho s}ds \quad i = 0, 1, ..., n-1$$

Now, for $i = 0, 1, ..., n-1$, we have

$$\sup_{u_{n-i}\in J}\{e^{-\rho u_{n-i}}\Phi_{\rho,i(u_{n-i})}\} = \frac{1}{i!}\sup_{u_{n-i}\in J}\Big\{\int_{s_0}^{u_{n-i}}(u_{n-i}-s)^i e^{\rho s}dse^{-\rho u_{n-i}}\Big\}$$
$$\Rightarrow \sup_{u_{n-i}\in J}\{e^{-\rho u_{n-i}}\Phi_{\rho,i(u_{n-i})}\} \leq \frac{1}{i!}\sup_{u_{n-i}\in J}\Big\{(S-s_0)^i\int_{s_0}^{u_{n-i}} e^{-\rho(u_{n-i}-s)}ds\Big\}$$
$$\Rightarrow \sup_{u_{n-i}\in J}\{e^{-\rho u_{n-i}}\Phi_{\rho,i(u_{n-i})}\} \leq \frac{1}{i!}\sup_{u_{n-i}\in J}\Big\{(S-s_0)^i\frac{1-e^{-\rho(u_{n-i}-s_0)}}{\rho}\Big\}$$
$$\Rightarrow \sup_{u_{n-i}\in J}\{e^{-\rho u_{n-i}}\Phi_{\rho,i(u_{n-i})}\} \leq \frac{1}{i!}(S-s_0)^i\frac{1-e^{-\rho(S-s_0)}}{\rho} \to 0 \ as\ \rho\to\infty$$

So, for $\rho > 0$, if we choose, $\max\{G_1, G_2, ..., G_n\}\sum_{i=0}^{n-1}(b-s_0)^i\frac{1-e^{-\rho(b-s_0)}}{\rho i!} \leq 1$

Then, we get

$$D_{n-1}(B_{n-1}x, B_{n-1}y) \leq D_{n-1}(x, y)\max\{G_1, G_2, ..., G_n\}\sum_{i=0}^{n-1}(b-s_0)^i\frac{1-e^{-\rho(b-s_0)}}{\rho i!}$$

This shows that D_{n-1} is a contraction mapping and hence $\exists$ a unique solution for (1).

Example 4.1 *Let us consider the following neutrosophic cauchy problem:*

$$\begin{aligned} y^n(s,I) = w_1 y(s,I) + w_2 y'(s,I) + ... + w_n y^{n-1}(s,I) + B(s) \quad s \in [s_0, S] \\ y(s_0,I) = l_1, \quad y'(s_0,I) = l_2, \quad ..., \quad y^{n-1}(s_0,I) = l_n \end{aligned} \tag{4}$$

Where $w_j, l_j; \quad j = 1,2,...,n$ *are NRN which are of the form* $w_{j1} + w_{j2}I$, $l_{j1} + l_{j2}I$ *respectively and* $w_{j1}, w_{j2}, l_{j1}, l_{j2} \in R$; $B \in C([s_0,S], R_N^n)$.

Let us define the function f *by*

$$f(s, y_1, y_2, ..., y_n) = w_1 y_1 + w_2 y_2 + ... + w_n y_n + B(s)$$

which satisfies (3).

Now,

$$\begin{aligned} & d(f(s, y_1, y_2, ..., y_n), f(s, z_1, z_2, ..., z_n)) \\ & = d(w_1 y_1 + w_2 y_2 + ... + w_n y_n + B(s), w_1 z_1 + w_2 z_2 + ... + w_n z_n + B(s)) \\ & \leq \sum_{j=1}^{n} d(w_j y_j, w_j z_j) \\ & \leq \sum_{j=1}^{n} |w_j| d(y_j, z_j) \quad \forall \quad s \in [s_0, S]; \quad y_j, z_j \in R_N^n. \end{aligned}$$

Thus from the above Theorem 4.2, we can conclude that the neutrosophic cauchy problem (4) has a unique solution in $C^n(J, R_N^n)$.

5. Application

Many scientific and engineering problems require the use of neutrosophic differential equations. These problems arise in various fields, including electronics, mechanics, medicine, communication, transportation, and the industrial sector. The equations are used to analyze the behavior of phenomena that are subject to ambiguity or uncertainty. The following are a few real-world applications of nth-order neutrosophic differential equations in different disciplines where we are planning to use our study.

(1) *In electrical engineering, nth-order neutrosophic differential equations are used to describe the behavior of circuits containing resistors, inductors, and capacitors.*

(2) *In telecommunications, nth-order neutrosophic differential equations can be used to represent how signals propagate along transmission lines, particularly for high-frequency transmissions.*

(3) *To ensure the stability and safety of structures like buildings and bridges, vibrations and oscillations are studied using nth-order neutrosophic differential equations.*

(4) *We can utilize nth-order neutrosophic differential equations to represent the motion of a mass linked to a spring and a damper and to characterize the system's response to outside influences.*

(5) *To evaluate the behavior of complex pendulum systems or multi-degree-of-freedom pendulum systems, nth-order neutrosophic differential equations are used.*

(6) *Temperature distribution is studied using nth-order neutrosophic differential equations, which are used to analyze the heat transport in materials over time.*

6. Conclusions

In this article, we defined the n-th order derivative of neutrosophic real functions with suitable examples. In addition, we have discussed the integral equation of form (2) to obtain the existence and uniqueness of the solution to the n-th order neutrosophic differential equation. This form of integral equation can be used to obtain the approximate solution to the given differential equation (1).

In the future, we are going to solve some real-life problems associated with uncertain data such as differential areas in mechanical, electrical, communication, and civil engineering fields.

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