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A Study on some Properties of Fourier Integrals Based on Neutrosophic Function

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Abstract. Fourier transforms is one of the oldest and a well-known technique in field of mathematic and engineering mathematical work. Fourier transform method represents the variable as a summation of complex exponentials. Fourier analysis has been used in signal processing and digital image processing for the analysis of a single image as a two-dimensional wave form, and many other type of form like Quantum mechanics, Signal processing, Image Processing. This analysis also represents filters, Transformation, representation, and encoding, Data Processing, Analysis and many more fields. In this article, some basics of Fourier Integrals have been discussed in terms of neutrosophic set. Dirichlet's Conditions, Fourier integral formula and it's five different forms are studied based on neutrosophic set. This article includes the F.T., F.S.T. and F.C.T. of a neutrosophic function and their inversion formulae. In this study, some properties of F.T. are discussed for a neutrosophic function. This study will help to get better results in signal processing, image processing, and in other fields also. This serves as an overview of the Fourier integral of a neutrosophic function.

Keywords: Fourier integral, F.T., neoutrosophic function.

Abreviation: F.T.= Fourier transform

F.S.T= Fourier sine transform

F.C.T.= Fourier cosine transform

N.R.N.= Neutrosophic real number

N.C.N.= Neutrosophic complex number

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1. Introduction:

In place of the present logic, F. Smarandache[1,2] put forward the neutrosophic concept to represent a mathematical model of undetermined, uncertainity, vagueness, unclearness, incompleteness, inconsistency, redundancy, in which the neutrosophy is a new concept brought in by Smarandache[1,2]. He defined the N.R.N. in standard form and the considerations for existence of N.R.N.s' division. The standard form of N.C.N.s is also deduced and the root index is found as $n \geq 2$ [3,4]. Studying the concept of neutrosophic probability [1,5] and the neutrosophic statistics [4,6], Professor Smarandache , for the first time studied the concept of preliminary calculus by introducing ideas of neutrosophic mereo-limit, mereo-continuity, mereo-derivative and mereo-integral [7,8].Madeleine Al-Taha produced results on single valued neutrosophic (weak) polygon [9]. Edalatpanah purport a new direct algorithm for the solution of neutrosophic linear programming in which the variables and right hand side are expressed with triangular neutrosophic numbers[10]. Pentagonal neutrosophic number is used in Networking problem and Shortest Path problem by Chakraborty [11,12].

A. Kharal[15] presents a method of multicriteria decision making using neutrosophic sets. A. A. Salama and F. Smarandache [16] used neutrosophic set to introduce new types of neutrosophic crisp sets with three types 1, 2, 3. D. Koundal, S. Gupta and S. Singh [17] demonstrates the use of neutrosophic theory in medical image denoising and segmentation, using which the performance is observed to be much better.

The concepts of Neutrosophic set have been used in different areas of Mathematics. Here we are using this concepts in Fourier integral and Fourier transform. Fourier integral represents a certain type of non periodic functions that are defined on either $(-\infty, \infty)$ or $(0, \infty)$. Fourier transform is a mathematical tool used to decompose a signal into its constituent frequency components. It breaks down signals into a combination of sines and cosines, which can be used to analyse the frequency content of a signal. Fourier analysis has been used in digital image and processing of image and for analysis of a single image into a two-dimensional wave form, and more recently has been used for magnetic resonance imaging, angiographic assessment, automated lung segmentation and image quality assessment and Mobile stethoscope [18]. Fourier transforms which is also used in frequency domain representation. Fourier analysis used as time series analysis proved its application in Quantum mechanics; Signal processing, Image Processing and filters, representation, Data Processing and Analysis and many more.

Fourier transforms are obviously very essential to conduct of Fourier spectroscopy, and that alone would justify its importance. Fourier transforms are very vital in other pursuits as well; such as electrical signal analysis, diffraction, optical testing, optical processing, imaging, holography, and also for remote sensing [13, 14]. Thus, knowledge of Fourier transforms can be a springboard to many other fields. The main idea behind Fourier transforms is that a function of direct time can be expressed as a complex valued function of reciprocal space, that is, frequency.

In this paper we are studying the Fourier integral and Fourier transform by using the concept of Neutrosophic set. The use of Neutrosophic concept in the Fourier transform will help to get the frequencies more accurately which constitude a signal or a sound and can be filter out the unwanted frequencies in more appropriately. This study will give better results in signal processing, image processing and filtering, noise filtering, etc.

This paper consist of 6 sections. The 1st section provides an introduction, in which review of Fourier transform has been given. In 2nd section, definition of neutrosophic real number is given and discussed about division of two neutrosophic real numbers. The 3rd section gives the knowledge of Fourier integral theorem for a neutrosophic function and different forms of neutrosophic Fourier integrals. In the 4th section, we have discussed about the F.T. of a neutrosophic function which includes Fourier sine transform and Fourier cosine transform of a neutrosophic function. In the 5th section we studied about the properties of F.T. of a neutrosophic function and their proofs. The 6th section is the conclusion of the article.

2. **Preliminaries:**

In this part, definitions of N.R.N. and division of two N.R.N.s are discussed.

2.1. *Neutrosophic Real Number[4]:*

If a number that can be written in the form $p_n + q_n I$, where p_n, q_n are real numbers and I is an indeterminate number such that $I.0 = 0$ and $I^N = I$, for all natural number **N**, is called N.R.N.. Here we denote the N.R.N. by w, and thus we can write $w = p_n + q_n I$ and it is knwon as the standard form of N.R.N..

2.2. *Division of two N.R.N.s[4]:*

Consider that w_1 and w_2 be two N.R.N.s where, $w_1 = p_{n1} + q_{n1} I$ and $w_2 = p_{n2} + q_{n2} I$. Then the division of these N.R.N.s, i.e. $(p_{n1} + q_{n1} I) \div (p_{n2} + q_{n2} I)$ is given by

$$\frac{p_{n1} + q_{n1} I}{p_{n2} + q_{n2} I} = \frac{p_{n1}}{p_{n2}} + \frac{p_{n2} q_{n1} - p_{n1} q_{n2}}{p_{n2}(p_{n2} + q_{n2})}.I$$

provided $p_{n2}(p_{n2} + q_{n2}) \neq 0$ or $p_{n2} \neq 0$ and $p_{n2} \neq -q_{n2}$

3. **Neutrosophic Dirichlet's Conditions:**

Dirichlet's conditions are those condition which must be satisfied by a function $f(x)$ to be expanded by using Fourier series. There are three Dirichlet's conditions, which must be satisfied by the function $f(x)$. In this article, we shall discuss the Dirichlet's conditions by using the concept of neutrosophic set.

Consider, $f^N(x, I)$ be any neutrosophic function satisfying the conditions given below:

(i) $f^N(x, I)$ is defined in the interval $-(h + I) < x < (h + I)$.

(ii) $f^N(x,I)$ and $f'^N(x,I)$ are sectionally continous in $-(h+I) < x < (h+I)$.
(iii) $f^N(x,I)$ is periodic with period $2(h+I)$.
Then the above conditions are called neutrosophic Dirichlet's conditions.
Here, I is called indeterminate number such that $I.0 = 0$ and $I^n = I, n \in \mathbf{N}$, the set of natural numbers.

4. **Fourier Integral Theorem (or Formula) for a neutrosophic function. Fourier Integral representation for a neutrosophic function:**

Consider, $f^N(x,I)$ be a neutrosophic function and it satisfy the following statements:
(a) $f^N(x,I)$ fulfill neutrosophic Dirichlet's conditions in each finite interval $-(h+I) \leq x \leq (h+I)$
(b) $\int_{-\infty}^{\infty} |f^N(x,I)|dx$ converges that means, $f^N(x,I)$ is absolutely integrable in $-\infty < x < \infty$.
Then Fourier integral theorem for a neutrosophic function will be

$$f^N(x,I) = \int_0^{\infty} \{A_N(s,I)coss(p_n+q_nI)x + B_N(s,I)sins(p_n+q_nI)x\}ds \tag{1}$$

$$where,\ A_N(s,I) = \frac{1}{\pi}\int_{-\infty}^{\infty} f^N(u,I)coss(p_n+q_nI)udu \tag{2}$$

$$and\ B_N(s,I) = \frac{1}{\pi}\int_{-\infty}^{\infty} f^N(u,I)sins(p_n+q_nI)udu \tag{3}$$

Thus Fourier integral theorem of a neutrosophic function is also re-written as

$$f^N(x,I) = \frac{1}{2\pi}\int_{s=-\infty}^{\infty}\int_{u=-\infty}^{\infty} f^N(u,I)coss\{(p_n+q_nI)(x-u)\}duds \tag{4}$$

5. **Different forms of Fourier integral theorem of a neutrosophic function:**

There are five different forms of Fourier integral theorem in classical method. These five forms are discussed in the sense of neutrosophic set as given below.

(i) General Form:

$$f^N(x,I) = \frac{1}{\pi}\int_{-\infty}^{\infty} f^N(u,I)\left\{\int_0^{\infty} coss\{(p_n+q_nI)(x-u)\}ds\right\}du \tag{5}$$

Proof: When x is a point of continuity of $f^N(x,I)$, we have

$$f^N(x,I) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f^N(u,I)coss((p_n+q_nI)(x-u))du\right\}ds$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f^N(u,I)\left\{\int_{-\infty}^{\infty} coss((p_n+q_nI)(x-u))ds\right\}du$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f^N(u,I)\left\{2\int_{0}^{\infty} coss((p_n+q_nI)(x-u))ds\right\}du,$$

$[using\ the\ property\ of\ definite\ integral$

$$\int_{-a}^{a} f^N(x,I)dx = 2\int_0^a f(x,I)dx\ if\ f^N(-x,I) = f^N(x,I)]$$
$$= \frac{1}{\pi}\int_{u=-\infty}^{\infty} f^N(u,I)\left\{\int_{s=0}^{\infty} coss((p_n+q_nI)(x-u))ds\right\}du$$

(ii) Another General Form: If $f^N(x,I)$ is continous at x, then

$$f^N(x,I) = \int_0^{\infty}\left\{A_N(s,I)coss(p_n+q_nI)x + B_N(s,I)sins(p_n+q_nI)x\right\}ds \tag{6}$$

$$where,\ A_N(s,I) = \frac{1}{\pi}\int_{-\infty}^{\infty} f^N(u,I)coss(p_n+q_nI)udu \tag{7}$$

$$and\ B_N(s,I) = \frac{1}{\pi}\int_{-\infty}^{\infty} f^N(u,I)sins(p_n+q_nI)udu \tag{8}$$

Proof: We have

$$f^N(x,I) = \frac{1}{\pi}\int_{-\infty}^{\infty} f^N(u,I)\left\{\int_{-\infty}^{\infty} coss(p_n+q_nI)(x-u)ds\right\}du$$
$$= \frac{1}{\pi}\int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} f^N(u,I)coss(p_n+q_nI)(x-u)du\right\}ds$$
$$= \frac{1}{\pi}\int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} f^N(u,I)coss((p_n+q_nI)(x-u))ds\right\}du,$$
$$= \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f^N(u,I)\{coss((p_n+q_nI)xcoss(p_n+q_nI)u$$
$$+ sins(p_n+q_nI)xsins(p_n+q_nI)u)du\}ds$$
$$= \int_0^{\infty}[coss(p_n+q_nI)x\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} f^N(u,I)coss(p_n+q_nI)udu\right\}$$
$$+ sins(p_n+q_nI)x\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} f^N(u,I)sins(p_n+q_nI)udu\right\}]ds$$
$$= \int_0^{\infty}\left\{A_N(s,I)coss(p_n+q_nI)x + B_N(s,I)sins(p_n+q_nI)x\right\}ds,\ [using (7) and (8)]$$

(iii) Fourier Sine Integral formula for a Neutrosophic Function:

If $f^N(x,I)$ is an odd neutrosophic function, then

$$f^N(x,I) = \frac{2}{\pi}\int_0^{\infty} sins(p_n+q_nI)x\left\{\int_0^{\infty} f^N(u,I)sins(p_n+q_nI)udu\right\}ds \tag{9}$$

Proof: Proceeding as in form (ii), we have

$$= \frac{1}{\pi}\int_0^\infty [coss(p_n+q_nI)x\int_{-\infty}^\infty f^N(u,I)coss(p_n+q_nI)udu$$
$$+sins(p_n+q_nI)x\int_{-\infty}^\infty f^N(u,I)sins(p_n+q_nI)udu]ds \tag{10}$$

Since, $f^N(x,I)$ is an odd function, it follows that $f^N(u,I)coss(p_n+q_nI)u$ is an odd function and $f^N(u,I)sins(p_n+q_nI)u$ is an even function and so by property of definite integral, we have

$$\int_{-\infty}^\infty f^N(u,I)coss(p_n+q_nI)udu = 0$$
$$and \int_{-\infty}^\infty f^N(u,I)sins(p_n+q_nI)udu = 2\int_0^\infty f^N(u,I)sins(p_n+q_nI)udu$$

Therefore (10) implies

$$f^N(x,I) = \frac{2}{\pi}\int_0^\infty sins(p_n+q_nI)x\left\{\int_0^\infty f^N(u,I)sins(p_n+q_nI)udu\right\}ds$$

(iv) Fourier Cosine Integral formula for a Neutrosophic Function:

If $f^N(x,I)$ is an even neutrosophic function, then

$$f^N(x,I) = \frac{2}{\pi}\int_0^\infty coss(p_n+q_nI)x\left\{\int_0^\infty f^N(u,I)coss(p_n+q_nI)udu\right\}ds \tag{11}$$

Proof: Proceeding as in (ii), we get

$$= \frac{1}{\pi}\int_0^\infty [coss(p_n+q_nI)x\int_{-\infty}^\infty f^N(u,I)coss(p_n+q_nI)udu$$
$$+sins(p_n+q_nI)x\int_{-\infty}^\infty f^N(u,I)sins(p_n+q_nI)udu]ds \tag{12}$$

Since, $f^N(x,I)$ is an even function, it follows that $f^N(u,I)coss(p_n+q_nI)u$ is an even function and $f^N(u,I)sins(p_n+q_nI)u$ is an odd function and hence we have

$$\int_{-\infty}^\infty f^N(u,I)coss(p_n+q_nI)udu = 2\int_0^\infty f^N(u,I)coss(p_n+q_nI)udu$$
$$and \int_{-\infty}^\infty f^N(u,I)sins(p_n+q_nI)udu = 0$$

Therefore (13) implies that

$$f^N(x,I) = \frac{2}{\pi}\int_0^\infty coss(p_n+q_nI)x\left\{\int_0^\infty f^N(u,I)coss(p_n+q_nI)udu\right\}ds$$

(v) Complex or Exponential form of Fourier Integral Formula for a Neutrosophic Function:

$$f^N(x,I) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-is(p_n+q_nI)x}\left\{\int_{-\infty}^\infty f^N(u,I)e^{is(p_n+q_nI)u}du\right\}ds \tag{14}$$

Proof: From general form (i) of Fourier integral theorem of a neutrosophic function, we have

$$f^N(x,I) = \frac{1}{\pi}\int_{-\infty}^{\infty} f^N(u,I)\left\{\int_0^{\infty} coss(p_n+q_nI)(x-u)du\right\}ds$$
$$= \frac{1}{\pi}\int_0^{\infty}\left\{\int_{-\infty}^{\infty} f^N(u,I)coss(p_n+q_nI)(x-u)ds\right\}du$$
$$(changing\ the\ order\ of\ integration)$$
$$= \frac{1}{\pi}\int_0^{\infty}\left\{\int_{-\infty}^{\infty} f^N(u,I)\frac{e^{is(p_n+q_nI)(x-u)}+e^{-is(p_n+q_nI)(x-u)}}{2}du\right\}ds$$
$$= \frac{1}{2\pi}\int_0^{\infty} e^{is(p_n+q_nI)x}\left\{\int_{-infty}^{\infty} f^N(u,I)e^{-is(p_n+q_nI)u}du\right\}ds$$
$$+ \frac{1}{2\pi}\int_0^{\infty} e^{-is(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty} f^N(u,I)e^{is(p_n+q_nI)u}du\right\}ds \quad (15)$$

substituting $s=-y$ and $ds=-dy$ in the first integral only of (15), we get

$$f^N(x,I) = \frac{1}{2\pi}\int_0^{-\infty} e^{-iy(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty} f^N(u,I)e^{iy(p_n+q_nI)u}du\right\}(-dy)$$
$$+ \frac{1}{2\pi}\int_0^{\infty} e^{-is(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty} f^N(u,I)e^{is(p_n+q_nI)u}du\right\}ds$$
$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{-iy(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty} f^N(u,I)e^{iy(p_n+q_nI)u}du\right\}(dy)$$
$$+ \frac{1}{2\pi}\int_0^{\infty} e^{-is(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty} f^N(u,I)e^{is(p_n+q_nI)u}du\right\}ds$$
$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{-is(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty} f^N(u,I)e^{iy(p_n+q_nI)u}du\right\}(ds)$$
$$+ \frac{1}{2\pi}\int_0^{\infty} e^{-is(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty} f^N(u,I)e^{is(p_n+q_nI)u}du\right\}ds$$
$$Therefore\ f^N(x,I) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-is(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty} f^N(u,I)e^{is(p_n+q_nI)u}du\right\}ds$$

Note: In (15), substituting $s=-y$ and $ds=-dy$ in the second integral only in place of first integral and proceeding as before, we get

$$f^N(x,I) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\left[\int_{-\infty}^{\infty} f^N(u,I)e^{-is(p_n+q_nI)u}du\right]ds$$

which is another form of complex form of Fourier integral formula in terms a neutrosophic function.

Example 1: Using cosine integral formula of a neutrosophic function, we get the following result:

$$\int_0^{\infty}\frac{cos(p_n+q_nI)\lambda x}{\lambda^2+1}d\lambda = \frac{\pi(p_n+q_nI)}{2}e^{-(p_n+q_nI)x}, x\geq 0$$

Solution: Fourier cosine integral formula for a neutrosophic function is

$$f^N(x,I) = \frac{2}{\pi}\int_0^{\infty} cos\lambda(p_n+q_nI)x\left\{\int_0^{\infty} f^N(u,I)cos\lambda(p_n+q_nI)udu\right\}d\lambda \quad \text{(i)}$$

Taking $f^N(x,I) = e^{-(p_n+q_nI)x}$ *so that* $f^N(u,I) = e^{-(p_n+q_nI)u}$

Using these in (i), we get

$$e^{-(p_n+q_nI)x} = \frac{2}{\pi}\int_0^\infty cos\lambda(p_n+q_nI)x\left\{\int_0^\infty e^{-(p_n+q_nI)u}cos\lambda(p_n+q_nI)udu\right\}d\lambda$$
$$= \frac{2}{\pi}\int_0^\infty cos\lambda(p_n+q_nI)x\left\{\int_0^\infty e^{-v}cos\lambda v\frac{dv}{(p_n+q_nI)}\right\}d\lambda$$
$$[Taking\ v = (p_n+q_nI)u\ so\ that\ dv = (p_n+q_nI)du$$
$$as\ u \to 0, v \to 0\ and\ u \to \infty, v \to \infty]$$
$$= \frac{2}{(p_n+q_nI)\pi}\int_0^\infty cos\lambda(p_n+q_nI)x\left\{\int_0^\infty e^{-v}cos\lambda vdv\right\}d\lambda$$
$$= \frac{2}{(p_n+q_nI)\pi}\int_0^\infty cos\lambda(p_n+q_nI)x\left\{\frac{1}{1+\lambda^2}\right\}d\lambda$$
$$= \frac{2}{(1+\lambda^2)(p_n+q_nI)\pi}\int_0^\infty cos\lambda(p_n+q_nI)xd\lambda$$
$$e^{-(p_n+q_nI)x} = \frac{2}{(p_n+q_nI)\pi}\int_0^\infty \frac{cos\lambda(p_n+q_nI)x}{(1+\lambda^2)}d\lambda$$
$$Therefore,\ \int_0^\infty \frac{cos\lambda(p_n+q_nI)x}{(1+\lambda^2)}d\lambda = \frac{\pi(p_n+q_nI)}{2}e^{-(p_n+q_nI)x}, x \geq 0$$

Example 2: Using the Fourier cosine integral formula, we get the following result:

$$e^{-(p_n+q_nI)x}cos(p_n+q_nI)x = \left(\frac{1}{p_n} - \frac{q_n}{p_n(p_n+q_n)}I\right)\frac{2}{\pi}\int_0^\infty \frac{(s^2+2)coss(p_n+q_nI)x}{s^2+4}ds$$

Solution: Fourier Cosine Integral formula is given by

$$f^N(x,I) = \frac{2}{\pi}\int_0^\infty coss(p_n+q_nI)x\left\{\int_0^\infty f(u,I)coss(p_n+q_nI)udu\right\}ds \qquad \text{(i)}$$

Solution: Taking $f^N(x,I) = e^{-(p_n+q_nI)x}cos(p_n+q_nI)x$ so that $f^N(u,I) = e^{-(p_n+q_nI)u}cos(p_n+q_nI)u$ Using these in (i), we get

$$
\begin{aligned}
& e^{-(p_n+q_nI)x}cos(p_n+q_nI)x \\
&= \frac{2}{\pi}\int_0^\infty coss(p_n+q_nI)x\left\{\int_0^\infty e^{-(p_n+q_nI)u}cos(p_n+q_nI)ucoss(p_n+q_nI)udu\right\}ds \\
&= \frac{1}{\pi}\int_0^\infty coss(p_n+q_nI)x\left\{\int_0^\infty e^{-(p_n+q_nI)u}(2cos(p_n+q_nI)ucoss(p_n+q_nI)u)du\right\}ds \\
&= \frac{1}{\pi}\int_0^\infty coss(p_n+q_nI)x\left[\int_0^\infty e^{-(p_n+q_nI)u}\{cos(p_n+q_nI)(s+1)u+cos(p_n+q_nI)(s-1)u\}du\right]ds \\
&= \frac{1}{\pi}\int_0^\infty coss(p_n+q_nI)x[\int_0^\infty e^{-(p_n+q_nI)u}cos(p_n+q_nI)(s+1)udu \\
&\quad +\int_0^\infty e^{-(p_n+q_nI)u}cos(p_n+q_nI)(s-1)udu]ds \\
&= \frac{1}{\pi}\int_0^\infty coss(p_n+q_nI)x\left[\frac{(p_n+q_nI)}{(p_n+q_nI)^2+(p_n+q_nI)^2(s+1)^2}+\frac{(p_n+q_nI)}{(p_n+q_nI)^2+(p_n+q_nI)^2(s-1)^2}\right]ds \\
&= \frac{1}{\pi}\int_0^\infty coss(p_n+q_nI)x.\frac{1}{(p_n+q_nI)}\left[\frac{1}{1+(s+1)^2}+\frac{1}{1+(s-1)^2}\right]ds \\
&= \frac{1}{(p_n+q_nI)\pi}\int_0^\infty coss(p_n+q_nI)x\left[\frac{(s^2-2s+2)+(s^2+2s+2)}{(s^2+2s+2)(s^2-2s+2)}\right]ds \\
&= \frac{1}{(p_n+q_nI)\pi}\int_0^\infty \frac{2(s^2+2)}{(s^2+2)^2-(2s)^2}coss(p_n+q_nI)xds \\
&= \frac{2}{(p_n+q_nI)\pi}\int_0^\infty \frac{2(s^2+2)coss(p_n+q_nI)x}{s^4+4}ds \\
&= \left(\frac{1}{p_n}-\frac{q_n}{p_n(p_n+q_n)}I\right)\frac{2}{\pi}\int_0^\infty \frac{2(s^2+2)coss(p_n+q_nI)x}{s^4+4}ds
\end{aligned}
$$

Example 3: Using Fourier sine integral formula, the following result is obtained

$$
e^{-(a_n+b_nI)x}-e^{-(c_n+d_nI)x} = \frac{2(p_n+q_nI)[(c_n+d_nI)^2-(a_n+b_nI)^2]}{\pi} \int_0^\infty \frac{sins(p_n+q_nI)xds}{\{(a_n+b_nI)^2+s^2(p_n+q_nI)^2\}\{(c_n+d_nI)^2+s^2(p_n+q_nI)^2\}}.
$$

Solution: Fourier sine integral formula is given by

$$
f^N(x,I) = \frac{2}{\pi}\int_0^\infty sins(p_n+q_nI)x\left\{\int_0^\infty f^N(u,I)sins(p_n+q_nI)udu\right\}ds \qquad \text{(i)}
$$

Taking $f^N(x,I) = e^{-(a_n+b_nI)x} - e^{-(c_n+d_nI)x}$
so that $f^N(u,I) = e^{-(a_n+b_nI)u} - e^{-(c_n+d_nI)u}$

Using these in (i), we get

$$e^{-(a_n+b_nI)x} - e^{-(c_n+d_nI)x}$$
$$= \frac{2}{\pi}\int_0^\infty sins(p_n+q_nI)x\left\{\int_0^\infty (e^{-(a_n+b_nI)u} - e^{-(c_n+d_nI)u})sins(p_n+q_nI)udu\right\}ds$$
$$= \frac{2}{\pi}\int_0^\infty sins(p_n+q_nI)x\left\{\int_0^\infty e^{-(a_n+b_nI)u}sins(p_n+q_nI)udu - \int_0^\infty e^{-(c_n+d_nI)u}sins(p_n+q_nI)udu\right\}ds$$
$$= \frac{2}{\pi}\int_0^\infty sins(p_n+q_nI)x\left[\frac{s(p_n+q_nI)}{(a_n+b_nI)^2+s^2(p_n+q_nI)^2} - \frac{s(p_n+q_nI)}{(c_n+d_nI)^2+s^2(p_n+q_nI)^2}\right]ds$$
$$= \frac{2}{\pi}\int_0^\infty sins(p_n+q_nI)x.(p_n+q_nI)s\left[\frac{(c_n+d_nI)^2-(a_n+b_nI)^2}{\{(a_n+b_nI)^2+s^2(p_n+q_nI)^2\}\{(c_n+d_nI)^2+s^2(p_n+q_nI)^2\}}\right]ds$$
$$= \frac{2(p_n+q_nI)[(c_n+d_nI)^2-(a_n+b_nI)^2]}{\pi}\int_0^\infty \frac{sins(p_n+q_nI)xds}{\{(a_n+b_nI)^2+s^2(p_n+q_nI)^2\}\{(c_n+d_nI)^2+s^2(p_n+q_nI)^2\}}.$$

6. F.T.s or Complex F.T. of a Neutrosophic Function

Fourier transform is a mathematical model which helps to transform the signals between two different domains, such as transforming signal from frequency domain to time domain or vice versa. It is an integral transform that converts a function into a form that describes the frequencies present in the original function. The output of the transform is a complex-valued function of frequencies.

Definition: Let $\mathscr{F}(x,I)$ be a function defined on $(-\infty,\infty)$ and be piecewise continuous in each partial interval and absolutely integrable in $(-\infty,\infty)$. Then the F.T. of $\mathscr{F}(x,I)$ is a function of a new variable $'s'$ and it is denoted and defined as

$$\mathscr{F}\{\mathscr{F}(x,I)\} = \overline{\mathscr{F}}(s,I) = f^N(s,I) = \int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx \tag{16}$$

The function $\mathscr{F}(x,I)$ is then known as inverse F.T.s of $\overline{\mathscr{F}}(s,I)$ or $f^N(s,I)$ and is denoted by

$$\mathscr{F}(x,I) = \mathscr{F}^{-1}\{\overline{\mathscr{F}}(s,I)\}$$
$$\mathscr{F}(x,I) = \mathscr{F}^{-1}\{f^N(s,I)\} \tag{17}$$

6.1. *Inversion Formula for F.T. or Complex F.T. of a Neutrosophic Function:*

If $\overline{\mathscr{F}}(s,I)$ is the F.T. of $\mathscr{F}(x,I)$ and if $\mathscr{F}(x,I)$ satiesfies the Dirichlet's conditions in every finite interval $(-l-I, l+I)$ and further $\mathscr{F}(x,I)$ is absolutely integrable in $(-\infty,\infty)$, then at every point of continuity of $\mathscr{F}(x,I)$,

$$\mathscr{F}(x,I) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\mathscr{F}}(s,I)e^{-is(p_n+q_nI)x}ds$$

Proof: Complex Fourier integral formula is given by

$$\mathscr{F}(x,I) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-is(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty}\mathscr{F}(u,I)e^{is(p_n+q_nI)u}du\right\}ds \tag{18}$$

Complex F.T. of $\mathscr{F}(x, I)$ is given by

$$\overline{\mathscr{F}}(s, I) = \int_{-\infty}^{\infty} e^{is(p_n+q_nI)x} \mathscr{F}(x, I)dx$$
$$= \int_{-\infty}^{\infty} \mathscr{F}(u, I)e^{is(p_n+q_nI)u} du \tag{19}$$

From (18) and (19) we get

$$\mathscr{F}(x, I) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\mathscr{F}}(s, I)e^{-is(p_n+q_nI)x} ds \tag{20}$$

Which is the required inversion formula for complex F.T. of a neutrosophic function.
Thus, we have

$$\mathscr{F}\{\mathscr{F}(x, I)\} = \overline{\mathscr{F}}(s, I)$$
$$= f^N(s, I)$$
$$= \int_{-\infty}^{\infty} e^{is(p_n+q_nI)x} \mathscr{F}(x, I)dx \tag{21a}$$

and

$$\mathscr{F}(x, I) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-is(p_n+q_nI)x} \overline{\mathscr{F}}(s, I)ds$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-is(p_n+q_nI)x} f^N(s, I)ds \tag{21b}$$

Theorem 1. If $f^N(s, I)$ is F.T. of $\mathscr{F}(x, I)$, then

$$(i)\ \mathscr{F}\{\mathscr{F}(-x, I)\} = f^N(-s, I)$$
$$(ii)\ \mathscr{F}\{\overline{\mathscr{F}(-x, I)}\} = \overline{f^N(s, I)}$$
$$(iii)\ \mathscr{F}\{\overline{\mathscr{F}(x, I)}\} = \overline{f^N(-s, I)}$$

Where bar over a quantity represents its complex conjugate.

Proof: By definition,

$$\mathscr{F}\{\mathscr{F}(x, I)\} = f^N(s, I) = \int_{-\infty}^{\infty} \mathscr{F}(x, I)e^{is(p_n+q_nI)x} dx \tag{22}$$

Now

$$
\begin{aligned}
(i)\ \mathscr{F}\{\mathscr{F}(-x,I)\} &= \int_{-\infty}^{\infty} \mathscr{F}(-x,I)e^{is(p_n+q_nI)x}dx \\
&= \int_{\infty}^{-\infty} \mathscr{F}(y,I)e^{-is(p_n+q_nI)y}(-dy) \\
&\quad \{putting\ y=-x\ \&\ dx=-dy \\
&= \int_{-\infty}^{\infty} \mathscr{F}(y,I)e^{-is(p_n+q_nI)y}(dy) \\
&= \int_{-\infty}^{\infty} \mathscr{F}(x,I)e^{i(-s)(p_n+q_nI)y}(-dy) \\
&= f^N(-s,I)\ \{by(22) \\
(ii)\ \mathscr{F}\{\mathscr{F}(-x,I)\} &= \int_{-\infty}^{\infty} \overline{\{\mathscr{F}(-x,I)\}}e^{is(p_n+q_nI)x}dx \\
&= \int_{-\infty}^{\infty} \overline{\{\mathscr{F}(y,I)\}}e^{-is(p_n+q_nI)y}(-dy) \\
&\quad \{putting\ y=-x\ \&\ dx=-dy \\
&= \int_{-\infty}^{\infty} \overline{\{\mathscr{F}(x,I)\}}e^{-is(p_n+q_nI)x}dx \\
&= \overline{\left\{\int_{-\infty}^{\infty} \mathscr{F}(x,I)e^{is(p_n+q_nI)x}dx\right\}} \\
&= \overline{f^N(s,I)}\ by(22) \\
(iii)\ \mathscr{F}\{\overline{\mathscr{F}(x,I)}\} &= \int_{-\infty}^{\infty} \{\overline{\mathscr{F}(x,I)}\}e^{is(p_n+q_nI)x}dx \\
&= \left\{\int_{-\infty}^{\infty} \mathscr{F}(x,I)e^{i(-s)(p_n+q_nI)x}dx\right\} \\
&= \overline{f^N(-s,I)},\ by(22)
\end{aligned}
$$

7. F.S.T. or Infinite F.S.T. for a Neutrosophic Function:

Definition: Let $\mathscr{F}(x,I)$ be a neutrosophic function defined on $(-\infty,\infty)$ and be piecewise contininous in each partial interval and absolutely integrable in $(-\infty,\infty)$. Then the infinte F.S.T. of $\mathscr{F}(x,I)$ is a function of a new variable 's' and it is denoted and defined as

$$\mathscr{F}_s\{\mathscr{F}(x,I)\} = \overline{\mathscr{F}_s}(x,I) = f_s^N(s,I) = \int_0^{\infty} \mathscr{F}(x,I)sins(p_n+q_nI)xdx \tag{23}$$

The function $\mathscr{F}(x,I)$ is then called inverse Fourier sine transfom of $\overline{\mathscr{F}_s}(s,I)$ or $f_s^N(s,I)$ and is denoted by

$$\mathscr{F}(x,I) = \mathscr{F}_s{}^{-1}\{\mathscr{F}_s(s,I)\}\ or\ \mathscr{F}(x,I) = \mathscr{F}_s{}^{-1}\{f_s^N(s,I)\} \tag{24}$$

7.1. *Inversion Formula for Infinite F.S.T. of a Neutrosophic Function:*

If $\overline{\mathscr{F}_s}(s,I)$ is the infinite F.S.T. of $\mathscr{F}(x,I)$ and if $\mathscr{F}(x,I)$ satisfies the Dirichlet conditions in every finite interval $(-l-I, l+I)$ and further if $\mathscr{F}(x,I)$ is absolutely integrable in $(-\infty,\infty)$, then at every point of continuity of $\mathscr{F}(x,I)$, then we have

$$\mathscr{F}(x,I) = \frac{2}{\pi}\int_0^\infty \overline{\mathscr{F}_s}(s,I)sins(p_n+q_nI)xds$$

Proof: Fourier sine integral formula is given by

$$\mathscr{F}(x,I) = \frac{2}{\pi}\int_0^\infty sins(p_n+q_nI)x\left\{\int_0^\infty \mathscr{F}(u,I)sins(p_n+q_nI)udu\right\}ds \tag{25}$$

Now, infinite F.S.T. of $\mathscr{F}(x,I)$ is given by

$$\begin{aligned}\overline{\mathscr{F}_s}(s,I) &= \int_0^\infty \mathscr{F}(x,I)sins(p_n+q_nI)xdx\\ &= \int_0^\infty \mathscr{F}(u,I)sins(p_n+q_nI)udu \qquad (26)\end{aligned}$$

Thus, from (25) and (26), we get

$$\mathscr{F}(x,I) = \frac{2}{\pi}\int_0^\infty \overline{\mathscr{F}_s}(s,I)sins(p_n+q_nI)xds \tag{27}$$

Which is the required inversion formula for infinite F.S.T..

Thus, we have

$$\begin{aligned}\mathscr{F}_s\{\mathscr{F}(x,I)\} &= \overline{\mathscr{F}_s}(s,I)\\ &= f_s^N(s,I)\\ &= \int_0^\infty \mathscr{F}(x,I)sins(p_n+q_nI)xds \qquad (28a)\\ and\ \mathscr{F}(x,I) &= \mathscr{F}_s{}^{-1}\{\overline{\mathscr{F}_s}(s,I)\}\\ &= \frac{2}{\pi}\int_0^\infty \overline{\mathscr{F}_s}(s,I)sins(p_n+q_nI)xds \qquad (28b)\end{aligned}$$

Equations (28a) and (28b) are also defined as

$$\begin{aligned}\mathscr{F}_s\{\mathscr{F}(x,I)\} &= \overline{\mathscr{F}_s}(s,I)\\ &= f_s^N(s,I)\\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \mathscr{F}(x,I)sins(p_n+q_nI)xds \qquad (28a^*)\\ and\ \mathscr{F}(x,I) &= \mathscr{F}_s{}^{-1}\{\overline{\mathscr{F}_s}(s,I)\}\\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \overline{\mathscr{F}_s}(s,I)sins(p_n+q_nI)xds \qquad (28b^*)\end{aligned}$$

8. **F.C.T. or Infinite F.C.T. of a Neutrosophic Function:**

Definition: Let $\mathscr{F}(x,I)$ be a neutrosophic function defined on $(-\infty,\infty)$ and be piecewise continuous in each partial interval and absolutely integrable in $(-\infty,\infty)$. Then the infinite F.C.T. of $\mathscr{F}(x,I)$ is a function of a new variable 's' and it is denoted and defined as

$$\mathscr{F}_c\{\mathscr{F}(x,I)\} = \overline{\mathscr{F}_c}(s,I) = f_c^N(s,I) = \int_0^\infty \mathscr{F}(x,I)coss(p_n+q_nI)xdx \tag{29}$$

The function $\mathscr{F}(x,I)$ is then called inverse F.C.T. of $\overline{\mathscr{F}_c}(s,I)$ or $f_c^N(s,I)$ and is denoted by

$$\mathscr{F}(x,I) = \mathscr{F}_c{}^{-1}\{\overline{\mathscr{F}_c}(s,I)\}$$
$$or\ \mathscr{F}(x,I) = \mathscr{F}_c{}^{-1}\{f_c^N(s,I)\} \tag{30}$$

8.1. *Inversion Formula for Infinite F.C.T. of a Neutrosophic Function:*

If $\overline{\mathscr{F}_c}(s,I)$ is the infinite F.C.T. of $\mathscr{F}(x,I)$ and if $\mathscr{F}(x,I)$ satisfies Dirichlet's conditions in every finite interval $(-l-I,l+I)$ and further if $\mathscr{F}(x,I)$ is absolutely integrable in $(-\infty,\infty)$ then at every point of continuity of $\mathscr{F}(x,I)$, we have

$$\mathscr{F}(x,I) = \frac{2}{\pi}\int_0^\infty \overline{\mathscr{F}_c}(s,I)coss(p_n+q_nI)xds$$

Proof: Fourier cosine integral formula is given by

$$\mathscr{F}(x,I) = \frac{2}{\pi}\int_0^\infty coss(p_n+q_nI)x\left\{\int_0^\infty \mathscr{F}(u,I)coss(p_n+q_nI)udu\right\}ds \tag{31}$$

Now, infinite F.C.T. of $\mathscr{F}(x,I)$ is given by

$$\overline{\mathscr{F}_c}(s,I) = \int_0^\infty \mathscr{F}(x,I)coss(p_n+q_nI)xdx$$
$$= \int_0^\infty \mathscr{F}(u,I)coss(p_n+q_nI)udu \tag{32}$$

Thus, from (31) *and* (32) *we get*

$$\mathscr{F}(x,I) = \frac{2}{\pi}\int_0^\infty \overline{\mathscr{F}_c}(s,I)coss(p_n+q_nI)xds \tag{33}$$

Which is the required inversion formula for infinite F.C.T. of a neutrosophic function.

Thus, we have

$$\mathscr{F}_c\{\mathscr{F}(x,I)\} = \overline{\mathscr{F}_c}(s,I) = f_c(s,I) = \int_0^\infty \mathscr{F}(x,I)coss(p_n+q_nI)xdx \tag{34a}$$

$$and\ \mathscr{F}(x,I) = \mathscr{F}_c{}^{-1}\{\overline{\mathscr{F}_c}(s,I)\} = \frac{2}{\pi}\int_0^\infty \overline{\mathscr{F}_c}(s,I)coss(p_n+q_nI)xds \tag{34b}$$

Equations (34a) and (34b) are also re-written in the following symmetric form as

$$\begin{aligned}\mathscr{F}_c\{\mathscr{F}(x,I)\} &= \overline{\mathscr{F}_c}(s,I)\\ &= f_c^N(s,I)\\ &= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\mathscr{F}(x,I)coss(p_n+q_nI)xds \qquad (34a^*)\\ and\ \mathscr{F}(x,I) &= \mathscr{F}_c{}^{-1}\{\overline{\mathscr{F}_c}(s,I)\}\\ &= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\overline{\mathscr{F}_c}(s,I)coss(p_n+q_nI)xds \qquad (34b^*)\end{aligned}$$

9. **Linearity Property of F.T. of a Neutrosophic Function:**

If c_1 and c_2 be constants, then

$$(i)\ \mathscr{F}\{c_1\mathscr{F}_1(x,I)+c_2\mathscr{F}_2(x,I)\} = c_1\mathscr{F}\{\mathscr{F}_1(x,I)\}+c_2\mathscr{F}\{\mathscr{F}_2(x,I)\}$$

$$(ii)\ \mathscr{F}_s\{c_1\mathscr{F}_1(x,I)+c_2\mathscr{F}_2(x,I)\} = c_1\mathscr{F}_s\{\mathscr{F}_1(x,I)\}+c_2\mathscr{F}_s\{\mathscr{F}_2(x,I)\}$$

$$(iii)\ \mathscr{F}_c\{c_1\mathscr{F}_1(x,I)+c_2\mathscr{F}_2(x,I)\} = c_1\mathscr{F}_c\{\mathscr{F}_1(x,I)\}+c_2\mathscr{F}_c\{\mathscr{F}_2(x,I)\}$$

Proof: (i) By definition (16), we have

$$\begin{aligned}\mathscr{F}\{c_1\mathscr{F}_1(x,I)+c_2\mathscr{F}_2(x,I)\} &= \int_{-\infty}^{\infty}e^{is(p_n+q_nI)x}\{c_1\mathscr{F}_1(x,I)+c_2\mathscr{F}_2(x,I)\}dx\\ &= c_1\int_{-\infty}^{\infty}e^{is(p_n+q_nI)x}\mathscr{F}_1(x,I)dx\\ &\quad + c_2\int_{-\infty}^{\infty}e^{is(p_n+q_nI)x}\mathscr{F}_2(x,I)dx\\ &= c_1\mathscr{F}\{\mathscr{F}_1(x,I)\}+c_2\mathscr{F}\{\mathscr{F}_2(x,I)\}\end{aligned}$$

(ii) By definition (23), we have

$$\begin{aligned}\mathscr{F}_s\{c_1\mathscr{F}_1(x,I)+c_2\mathscr{F}_2(x,I)\} &= \int_0^{\infty}\{c_1\mathscr{F}_1(x,I)+c_2\mathscr{F}_2(x,I)\}sins(p_n+q_nI)xdx\\ &= c_1\int_0^{\infty}\mathscr{F}_1(x,I)sins(p_n+q_nI)xdx\\ &\quad + c_2\int_0^{\infty}\mathscr{F}_2(x,I)sins(p_n+q_nI)xdx\\ &= c_1\mathscr{F}_s\{\mathscr{F}_1(x,I)\}+c_2\mathscr{F}_s\{\mathscr{F}_2(x,I)\}\end{aligned}$$

(iii) By definition (29), we have

$$\begin{aligned}\mathscr{F}_c\{c_1\mathscr{F}_1(x,I)+c_2\mathscr{F}_2(x,I)\} =& \int_0^\infty \{c_1\mathscr{F}_1(x,I)+c_2\mathscr{F}_2(x,I)\}coss(p_n+q_nI)xdx\\ =& c_1\int_0^\infty \mathscr{F}_1(x,I)coss(p_n+q_nI)xdx\\ &+c_2\int_0^\infty \mathscr{F}_2(x,I)coss(p_n+q_nI)xdx\\ =& c_1\mathscr{F}_c\{\mathscr{F}_1(x,I)\}+c_2\mathscr{F}_c\{\mathscr{F}_2(x,I)\}\end{aligned}$$

10. **Change of Scale Property of a Neutrosophic Function:**

(i) If $f^N(s,I)$ is the F.T. of $\mathscr{F}(x,I)$, then $\frac{1}{|\alpha|}f^N(\frac{s}{\alpha},I)$ is the F.T. of $\mathscr{F}(\alpha x,I)$, where $\alpha \neq 0$. i.e., If $\mathscr{F}\{\mathscr{F}(x,I)\} = f^N(s,I)$, then $\mathscr{F}\{\mathscr{F}(\alpha x,I)\} = \frac{1}{|\alpha|}f^N(\frac{s}{\alpha},I)$, where $\alpha \neq 0$.
(ii) If $\mathscr{F}_s\{\mathscr{F}(x,I)\} = f_s^N(s,I)$, then $\mathscr{F}_s\{\mathscr{F}(\alpha x,I)\} = \frac{1}{\alpha}f_s^N(\frac{s}{\alpha},I)$
(iii) If $\mathscr{F}_c\{\mathscr{F}(x,I)\} = f_c^N(s,I)$, then $\mathscr{F}_c\{\mathscr{F}(\alpha x,I)\} = \frac{1}{\alpha}f_c^N(\frac{s}{\alpha},I)$

Proof: (i) The following two cases arise:

Case I: Let $\alpha > 0$. Then by definition of F.T., we get,

$$\begin{aligned}\mathscr{F}\{\mathscr{F}(\alpha x,I)\} &= \int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(\alpha x,I)dx\\ &= \frac{1}{\alpha}\int_{-\infty}^{\infty} e^{(\frac{s}{\alpha})(p_n+q_nI)t}\mathscr{F}(t,I)dt\\ &[putting\ \alpha x = t\ so\ that\ dx = \frac{1}{\alpha}dt]\\ &= (\frac{1}{\alpha})f^N(\frac{s}{\alpha},I),\ by\ definition\ of\ Fourier\ Transform.\end{aligned}$$

Case II: Let $\alpha < 0$. Let $\beta > 0$ such that $\alpha = -\beta$. Then by definition, we have

$$\begin{aligned}\mathscr{F}\{\mathscr{F}(\alpha x,I)\} &= \int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(\alpha x,I)dx\\ &= \int_{-\infty}^{\infty} e^{is(p_n+q_nI)t}\mathscr{F}(-\beta x,I)dx\\ &[as\ \alpha = -\beta]\\ &= \int_{\infty}^{-\infty} e^{i(\frac{s}{\alpha})t}\frac{dt}{-\beta},\ [putting\ -\beta x = t\ so\ that\ dx = -\frac{1}{\beta}dt]\\ &= \frac{1}{\beta}\int_{-\infty}^{\infty} e^{i(\frac{s}{\alpha})t}\mathscr{F}(t,I)dt\\ &= \frac{1}{-\alpha}f^N(\frac{s}{\alpha},I),\ by\ definition\ of\ Fourier\ Transform.\end{aligned}$$

Combining the above two cases, we get

$$\mathscr{F}\{\mathscr{F}(\alpha x,I)\} = \frac{1}{|\alpha|}f^N(\frac{s}{\alpha},I)$$

. (ii) By definition (23), we get

$$\int_0^\infty \mathscr{F}(x,I) sins(p_n+q_nI)xdx = f_s^N(s,I) \qquad \text{(i)}$$

$$
\begin{aligned}
Now,\ \mathscr{F}_s\{\mathscr{F}(\alpha x,I)\} &= \int_0^\infty \mathscr{F}(\alpha x,I) sins(p_n+q_nI)xdx \\
&= \frac{1}{\alpha}\int_0^\infty \mathscr{F}(t,I) sin\frac{s(p_n+q_nI)t}{\alpha}dt,\ [putting\ \alpha x=t\ so\ that\ dx=\frac{dt}{\alpha}] \\
&= \frac{1}{\alpha}\int_0^\infty \mathscr{F}(x,I) sin\{\frac{s}{\alpha}(p_n+q_nI)x\}dx \\
&= \frac{1}{\alpha} f_s^N(\frac{s}{\alpha},I),\ on\ replacing\ s\ by\ \frac{s}{\alpha}\ in\ (i)
\end{aligned}
$$

(iii) By definition (29), we get

$$\int_0^\infty \mathscr{F}(x,I) coss(p_n+q_nI)xdx = f_c^N(s,I) \qquad \text{(i)}$$

$$
\begin{aligned}
Now,\ \mathscr{F}_c\{\mathscr{F}(\alpha x,I)\} &= \int_0^\infty \mathscr{F}(\alpha x,I) coss(p_n+q_nI)xdx \\
&= \frac{1}{\alpha}\int_0^\infty \mathscr{F}(t,I) cos\frac{s(p_n+q_nI)t}{\alpha}dt,\ [putting\ \alpha x=t\ so\ that\ dx=\frac{dt}{\alpha}] \\
&= \frac{1}{\alpha}\int_0^\infty \mathscr{F}(x,I) cos\{\frac{s}{\alpha}(p_n+q_nI)x\}dx \\
&= \frac{1}{\alpha} f_c^N(\frac{s}{\alpha},I),\ on\ replacing\ s\ by\ \frac{s}{\alpha}\ in\ (i)
\end{aligned}
$$

Example: If $f_c^N(s,I)$ is F.C.T. of $\mathscr{F}(x,I)$, show that F.C.T. of $\mathscr{F}(\frac{x}{\alpha},I)$ is $\alpha f_s^N(\alpha s,I)$.

Solution: By definition (29), we get

$$\int_0^\infty \mathscr{F}(x,I) coss(p_n+q_nI)xdx = f_c^N(s,I) \qquad \text{(i)}$$

$$
\begin{aligned}
Now,\ \mathscr{F}_c\{\mathscr{F}(\frac{x}{\alpha},I)\} &= \int_0^\infty \mathscr{F}(\frac{x}{\alpha},I) coss(p_n+q_nI)xdx \\
&= \alpha\int_0^\infty \mathscr{F}(t,I) cos(\alpha st)(p_n+q_nI)dt,\ [on\ putting\ \frac{x}{\alpha}=t\ so\ that\ dx=\alpha dt] \\
&= \alpha\int_0^\infty \mathscr{F}(x,I) cos\{\alpha s(p_n+q_nI)\}xdx \\
&= \alpha f_c^N(\alpha s,I)\ \ [replacing\ s\ by\ \alpha s\ in\ (i)]
\end{aligned}
$$

11. **Shifting Property for F.T. of a Neutrosophic Function:**

If $f^N(s,I)$ is the complex F.T. of $\mathscr{F}(x,I)$, then complex F.T. of $\mathscr{F}(x-\alpha,I)$ is $e^{is\alpha(p_n+q_nI)}f^N(s,I)$, i.e., if $\mathscr{F}\{\mathscr{F}(x,I)\} = f^N(s,I)$, then

$$\mathscr{F}\{\mathscr{F}(x-\alpha,I)\} = e^{is\alpha(p_n+q_nI)}f^N(s,I).$$

Proof: By definition (16), we have

$$\int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx = f^N(s,I) \tag{i}$$

Now,

$$\begin{aligned}\mathscr{F}\{\mathscr{F}(x-\alpha,I)\} &= \int_{\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(x-\alpha,I)dx\\ &= \int_{\infty}^{\infty} e^{is(p_n+q_nI)(t+\alpha)}\mathscr{F}(t,I)dt,\ [on\ putting\ x-\alpha=t\ so\ that\ dx=dt]\\ &= e^{is\alpha(p_n+q_nI)\alpha}\int_{-\infty}^{\infty} e^{is(p_n+q_nI)t}\mathscr{F}(t,I)dt\\ &= e^{is(p_n+q_nI)\alpha}\int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx\\ &= e^{is(p_n+q_nI)\alpha}f^N(s,I),\ \{using\ (i)\end{aligned}$$

12. **Modulation Theorem for a Neutrosophic Function:**

If $f^N(s,I)$ is the neutrosophic F.T. of a neutrosophic function $f^N(x,I)$, then $\frac{[f^N(s+\alpha,I)+f^N(s-\alpha,I)]}{2}$ is the F.T. of $\mathscr{F}(x,I)cos\alpha(p_n+q_nI)x$, i.e.,

(i)If $\mathscr{F}\{\mathscr{F}(x,I)\} = f^N(s,I)$, then
$\mathscr{F}_c\{\mathscr{F}(x,I)cos\alpha(p_n+q_nI)x\} = \frac{[f^N(s+\alpha,I)+f^N(s-\alpha,I)]}{2}$
(ii) If $\mathscr{F}_s\{\mathscr{F}(x,I)\} = f^N_s(s,I)$, then
$\mathscr{F}_s\{\mathscr{F}(x,I)cos\alpha(p_n+q_nI)x\} = \frac{[f^N_s(s+\alpha,I)+f^N_s(s-\alpha,I)]}{2}$
(iii) If $\mathscr{F}_s\{\mathscr{F}(x,I)\} = f^N_s(s,I)$, then
$\mathscr{F}_c\{\mathscr{F}(x,I)sin\alpha(p_n+q_nI)x\} = \frac{[f^N_s(s+\alpha,I)+f^N_s(s-\alpha,I)]}{2}$
(iv) If $\mathscr{F}_c\{\mathscr{F}(x,I)\} = f^N_c(s,I)$, then
$\mathscr{F}_s\{\mathscr{F}(x,I)sin\alpha(p_n+q_nI)x\} = \frac{[f^N_c(s-\alpha,I)-f^N_c(s+\alpha,I)]}{2}$

Proof (i) By definition(16),

$$\int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx = f^N(s,I) \tag{i}$$

$$\begin{aligned}Now,\ \mathscr{F}\{\mathscr{F}(x,I)cos\alpha x\} &= \int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(x,I)cos\alpha(p_n+q_nI)xdx\\ &= \int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(x,I)\frac{e^{i\alpha(p_n+q_nI)x}+e^{-i\alpha(p_n+q_nI)x}}{2}dx\\ &= \frac{1}{2}\int_{-\infty}\infty e^{i(s+\alpha)(p_n+q_nI)x}\mathscr{F}(x,I)dx + \frac{1}{2}\int_{-\infty}\infty e^{i(s-\alpha)(p_n+q_nI)x}\mathscr{F}(x,I)dx\\ &= \frac{1}{2}f^N(s+\alpha,I)+\frac{1}{2}f^N(s-\alpha,I),\ \{using\ (i)\\ &= \frac{[f^N(s+\alpha,I)+f^N(s-\alpha,I)]}{2}\end{aligned}$$

(ii) By definition(23), we have

$$\int_0^\infty \mathscr{F}(x,I)sins(p_n+q_nI)xdx = f_s^N(s,I) \tag{i}$$

Now,

$$\begin{aligned}\mathscr{F}_s\{\mathscr{F}(x,I)cos\alpha(p_n+q_nI)x\} = &\int_{-\infty}^\infty \mathscr{F}(x,I)cos\alpha(p_n+q_nI)xsins(p_n+q_nI)xdx, \{By\ defn.\ (i)\\ =&\frac{1}{2}\int_0^\infty \mathscr{F}(x,I)2sins(p_n+q_nI)xcos\alpha(p_n+q_nI)xdx\\ =&\frac{1}{2}\int_0^\infty \mathscr{F}(x,I)[sin(p_n+q_nI)(s+\alpha)x+sin(p_n+q_nI)(s-\alpha)x]dx\\ =&\frac{1}{2}[\int_0^\infty \mathscr{F}(x,I)sin(s+\alpha)(p_n+q_nI)xdx\\ &+\int_0^\infty \mathscr{F}(x,I)sin(s-\alpha)(p_n+q_nI)xdx]\\ =&\frac{1}{2}[f_s^N(s+\alpha,I)+f_s^N(s-\alpha,I)],\ \{using\ (i)\end{aligned}$$

(iii)

$$\int_0^\infty \mathscr{F}(x,I)sins(p_n+q_nI)xdx = f_s^N(s,I) \tag{i}$$

Now,

$$\begin{aligned}\mathscr{F}_c\{\mathscr{F}(x,I)sin\alpha(p_n+q_nI)x\} = &\int_{-\infty}^\infty \mathscr{F}(x,I)sin\alpha(p_n+q_nI)xcoss(p_n+q_nI)xdx, \{By\ defn.\ (i)\\ =&\frac{1}{2}\int_0^\infty \mathscr{F}(x,I)2coss(p_n+q_nI)xsin\alpha(p_n+q_nI)xdx\\ =&\frac{1}{2}\int_0^\infty \mathscr{F}(x,I)[sin(p_n+q_nI)(s+\alpha)x-sin(p_n+q_nI)(s-\alpha)x]dx\\ =&\frac{1}{2}[\int_0^\infty \mathscr{F}(x,I)sin(s+\alpha)(p_n+q_nI)xdx\\ &-\int_0^\infty \mathscr{F}(x,I)sin(s-\alpha)(p_n+q_nI)xdx]\\ =&\frac{1}{2}[f_s^N(s+\alpha,I)-f_s^N(s-\alpha,I)],\ \{using\ (i)\end{aligned}$$

(iv) By definition (28)

$$\int_0^\infty \mathscr{F}(x,I)coss(p_n+q_nI)xdx = f_c^N(s,I) \tag{i}$$

Now,

$$\mathscr{F}_s\{\mathscr{F}(x,I)sin\alpha(p_n+q_nI)x\} = \int_{-\infty}^{\infty}\mathscr{F}(x,I)sin\alpha(p_n+q_nI)xsins(p_n+q_nI)xdx, \{By\ defn.\ (i)$$

$$=\frac{1}{2}\int_0^{\infty}\mathscr{F}(x,I)2sins(p_n+q_nI)xsin\alpha(p_n+q_nI)xdx$$

$$=\frac{1}{2}\int_0^{\infty}\mathscr{F}(x,I)[cos(p_n+q_nI)(s-\alpha)x - cos(p_n+q_nI)(s+\alpha)x]dx$$

$$=\frac{1}{2}[\int_0^{\infty}\mathscr{F}(x,I)cos(s+\alpha)(p_n+q_nI)xdx$$

$$-\int_0^{\infty}\mathscr{F}(x,I)cos(s-\alpha)(p_n+q_nI)xdx]$$

$$=\frac{1}{2}[f_c^N(s+\alpha,I) - f_c^N(s-\alpha,I)],\ \{using\ (i)$$

13. **Solved examples based on Infinite F.T.:**

Example 1. To find the Fourier complex transform of $\mathscr{F}(x,I)$, if

$$\mathscr{F}(x,I) = \begin{cases} e^{iw(p_n+q_nI)x} & \text{if } \alpha < x < \beta \\ 0 & \text{if } x < \alpha \text{ or } x > \beta \end{cases}$$

Solution: By definition (16), we have

$$\mathscr{F}\{\mathscr{F}(x,I)\} = \int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx$$

$$= \int_{-\infty}^{\alpha} e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx + \int_{\alpha}^{\beta} e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx + \int_{\beta}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx$$

$$= 0 + \int_{\alpha}^{\beta} e^{is(p_n+q_nI)x}.e^{iw(p_n+q_nI)x}dx + 0,\ \{\text{using the given values of } \mathscr{F}(x,I).$$

$$= \int_{\alpha}^{\beta} e^{i(s+w)(p_n+q_nI)x}dx$$

$$= \left[\frac{e^{i(s+w)(p_n+q_nI)x}}{i(s+w)(p_n+q_nI)}\right]_{\alpha}^{\beta}$$

$$= i.\frac{e^{i(s+w)(p_n+q_nI)\beta} - e^{i(s+w)(p_n+q_nI)\alpha}}{i^2(s+w)(p_n+q_nI)}$$

$$= \frac{i}{(s+w)(p_n+q_nI)}\left[e^{i(s+w)(p_n+q_nI)\alpha} - e^{i(s+w)(p_n+q_nI)\beta}\right],\ \{since\ i^2 = 1$$

$$= (\frac{1}{p_n} - \frac{q_n}{p_n(p_n+q_n)}I)\frac{i}{(s+w)}\left[e^{i(s+w)(p_n+q_nI)\alpha} - e^{i(s+w)(p_n+q_nI)\beta}\right]$$

Example 2. To find the F.T. of

$$f^N(x,I) = \begin{cases} x+I, & \text{if } |x| \leq \alpha \\ 0, & \text{if } |x| > \alpha \end{cases}$$

Solution: By definition (16), we have

$$\begin{aligned}
\mathscr{F}\{f^N(x,I)\} &= \int_{-\infty}^{\infty} f^N(x,I)e^{is(p_n+q_nI)x}dx \\
&= \int_{-\infty}^{-\alpha} f^N(x,I)e^{is(p_n+q_nI)x}dx + \int_{-\alpha}^{\alpha} f^N(x,I)e^{is(p_n+q_nI)x}dx + \int_{\alpha}^{\infty} f^N(x,I)e^{is(p_n+q_nI)x}dx \\
&= \int_{-\alpha}^{\alpha} (x+I)e^{is(p_n+q_nI)x}dx, \ \{\text{using the given values of } f^N(x,I). \\
&= \left[(x+I).\frac{e^{is(p_n+q_nI)x}}{is(p_n+q_nI)}\right]_{-\alpha}^{\alpha} - \int_{-\alpha}^{\alpha} \frac{e^{is(p_n+q_nI)x}}{is(p_n+q_nI)}dx, \ \{\text{using integration by parts.} \\
&= \frac{\alpha+I}{is(p_n+q_nI)}.e^{is(p_n+q_nI)\alpha} - \frac{-\alpha+I}{is(p_n+q_nI)}.e^{is(p_n+q_nI)(-\alpha)} - \left[\frac{.e^{is(p_n+q_nI)x}}{\{is(p_n+q_nI)\}^2}\right]_{-\alpha}^{\alpha} \\
&= \frac{1}{is(p_n+q_nI)}\left[(\alpha+I)e^{is(p_n+q_nI)\alpha} - (I-\alpha)e^{-is(p_n+q_nI)\alpha}\right] \\
&+ \frac{1}{s^2(p_n+q_nI)^2}\left[e^{is(p_n+q_nI)\alpha} - e^{-is(p_n+q_nI)\alpha}\right] \\
&= \frac{2I}{is(p_n+q_nI)}\left[\frac{e^{is(p_n+q_nI)\alpha} - e^{-is(p_n+q_nI)\alpha}}{2i}\right] \\
&+ \frac{2\alpha i}{i^2s(p_n+q_nI)}\left[\frac{e^{is(p_n+q_nI)\alpha} + e^{-is(p_n+q_nI)\alpha}}{2i}\right] \\
&+ \frac{2i}{s^2(p_n+q_nI)}\left[\frac{e^{is(p_n+q_nI)\alpha} - e^{-is(p_n+q_nI)\alpha}}{2i}\right] \\
&= \frac{2I}{s(p_n+q_nI)}sins(p_n+q_nI)\alpha - \frac{2\alpha i}{s(p_n+q_nI)}coss(p_n+q_nI)\alpha \\
&+ \frac{2i}{s^2(p_n+q_nI)^2}sins(p_n+q_nI)\alpha \\
&= (\frac{1}{p_n} - \frac{q_n}{p_n(p_n+q_n)}I).\frac{2}{s}sins(p_n+q_nI)\alpha - \frac{2\alpha i}{s}(\frac{1}{p_n} - \frac{q_n}{p_n(p_n+q_n)}I)coss(p_n+q_nI)\alpha \\
&+ \frac{2i}{s^2}(\frac{1}{p_n} - \frac{q_n}{p_n(p_n+q_n)}I)^2 sins(p_n+q_nI)\alpha.
\end{aligned}$$

Example 3 To find the F.T. of $f^N(x,I)$ if

$$f^N(x,I) = \begin{cases} x^2+I, & \text{if } |x| \leq \alpha \\ 0, & \text{if } |x| > \alpha \end{cases}$$

Solution: By definition (16), we have

$$\begin{aligned}
\mathscr{F}\{f^N(x,I)\} &= \int_{-\infty}^{\infty} f^N(x,I)e^{is(p_n+q_nI)x}dx \\
&= \int_{-\infty}^{-\alpha} e^{is(a+bIx)}dx + \int_{-\alpha}^{\alpha} f^N(x,I)e^{is(p_n+q_nI)x}dx + \int_{\alpha}^{\infty} f^N(x,I)e^{is(p_n+q_nI)x}dx \\
&= \int_{-\alpha}^{\alpha} (x^2+I)e^{is(p_n+q_nI)x}dx, \ \{\text{using the given values of } f^N(x,I) \\
&= \left[(x^2+I).\frac{e^{is(p_n+q_nI)x}}{is(p_n+q_nI)}\right]_{-\alpha}^{\alpha} - \int_{-\alpha}^{\alpha} 2x.\frac{e^{is(p_n+q_nI)x}}{is(p_n+q_nI)}dx, \ \{\text{using integration by parts.} \\
&= \frac{(\alpha^2+I)}{is(p_n+q_nI)}.e^{is(p_n+q_nI)\alpha} - \frac{(\alpha^2+I)}{is(p_n+q_nI)}.e^{-is(p_n+q_nI)\alpha} \\
&- \frac{2}{is(p_n+q_nI)}.\left[x.\frac{e^{is(p_n+q_nI)x}}{is(p_n+q_nI)}\right]_{-\alpha}^{\alpha} \\
&- \frac{2}{is(p_n+q_nI)}\int_{-\alpha}^{\alpha} 1.\frac{e^{is(p_n+q_nI)x}}{is(p_n+q_nI)}dx \\
&= \frac{(\alpha^2+I)}{is(p_n+q_nI)}.2isins(p_n+q_nI)\alpha - \frac{2}{i^2s^2(p_n+q_nI)^2}\left[\alpha e^{is(p_n+q_nI)\alpha} + \alpha e^{-is(p_n+q_nI)\alpha}\right] \\
&+ \frac{2}{i^2s^2(p_n+q_nI)^2}.\left[\frac{e^{is(p_n+q_nI)x}}{is(p_n+q_nI)}\right]_{-\alpha}^{\alpha} \\
&= \frac{2(\alpha^2+I)}{s(p_n+q_nI)}.sins(p_n+q_nI)\alpha + \frac{4\alpha}{s^2(p_n+q_nI)^2}coss(p_n+q_nI)\alpha \\
&- \frac{4}{s^3(p_n+q_nI)^3}sins(p_n+q_nI)\alpha
\end{aligned}$$

14. Conclusion:

In this article, some basics of Fourier Integrals have been discussed in terms of neutrosophic set. Dirichlet's Conditions, Fourier integral formula and it's five different forms are studied based on neutrosophic set. This article includes the F.T., F.S.T. and F.C.T. of a neutrosophic function and their inversion formulae. In this study, some properties of F.T. are discussed for a neutrosophic function. The purpose of this work is to extend the classical analysis of Fourier integrals and Fourier transform to Neutrosophic form. This study will help to get better results in signal processing, image processing, noise filtering and in some other fields also. Presently, in classical form, there are many applications of Fourier transform in different fields. In future, by using the concept of Neutrosophic in Fourier integrals and Fourier transform it can be axpected to get better results than the present applications of Fourier integrals and Fourier transform.

References

[1] F. Smarandache, Neutrosophy/ Neutrosophic Probability, Set, and Logic, American Research Press, Rehoboth, USA, 1998.

[2] F.Smarandache, Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics, University of New Mexico, Gallup, NM 87301, USA 2002.

[3] F. Smarandache, Finite N.C.N.s, by W. B. Vasantha Kandasamy, Zip Publisher, Columbus, Ohio, USA, PP: 1-16, 2011.

[4] F.Smarandache, Introduction to Neutrosophic statistics, Sitech-Education Publisher, PP:34-44, 2014.

[5] F.Smarandache, A Unifying Field in Logics: Neutrosophic Logic, Preface by Charles Le, American Research Press, Rehoboth, 1999, 2000. Second edition of the Proceedings of the First International Conference on Neutrosophy, Neutrosophic Logic, Neutrosophic Set, Neutrosophic Probability and Statistics, University of New Mexico, Gallup, 2001.

[6] F.Smarandache, Proceedings of the First International Conference on Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics, University of New Mexico, 2001.

[7] F. Smarandache, Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability, Sitech-Education Publisher, Craiova-Columbus, 2013.

[8] F.Smarandache, Neutrosophic Precalculus and Neutrosophic Calculus, book, 2015.

[9] Madeleine Al- Tahan, Some Results on Single Valued Neutrosophic (Weak) Polygroups, International Journal of Neutrosophic Science, Volume 2 , Issue 1, pp. 38-46 , 2020.

[10] S. A. Edalatpanah, A Direct Model for Triangular Neutrosophic Linear Programming, International Journal of Neutrosophic Science, Volume 1 , Issue 1, pp. 19-28 , 2020.

[11] A. Chakraborty, A New Score Function of Pentagonal Neutrosophic Number and its Application in Networking Problem, International Journal of Neutrosophic Science, Volume 1 , Issue 1, pp.40-51 , 2020.

[12] A. Chakraborty, Application of Pentagonal Neutrosophic Number in Shortest Path Problem, International Journal of Neutrosophic Science, Volume 3 , Issue 1, pp. 21-28 , 2020.

[13] S. Mustafi; T. Latychevskaia, Fourier Transform Holography: A Lensless Imaging Technique, Its Principles and Applications,Multidisciplinary Digital Publishing Institude, 2023.

[14] U. M. Pirzada1 · D. C. Vakaskar2, Fuzzy solution of homogeneous heat equation having solution in Fourier series form. https://doi.org/10.1007/s40324-018-0169.

[15] A.Kharal, a Neutrosophic Multicriteria Decision Making Method, 2011.

[16] A. A. Salama, F.Smarandache, Neutrosophic Crisp Set Theory, Educational Publisher, Columbus, 2015

[17] D. Koundal; S. Gupta; S. Singh, Applications of Neutrosophic Sets in Medical Image Denoising and Segmentation, New Trends in Neutrosophic Theory and Applications, pp. 257-273.

[18] N. Chittora and D. Babel, A brief study on Fourier transform and its applications, International Research Journal of Engineering and Technology, Volume 05, Issue 12, pp. 1127-1131 , 2018.

[19] U. Kadak; F. Basar, On Fourier Series of Fuzzy Valued Functions, The Scientific World of Journal, 2014.

[20] F. Smarandache, New Types of Topologies and Neutrosophic Topologies, Neutrosophic Sysytems with Applications, Vol. 1, 2023, https://doi.org/10.61356/j.nswa.2023.1.

[21] S. N. I. Rosli; M. I. E. Zulkifly, Neutrosophic Bicubic B-spline Surface Interpolation Model for Uncertainty Data, Neutrosophic Systems with Applications, Vol. 10, 2023, https://doi.org/10.61356/j.nswa.2023.69.

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