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Topological Aspects of Set-Valued Mappings Defned on Neutrosophic Normed Spaces

Vakeel A Khan[1] and Mohd Kamran[2]

[1] Department of Mathematics, Aligarh Muslim University Aligarh. India; vakhanmaths@gmail.com

[2] Department of Mathematics, Aligarh Muslim University Aligarh. India; kamshafiq@gmail.com

Abstract. Samarandache [43] introduced neutrosophoic sets to generalize the theory of fuzzy sets. In this paper we modify the definition of neutrosophic normed space with help of continuous neutrosophic t-representable norm. Then we study the statistical graphical convergence and pointwise convergence of the sequences of set-valued functions defined on modified version of neutrosophic normed space and give some related theorems. We also introduce neutrosophic upper and lower semi continuities of set valued maps to develop the link between these convergences.

Keywords: NNS; t-norm; t-conorm; t-representable norm; Statistical graph convergence; Set valued maps; neutrosophic set.

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1. Introduction

The landscape of both pure and applied mathematics has been substantially enriched through the insights provided by fuzzy theory. The foundational work was established by L.A. Zadeh in 1965 [38]. In 1975, Kramosil and Michálek introduced the concept of fuzzy metric space [25], which served as a generalization of the conventional metric space. Later, George and Veeramani refined this concept in 1994 [16]. Katsaras proposed the idea of fuzzy normed space in 1984 [21], which was subsequently modified by Bag and Samanta [8]. Saadati and Vaezpour expanded upon this notion, further advancing the field of fuzzy normed spaces [33].

In the realm of fuzzy mathematics, Atanassov introduced intuitionistic fuzzy set theory, broadening the scope beyond traditional fuzzy set theory [2]. Building upon this foundation, Park extended the concept into intuitionistic fuzzy metric spaces [29]. Saadati and Park further generalized the notion to intuitionistic fuzzy normed spaces [34]. Recognizing the need for refinement, additional conditions were incorporated into the concept by Hosseini et al. [18]. In a recent exploration, Jakhar et al. applied fixed point and direct methods to investigate the intuitionistic fuzzy stability of 3-dimensional cubic functional equations [19]. Also Vakeel A. Khan and S. K. Ashadul Rahaman [44] explopred statistical graph and pointwise convergence of sequences of set-valued functions on intuitionistic fuzzy normed spaces.

Beyond intuitionistic fuzzy sets, the emergence of neutrosophic sets marked a significant generalization in dealing with uncertainty and imprecision. Smarandache introduced neutrosophic sets, providing a broader framework for this purpose [43]. Advancing the field, Kirisci and Simsek defined a metric and a norm specifically tailored for neutrosophic sets, while also delving into their topological properties 41, 42. These advancements not only expanded the toolkit for handling complex and uncertain data but also offered fresh perspectives on fuzzy systems.

Jemima and colleagues investigated the convergence patterns within a Kothe sequence space, whereas Al-Marzouki's study delved into the statistical characteristics of the type II Topp Leone inverse exponential distribution [1, 20]. The understanding of sequence set limits is pivotal not just in set-valued analysis [4] but also in variational analysis [32]. In 1902, Painlevé initially proposed the notions of upper and lower limits for sequences of sets, advancing the idea of Kuratowski convergence by aligning the corresponding upper and lower limits [26]. The formalization of upper and lower limits for a sequence of subsets within a metric space $(\mathcal{X}, d)$ was first introduced in [27]. Building upon these foundational ideas, Beer introduced the concept of topological convergence [11]. Later, Kowalczyk utilized the notion of sequence set convergence within a topological space [24]. For those interested in further exploration of sequence set convergence, additional resources include articles such as [12, 37]. informative.

The convergence of function sequences holds paramount significance across both analysis and topology, particularly concerning the convergence of graphs associated with such sequences. When discussing the convergence of sequences of real-valued functions, terms such as pointwise convergence and uniform convergence frequently emerge. In 1983, Beer elucidated the conditions under which topological convergence and uniform convergence of sequences of continuous functions from one metric space to another coincide [10]. In 2008, Grande explored the graph convergence of single-valued functions defined from one topological space to another, drawing comparisons between graph convergence, pointwise convergence, and uniform convergence [17]. For those wishing to delve further into the graph convergence of single-valued functions, additional resources include articles such as [28, 30]. informative.

The convergence of sequences of set-valued functions plays a fundamental role in various mathematical contexts. Attouch extensively examined the graph convergence of sequences of maximal monotone set-valued operators in his seminal work [3]. Aubin and Frankowska introduced the concept of graph convergence for sequences of set-valued functions utilizing the notions of upper and lower limits of sets in the Kuratowski sense [6]. Similarly, Kowalczyk addressed the convergence of sequences of set-valued functions employing Kuratowski limits, while also introducing the equicontinuity of set-valued functions to establish connections between different types of convergence [24]. Delgado and colleagues investigated the interplay between pointwise convergence and graph convergence of sequences of set-valued functions, introducing the concept of outer-semicontinuity for set-valued functions [31]. For further insights and applications concerning the graph convergence of sequences of set-valued functions, interested readers may consult papers and books such as [7, 9, 14].

The primary aim of this article is to conduct an analysis of sequences of set-valued functions originating from a modified variant of intuitionistic fuzzy normed space and extending to another space. This analysis will delve into the phenomena of both graph convergence and pointwise convergence exhibited by these sequences, shedding light on their properties and implications within the framework of fuzzy normed spaces.

2. **Preliminaries**

Throughout the entirety of this investigation, we will consistently use the symbols $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{Q}$ to represent the sets of natural numbers, real numbers, and rational numbers, respectively. Here are some fundamentals to review:

Consider U and V as two arbitrary non-void sets. A set - valued mapping $\varphi : U \longrightarrow P(V)$ is a mapping from U to power set of V i.e. $P(V)$, such that $\forall\ u \in U$, $\varphi(u) \subseteq V$.
The domain of the function φ is defined as

$$\mathbf{Dom}(\varphi) = \big\{u \in U : \varphi(u) \neq \emptyset\big\}.$$

and the image of the function φ is defined as

$$\mathbf{Im}(\varphi) = \bigcup_{u \in U} \varphi(u)$$

Consider U as an arbitrary non-void set and V be a linear space over the field F. Let $\varphi_1, \varphi_2 : X \longrightarrow P(V)$ to be the set - valued functions. The addition and scalar multiplication of φ_1 and φ_2 are therefore defined below:

$$(\varphi_1 + \varphi_2)(u) = \varphi_1(u) + \varphi_2(u)$$
$$= \Bigg\{v_1 + v_2 : v_1 \in \varphi_1(u) \text{ and } v_2 \in \varphi_2(u)\Bigg\};$$

$$(\alpha\varphi_1)(u) = \alpha\varphi_1(u) = \Bigg\{\alpha v : v \in \varphi_1(u), \alpha \in K\Bigg\}.$$

Definition 2.1. [32] Let U, V be topological spaces and $\varphi_n : U \longrightarrow P(V)$ be a sequence of set - valued functions. Then the pointwise lower limit and pointwise upper limit of the sequence $(\varphi_n)_{n=1}^{\infty}$ are the functions $p - \varphi_n^l$ and $p - \varphi_n^{\mathfrak{u}}$, respectively, defined by

$$(p - \varphi_n^l)(u) = \liminf_{n\to\infty} \varphi_n(u), \ \forall u \in U.$$

and

$$(p - \varphi_n^{\mathfrak{u}})(u) = \limsup_{n\to\infty} \varphi_n(u), \ \forall u \in U$$

If $(p - \varphi_n^{\mathfrak{u}})(u) = (p - \varphi_n^l)(u) = \varphi(u)$, $\forall u \in U$, then $\varphi(u)$ is called pointwise limit and we denote $\lim_{n\to\infty} \varphi_n(u) = \varphi(u)$, $\forall u \in U$ then we say, the sequence $(\varphi_n)_{n=1}^{\infty}$ is pointwise convergent to $\varphi(u)$.

Define the notions as follows:

$$\mathscr{M}_f = \{M \subseteq \mathbb{N} : m^c \text{ is finite } \}$$

$$\mathscr{M}_\infty = \{M' \subseteq \mathbb{N} : M' \text{ is infinite } \}.$$

In general, we denote $\lim_{n\to\infty}$ when n approaches to ∞ along $\mathbb{N}$. Throughout this paper, we will denote $\lim_{n\in M}$ and $\lim_{n\in M'}$ when n approaches to ∞ along the subsets M and M' of $\mathbb{N}$, respectively.

Definition 2.2. [32] Let U, V be topological spaces and $\varphi_n : U \longrightarrow P(V)$ be a sequence of set-valued functions. The graphical lower limit of the sequence $(\varphi_n)_{n=1}^{\infty}$ is the function $Gr(\varphi_n^l) = \liminf_{n\to\infty} (Gr(\varphi_n))$, where $v \in \varphi^l(u)$ if and only if $(u, v) \in Gr(\varphi_n^l)$, i.e., there exists $M \in \mathscr{M}_f$ such that

$$\lim_{n\in M} u_n = u, \lim_{n\in M} v_n = v \text{ for } v_n \in \varphi_n(u_n).$$

and the graphical upper limit of the sequence $(\varphi_n)_{n=1}^{\infty}$ is the function $Gr(\varphi_n^{\mathfrak{u}}) = \limsup_{n\to\infty} (Gr(\varphi_n))$, where $v \in \varphi^{\mathfrak{u}}(u)$ if and only if $(u, v) \in Gr(\varphi_n^{\mathfrak{u}})$, i.e., there exists $M' \in \mathscr{M}_\infty$ such that

$$\lim_{n\in M'} u_n = u, \lim_{n\in M'} v_n = v \ \text{ for } v_n \in \varphi_n(u_n)$$

If $Gr(\varphi_n^{\mathfrak{u}}) = Gr(\varphi_n^l)$, then the limit is known as the graphical limit denoted by $\lim_{n\to\infty} Gr(\varphi_n)$. The sequence $(\varphi_n)_{n=1}^{\infty}$ is graph convergent to a function $\varphi : U \longrightarrow P(V)$ if

$$\limsup_{n\to\infty} (Gr(\varphi_n)) \subseteq Gr(\varphi) \subseteq \liminf_{n\to\infty} (Gr(\varphi_n)).$$

Lemma 2.3. *[39] Consider $(\mathfrak{D}^\star, \leq_{\mathfrak{D}^\star})$ to be partially ordered set, defined as*

$$\mathfrak{D}^\star = \big\{(\zeta_1, \zeta_2, \zeta_3) : \zeta_1, \zeta_2, \zeta_3 \in [0, 1]\big\},$$

$$(\zeta_1, \zeta_2, \zeta_3) \leq_{\mathfrak{D}^\star} (\eta_1, \eta_2, \eta_3) \text{ if and only if } \zeta_1 \leq \eta_1, \zeta_2 \geq \eta_2, \zeta_3 \geq \eta_3$$

for all $(\zeta_1, \zeta_2, \zeta_3), (\eta_1, \eta_2, \eta_3) \in \mathfrak{D}^\star$. Then $(\mathfrak{D}^\star, \leq_{\mathfrak{D}^\star})$ is complete lattice.

$0_{\mathfrak{D}^\star} = (0, 1, 1)$ *and* $1_{\mathfrak{D}^\star} = (1, 0, 0)$ *are its units.*

Definition 2.4. [40] In the context of a non-empty set U, a single-valued neutrosophic set M is defined by three essential functions: the truth membership function $T_M(u)$, the indeterminacy-membership function $I_M(u)$, and the false-membership function $F_M(u)$. Therefore, a single-valued neutrosophic set M can be represented as

$$M = \{(u, T_M(u), I_M(u), F_M(u)); u \in U\}$$

where $T_M(u), I_M(u), F_M(u) \in [0,1]$ and $\forall\ u \in U$, they satisfy the condition $0 \leq T_M(u) + I_M(u) + F_M(u) \leq 3$.

In the traditional context, a triangular norm denoted by $*$ on the interval $[0,1]$ refers to a function $* : [0,1]^2 \longrightarrow [0,1]$ that exhibits properties of being increasing, commutative, and associative, satisfying the condition $1 * \zeta = \zeta$ for all $\zeta \in [0,1]$. Conversely, a triangular conorm denoted by $\diamond$ on $[0,1]$ is a function $\diamond : [0,1]^2 \longrightarrow [0,1]$ with similar properties, such as being increasing, commutative, and associative, and satisfying $0 \diamond \zeta = \zeta$ for all $\zeta \in [0,1]$ (refer to [23], [22]). This terminology is utilized within the lattice $(\mathfrak{D}^\star, \leq_{\mathfrak{D}^\star})$.

, these definitions can be extended as follows.

Definition 2.5. [39] A mapping $\Gamma : \mathfrak{D}^\star \times \mathfrak{D}^\star \longrightarrow \mathfrak{D}^\star$ is said to be a neutrosophic t-norm on $\mathfrak{D}^\star$ if it adheres to the following conditions:

(1) $\Gamma(\zeta, 1_{\mathfrak{D}^\star}) = \zeta$ for all $\zeta \in \mathfrak{D}^\star$,

(2) $\Gamma(\zeta_1, \zeta_2) = \Gamma(\zeta_2, \zeta_1)$ for all $\zeta_1, \zeta_2, \in \mathfrak{D}^\star$,

(3) $\Gamma(\zeta_1, \Gamma(\zeta_2, \zeta_3)) = \Gamma(\Gamma(\zeta_1, \zeta_2), \zeta_3)$ for all $\zeta_1, \zeta_2, \zeta_3 \in \mathfrak{D}^\star$,

(4) $\zeta_1 \leq_{\mathfrak{D}^\star} \eta_1$ and $\zeta_2 \leq_{\mathfrak{D}^\star} \eta_2$ implies $\Gamma(\zeta_1, \zeta_2) \leq_{\mathfrak{D}^\star} \Gamma(\eta_1, \eta_2)$ for all $\zeta_1, \zeta_2, \eta_1, \eta_2 \in \mathfrak{D}^\star$.

Definition 2.6. [39] A mapping $\Gamma : \mathfrak{D}^\star \times \mathfrak{D}^\star \longrightarrow \mathfrak{D}^\star$ is said to be a neutrosophic t-conorm on $\mathfrak{D}^\star$ if it adheres to the following conditions:

(1) $\Gamma(\zeta, 0_{\mathfrak{D}^\star}) = \zeta$ for all $\zeta \in \mathfrak{D}^\star$,

(2) $\Gamma(\zeta_1, \zeta_2) = \Gamma(\zeta_2, \zeta_1)$ for all $\zeta_1, \zeta_2, \in \mathfrak{D}^\star$,

(3) $\Gamma(\zeta_1, \Gamma(\zeta_2, \zeta_3)) = \Gamma(\Gamma(\zeta_1, \zeta_2), \zeta_3)$ for all $\zeta_1, \zeta_2, \zeta_3 \in \mathfrak{D}^\star$,

(4) $\zeta_1 \leq_{\mathfrak{D}^\star} \eta_1$ and $\zeta_2 \leq_{\mathfrak{D}^\star} \eta_2$ implies $\Gamma(\zeta_1, \zeta_2) \leq_{\mathfrak{D}^\star} \Gamma(\eta_1, \eta_2)$ for all $\zeta_1, \zeta_2, \eta_1, \eta_2 \in \mathfrak{D}^\star$.

Definition 2.7. [39] A continuous neutrosophic t-norm Γ defined on $\mathfrak{D}^\star$ is classified as continuous t-representable if there exists both a continuous t-norm denoted by $*$ and a continuous t-conorm represented by $\diamond$ on the interval $[0,1]$ such that,

$$\Gamma(\zeta, \eta) = \big(\zeta_1 * \eta_1,\ \zeta_2 \diamond \eta_2,\ \zeta_3 \diamond \eta_3\big)$$

for all $\zeta = (\zeta_1, \zeta_2, \zeta_3), \eta = (\eta_1, \eta_2, \eta_3) \in \mathfrak{D}^\star$.

For example, $\Gamma(\zeta, \eta) = \big(\zeta_1\eta_1,\ \min\{\zeta_2 + \eta_2, 1\},\ \min\{\zeta_2 + \eta_2, 1\}\big)$ for all $\zeta = (\zeta_1, \zeta_2, \zeta_3), \eta = (\eta_1, \eta_2, \eta_3) \in \mathfrak{D}^\star$, is a continuous t-representable norm.

Definition 2.8. [18] Let Ψ, Φ and Π are fuzzy sets from $U \times (0, \infty)$ to $[0, 1]$ such that $0 \le \Psi(u, r) + \Phi(u, r) + \Pi(u, r) \le 3$ for all $u \in U$ and $r > 0$. The tuple $(U, \mathfrak{I}_{\Psi,\Phi,\Pi}, \Gamma)$ is called neutrosophic normed space (NNS) if U is a linear space over $F(\mathbb{R}$ or $\mathbb{C})$, Γ is continuous t-representable norm and $\mathfrak{I}_{\Psi,\Phi,\Pi} : U \times (0, \infty) \longrightarrow \mathfrak{D}^\star$ is a mapping such that for all $u, v \in U$ and $r, s > 0$ the following conditions hold:

(a) $\mathfrak{I}_{\Psi,\Phi,\Pi}(u, r) >_{\mathfrak{D}^\star} 0_{\mathfrak{D}^\star}$,

(b) $\mathfrak{I}_{\Psi,\Phi,\Pi}(u, r) = 1_{\mathfrak{D}^\star}$ if and only if $u = 0$,

(c) $\mathfrak{I}_{\Psi,\Phi,\Pi}(au, r) = \mathfrak{I}_{\Psi,\Phi,\Pi}\big(u, \frac{r}{|a|}\big)$ for any $0 \neq a \in F$,

(d) $\Gamma(\mathfrak{I}_{\Psi,\Phi,\Pi}(u, r), \mathfrak{I}_{\Psi,\Phi,\Pi}(v, s)) \le_{\mathfrak{D}^\star} \mathfrak{I}_{\Psi,\Phi,\Pi}(u + v, r + s)$,

(e) $\mathfrak{I}_{\Psi,\Phi,\Pi}(u, .) : (0, \infty) \longrightarrow \mathfrak{D}^\star$ is continuous,

(f) $\lim_{r\to\infty} \mathfrak{I}_{\Psi,\Phi,\Pi}(u, r) = 1_{\mathfrak{D}^\star}$ and $\lim_{r\to 0} \mathfrak{I}_{\Psi,\Phi,\Pi}(u, r) = 0_{\mathfrak{D}^\star}$.

Here, $\mathfrak{I}_{\Psi,\Phi,\Pi}$ is referred to as the neutrosophic norm (NN) on U and

$$\mathfrak{I}_{\Psi,\Phi,\Pi}(u, r) = \big(\Psi(u, r), \Phi(u, r), \Pi(u, r)\big)$$

Example 2.9. Let $(X, ||.||)$ be a normed linear space and let $\Gamma(\zeta, \eta) = \big(\zeta_1\eta_1,\ \min\{\zeta_2 + \eta_2, 1\},\ \min\{\zeta_2 + \eta_2, 1\}\big)$ for all $\zeta = (\zeta_1, \zeta_2, \zeta_3), \eta = (\eta_1, \eta_2, \eta_3) \in \mathfrak{D}^\star$. Now let Ψ, Φ and Π are fuzzy sets from $X \times (0, \infty)$ to $[0, 1]$ and define,

$$\mathfrak{I}_{\Psi,\Phi,\Pi}(u, r) = \big(\Psi(u, r), \Phi(u, r), \Pi(u, r)\big) = \Big(\frac{r}{r + ||u||}, \frac{||u||}{r + ||u||}, \frac{||u||}{r}\Big)$$

for all $u \in U$and $r > 0$ then $(U, I_{\Psi,\Phi,\Pi}, \Gamma)$ is neutrosophic normed space.

Definition 2.10. Consider $(U, \mathfrak{I}_{\Psi,\Phi,\Pi}, \Gamma)$ as a NNS. The open ball centered at $u \in U$ of radius $r > 0$ with respect to $\zeta \in (0, 1)$ is the set

$$\mathcal{B}_u(r, \zeta) = \Big\{v \in U : \mathfrak{I}_{\Psi,\Phi,\Pi}(u - v,\ r) >_{\mathfrak{D}^\star} (1 - \zeta, \zeta, \zeta)\Big\}.$$

Consider the set

$$\mathcal{T}_{\mathfrak{I}_{\Psi,\Phi,\Pi}} = \Big\{P \subset U :\ \text{for any } u \in P,\ \text{there exist}\ \zeta \in (0, 1) \text{ and } r > 0 \text{ so that } \mathcal{B}_u(r, \zeta) \subseteq P\Big\}.$$

Then $\mathcal{T}_{\mathfrak{I}_{\Psi,\Phi,\Pi}}$ defines a topology on X, induced by $\mathfrak{I}_{\Psi,\Phi,\Pi}$ and the collection

$$\Big\{\mathcal{B}_x(r, \zeta) : x \in X, r > 0,\ \zeta \in (0, 1)\Big\}$$

is the base for the topology $\mathcal{T}_{\mathfrak{I}_{\Psi,\Phi,\Pi}}$ on U.

Definition 2.11. Consider $(U, \mathfrak{I}_{\Psi,\Phi,\Pi}, \Gamma)$ as a *NNS*. A sequence $(u_n)_{n=1}^{\infty}$ in U is termed as Cauchy if for any $r > 0$ and $\zeta \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $\mathfrak{I}_{\Psi,\Phi,\Pi}(u_n - u_m, r) > \mathfrak{D}^{\star}(1-\zeta, \zeta, \zeta)$ for each $m, n \geq n_0$. Additionally, the sequence $(u_n)_{n=1}^{\infty}$ in U is termed as convergent to $u \in U$ if $\lim_{n\to\infty} \mathfrak{I}_{\Psi,\Phi,\Pi}(u_n - u, r) = 1_{\mathfrak{D}^{\star}}$ for every $r > 0$. In this scenario, we denote the limit as $\mathfrak{I}_{\Psi,\Phi,\Pi} - \lim_{n\in\mathbb{N}} u_n = u$.

Definition 2.12. Consider $(U, \mathfrak{I}_{\Psi,\Phi,\Pi}, \Gamma)$ as a *NNS*. A sequence $(u_n)_{n=1}^{\infty}$ in U is termed as statistically convergent to some $u \in U$ with respect to $\mathfrak{I}_{\Psi,\Phi,\Pi}$ if, for every $\zeta \in (0,1)$ and $r > 0$,

$$\delta\big\{k \in \mathbb{N} : \mathfrak{I}_{\Psi,\Phi,\Pi}(u_k - u, s) \not>_{\mathfrak{D}^{\star}} (1-\zeta, \zeta, \zeta)\big\} = 0,$$

or equivalently

$$\delta\big\{k \in \mathbb{N} : \mathfrak{I}_{\Psi,\Phi,\Pi}(u_k - u, r) >_{\mathfrak{D}^{\star}} (1-\zeta, \zeta, \zeta)\big\} = 1.$$

We write the limit as $\mathfrak{I}^{\mathfrak{st}}_{\Psi,\Phi,\Pi} - \lim_{n\in\mathbb{N}} u_n = u$.

3. Main Results

Within this section, we present the concepts of statistical graph convergence and statistical pointwise convergence pertaining to sequences of set-valued functions originating from one neutrosophic normed space and extending to another.

Let's characterize the collections of subsets of $\mathbb{N}$ in the following manner:

$$\mathscr{M} = \{M \subseteq \mathbb{N} : \ \delta(M) = 1\};$$

$$\mathscr{M}^{*} = \{M' \subseteq \mathbb{N} : \ \delta(M') \neq 0\}.$$

Consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s with respect to NN_s $\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}$ and $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}$, respectively, where U and V are linear spaces over the field of $\mathbb{R}$. Let

$$\mathfrak{D} = \big\{\varphi \mid \varphi : U \longrightarrow P(V) \text{ is a set valued function } \big\} \quad (1)$$

is the collection of all set - valued functions from $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ to $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$. Now we are going to introduce some definitions:

Definition 3.1. Consider $(\varphi_n)_{n=1}^{\infty}$ as a sequence in $\mathfrak{D}$. The statistical pointwise lower limit and statistical pointwise upper limit of $(\varphi_n)_{n=1}^{\infty}$, denoted by $\mathfrak{st}_p - \varphi_n^{l}$ and $\mathfrak{st}_p - \varphi_n^{\mathfrak{u}}$, respectively, are the set - valued functions from $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ to $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$, defined at each $u \in U$ by

$$\big(\mathfrak{st}_p - \varphi_n^l\big)(u) = \Big\{ w \in V \mid \text{ there exist } M \in \mathscr{M} \text{ and } w_n \in \varphi_n(u)(n \in M) \text{ such that } \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n \in M} w_n = w \Big\}. \tag{2}$$

and

$$\big(\mathfrak{st}_p - \varphi_n^{\mathfrak{u}}\big)(u) = \Big\{ v \in V \mid \text{ there exist } M' \in \mathscr{M}^* \text{ and } v_n \in \varphi_n(u)(n \in M') \text{ such that } \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n \in M'} v_n = v \Big\} \tag{3}$$

Definition 3.2. Consider $(\varphi_n)_{n=1}^{\infty}$ as a sequence in $\mathfrak{D}$. The statistical graphical lower limit of $(\varphi_n)_{n=1}^{\infty}$, denoted by $\mathfrak{st}_g - \varphi_n^l$, is a set - valued function from $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ to $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ with its graph $Gr\big(\mathfrak{st}_g - \varphi_n^l\big)$ such that for each $u \in U$,

$$\big(\mathfrak{st}_g - \varphi_n^l\big)(u) = \Big\{ w \in V \mid \text{ there exists } M \in \mathscr{M} : \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1} - \lim_{n \in M} u_n = u, \ \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n \in M} w_n = w, w_n \in \varphi_n(u_n) \Big\}. \tag{4}$$

and the statistical graphical uper limit, denoted by $\mathfrak{st}_g - \varphi_n^{\mathfrak{u}}$, is a set - valued function from $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ to $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ with its graph $Gr\big(\mathfrak{st}_g - \varphi_n^{\mathfrak{u}}\big)$ such that for each $u \in U$,

$$\big(\mathfrak{st}_g - \varphi_n^{\mathfrak{u}}\big)(u) = \Big\{ v \in V \mid \text{ there exists } M' \in \mathscr{M}^* : \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1} - \lim_{n \in M'} u_n = u, \ \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n \in M'} v_n = v, \ v_n \in \varphi_n(u_n) \Big\} \tag{5}$$

Remark 3.3. From Definition 3.1 and Definition 3.2, it is clear that $(\mathfrak{st}_p - \varphi_n^l)(u)$, $(\mathfrak{st}_p - \varphi_n^{\mathfrak{u}})(u)$, $\big(\mathfrak{st}_g - \varphi_n^l\big)(u)$ and $\big(\mathfrak{st}_g - \varphi_n^{\mathfrak{u}}\big)(u)$ are closed subsets of V, for every $u \in U$. Since $\mathscr{M} \subset \mathscr{M}^*$, we get

$$\mathfrak{st}_p - \varphi_n^l \subseteq \mathfrak{st}_p - \varphi_n^{\mathfrak{u}} \text{ and } \mathfrak{st}_g - \varphi_n^l \subseteq \mathfrak{st}_g - \varphi_n^{\mathfrak{u}}. \tag{6}$$

Now, we itntroduce the definitions of statistical graph convergence and statistical pointwise convergence of a sequence $(\varphi_n)_{n=1}^{\infty}$ of members of $\mathfrak{D}$, by using the concept of lower limit and upper limit introduced above as follows:

Definition 3.4. Consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s and $(\varphi_n)_{n=1}^{\infty}$ be a sequence in $\mathfrak{D}$. Then $(\varphi_n)_{n=1}^{\infty}$ is termed as statistically pointwise convergent, if there exists $\varphi \in \mathfrak{D}$ such that

$$\big(\mathfrak{st}_p - \varphi_n^{\mathfrak{u}}\big)(u) = \big(\mathfrak{st}_p - \varphi_n^{l}\big)(u) = \varphi(u), \ \forall u \in U$$

.

In such an instance, φ is termed as the statistical pointwise limit of the sequence $(\varphi_n)_{n=1}^{\infty}$, denoted by $st - \lim_{n} \varphi_n(u) = \varphi(u)$, $\forall u \in U$.

Definition 3.5. Consider $(X, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, T)$ and $(Y, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, T)$ as two NNS_s and $(\varphi_n)_{n=1}^{\infty}$ be a sequence in $\mathfrak{D}$. Then $(\varphi_n)_{n=1}^{\infty}$ is termed as statistically graph convergent to some $\varphi \in \mathfrak{D}$ or $Gr(\varphi_n)$ is statistically convergent to $Gr(\varphi)$, if

$$Gr\big(\mathfrak{st}_g - \varphi_n^{\mathfrak{u}}\big) = Gr\big(\mathfrak{st}_g - \varphi_n^{l}\big) = Gr(\varphi).$$

In such an instance, φ is termed as the statistical graphical limit of the sequence $(\varphi_n)_{n=1}^{\infty}$, denoted by $st - \lim_{n} Gr(\varphi_n) = Gr(\varphi)$.

Definition 3.6. Consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s. A sequence $(\varphi_n)_{n=1}^{\infty}$ in $\mathfrak{D}$ is termed as statistically pointwise bounded if both $\big(\mathfrak{st}_p - \varphi_n^{\mathfrak{u}}\big)(u)$ and $\big(\mathfrak{st}_p - \varphi_n^{l}\big)(u)$ exist for each $u \in U$.

For a sequence of real or complex numbers to converge in the ordinary sense, it must be bounded. Similarly, we observe that for a sequence $(\varphi_n)_{n=1}^{\infty}$ in $\mathfrak{D}$ to be statistically graph convergent, it must be statistically bounded in terms of the graph. However, it's important to note that statistical boundedness alone is not adequate for statistical graph convergence. Consequently, we introduce the definition of statistical graph boundedness for $(\varphi_n)_{n=1}^{\infty}$ as follows.

Definition 3.7. Consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s. A sequence $(\varphi_n)_{n=1}^{\infty}$ in $\mathfrak{D}$ is termed as statistically graph bounded if both $Gr\big(\mathfrak{st}_g - \varphi_n^{\mathfrak{u}}\big)$ and $Gr\big(\mathfrak{st}_g - \varphi_n^{l}\big)$ exist .

Remark 3.8. Consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s and let $(\varphi_n)_{n=1}^{\infty}$ represent an infinite sequence in $\mathfrak{D}$. Assuming $\mathscr{M}$ equals $\mathscr{M}_f$ and $\mathscr{M}^*$ equals $\mathscr{M}_\infty$, the concepts of statistical pointwise limits and statistical graphical limits of $(\varphi_n)_{n=1}^{\infty}$ correspond to the pointwise limits and graphical limits of the same sequence. Under these conditions, the statistical pointwise convergence of $(\varphi_n)_{n=1}^{\infty}$ aligns with its pointwise convergence, and the statistical graph convergence of $(\varphi_n)_{n=1}^{\infty}$ aligns with its graph convergence.

Theorem 3.9. *Consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s. Suppose $(\varphi_n)_{n=1}^{\infty}$ and $(\Omega_n)_{n=1}^{\infty}$ are two sequences in $\mathfrak{D}$ such that $(\varphi_n)_{n=1}^{\infty}$ and $(\Omega_n)_{n=1}^{\infty}$ are pointwise convergent and graph convergent, respectively. Then $(\varphi_n)_{n=1}^{\infty}$ and $(\Omega_n)_{n=1}^{\infty}$ are statistically pointwise convergent and statistically graph convergent, respectively.*

Proof. It is evident that $\mathscr{M}_f \subset \mathscr{M}$ and $\mathscr{M}^* \subset \mathscr{M}_\infty$.
Suppose $(\varphi_n)_{n=1}^{\infty}$ be a sequence in $\mathfrak{D}$. Then

$$p - \varphi_n^l \subset \mathfrak{st}_p - \varphi_n^l \text{ and}$$

$$\mathfrak{st}_p - \varphi_n^{\mathfrak{u}} \subset p - \varphi_n^{\mathfrak{u}}.$$

Hence, by (6), we get

$$p - \varphi_n^l \subset \mathfrak{st}_p - \varphi_n^l \subseteq \mathfrak{st}_p - \varphi_n^{\mathfrak{u}} \subset p - \varphi_n^{\mathfrak{u}}.$$

Let $(\varphi_n)_{n=1}^{\infty}$ is pointwise convergent. Then

$$(p - \varphi_n^l)(u) = (p - \varphi_n^{\mathfrak{u}})(u) \ \forall u \in U.$$

Therefore

$$(\mathfrak{st}_p - \varphi_n^l)(u) = (\mathfrak{st}_p - \varphi_n^{\mathfrak{u}})(u) \ \forall u \in U.$$

Thus, $(\varphi_n)_{n=1}^{\infty}$ is statistically pointwise convergent.

Let $(\Omega_n)_{n=1}^{\infty}$ be a sequence in $\mathfrak{D}$ such that $(\Omega_n)_{n=1}^{\infty}$ is graph convergent. Similarly, it can be seen that $(\Omega_n)_{n=1}^{\infty}$ is statistically graph convergent. □

The converse of the Theorem 3.9 is not necessarily true. To illustrate this, let's consider the following example:

Example 3.10. Consider $\Gamma(\zeta, \eta) = \big(\zeta_1\eta_1, \ \min(\zeta_2 + \eta_2, 1), \min(\zeta_3 + \eta_3, 1)\big)$ for all $\zeta = (\zeta_1, \zeta_2, \zeta_3), \eta = (\eta_1, \eta_2, \eta_3) \in \mathfrak{D}^\star$. Define fuzzy sets Ψ, Φ and Π on $\mathbb{R} \times (0, \infty)$ by

$$\Psi(u, s) = e^{\left(-\frac{|u|}{s}\right)}, \ \Phi(u, s) = 1 - e^{\left(-\frac{|u|}{s}\right)} \text{ and } \Pi(u, s) = 1 - e^{\left(-\frac{|u|}{s}\right)}$$

for every $u \in \mathbb{R}$ and for all $s \in (0, \infty)$. Then $(\mathbb{R}, \mathfrak{I}_{\Psi,\Phi,\Pi}, \Gamma)$ is a *NNS*, where $\mathfrak{I}_{\Psi,\Phi,\Pi}(u, s) = \big(\Psi(u, s), \Phi(u, s), \Pi(u, s)\big)$.
Now define $\varphi_n : \mathbb{R} \longrightarrow P(\mathbb{R})$ by

$$\varphi_n(u) = \begin{cases} [0, \frac{1}{2}], & \text{if } n = p \\ [-\frac{1}{2}, 0], & \text{if } n \neq p \end{cases} \text{ p is prime.}$$

for each $u \in \mathbb{R}$.
Then

$$\big(\mathfrak{st}_p - \varphi_n^{\mathfrak{u}}\big)(u) = \big(\mathfrak{st}_p - \varphi_n^l\big)(u) = \big[-\frac{1}{2}, 0\big]$$

for each $u \in \mathbb{R}$.

Also,

$$Gr\big(\mathfrak{st}_g - \varphi_n^{\mathfrak{u}}\big) = Gr\big(\mathfrak{st}_g - \varphi_n^{l}\big)$$
$$= \big\{(u, v) : u \in \mathbb{R}, -\frac{1}{2} \leq v \leq 0\big\}.$$

Thus, $(\varphi_n)_{n=1}^{\infty}$ is both statistically pointwise convergent and statistically graph convergent. However, on the other hand,

$$(p - \varphi_n^{\mathfrak{u}})(u) = \big[-\frac{1}{2}, \frac{1}{2}\big] \text{ and } (p - \varphi_n^{l})(u) = \{0\}$$

for each $u \in \mathbb{R}$. Thus $(\varphi_n)_{n=1}^{\infty}$ is not a pointwise convergent sequence. Also,

$$Gr(\varphi_n^{\mathfrak{u}}) = \big\{(u, v) : u \in \mathbb{R}, -\frac{1}{2} \leq v \leq \frac{1}{2}\big\}$$

but $Gr(\varphi_n^{l})$, the graphical lower limit of $(\varphi_n)_{n=1}^{\infty}$ does not exist. Hence $(\varphi_n)_{n=1}^{\infty}$ is not a graph convergent sequence.

Theorem 3.11. *Consider $(U, \Im_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \Im_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS$_s$. Assume $(\varphi_n)_{n=1}^{\infty}$ and $(\Omega_n)_{n=1}^{\infty}$ are two sequences in $\mathfrak{D}$ such that $st - \lim_n \varphi_n(u) = \varphi(u)$ for each $u \in U$ and $st - \lim_n Gr(\Omega_n) = Gr(\Omega)$. Then φ and Ω are unique.*

Proof. Let, $\exists\ \Omega^0 \in \mathfrak{D}$ such that

$$st - \lim_n Gr(\Omega_n) = Gr(\Omega^0)$$

. Then

$$Gr\big(\mathfrak{st}_g - \Omega_n^{u}\big) = Gr\big(\mathfrak{st}_g - \Omega_n^{l}\big) = Gr(\Omega^0) = Gr(\Omega).$$

Hence $\Omega = \Omega^0$. Similarly, it can be easily seen that pointwise limit is also unique. □

Proposition 3.12. *Consider $(U, \Im_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \Im_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS$_s$. Assume $(\varphi_n)_{n=1}^{\infty}$ and $(\Omega_n)_{n=1}^{\infty}$ are statistically pointwise convergent sequences in $\mathfrak{D}$ such that $st - \lim_n \varphi_n(u) = \varphi(u)$ and $st - \lim_n \Omega_n(u) = \Omega(u)$ for each $u \in U$. Then the sum of the sequences $(\varphi_n)_{n=1}^{\infty}$ and $(\Omega_n)_{n=1}^{\infty}$ is statistically pointwise convergent with $st - \lim_n(\varphi_n + \Omega_n)(u) = (\varphi + \Omega)(u)$ for each $u \in U$.*

Proof. Let $(\varphi_n)_{n=1}^{\infty}$ and $(\Omega_n)_{n=1}^{\infty}$ be two sequences in $\mathfrak{D}$ such that $st - \lim_n \varphi_n(u) = \varphi(u)$ and $st - \lim_n \Omega_n(u) = \Omega(u)$ for each $u \in U$. Then

$$\big(\mathfrak{st}_p - \varphi_n^{\mathfrak{u}}\big)(u) = \big(\mathfrak{st}_p - \varphi_n^{l}\big)(u) = \varphi(u),\ \forall u \in U$$

and

$$\big(\mathfrak{st}_p - \Omega_n^{u}\big)(u) = \big(\mathfrak{st}_p - \Omega_n^{l}\big)(u) = \Omega(u),\ \forall u \in U.$$

Let v be an arbitrary element of $\varphi(u)$. Since $\left(\mathfrak{st}_p - \varphi_n^{\mathfrak{u}}\right)(u) = \varphi(u),\ \forall u \in U$, there exist $M_1' \in \mathscr{M}^*$ and $v_n \in \varphi_n(u)$ $(n \in M_1')$ such that $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n\in M_1'} v_n = v$. Hence $\lim_{n\in M_1'} \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}(v_n - v, r) = 1_{\mathfrak{D}^\star}$ for every $r > 0$, i.e.,

$$\lim_{n\in M_1'} \Psi_2\left(v_n - v,\ \frac{r}{2}\right) = 1, \lim_{n\in M_1'} \Phi_2\left(v_n - v,\ \frac{r}{2}\right) = 0, \lim_{n\in M_1'} \Pi_2\left(v_n - v,\ \frac{r}{2}\right) = 0. \qquad (7)$$

Let w be an arbitrary element of $\Omega(u)$. Also, $\left(\mathfrak{st}_p - \Omega_n^{u}\right)(u) = \Omega(u),\ \forall u \in U$. Hence, there exist $M_2' \in \mathscr{M}^*$ and $w_n \in \Omega_n(u)(n \in M_2)$ such that $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n\in M_2'} w_n = w$. Hence $\lim_{n\in M_2'} \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}(w_n - w, r) = 1_{\mathfrak{D}^\star}$ for every $r > 0$, i.e.,

$$\lim_{n\in M_2'} \Psi_2\left(w_n - w,\ \frac{r}{2}\right) = 1, \lim_{n\in M_2'} \Phi_2\left(w_n - w,\ \frac{r}{2}\right) = 0, \lim_{n\in M_2'} \Pi_2\left(w_n - w,\ \frac{r}{2}\right) = 0.. \qquad (8)$$

Because $\left(\mathfrak{st}_p - \varphi_n^{\mathfrak{u}}\right)(u) = \left(\mathfrak{st}_p - \varphi_n^{l}\right)(u)$ and $\left(\mathfrak{st}_p - \Omega_n^{u}\right)(u) = \left(\mathfrak{st}_p - \Omega_n^{l}\right)(u),\ \forall u \in U$, (7) and (8) hold for $n \in M_2'$ and $n \in M_1'$, respectively. Now, in the choice of $M_1', M_2' \in \mathscr{M}^*$, we have the following two possibilities:

(1) $M_1' \cap M_2' = \emptyset$ or $\delta(M_1' \cap M_2') = 0$.

(2) $M_1' \cap M_2' \in \mathscr{M}^*$.

If $M_1' \cap M_2' = \emptyset$ or $\delta(M_1' \cap M_2') = 0$, take $M' = M_1'$ or $M' = M_2'$. If $M_1' \cap M_2' \in \mathscr{M}^*$, put $M' = M_1' \cap M_2'$. Thus for $n \in M'$, we get

$$\begin{aligned}
&\Psi_2\left((v_n + w_n) - (v + w),\ r\right)\\
&\quad = \Psi_2\left((v_n - v) + (w_n - w),\ r\right)\\
&\quad \geq \Psi_2\left(v_n - v,\ \frac{r}{2}\right) * \Psi_2\left(w_n - w,\ \frac{r}{2}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
&\lim_{n\in M'} \Psi_2\left((v_n + w_n) - (v + w),\ r\right)\\
&\quad \geq \lim_{n\in M'} \Psi_2\left(v_n - v,\ \frac{r}{2}\right) * \lim_{n\in M'} \varphi_2\left(w_n - w,\ \frac{r}{2}\right)\\
&\quad = 1 * 1\\
&\quad = 1.
\end{aligned}$$

Also for $n \in M'$,

$$\Phi_2\bigg((v_n + w_n) - (v + w),\ r\bigg)$$
$$= \Phi_2\bigg((v_n - v) + (w_n - w),\ r\bigg)$$
$$\leq \Phi_2\bigg(v_n - v,\ \frac{r}{2}\bigg) \diamond \Phi_2\bigg(w_n - w,\ \frac{r}{2}\bigg).$$

Hence

$$\lim_{n\in M'} \Phi_2\bigg((v_n + w_n) - (v + w),\ r\bigg)$$
$$\leq \lim_{n\in M'} \Phi_2\bigg(v_n - v,\ \frac{r}{2}\bigg) \diamond \lim_{n\in M'} \Phi_2\bigg(w_n - w,\ \frac{r}{2}\bigg)$$
$$= 0 * 0$$
$$= 0.$$

Similarly, for $n \in M'$,

$$\Pi_2\bigg((v_n + w_n) - (v + w),\ r\bigg)$$
$$= \Pi_2\bigg((v_n - v) + (w_n - w),\ r\bigg)$$
$$\leq \Pi_2\bigg(v_n - v,\ \frac{r}{2}\bigg) \diamond \Pi_2\bigg(w_n - w,\ \frac{r}{2}\bigg).$$

Hence

$$\lim_{n\in M'} \Pi_2\bigg((v_n + w_n) - (v + w),\ r\bigg)$$
$$\leq \lim_{n\in M'} \Pi_2\bigg(v_n - v,\ \frac{r}{2}\bigg) \diamond \lim_{n\in M'} \Pi_2\bigg(w_n - w,\ \frac{r}{2}\bigg)$$
$$= 0 * 0$$
$$= 0.$$

Thus $\lim_{n\in M'} \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}\big((v_n + w_n) - (v + w), r\big) = 1_{\mathfrak{D}^\star}$ for every $r > 0$ and hence

$$\bigg(\mathfrak{st}_p - \big(\varphi_n + \Omega_n\big)^{\mathfrak{u}}\bigg)(u) = \big(\varphi + \Omega\big)(u),\ \forall u \in U.$$

Again, since $\big(\mathfrak{st}_p - \varphi_n^l\big)(u) = \varphi(u)$ for each $u \in U$ and $v \in \varphi(u)$, there exist $M_1 \in \mathscr{M}$ and $c_n \in \varphi_n(u)$ $(n \in M_1)$ such that $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n\in M_1} c_n = v$, i.e., for every $s > 0$,

$$\lim_{n\in M_1} \Psi_2\bigg(c_n - v,\ \frac{s}{2}\bigg) = 1, \lim_{n\in M_1} \Phi_2\bigg(c_n - v,\ \frac{s}{2}\bigg) = 0, \lim_{n\in M_1} \Pi_2\bigg(c_n - v,\ \frac{s}{2}\bigg) = 0.$$

Also, $\big(\mathfrak{st}_p - \Omega_n^l\big)(u) = \Omega(u)$ for each $u \in U$ and $w \in \Omega(u)$. Hence, there exist $M_2 \in \mathscr{M}$ and $d_n \in \Omega_n(x)(n \in M_2)$ such that $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n\in M_2} d_n = w$, i.e.,

$$\lim_{n\in M_2} \Psi_2\left(d_n - w,\ \frac{s}{2}\right) = 1, \lim_{n\in M_2} \Phi_2\left(d_n - w,\ \frac{s}{2}\right) = 0, \lim_{n\in M_2} \Pi_2\left(d_n - w,\ \frac{s}{2}\right) = 0,$$

for every $s > 0$. Put $M = M_1 \cap M_2$. Clearly, $M \in \mathscr{M}$. Then, similar to above, for every $s > 0$ and $n \in M$, we have

$$\lim_{n\in M} \Psi_2\Big((c_n + d_n) - (v + w),\ s\Big) = 1,$$

$$\lim_{n\in M} \Phi_2\Big((c_n + d_n) - (v + w),\ s\Big) = 0,$$

$$\lim_{n\in M} \phi_2\Big((c_n + d_n) - (v + w),\ s\Big) = 0$$

and thus $\lim_{n\in M} \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}\big((c_n + d_n) - (v + w), s\big) = 1_{\mathfrak{D}^\star}$ for every $s > 0$. Since v and w are arbitrary members of $\varphi(u)$ and $\Omega(u)$, respectively, we get

$$\Big(\mathfrak{st}_p - \big(\varphi_n + \Omega_n\big)^l\Big)(u) = \big(\varphi + \Omega\big)(u),\ \forall u \in U.$$

Therefore the sum of the sequence $(\varphi_n)_{n=1}^{\infty}$ and $(\Omega_n)_{n=1}^{\infty}$ is statistically pointwise convergent with $st - \lim_{n}(\varphi_n + \Omega_n)(u) = (\varphi + \Omega)(u)$ for each $u \in U$. □

Lemma 3.13. *consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\phi_2}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s. Suppose $(\varphi_n)_{n=1}^{\infty}$ is a sequence in $\mathfrak{D}$. Then the following hold:*

(1) $\bigcup_{u\in U} \{u\} \times \big(\mathfrak{st}_p - \varphi_n^l\big)(u) \subseteq Gr\big(\mathfrak{st}_g - \varphi_n^l\big)$,

(2) $\bigcup_{u\in U} \{u\} \times \big(\mathfrak{st}_p - \varphi_n^u\big)(u) \subseteq Gr\big(\mathfrak{st}_g - \varphi_n^u\big)$.

Proof. Part (1) , let $u \in U$ and $v \in \big(\mathfrak{st}_p - \varphi_n^l\big)(u)$. Then there exist $M \in \mathscr{M}$ and $v_n \in \varphi_n(u)$ $(n \in M)$ such that $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n\in M} v_n = v$, i.e.,

$$\lim_{n\in M} \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}(v_n - v, r) = 1_{\mathfrak{D}^\star},\ \text{for every } r > 0. \tag{9}$$

Now, consider the constant sequence $(u_n)_{n=1}^{\infty} = \{u\}$ in U. Then $\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1} - \lim_{n\in M} u_n = u$, i.e., for every $r > 0$,

$$\lim_{n\in M} \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}(u_n - u, r) = 1_{\mathfrak{D}^\star}.$$

Thus, $v_n \in \varphi_n(u_n) = \varphi_n(u)$ $(n \in M)$ satisfies (9). Therefore $v \in \big(\mathfrak{st}_g - \varphi_n^l\big)(u)$ and hence $(u, v) \in Gr\big(\mathfrak{st}_g - \varphi_n^l\big)$.

Similarly, part (2) can be proved. □

Corollary 3.14. *Consider* $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ *and* $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ *as two NNS$_s$ and* $(\varphi_n)_{n=1}^{\infty}$ *be a sequence in* $\mathfrak{D}$*. If statistical pointwise limit* φ *and statistical graphical limit* Ω *of* $(\varphi_n)_{n=1}^{\infty}$ *both exist, then for all elements* u *in* U*, it follows that* $\varphi(u)$ *is a subset of* $\Omega(u)$*.*

It is being established that the presence of a statistical pointwise limit for a sequence $(\varphi_n)_{n=1}^{\infty}$ in $\mathfrak{D}$ does not guarantee the existence of the statistical graphical limit, and vice versa. Even in cases where both limits exist, they may not be directly comparable. However, there exists a specific condition under which these limits coincide. To formalize this, we introduce the following theorem:

Theorem 3.15. *Consider* $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ *and* $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ *as two NNS$_s$. Let* $\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}$ *induces the discrete topology* $\mathcal{T}_{\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}}$ *on* U*. Then a sequence* $(\varphi_n)_{n=1}^{\infty}$ *in* $\mathfrak{D}$ *is statistically pointwise convergent if and only if it is statistically graph convergent and both the statistical pointwise limit and the statistical graphical limit of* $(\varphi_n)_{n=1}^{\infty}$ *are equivalent.*

Proof. Let $(\varphi_n)_{n=1}^{\infty} \in \mathfrak{D}$ such that $(\varphi_n)_{n=1}^{\infty}$ is statistically pointwise convergent. Then there exists $\varphi \in \mathfrak{D}$ such that

$$\big(\mathfrak{st}_p - \varphi_n^u\big)(u) = \big(\mathfrak{st}_p - \varphi_n^l\big)(u) = \varphi(u),\ \forall u \in U.$$

Thus, by using part (1) of Lemma 3.13, we obtain

$$Gr(\varphi) \subset Gr\big(\mathfrak{st}_g - \varphi_n^l\big). \tag{10}$$

Now we claim to show that $Gr\big(\mathfrak{st}_g - \varphi_n^u\big) \subset Gr(\varphi)$.

Let $(u, v) \in Gr\big(\mathfrak{st}_g - \varphi_n^u\big)$. Hence $v \in \big(\mathfrak{st}_g - \varphi_n^u\big)(u)$. Then, there exists $M' \in \mathscr{M}^*$ such that $\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1} - \lim\limits_{n\in M'} u_n = u$ and $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim\limits_{n\in M'} v_n = v$ for $v_n \in \varphi_n(u_n)$. Since $\mathcal{T}_{\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}}$ is the discrete topology on U, we have $u_n = u$ and $\varphi_n(u_n) = \varphi_n(u)$, for all $n \in M'$. This implies that $v \in \big(\mathfrak{st}_p - \varphi_n^u\big)(u) = \varphi(u)$. Thus $(u, v) \in Gr(\varphi)$ and hence

$$Gr\big(\mathfrak{st}_g - \varphi_n^u\big) \subset Gr(\varphi). \tag{11}$$

Consequently, by (10) and (11), we obtain

$$Gr\big(\mathfrak{st}_g - \varphi_n^l\big) = Gr\big(\mathfrak{st}_g - \varphi_n^u\big) = Gr(\varphi).$$

Thus the sequence $(\varphi_n)_{n=1}^{\infty}$ is statistically graph convergent with $st - \lim\limits_n Gr(\varphi_n) = Gr(\varphi)$.

Conversely, suppose that $(\varphi_n)_{n=1}^{\infty}$ is statistically graph convergent. Then there exists $\varphi \in \mathfrak{D}$ such that

$$Gr\big(\mathfrak{st}_g - \varphi_n^{\mathrm{u}}\big) = Gr\big(\mathfrak{st}_g - \varphi_n^l\big) = Gr(\varphi).$$

Now, by using part (2) of Lemma 3.13, we obtain

$$\big(\mathfrak{st}_p - \varphi_n^{\mathrm{u}}\big)(u) \subset \varphi(u),\ \forall u \in U. \tag{12}$$

Now we claim to show that $\varphi(u) \subset \big(\mathfrak{st}_p - \varphi_n^l\big)(u)$, $\forall u \in U$.

Let $w \in \varphi(u)$. Then $(u,w) \in Gr(\varphi) = Gr\big(\mathfrak{st}_g - \varphi_n^l\big)$. Hence, there exists $M \in \mathscr{M}$ such that $\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1} - \lim_{n\in M} u_n = u$ and $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n\in M} w_n = w$ for $w_n \in \varphi_n(u_n)$. Since $\mathcal{T}_{\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}}$ is discrete topology on U, we have $u_n = u$ and $\varphi_n(u_n) = \varphi_n(u)$, for all $n \in M$. Therefore, $(u,w) \in \big(\mathfrak{st}_p - \varphi_n^l\big)$. Consequently, we get

$$\varphi(u) \subset \big(\mathfrak{st}_p - \varphi_n^l\big)(u),\ \forall u \in U. \tag{13}$$

From (12) and (13), we get

$$\big(\mathfrak{st}_p - \varphi_n^l\big)(u) = \big(\mathfrak{st}_p - \varphi_u^l\big)(u) = \varphi(u),\ \forall u \in U.$$

Thus $(\varphi_n)_{n=1}^{\infty}$ is statistically pointwise convergent with $st - \lim_n \varphi(u) = \varphi(u)$ for each $u \in U$. □

Within $\mathfrak{D}$, there exist specific collections of sequences where the statistical graphical limit differs from the statistical pointwise limit, and conversely. Before delving into such cases, let's introduce the concept of semicontinuity of sequences in $\mathfrak{D}$ as follows:

Definition 3.16. Consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s. A set–valued function $\varphi : U \longrightarrow P(V)$ is termed as neutrosophic lower semicontinuous ($NLSC$) at $u_0 \in U$ if and only if for any $r > 0$, $\zeta \in (0,1)$ and $v \in V$ with $\mathcal{B}_v(r,\zeta)$ in V such that $\varphi(u_0) \cap \mathcal{B}_v(r,\zeta) \neq \emptyset$, there exists $\mathcal{B}_{u_0}(r^0,\zeta^0)$ in U for some $r^0 > 0$ and $\zeta^0 \in (0,1)$ such that $\varphi(w) \cap \mathcal{B}_v(r,\zeta) \neq \emptyset$, for each $w \in \mathcal{B}_{u_0}(r^0,\zeta^0)$.

The set–valued function $\varphi : U \longrightarrow P(V)$ is called $NLSC$ on U, if it is $NLSC$ at every $u \in U$.

Definition 3.17. Consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s. A set–valued function $\varphi : U \longrightarrow P(V)$ is termed as neutrosophic upper semicontinuous ($NUSC$) at $u_0 \in U$ if and only if for any $r > 0$ and $\zeta \in (0,1)$ with $\mathcal{B}_{\varphi(u_0)}(r,\zeta)$ in V, there exists $\mathcal{B}_{u_0}(r^0,\zeta^0)$ in U for some $r^0 > 0$ and $\zeta^0 \in (0,1)$ such that

$$\varphi\big(\mathcal{B}_{u_0}(r^0,\zeta^0)\big) = \bigcup_{w\in\mathcal{B}_{u_0}(r^0,\zeta^0)} \varphi(w) \subseteq \mathcal{B}_{\varphi(u_0)}(r,\zeta).$$

The set–valued function $\varphi : U \longrightarrow P(V)$ is called $NUSC$ on U, if it is $NUSC$ at every $u \in U$.

Definition 3.18. Consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s. A set–valued function $\varphi : U \longrightarrow P(V)$ is termed as neutrosophic continuous (NC) on U, if it is $NUSC$ as well as $NLSC$ on U.

Definition 3.19. Consider $(U, \mathfrak{I}_{\Psi,\Phi,\Pi}, \Gamma)$ as a *NNS* and $C \subset U$. Then the statistical closure of set C with respect to $\mathfrak{I}_{\Psi,\Phi,\Pi}$ denoted by $\overline{C_{st}}$, is defined as

$$\overline{C_{st}} = \Big\{ \mathcal{L} \in U \mid \text{there exist } M \in \mathscr{M} \text{ and } (u_n) \text{ in } E$$
$$\text{such that } \mathfrak{I}_{\Psi,\Phi,\Pi} - \lim_{n \in N} x_n = \mathcal{L} \Big\}.$$

We define C as a statistically closed subset of U if C coincides with its statistical closure, denoted as $\overline{C_{st}}$. It's evident that every closed subset of U is also statistically closed.

For any two topological spaces U and V, we denote $\mathcal{C}(U, P(V))$ as the collection of all continuous set–valued functions from U to V with closed values.

Theorem 3.20. *Consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s such that $\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}$ induces the non–discrete topology $\mathcal{T}_{\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}}$ on U and $\mathcal{C}\big([0,1], P(V)\big)$ is non–trivial, where $[0,1]$ is equipped with the usual topology. Then there exists $\{\varphi, \varphi_n : n \in \mathbb{N}\} \in \mathcal{C}(U, P(V))$ such that $st - \lim_n \varphi_n(u) = \varphi(u)$ for each $u \in U$ but $st - \lim_n Gr(\varphi_n) \neq Gr(\varphi)$.*

Proof. Consider $u_0 \in U$ be a non–isolated point. Then for every neighborhood $\mathbb{B}_{u_0}$ of u_0 in U, we have $\mathbb{B}_{u_0} \neq \{u_0\}$. Without loss of generality, consider the countable collection

$$\mathfrak{B}_{u_0} = \left\{ \mathbb{B}^j_{u_0} : \ \mathbb{B}^j_{u_0} = \mathcal{B}_{u_0}\left(\frac{1}{j}, \frac{1}{j}\right) : j \in \mathbb{N} \right\}$$

of neighborhoods of u_0 in U. For fixed $j \in \mathbb{N}$, choose $\mathbb{B}^j_{u_0} \in \mathfrak{B}_{u_0}$ and let $u_j \in \mathbb{B}^j_{u_0}$ such that $u_j \neq u_0$. Since $\big(U, \mathcal{T}_{\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}}\big)$ is completely regular Hausdorff space, there exists a continuous function $f_j : U \longrightarrow [0,1]$ corresponding to $\mathbb{B}^j_{u_0}$ such that $f_j(u_j) = 1$ and

$$f_j(u) = 0, \text{ for all } u \in (\mathbb{B}^j_{u_0})^c \bigcup \{u_0\}.$$

As $\mathcal{C}\big([0,1], P(V)\big)$ is non–trivial, there exists $h \in \mathcal{C}\big([0,1], P(V)\big)$ such that $h(1) \neq h(0)$. Therefore, for every $j \in \mathbb{N}$, $\varphi_j = h \circ f_j \in \mathcal{C}\big(U, P(V)\big)$. Thus, the sequence $\big\{\varphi_j : \mathbb{B}^j_{u_0} \in \mathfrak{B}_{u_0}, j \in \mathbb{N}\big\}$ belongs to $\mathcal{C}\big(U, P(V)\big)$. Let $u \in U$ be arbitrary. If $u = u_0$ or $u \notin \mathbb{B}^j_{u_0}$ for all $j \in \mathbb{N}$, then

$$F_j(u) = h(f_j(u)) = h(0), \text{ for all } j.$$

If $u \neq u_0$ or $u \in \mathbb{B}^j_{u_0}$ for some fixed $j \in \mathbb{N}$, then there is $M_1 \in \mathscr{M}$ such that

$$\varphi_n(u) = h(f_n(u)) = h(0), \text{ for all } n \in M_1.$$

Take $\varphi(u) = h(0)$ for each $u \in U$. Hence the sequence $\big\{\varphi_j : \mathbb{B}^j_{u_0} \in \mathfrak{B}_{u_0}, j \in \mathbb{N}\big\}$ statistically pointwise convergent with $st - \lim_j \varphi_j(u) = \varphi(u)$ for each $u \in U$.

Alternatively, let $u_j \in \mathbb{B}^j_{u_0}$ such that $u_j \neq u_0$ for every $j \in \mathbb{N}$. Consequently, there exists $M_2 \in \mathscr{M}$ such that $\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1} - \lim_{j \in M_2} u_j = u_0$ and $f_j(u_j) = 1$ for each j. This implies $\varphi_j(u_j) =$

$h(1)$ for every j. Suppose $v \in h(1) \setminus h(0)$. Then $(u_{\ j}, v) \in Gr(\varphi_j)$ for every j, and $(u_0, v) \in Gr(\mathfrak{st}_g - \varphi_j^l) \setminus Gr(\varphi)$. Consequently, $st - \lim_j Gr(\varphi_j) \neq Gr(\varphi)$. □

Theorem 3.21. *Consider* $(U, \Im_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ *and* $(V, \Im_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ *as two* NNS_s *such that* $\Im_{\Psi_1,\Phi_1,\Pi_1}$ *induces the non–discrete topology* $\mathcal{T}_{\Im_{\Psi_1,\Phi_1,\Pi_1}}$ *on* U *and* $\mathcal{C}\big([0,1], P(V)\big)$ *is non–trivial, where* $[0,1]$ *is equipped with the usual topology. Then there exists* $\{\varphi, \varphi_n : n \in \mathbb{N}\} \in \mathcal{C}(U, P(V))$ *such that* $st - \lim_n Gr(\varphi_n) = Gr(\varphi)$ *but* $st - \lim_n \varphi_n(u) \neq \varphi(u)$ *for each* $u \in U$.

Proof. let $u_0 \in U$ be a non–isolated point and $h_1 \in \mathcal{C}([0,1], P(V))$ such that $h_1(1) \neq h_1(0)$. Then there exist a non constant sequence $(u_n)_{n=1}^{\infty}$ in U and $M \in \mathscr{M}$ such that $\Im_{\Psi_1,\Phi_1,\Pi_1} - \lim_{n\in M} u_n = u_0$. Now define $h_2 : [0,1] \longrightarrow P(V)$ by

$$h_2(\zeta) = \overline{\bigcup_{\eta\leq\zeta} h_1(\eta)},\ \zeta \in [0,1].$$

It is certain that $h_2 \in \mathcal{C}([0,1], P(V))$ and $h_2(1) \neq h_2(0)$. Now, for every $\zeta \in [0,1]$, we have $h_2(\zeta) \subset h_2(1)$. let $r > 0$ and $u \in U$. Set $\zeta = \varphi_1(u - u_0, r)$ and $\zeta_n = \varphi_1(u_n - u_0, r)$. Clearly, for each $n \in \mathbb{N}$, $\zeta_{n+1} \geq \zeta_n$ and $\lim_{n\in M} \zeta_n = 1$. Now define the set–valued functions φ, φ_n from U to V as follows:

$$\varphi_n(u) = \begin{cases} h_2\bigg(\frac{1-\zeta}{1-\zeta_n} \times \frac{\zeta_n}{\zeta}\bigg), & \text{if } \zeta > \zeta_n, \\ h_2(1), & \text{if } \zeta_n \geq \zeta, \end{cases} \quad n \in \mathbb{N}$$

$$\varphi(u) = h_2(1),\ \text{for all } u \in U.$$

From the definition of h_2, it is clear that $\{\varphi, \varphi_n : n \in \mathbb{N}\} \in \mathcal{C}(U, P(V))$. Let $(u_1, v_1) \in Gr\big(\mathfrak{st}_g - \varphi_n^u\big)$. Then, there exists $M' \in \mathscr{M}^*$ such that $\Im_{\Psi_1,\Phi_1,\Pi_1} - \lim_{n\in M'} u_n^{'} = u_1$ and $\Im_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n\in M'} v_n^{'} = v_1$ for $v_n^{'} \in \varphi_n(u_n^{'}) = h_2(1)$. Hence $\Im_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n\in M'} v_n^{'} = v_1$ for $v_n^{'} \in \varphi(u_1)$. Since $Gr(\varphi)$ is closed, we get $v_1 \in \varphi(u_1)$ and thus $(u_1, v_1) \in Gr(\varphi)$. Therefore,

$$Gr\big(\mathfrak{st}_g - \varphi_n^u\big) \subset Gr(\varphi). \tag{14}$$

Now, let $(u_2, v_2) \in Gr(\varphi)$. Hence $v_2 \in \varphi(u_2) = h_2(1)$. If $u_2 = u_0$, then $\Im_{\Psi_1,\Phi_1,\Pi_1} - \lim_{n\in M} u_n = u_2$ and $\varphi_n(u_n) = h_2(1)$, for all $n \in M$. Since $h_2(1)$ is closed and hence statistically closed, there exists $v_n \in h_2(1) = \varphi_n(u_n)$ such that $\Im_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n\in N} v_n = v_2$. If $u_2 \neq u_0$, there exist $N \in \mathscr{M}$ and a sequence $(u_n^{''})_{n=1}^{\infty}$ in U and such that $\Im_{\Psi_1,\Phi_1,\Pi_1} - \lim_{n\in N} u_n^{''} = u_2$ and $\Im_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n\in N} v_n^{''} = h_2(1)$ for $v_n^{''} \in \varphi_n(u_n^{''})$. Since $v_2 \in h_2(1)$, we get $(u_2, v_2) \in Gr(\mathfrak{st}_g - \varphi_n^l)$. Hence

$$Gr(\varphi) \subset Gr(\mathfrak{st}_g - \varphi_n^l) \tag{15}$$

From (14) and (15), we get $st - \lim_n Gr(\varphi_n) = Gr(\varphi)$.
Alternatively, $\varphi_n(u_0) = h_2(0) \neq h_2(1)$, for every $n \in \mathbb{N}$. Thus $st - \lim_n \varphi_n(u_0) \neq \varphi(u_0)$.
Hence proof of the theorem is completed. □

4. Conclusions

This articles introduces the concepts of statistical pointwise and statistical graphical limits of sequences of set–valued functions defined from a neutrosophic normed space to another, certan theorems about statistical point wise and statistical graphical covergence are proved and lastly the idea of neutrosophic upper and lower semi continuities of set valued maps is given and used to develop link between these convergences.

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