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Numerical Techniques for solving ordinary differential equations with uncertainty

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Abstract. In this article, we derive a numerical solution to the ordinary differential equation with a neutrosophic number as the initial condition.The Adams-Bashforth, Adams-Moulton, and predictorcorrector algorithms are used to solve the differential equation with hexagonal neutrosophic number as the initial condition. The convergence and stability of the methods are also investigated.

Keywords: Neutrosophic number, Adams Bashforth, Adams-Moulton and predictor corrector methods.

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1. Introduction

Differential equations may be used to solve various scientific and technological challenges. Analytical approaches for solving differential equations are only relevant to specific types of equations. Many physical problems require numerical approaches to solve differential equations that do not fit into the conventional forms.These strategies are more crucial now that computing devices can significantly reduce numerical effort.

The linear multi-step technique is one of the ways of acquiring an approximation solution to a certain differential equation where the precise or analytical can be determined or not. The linear multi-step technique improves efficiency by combining information from previous approximations. Adams-Bashforth method is a linear multi-step method. Numerical methods are classified as explicit and implicit types.The explicit technique calculates the future system state based on the current system status. The implicit technique derives the future system status from the current and future system states. The explicit type is called the Adams-Bashforth [5] (AB) method which was introduced by John Couch Adams to solve a differential equation modelling capillary action while the implicit type is called the Adams-Moulton (AM) method developed by Ray Moulton [10].

Fuzzy set (FS) [15] theory is one of the efficient tool to represent uncertainty. The study of fuzzy differential equations [7] is fast emerging in many fields. Over the past few years, there has been intense discussion in theoretical aspects and its applications. Intutionistic fuzzy set(IFS) [1] is a generalization of fuzzy set. Classical set and FS were expanded in IFS context.The analytical and numerical solution of differential equations with fuzzy number and IF number have been discussed in [6, 8, 11, 13] and [3]. Neutrosophic logic [12] is a useful technique for dealing with incomplete, uncertain, and inconsistent data, which is an extension of FS and IFS. Many research have been conducted in the theoretical foundations and practical implications of Neutrosophic logic. Neutrosophic number involved in differential equations have been discussed in [2, 14] and [9] . In this research,

- The solution of ODE with hexagonal neutrosophic number as initial condition is discussed.
- Adam-Bashforth , Adam Moulton and Predictior-Corrector methods are applied to solve the ODE
- The stability and convergence criteria is investigated.

The paper is organized as follows: The preliminaries section presents the key concepts that are relevant to this field of study. In the next section we have defined the basic operations and interpolation of neutrosophic number followed by Adam-Bashforth Adam Moulton and Predictior-Corrector methods. Finally, we have highlighted our important findings and proposed areas for further investigation.

2. **Preliminaries**

Definition 2.1. [4] A neutrosophic set is defined as follows,
S= $\left\{ \langle \kappa, \lambda, \mu \rangle \,/\, \langle \kappa, \lambda, \mu \rangle \in [0,1]^3 \; and \; 0 \leq \kappa(x) + \lambda(x) + \mu(x) \leq 3 \right\}$

Definition 2.2. [4] The support of a Neutrosophic set $S = \{\langle x, \kappa_s, \lambda_s, \mu_s)\rangle\}$ is defined as
Supp(S)= $\{x \in U/ \; \kappa_s \neq 0, \; \lambda_s \neq 1, \; \mu_s \neq 1\}$.

Definition 2.3. [4] A set S is said to be normal if
H(S)= $\{x \in U/\kappa_s = h_1\left(S\right) = 1, \lambda_s = h_2\left(S\right) = 0, \mu_s = h_3\left(S\right) = 0\}$,
where $h_1\left(S\right) = \sup_{x\in U} T_s\left(x\right)$, $h_2\left(S\right) = \inf_{x\in U} I_s\left(x\right)$ and $h_3\left(S\right) = \inf_{x\in U} \mu_s\left(x\right)$.

Definition 2.4. [4] To qualify as a neutrosophic number, a neutrosophic set S on $\mathcal{R}$ must satisfy the following three properties:

(1) The neutrosophic set S should be normal;
(2) $S_{(\theta,\beta,\gamma)}$ should be closed for every $\theta \in (0, 1]$, $\beta \in [0, 1)$ and $\gamma \in [0, 1)$;
(3) The support of S must be bounded.

Definition 2.5. [4] GNHNNA is defined as, $\eta_{GNHNNA} = \Big\{ T\left(j_1, j_2, j_3, j_4, j_5, j_6; r, s; \omega\right)_{(n_1,n_2,n_3,n_4)},$
$I\left(o_1, o_2, o_3, o_4, o_5, o_6; r_1, s_1; \rho\right)_{(m_1,m_2,m_3,m_4)}, F\left(q_1, q_2, q_3, q_4, q_5, q_6; r_2, s_2; \delta\right)_{(p_1,p_2,p_3,p_4)} \Big\}.$

Where the membership function, $T_{\eta GNHNNA} = \begin{cases} r\left(\frac{x-j_1}{j_2-j_1}\right)^{n_1} & , \text{ if } j_1 \leq x \leq j_2 \\ r+(\omega-r)\left(\frac{x-j_2}{j_3-j_2}\right)^{n_2} & , \text{ if } j_2 \leq x \leq j_3 \\ \omega & , \text{ if } j_3 \leq x \leq j_4 \\ s+(\omega-s)\left(\frac{x-j_5}{j_4-j_5}\right)^{n_3} & , \text{ if } j_4 \leq x \leq j_5 \\ s\left(\frac{x-j_6}{j_5-j_6}\right)^{n_4} & , \text{ if } j_5 \leq x \leq j_6 \\ 0 & , \text{ otherwise.} \end{cases}$

Indeterminacy function, $I_{\eta GNHNNA} = \begin{cases} 1-r_1\left(\frac{x-o_1}{o_2-o_1}\right)^{m_1} & , \text{ if } o_1 \leq x \leq o_2 \\ 1-r_1+(r_1-\rho)\left(\frac{x-o_2}{o_3-o_2}\right)^{m_2} & , \text{ if } o_2 \leq x \leq o_3 \\ 1-\rho & , \text{ if } o_3 \leq x \leq o_4 \\ 1-s_1+(s_1-\rho)\left(\frac{x-o_5}{o_4-o_5}\right)^{m_3} & , \text{ if } o_4 \leq x \leq o_5 \\ 1-s_1\left(\frac{x-o_6}{o_5-o_6}\right)^{m_4} & , \text{ if } o_5 \leq x \leq o_6 \\ 1 & , \text{ otherwise.} \end{cases}$

Non-membership function, $\mu_{\eta_{GNHNNA}} = \begin{cases} 1-r_2\left(\frac{x-q_1}{q_2-q_1}\right)^{p_1} & , \text{ if } q_1 \leq x \leq q_2 \\ 1-r_2+(r_2-\delta)\left(\frac{x-q_2}{q_3-q_2}\right)^{p_2} & , \text{ if } q_2 \leq x \leq q_3 \\ 1-\delta & , \text{ if } q_3 \leq x \leq q_4 \\ 1-s_2+(s_2-\delta)\left(\frac{x-q_5}{q_4-q_5}\right)^{p_3} & , \text{ if } q_4 \leq x \leq q_5 \\ 1-s_2\left(\frac{x-q_6}{q_5-q_6}\right)^{p_4} & , \text{ if } q_5 \leq x \leq q_6 \\ 1 & , \text{ otherwise.} \end{cases}$

where $j_1 < j_2 < j_3 < j_4 < j_5 < j_6$, $o_1 < o_2 < o_3 < o_4 < o_5 < o_6$ and $q_1 < q_2 < q_3 < q_4 < q_5 < q_6$ $\forall$ $j_i,\ o_i$ and $q_i\,(i = 1, ..., 6)$ are real constants and $0 < r, s < \omega$, $1-\rho < r_1, s_1 < 1$ and $1-\delta < r_2, s_2 < 1, \omega,\ \rho,\ \delta \in [0, 1]$.

Definition 2.6. The (θ, β, γ)-cut of GNHNNA,

$\eta_{(\alpha,\beta,\gamma)} =$

$\{x \in X/\, T_{\eta_{GNHNNA}} \geq \theta,\ I_{\eta_{GNHNNA}} \leq \beta,\ \mu_{\eta_{GNHNNA}} \leq \gamma\}$.

Let $T_\theta = \{x \in X/\, T_{\eta_{GNHNNA}} \geq \theta\}$ where $\theta \in (0, \omega]$.

If $r \leq s$ then,

$$T_\theta = \begin{cases} \left[j_1 + \left(\frac{\theta}{r}\right)^{\frac{1}{n_1}}(j_2-j_1),\, j_6 + \left(\frac{\theta}{s}\right)^{\frac{1}{n_4}}(j_5-j_6)\right] & , \text{ if } 0 < \theta \leq r \\ \left[j_2 + \left(\frac{\theta-r}{\omega-r}\right)^{\frac{1}{n_2}}(j_3-j_2),\, j_6 + \left(\frac{\theta}{s}\right)^{\frac{1}{n_4}}(j_5-j_6)\right] & , \text{ if } r \leq \theta \leq s \\ \left[j_2 + \left(\frac{\theta-r}{\omega-r}\right)^{\frac{1}{n_2}}(j_3-j_2),\, j_5 + \left(\frac{\theta-s}{\omega-s}\right)^{\frac{1}{n_3}}(j_4-j_5)\right] & , \text{ if } s \leq \theta \leq \omega \\ [j_3, j_4] & , \text{ if } \theta = \omega. \end{cases}$$

If $s \leq r$, then,

$$T_\theta=\begin{cases}\left[j_1+\left(\frac{\theta}{r}\right)^{\frac{1}{n_1}}(j_2-j_1),j_6+\left(\frac{\theta}{s}\right)^{\frac{1}{n_4}}(j_5-j_6)\right] & \text{, if } 0<\theta\le s\\ \left[j_1+\left(\frac{\theta}{r}\right)^{\frac{1}{n_1}}(j_2-j_1),j_5+\left(\frac{\theta-s}{\omega-s}\right)^{\frac{1}{n_3}}(j_4-j_5)\right] & \text{, if } s\le\theta\le r\\ \left[j_2+\left(\frac{\theta-r}{\omega-r}\right)^{\frac{1}{n_2}}(j_3-j_2),j_5+\left(\frac{\theta-s}{\omega-s}\right)^{\frac{1}{n_3}}(j_4-j_5)\right] & \text{, if } r\le\theta\le\omega\\ [j_3,j_4] & \text{, if } \theta=\omega.\end{cases}$$

Let $I_\beta=\{x\in X/\ I_{\eta_{GNHNNA}}\le\beta\}$,
where $\beta\in[1-\rho,1)$. If $r_1\le s_1$,

$$I_\beta=\begin{cases}[o_3,o_4] & \text{, if } \beta=1-\rho\\ \left[o_2+\left(\frac{1-\beta-r_1}{\rho-r_1}\right)^{\frac{1}{m_2}}(o_3-o_2),o_5+\left(\frac{1-\beta-s_1}{\rho-s_1}\right)^{\frac{1}{m_3}}(o_4-o_5)\right] & \text{, if } 1-\rho\le\beta\le1-s_1\\ \left[o_2+\left(\frac{1-\beta-r_1}{\rho-r_1}\right)^{\frac{1}{m_2}}(o_3-o_2),o_6+\left(\frac{1-\beta}{s_1}\right)^{\frac{1}{m_4}}(o_5-o_6)\right] & \text{, if } 1-s_1\le\beta\le1-r_1\\ \left[o_1+\left(\frac{1-\beta}{r_1}\right)^{\frac{1}{m_1}}(o_2-o_1),o_6+\left(\frac{1-\beta}{s_1}\right)^{\frac{1}{m_4}}(o_5-o_6)\right] & \text{, if } 1-r_1\le\beta<1.\end{cases}$$

If $s_1\le r_1$, then,

$$I_\beta=\begin{cases}[o_3,o_4] & \text{, if } \beta=1-\rho\\ \left[o_2+\left(\frac{1-\beta-r_1}{\rho-r_1}\right)^{\frac{1}{m_2}}(o_3-o_2),o_5+\left(\frac{1-\beta-s_1}{\rho-s_1}\right)^{\frac{1}{m_3}}(o_4-o_5)\right] & \text{, if } 1-\rho\le\beta\le1-r_1\\ \left[o_1+\left(\frac{1-\beta}{r_1}\right)^{\frac{1}{m_1}}(o_2-o_1),o_5+\left(\frac{1-\beta-s_1}{\rho-s_1}\right)^{\frac{1}{m_3}}(o_4-o_5)\right] & \text{, if } 1-r_1\le\beta\le1-s_1\\ \left[o_1+\left(\frac{1-\beta}{r_1}\right)^{\frac{1}{m_1}}(o_2-o_1),o_6+\left(\frac{1-\beta}{s_1}\right)^{\frac{1}{m_4}}(o_5-o_6)\right] & \text{, if } 1-s_1\le\beta<1.\end{cases}$$

Let $\mu_\gamma=\{x\in X/\mu_{\eta_{GNHNNA}}\le\gamma\}$, where $\gamma\in[1-\delta,1)$. If $r_2\le s_2$, then,

$$\mu_\gamma=\begin{cases}[q_3,q_4] & \text{, if } \gamma=1-\delta\\ \left[q_2+\left(\frac{1-\gamma-r_2}{\delta-r_2}\right)^{\frac{1}{p_2}}(q_3-q_2),q_5+\left(\frac{1-\gamma-s_2}{\delta-s_2}\right)^{\frac{1}{p_3}}(q_4-q_5)\right] & \text{, if } 1-\delta\le\gamma\le1-s_2\\ \left[q_2+\left(\frac{1-\gamma-r_2}{\delta-r_2}\right)^{\frac{1}{p_2}}(q_3-q_2),q_6+\left(\frac{1-\gamma}{s_2}\right)^{\frac{1}{p_4}}(q_5-q_6)\right] & \text{, if } 1-s_2\le\gamma\le1-r_2\\ \left[q_1+\left(\frac{1-\gamma}{r_2}\right)^{\frac{1}{p_1}}(q_2-q_1),q_6+\left(\frac{1-\gamma}{s_2}\right)^{\frac{1}{p_4}}(q_5-q_6)\right] & \text{, if } 1-r_2\le\gamma<1.\end{cases}$$

If $s_2\le r_2$, then

$$\mu_\gamma=\begin{cases}[q_3,q_4] & \text{, if } \gamma=1-\delta\\ \left[q_2+\left(\frac{1-\gamma-r_2}{\delta-r_2}\right)^{\frac{1}{p_2}}(q_3-q_2),q_5+\left(\frac{1-\gamma-s_2}{\delta-s_2}\right)^{\frac{1}{p_3}}(q_4-q_5)\right] & \text{, if } 1-\delta\le\gamma\le1-r_2\\ \left[q_1+\left(\frac{1-\gamma}{r_2}\right)^{\frac{1}{p_1}}(q_2-q_1),q_5+\left(\frac{1-\gamma-s_2}{\delta-s_2}\right)^{\frac{1}{p_3}}(q_4-q_5)\right] & \text{, if } 1-r_2\le\gamma\le1-s_2\\ \left[q_1+\left(\frac{1-\gamma}{r_2}\right)^{\frac{1}{p_1}}(q_2-q_1),q_6+\left(\frac{1-\gamma}{s_2}\right)^{\frac{1}{p_4}}(q_5-q_6)\right] & \text{, if } 1-s_2\le\gamma<1.\end{cases}$$

3. Interpolation of Neutrosophic Number

Definition 3.1. We define the upper, middle and lower θ-cuts of $\langle\kappa,\lambda,\mu\rangle\in$ NS, with $\theta\in$ [0,1] by, $\langle\kappa,\lambda,\mu\rangle^\theta$=$\{x\in\Re/T(x)\ge\theta\}$, $\langle\kappa,\lambda,\mu\rangle\ \theta$=$\{x\in\Re/I(x)\le\theta\}$, and $\langle\kappa,\lambda,\mu\rangle_\theta$=$\{x\in\Re/F(x)\le\theta\}$. Where $\Re$ is a real number.

Definition 3.2. The neutrosophic zero in a neutrosophic set is defined by

$$0(\delta) = \begin{cases} (1,0,0), & \text{if } t = 0, \\ (0,0,1), & \text{otherwise.} \end{cases}$$

Definition 3.3. Let $\langle \kappa, \lambda, \mu \rangle$ and $\langle \kappa', \lambda', \mu' \rangle \in$ Neutrosophic Number, and let $\theta \in \mathbb{R}$. We define the following operations:

(1) $\langle \kappa, \lambda, \mu \rangle \bigoplus \langle \kappa', \lambda', \mu' \rangle (x) =$

$$\left(\sup_{x=y+z} \min(\kappa(x), \kappa'(x)), \inf_{x=y+z} \max(\lambda(x), \lambda'(x)), \inf_{x=y+z} \max(\mu(x), \mu'(x)) \right)$$

(2) $\theta \langle \kappa, \lambda, \mu \rangle =$

$$\begin{cases} \langle \theta\kappa, \theta\lambda, \theta\mu \rangle, & \text{if } \theta \neq 0 \\ 0(\delta), & \text{if } \theta = 0 \end{cases}$$

The operations of addition and scalar multiplication for Neutrosophic numbers are defined based on the extension principle as follows:
$[\langle \kappa, \lambda, \mu \rangle \bigoplus \left\langle \kappa^{'}, \lambda^{'}, \mu^{'} \right\rangle]^{\theta}$= $[\langle \kappa, \lambda, \mu \rangle]^{\theta}$ + $[\left\langle \kappa^{'}, \lambda^{'}, \mu^{'} \right\rangle]^{\theta}$, $[\lambda \langle \kappa, \lambda, \mu \rangle]^{\theta}$= λ $[\langle \kappa, \lambda, \mu \rangle]^{\theta}$
$[\langle \kappa, \lambda, \mu \rangle \bigoplus \left\langle \kappa^{'}, \lambda^{'}, \mu^{'} \right\rangle]\theta$= $[\langle \kappa, \lambda, \mu \rangle]\theta$ + $[\left\langle \kappa^{'}, \lambda^{'}, \mu^{'} \right\rangle]\theta$, $[\lambda \langle \kappa, \lambda, \mu \rangle]\theta$= λ $[\langle \kappa, \lambda, \mu \rangle]\theta$
$[\langle \kappa, \lambda, \mu \rangle \bigoplus \left\langle \kappa^{'}, \lambda^{'}, \mu^{'} \right\rangle]_{\theta}$= $[\langle \kappa, \lambda, \mu \rangle]_{\theta}$ + $[\left\langle \kappa^{'}, \lambda^{'}, \mu^{'} \right\rangle]_{\theta}$, $[\lambda \langle \kappa, \lambda, \mu \rangle]_{\theta}$= λ $[\langle \kappa, \lambda, \mu \rangle]_{\theta}$

Definition 3.4. Let $\langle \kappa, \lambda, \mu \rangle$ be an element of the Neutrosophic Set (NS), and let $\theta \in [0, 1]$. We define the following sets:
$[\langle \kappa, \lambda, \mu \rangle]_l^-(\theta) = inf\,\{x \in \Re/\kappa(x) \geq \theta\}$, $[\langle \kappa, \lambda, \mu \rangle]_l^+(\theta) = sup\,\{x \in \Re/\lambda(x) \geq \theta\}$,
$[\langle \kappa, \lambda, \mu \rangle]_m^-(\theta) = inf\,\{x \in \Re/\lambda(x) \leq 1-\theta\}$, $[\langle \kappa, \lambda, \mu \rangle]_m^+(\theta) = sup\,\{x \in \Re/\lambda(x) \leq 1-\theta\}$,
$[\langle \kappa, \lambda, \mu \rangle]_u^-(\theta) = inf\,\{x \in \Re/\mu(x) \leq 1-\theta\}$, $[\langle \kappa, \lambda, \mu \rangle]_u^+(\theta) = sup\,\{x \in \Re/\mu(x) \leq 1-\theta\}$.

Remark 3.5. $\langle \kappa, \lambda, \mu \rangle^{\theta}$=$[[\langle \kappa, \lambda, \mu \rangle]_l^-(\theta), [\langle \kappa, \lambda, \mu \rangle]_l^+(\theta)]$,
$\langle \kappa, \lambda, \mu \rangle\, \theta$=$[[\langle \kappa, \lambda, \mu \rangle]_m^-(\theta), [\langle \kappa, \lambda, \mu \rangle]_m^+(\theta)]$, and
$\langle \kappa, \lambda, \mu \rangle_{\theta}$=$[[\langle \kappa, \lambda, \mu \rangle]_u^-(\theta), [\langle \kappa, \lambda, \mu \rangle]_u^+(\theta)]$.

The interpolation problem for Neutrosophic sets can be described as follows: Consider a Neutrosophic set with specified properties, denoted by $\Phi(\xi)$, which encapsulates information at different points. The aim is to approximate the function $\Phi(\xi)$ for each ξ within its domain. Suppose $\langle \kappa_0, \lambda_0, \mu_0 \rangle$, $\langle \kappa_1, \lambda_1, \mu_1 \rangle$, ..., $\langle \kappa_s, \lambda_s, \mu_s \rangle$ represent $s + 1$ Neutrosophic fuzzy sets, and let $\xi_0 < \xi_1 < \cdots < \xi_s$ be $s + 1$ distinct points in $\mathbb{R}$. A Neutrosophic continuous function $f : I \to N$ that fulfills the following criteria is termed a Neutrosophic polynomial interpolation of the data.

(1) $\varphi(\xi_i) = \langle \kappa_i, \lambda_i, \mu_i \rangle$
(2) If the input data is crisp, then the interpolation φ is a crisp polynomial.

To construct a function φ that satisfies the specified conditions, consider the following approach. For each $\eta = (\eta_0, \eta_1, \ldots, \eta_s) \in \mathbb{R}^{s+1}$, the unique polynomial of degree $\leq s$, denoted by H_η, is defined as follows:

(1) $H_\eta(q) = \eta_q$ for $q = 0, 1, \ldots, s$
(2) $P_\eta(\xi) = \sum_{q=0}^{s} \eta_q \prod_{q \neq r} \frac{\xi - \xi_r}{\xi_q - \xi_r}$

For any $\xi \in \mathbb{R}$, the membership, indeterminacy and non-membership functions $\Phi(\xi)$ can be expressed using the extension principle as follows:

$$\kappa_{\Phi(\xi)}(\delta) = \begin{cases} \sup_{\delta = H^{\eta_0,\eta_1,\ldots,\eta_s}(\xi)} \min_{i=0,1,\ldots,n} \kappa_{\kappa_i}(\eta_i), & \text{if } H^{-1}_{\eta_0,\eta_1,\ldots,\eta_s}(\delta) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

where κ_{κ_i} is the membership function of κ_i.

$$\lambda_{\Phi(\xi)}(\delta) = \begin{cases} \inf_{\delta = H^{\eta_0,\eta_1,\ldots,\eta_s}(\xi)} \max_{i=0,1,\ldots,s} \lambda_{\lambda_q}(\eta_q), & \text{if } H^{-1}_{\eta_0,\eta_1,\ldots,\eta_s}(\delta) \neq \emptyset \\ 1, & \text{otherwise} \end{cases}$$

where λ_{λ_q} is the indeterminacy function of λ_q.

$$\mu_{\Phi(\xi)}(\delta) = \begin{cases} \inf_{\delta = H^{\eta_0,\eta_1,\ldots,\eta_s}(\xi)} \max_{i=0,1,\ldots,s} \mu_{\mu_q}(\eta_q), & \text{if } H^{-1}_{\eta_0,\eta_1,\ldots,\eta_s}(\delta) \neq \emptyset \\ 1, & \text{otherwise} \end{cases}$$

where μ_{μ_q} is the non-membership function of μ_q.

Let $\chi_l^q(\theta) = \langle \kappa, \lambda, \mu \rangle_\theta$, $\chi_m^q(\theta) = \langle \kappa, \lambda, \mu \rangle \theta$, and $\chi_u^q(\theta) = \langle \kappa, \lambda, \mu \rangle^\theta$ for any $\theta \in [0, 1]$ and $q = 0, 1, \ldots, s$. The lower, middle, and upper θ-cuts of $\langle \kappa, \lambda, \mu \rangle$ and $\Phi(\xi)$ are denoted by $[\Phi(\xi)]_\theta$, $[\Phi(\xi)]\theta$, and $[\Phi(\xi)]^\theta$, respectively. Therefore,

$$[\Phi(\xi)]_\theta = \left\{\delta \in \mathbb{R} \mid \kappa_{\Phi(\xi)}(\delta) \geq \theta\right\} = \left\{\delta \in \mathbb{R} \mid \exists \eta_0, \eta_1, \ldots, \eta_s : \kappa_{\kappa_q}(\eta_q) \geq \theta,\ q = 0, \ldots, s \text{ and } H_{\eta_0,\eta_1,\ldots,\eta_s}(\xi) = \delta\right\} = \left\{\delta \in \mathbb{R} \mid \exists \eta \in \prod_{q=0}^{s} \chi_u^q(\theta) : H_{\eta_0,\eta_1,\ldots,\eta_s}(\xi) = \delta\right\}$$

$$[\Phi(\xi)]\theta = \left\{\delta \in \mathbb{R} \mid \lambda_{\Phi(\xi)}(\delta) \leq 1-\theta\right\} = \left\{\delta \in \mathbb{R} \mid \exists \eta_0, \eta_1, \ldots, \eta_s : \lambda_{\lambda_q}(\eta_q) \leq 1-\theta,\ q = 0, \ldots, s \text{ and } H_{\eta_0,\eta_1,\ldots,\eta_s}(\xi) = \delta\right\} = \left\{\delta \in \mathbb{R} \mid \exists \eta \in \prod_{q=0}^{s} \chi_m^q(\theta) : H_{\eta_0,\eta_1,\ldots,\eta_s}(\xi) = \delta\right\}$$

$$[\Phi(\xi)]^\theta = \left\{\delta \in \mathbb{R} \mid \mu_{\Phi(\xi)}(\delta) \leq 1-\theta\right\} = \left\{\delta \in \mathbb{R} \mid \exists \eta_0, \eta_1, \ldots, \eta_s : \mu_{\mu_q}(\eta_q) \leq 1-\theta,\ q = 0, \ldots, s \text{ and } P_{\eta_0,\eta_1,\ldots,\eta_s}(\xi) = \delta\right\} = \left\{\delta \in \mathbb{R} \mid \exists \eta \in \prod_{q=0}^{s} \chi_l^q(\theta) : H_{\eta_0,\eta_1,\ldots,\eta_s}(\xi) = \delta\right\}$$

Finally, for each $\xi \in \mathbb{R}$ and all $\delta \in \mathbb{R}$, $\Phi(\xi)$ is defined in terms of NS by:

$$\Phi(\xi)(\delta) = \left(\sup\left\{ \theta \in (0,1) \mid \exists \eta \in \prod_{i=0}^{s} r_q^u(\theta) : H_\eta(\xi) = \delta \right\},\ 1 - \sup\left\{ \theta \in (0,1) \mid \exists \eta \in \prod_{q=0}^{s} \chi_m^q(\theta) : H_\eta(x) = \delta \right\}, \right.$$
$$\left. 1 - \sup\left\{ \theta \in (0,1) \mid \exists \eta \in \prod_{q=0}^{s} \chi_m^q(\theta) : H_\eta(\xi) = \delta \right\} \right)$$

where $\eta = (\eta_0, \eta_1, \ldots, \eta_s) \in \mathbb{R}^{s+1}$.

The interpolation polynomial can be expressed in terms of level sets as:

$$[\Phi(\xi)]_\theta = \{\eta \in \mathbb{R} \mid \eta = H_{\eta_0,\eta_1,\ldots,\eta_s}(\xi),\ \eta_q \in [\langle \kappa_q, \lambda_q, \mu_q \rangle]_\theta,\ q = 0, \ldots, s\} \text{ for } \theta \in (0,1]$$

$$[\Phi(\xi)]\theta = \{\eta \in \mathbb{R} \mid \eta = H_{\eta_0,\eta_1,\ldots,\eta_s}(\xi),\ \eta_q \in [\langle \kappa_q, \lambda_q, \mu_q \rangle]\theta,\ q = 0, \ldots, s\} \text{ for } \theta \in (0,1]$$
$$[\Phi(\xi)]^\theta = \left\{\eta \in \mathbb{R} \mid \eta = H_{\eta_0,\eta_1,\ldots,\eta_s}(\xi),\ \eta_q \in [\langle \kappa_q, \lambda_q, \mu_q \rangle]^\theta,\ q = 0, \ldots, s\right\} \text{ for } \theta \in (0,1]$$

According to the Lagrange interpolation formula, we have:

$$[\Phi(\xi)]_\theta = \sum_{q=0}^{s} o_q(x) \chi_u^w(\theta), \quad [\Phi(\xi)]\theta = \sum_{q=0}^{s} o_q(x) \chi_m^q(\theta), \quad [\Phi(\xi)]^\theta = \sum_{q=0}^{s} o_q(x) \chi_l^q(\theta)$$

where $o_q(x)$ represents the Lagrange polynomials.

When the data $\langle \kappa, \lambda, \mu \rangle$ is represented as hexagonal neutrosophic numbers, the values of the interpolation polynomial are also hexagonal neutrosophic numbers. Consequently, $\Phi(\xi)$ takes a particularly simple form that is convenient for computation. Define $\chi_u^q(\theta) = [c_u^-(\theta), d_u^+(\theta)], \chi_m^q(\theta) = [c_m^-(\theta), d_m^+(\theta)]$ and $\chi_l^q(\theta) = [c_l^-(\theta), d_l^+(\theta)]$. The upper endpoint of $[\Phi(\xi)]^\theta$ is obtained by solving the following optimization problem:

$$\text{Maximize } H_{\eta_0,\eta_1,\ldots,\eta_s}(\xi) \text{ subject to } c_u^-(\theta) \le \eta_q \le d_u^+(\theta), \quad q = 0, 1, \ldots, s.$$

The optimal solution is:

$$\eta_q = \begin{cases} d_u^-(\theta) & \text{if } o_q(\xi) \ge 0, \\ c_u^+(\theta) & \text{if } o_q(\xi) < 0. \end{cases}$$

Similarly, the lower endpoint is obtained by:

$$\eta_q = \begin{cases} d_u^-(\theta) & \text{if } o_q(\xi) < 0, \\ c_u^+(\theta) & \text{if } o_q(\xi) \ge 0. \end{cases}$$

$[\Phi(\xi)]\theta$ and $[\Phi(\xi)]_\theta$ can be obtained in a similar manner. Hence, if $\langle \kappa_q, \lambda_q, \mu_q \rangle$ is a neutrosophic number for all q, then $\Phi(\xi)$ is also a neutrosophic number for each ξ. Specifically, if

$$\langle \kappa_i, \lambda_i, \mu_i \rangle = \langle t_i^a, t_i^b, t_i^c, t_i^d, t_i^e, t_i^f, i_i^a, i_i^b, i_i^c, i_i^d, i_i^e, i_i^f, f_i^a, f_i^b, f_i^c, f_i^d, f_i^e, f_i^f \rangle,$$

then:

$$\Phi(\xi) =< \tilde{\varphi}^a(\xi), \tilde{\varphi}^b(\xi), \tilde{\varphi}^c(\xi), \tilde{\varphi}^d(\xi), \tilde{\varphi}^e(\xi), \tilde{\varphi}^f(\xi), \varphi^a(\xi), \varphi^b(\xi), \varphi^c(\xi), \varphi^d(\xi), \varphi^e(\xi), \varphi^f(\xi),$$
$$\underset{\sim}{\varphi}^a(\xi), \underset{\sim}{\varphi}^b(\xi), \underset{\sim}{\varphi}^c(\xi), \underset{\sim}{\varphi}^d(\xi), \underset{\sim}{\varphi}^e(\xi), \underset{\sim}{\varphi}^f(\xi) >$$

where:

$$\tilde{\varphi}^a(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi) t_i^a + \sum_{o_q(\xi)<0} o_q(\xi) t_i^f; \quad \tilde{\varphi}^b(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi) t_i^b + \sum_{o_q(\xi)<0} o_q(\xi) t_i^e,$$
$$\tilde{\varphi}^c(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi) t_i^c + \sum_{o_q(\xi)<0} o_q(\xi) t_i^d,; \quad \tilde{\varphi}^d(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi) t_i^d + \sum_{o_q(\xi)<0} o_q(\xi) t_i^c,$$
$$\tilde{\varphi}^e(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi) t_i^e + \sum_{o_q(\xi)<0} o_q(\xi) t_i^b; \quad \tilde{\varphi}^f(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi) t_i^f + \sum_{o_q(\xi)<0} o_q(\xi) t_i^a.$$
$$\varphi^a(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi) i_i^a + \sum_{o_q(\xi)<0} o_q(\xi) i_i^f; \quad \varphi^b(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi) i_i^b + \sum_{o_q(\xi)<0} o_q(\xi) i_i^e,$$
$$\varphi^c(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi) i_i^c + \sum_{o_q(\xi)<0} o_q(\xi) i_i^d; \quad \varphi^d(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi) i_i^d + \sum_{o_q(\xi)<0} o_q(\xi) i_i^c,$$
$$\varphi^e(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi) i_i^e + \sum_{o_q(\xi)<0} o_q(\xi) i_i^b; \quad \varphi^f(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi) i_i^f + \sum_{o_q(\xi)<0} o_q(\xi) i_i^a.$$
$$\underset{\sim}{\varphi}^a(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi) f_i^a + \sum_{o_q(\xi)<0} o_q(\xi) f_i^f; \quad \underset{\sim}{\varphi}^b(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi) f_i^b + \sum_{o_q(\xi)<0} o_q(\xi) f_i^e,$$
$$\varphi^c(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi) f_i^c + \sum_{o_q(\xi)<0} o_q(\xi) f_i^d; \quad \underset{\sim}{\varphi}^d(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi) f_i^d + \sum_{o_q(\xi)<0} o_q(\xi) f_i^c,$$
$$\underset{\sim}{\varphi}^e(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi) f_i^e + \sum_{o_q(\xi)<0} o_q(\xi) f_i^b; \quad \underset{\sim}{\varphi}^f(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi) f_i^f + \sum_{o_q(\xi)<0} o_q(\xi) f_i^a.$$

4. **Adams-Bashforth Methods**

Now, we are going to solve the Neutrosophic initial value problem $\xi'(\tau) = \varphi(\delta, \xi(\delta))$ using the Adams-Bashforth three-step method. Let the Neutrosophic initial values be $\xi(\delta_{q-1}), \xi(\delta_q), \xi(\delta_{q+1})$, i.e., $\varphi(\delta_{q-1}, \xi(\delta_{q-1})), \varphi(\delta_q, \xi(\delta_q)), \varphi(\delta_{q+1}, \xi(\delta_{q+1}))$, which are represented by hexagonal neutrosophic numbers. The truth membership,

$$\{\tilde{\varphi}^a(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}^b(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}^c(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}^d(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}^e(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}^f(\delta_{q-1}, \xi(\delta_{q-1})),$$
$$\tilde{\varphi}_a(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_b(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_c(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_d(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_e(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_f(\delta_{q-1}, \xi(\delta_{q-1})),$$
$$\tilde{\varphi}_a(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_b(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_c(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_d(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_e(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_f(\delta_{q-1}, \xi(\delta_{q-1}))\}$$

the indeterminacy,

$$\{\varphi^a(\delta_q,\xi(\delta_q)),\varphi^b(\delta_q,\xi(\delta_q)),\varphi^c(\delta_q,\xi(\delta_q)),\varphi^d(\delta_q,\xi(\delta_q)),\varphi^e(\delta_q,\xi(\delta_q)),\varphi^f(\delta_q,\xi(\delta_q)),$$
$$\varphi_a(\delta_q,\xi(\delta_q)),\varphi_b(\delta_q,\xi(\delta_q)),\varphi_c(\delta_q,\xi(\delta_q)),\varphi_d(\delta_q,\xi(\delta_q)),\varphi_e(\delta_q,\xi(\delta_q)),\varphi_f(\delta_q,\xi(\delta_q)),$$
$$\varphi_a(\delta_q,\xi(\delta_q)),\varphi_b(\delta_q,\xi(\delta_q)),\varphi_c(\delta_q,\xi(\delta_q)),\varphi_d(\delta_q,\xi(\delta_q)),\varphi_e(\delta_q,\xi(\delta_q)),\varphi_f(\delta_q,\xi(\delta_q))\}$$

the falsity,

$$\{\underset{\sim}{\varphi}^a(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}^b(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}^c(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}^d(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}^e(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}^f(\delta_{q+1},\xi(\delta_{q+1})),$$
$$\underset{\sim}{\varphi}a(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}b(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}c(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}d(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}e(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}f(\delta_{q+1},\xi(\delta_{q+1})),$$
$$\underset{\sim_a}{\varphi}(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim_b}{\varphi}(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim_c}{\varphi}(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim_d}{\varphi}(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim_e}{\varphi}(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim_f}{\varphi}(\delta_{q+1},\xi(\delta_{q+1}))\}$$

Also

$$\xi(\delta_{i+2})=\xi(\delta_{q+1})+\int_{\delta_{q+1}}^{\delta_{q+2}}\varphi(\delta,\xi(\delta))d\delta$$

By neutrosophic interpolation of $\varphi(\delta,\xi(\delta_{q-1})),\varphi(\delta,\xi(\delta_q)),\varphi(\delta,\xi(\delta_{q+1}))$, we have:

$$\tilde{\varphi}^a(\delta,\xi(\delta))=\sum_{r=q-1,o_j(\delta)\geq 0}^{q+1}o_r(\delta)\tilde{\varphi}^a(\delta_j,\xi(\delta_j))+\sum_{r=q-1,o_j(\delta)<0}^{q+1}o_j(\delta)\tilde{\varphi}^f(\delta_j,\xi(\delta_j))$$

$$\tilde{\varphi}^b(\delta,\xi(\delta))=\sum_{r=q-1,o_j(\delta)\geq 0}^{q+1}o_r((\delta)\tilde{\varphi}^b(\delta_j,\xi(\delta_j))+\sum_{r=q-1,o_j(\delta)<0}^{q+1}o_j(\delta)\tilde{\varphi}^e(\delta_j,\xi(\delta_j))$$

$$\tilde{\varphi}^c(\delta,\xi(\delta))=\sum_{r=q-1,o_j(\delta)\geq 0}^{q+1}o_r((\delta)\tilde{\varphi}^c(\delta_j,\xi(\delta_j))+\sum_{r=q-1,o_j(\delta)<0}^{q+1}o_j(\delta)\tilde{\varphi}^d(\delta_j,\xi(\delta_j))$$

$$\tilde{\varphi}^d(\delta,\xi(\delta))=\sum_{r=q-1,o_j(\delta)\geq 0}^{q+1}o_r((\delta)\tilde{\varphi}^d(\delta_j,\xi(\delta_j))+\sum_{r=q-1,o_j(\delta)<0}^{q+1}o_j(\delta)\tilde{\varphi}^c(\delta_j,\xi(\delta_j))$$

$$\tilde{\varphi}^e(\delta,\xi(\delta))=\sum_{r=q-1,o_j(\delta)\geq 0}^{q+1}o_r((\delta)\tilde{\varphi}^e(\delta_j,\xi(\delta_j))+\sum_{r=q-1,o_j(\delta)<0}^{q+1}o_j(\delta)\tilde{\varphi}^b(\delta_j,\xi(\delta_j))$$

$$\tilde{\varphi}^f(\delta,\xi(\delta))=\sum_{r=q-1,o_j(\delta)\geq 0}^{q+1}o_r((\delta)\tilde{\varphi}^f(\delta_j,\xi(\delta_j))+\sum_{r=q-1,o_j(\delta)<0}^{q+1}o_j(\delta)\tilde{\varphi}^a(\delta_j,\xi(\delta_j))$$

$$\varphi^a(\delta,\xi(\delta))=\sum_{r=q-1,o_j(\delta)\geq 0}^{q+1}o_r(\delta)\varphi^a(\delta_j,\xi(\delta_j))+\sum_{r=q-1,o_j(\delta)<0}^{q+1}o_j(\delta)\varphi^f(\delta_j,\xi(\delta_j))$$

$$\varphi^b(\delta,\xi(\delta))=\sum_{r=q-1,o_j(\delta)\geq 0}^{q+1}o_r((\delta)\varphi^b(\delta_j,\xi(\delta_j))+\sum_{r=q-1,o_j(\delta)<0}^{q+1}o_j(\delta)\varphi^e(\delta_j,\xi(\delta_j))$$

$$\varphi^c(\delta,\xi(\delta))=\sum_{r=q-1,o_j(\delta)\geq 0}^{q+1}o_r((\delta)\varphi^c(\delta_j,\xi(\delta_j))+\sum_{r=q-1,o_j(\delta)<0}^{q+1}o_j(\delta)\varphi^d(\delta_j,\xi(\delta_j))$$

$$\varphi^{d}(\delta,\xi(\delta)) = \sum_{r=q-1,o_j(\delta)\geq 0}^{q+1} o_r((\delta)\varphi^{d}(\delta_j,\xi(\delta_j)) + \sum_{r=q-1,o_j(\delta)<0}^{q+1} o_j(\delta)\varphi^{c}(\delta_j,\xi(\delta_j))$$

$$\varphi^{e}(\delta,\xi(\delta)) = \sum_{r=q-1,o_j(\delta)\geq 0}^{q+1} o_r((\delta)\varphi^{e}(\delta_j,\xi(\delta_j)) + \sum_{r=q-1,o_j(\delta)<0}^{q+1} o_j(\delta)\varphi^{b}(\delta_j,\xi(\delta_j))$$

$$\varphi^{f}(\delta,\xi(\delta)) = \sum_{r=q-1,o_j(\delta)\geq 0}^{q+1} o_r((\delta)\varphi^{f}(\delta_j,\xi(\delta_j)) + \sum_{r=q-1,o_j(\delta)<0}^{q+1} o_j(\delta)\varphi^{a}(\delta_j,\xi(\delta_j))$$

$$\underset{\sim a}{\varphi}(\delta,\xi(\delta)) = \sum_{r=q-1,o_j(\delta)\geq 0}^{q+1} o_r((\delta)\underset{\sim a}{\varphi}(\delta_j,\xi(\delta_j)) + \sum_{r=q-1,o_j(\delta)<0}^{q+1} o_j(\delta)\underset{\sim f}{\varphi}(\delta_j,\xi(\delta_j))$$

$$\underset{\sim b}{\varphi}(\delta,\xi(\delta)) = \sum_{r=q-1,o_j(\delta)\geq 0}^{q+1} o_r((\delta)\underset{\sim b}{\varphi}(\delta_j,\xi(\delta_j)) + \sum_{r=q-1,o_j(\delta)<0}^{q+1} o_j(\delta)\underset{\sim e}{\varphi}(\delta_j,\xi(\delta_j))$$

$$\underset{\sim c}{\varphi}(\delta,\xi(\delta)) = \sum_{r=q-1,o_j(\delta)\geq 0}^{q+1} o_r((\delta)\underset{\sim c}{\varphi}(\delta_j,\xi(\delta_j)) + \sum_{r=q-1,o_j(\delta)<0}^{q+1} o_j(\delta)\underset{\sim d}{\varphi}(\delta_j,\xi(\delta_j))$$

$$\underset{\sim d}{\underset{\sim}{\varphi}}(\delta,\xi(\delta)) = \sum_{r=q-1,o_j(\delta)\geq 0}^{q+1} o_r((\delta)\underset{\sim d}{\varphi}(\delta_j,\xi(\delta_j)) + \sum_{r=q-1,o_j(\delta)<0}^{q+1} o_j(\delta)\underset{\sim c}{\varphi}(\delta_j,\xi(\delta_j))$$

$$\underset{\sim e}{\varphi}(\delta,\xi(\delta)) = \sum_{r=q-1,o_j(\delta)\geq 0}^{q+1} o_r((\delta)\underset{\sim e}{\varphi}(\delta_j,\xi(\delta_j)) + \sum_{r=q-1,o_j(\delta)<0}^{q+1} o_j(\delta)\underset{\sim b}{\varphi}(\delta_j,\xi(\delta_j))$$

$$\underset{\sim f}{\varphi}(\delta,\xi(\delta)) = \sum_{r=q-1,o_j(\delta)\geq 0}^{q+1} o_r((\delta)\underset{\sim f}{\varphi}(\delta_j,\xi(\delta_j)) + \sum_{r=q-1,o_j(\delta)<0}^{q+1} o_j(\delta)\underset{\sim a}{\varphi}(\delta_j,\xi(\delta_j))$$

For $\delta_{q+1} \leq \delta \leq \delta_{i+2}$, the interpolation polynomials $o_j(\delta)$ are:

$$o_{q-1}(\delta) = \frac{(\delta-\delta_q)(\delta-\delta_{q+1})}{(\delta_{q-1}-\delta_q)(\delta_{q-1}-\delta_{q+1})} \geq 0$$

$$o_{q}(\delta) = \frac{(\delta-\delta_{q-1})(\delta-\delta_{q+1})}{(\delta_{q}-\delta_{q-1})(\delta_{q}-\delta_{q+1})} \leq 0$$

$$o_{q+1}(\delta) = \frac{(\delta-\delta_{q-1})(\delta-\delta_{q})}{(\delta_{q+1}-\delta_{q-1})(\delta_{q+1}-\delta_{q})} \geq 0$$

Therefore, the following results are obtained:

$$\varphi^{a}(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^{a}(\delta_{q-1},\xi(\delta_{q-1})) + o_q(\delta)\varphi^{f}(\delta_i,\xi(\delta_i)) + o_{q+1}(\delta)\varphi^{a}(\delta_{q+1},\xi(\delta_{q+1}))$$

$$\varphi^{b}(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^{b}(\delta_{q-1},\xi(\delta_{q-1})) + o_q(\delta)\varphi^{e}(\delta_i,\xi(\delta_i)) + o_{q+1}(\delta)\varphi^{b}(\delta_{q+1},\xi(\delta_{q+1}))$$

$$\varphi^{c}(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^{c}(\delta_{q-1},\xi(\delta_{q-1})) + o_q(\delta)\varphi^{d}(\delta_i,\xi(\delta_i)) + o_{q+1}(\delta)\varphi^{c}(\delta_{q+1},\xi(\delta_{q+1}))$$

$$\varphi^{d}(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^{d}(\delta_{q-1},\xi(\delta_{q-1})) + o_q(\delta)\varphi^{c}(\delta_i,\xi(\delta_i)) + o_{q+1}(\delta)\varphi^{d}(\delta_{q+1},\xi(\delta_{q+1}))$$

$$\varphi^{e}(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^{e}(\delta_{q-1},\xi(\delta_{q-1})) + o_q(\delta)\varphi^{b}(\delta_i,\xi(\delta_i)) + o_{q+1}(\delta)\varphi^{e}(\delta_{q+1},\xi(\delta_{q+1}))$$

$$\varphi^{f}(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^{f}(\delta_{q-1},\xi(\delta_{q-1})) + o_q(\delta)\varphi^{a}(\delta_i,\xi(\delta_i)) + o_{q+1}(\delta)\varphi^{f}(\delta_{q+1},\xi(\delta_{q+1}))$$

Also

$$[\xi(\delta_{q+2})]_\theta = \left[[\xi(\delta_{q+2})]^-_\kappa(\theta), [\xi(\delta_{q+2})]^+_\kappa(\theta)\right]$$

,

$$[\xi(\delta_{q+2})]\theta = \left[[\xi(\delta_{q+2})]^-_\lambda(\theta), [\xi(\delta_{q+2})]^+_\lambda(\theta)\right],$$

, and

$$[\xi(\delta_{q+2})]_\theta = \left[[\xi(\delta_{q+2})]^-_\mu(\theta), [\xi(\delta_{q+2})]^+_\mu(\theta)\right]$$

where

$$[\xi(\delta_{q+1})]^-_\kappa(\theta) = \begin{cases} [\xi(\delta_{q-1})]^-_\kappa(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^a(\delta,\xi(\delta)) + \left(\frac{\theta}{o}\right)\left(\varphi^b(\delta,\xi(\delta)) - \varphi^a(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0 < \theta < o \\ [\xi(\delta_{q-1})]^-_\kappa(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^b(\delta,\xi(\delta)) + \left(\frac{\theta-o}{1-o}\right)\left(\varphi^c(\delta,\xi(\delta)) - \varphi^b(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } o < \theta < 1 \end{cases}$$

$$[\xi(\delta_{q+1})]^+_\kappa(\theta) = \begin{cases} [\xi(\delta_{q-1})]^+_\kappa(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^f(\delta,\xi(\delta)) + \left(\frac{\theta}{o}\right)\left(\varphi^e(\delta,\xi(\delta)) - \varphi^f(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0 < \theta < o \\ [\xi(\delta_{q-1})]^+_\kappa(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^c(\delta,\xi(\delta)) + \left(\frac{\theta-o}{1-o}\right)\left(\varphi^d(\delta,\xi(\delta)) - \varphi^e(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } o < \theta < 1 \end{cases}$$

$$[\xi(\delta_{q+1})]^-_\lambda(\theta) = \begin{cases} [\xi(\delta_{q-1})]^-_\lambda(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^b(\delta,\xi(\delta)) + \left(\frac{1-\theta-o}{1-o}\right)\left(\varphi^c(\delta,\xi(\delta)) - \varphi^b(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0 < \theta < 1-o \\ [\xi(\delta_{q-1})]^-_\lambda(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^a(\delta,\xi(\delta)) + \left(\frac{1-\theta}{o}\right)\left(\varphi^b(\delta,\xi(\delta)) - \varphi^a(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 1-o < \theta < 1 \end{cases}$$

$$[\xi(\delta_{q+1})]^+_\lambda(\theta) = \begin{cases} [\xi(\delta_{q-1})]^+_\lambda(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^e(\delta,\xi(\delta)) + \left(\frac{1-\theta-o}{1-o}\right)\left(\varphi^d(\delta,\xi(\delta)) - \varphi^e(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0 < \theta < 1-o \\ [\xi(\delta_{q-1})]^+_\lambda(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^f(\delta,\xi(\delta)) + \left(\frac{1-\theta}{o}\right)\left(\varphi^e(\delta,\xi(\delta)) - \varphi^f(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 1-o < \theta < 1 \end{cases}$$

$$[\xi(\delta_{q+1})]^-_\mu(\theta) = \begin{cases} [\xi(\delta_{q-1})]^-_\mu(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^b(\delta,\xi(\delta)) + \left(\frac{1-\theta-o}{1-o}\right)\left(\varphi^c(\delta,\xi(\delta)) - \varphi^b(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0 < \theta < 1-o \\ [\xi(\delta_{q-1})]^-_\mu(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^a(\delta,\xi(\delta)) + \left(\frac{1-\theta}{o}\right)\left(\varphi^b(\delta,\xi(\delta)) - \varphi^a(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 1-o < \theta < 1 \end{cases}$$

$$[\xi(\delta_{q+1})]^+_\mu(\theta) = \begin{cases} [\xi(\delta_{q-1})]^+_\mu(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^e(\delta,\xi(\delta)) + \left(\frac{1-\theta-o}{1-o}\right)\left(\varphi^d(\delta,\xi(\delta)) - \varphi^e(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0 < \theta < 1-o \\ [\xi(\delta_{q-1})]^+_\mu(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^f(\delta,\xi(\delta)) + \left(\frac{1-\theta}{o}\right)\left(\varphi^e(\delta,\xi(\delta)) - \varphi^f(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 1-o < \theta < 1 \end{cases}$$

For the integral $[\xi(\delta_{r+2})]^-_\kappa(\theta)$, the Adams-Bashforth method gives the following expressions:

Case 1: $[\xi(\delta_{s+2})]^-_\kappa(\theta)$ For $0 < \theta < r$:

$$[\xi(\delta_{s+2})]^-_\kappa(\theta) = [\xi(\delta_{q+1})]^-_\kappa(\theta) + \frac{h}{12}\left[5\varphi_1^T(\delta_{q-1}) - 16\varphi_1^T(\delta_s) + 23\varphi_1^T(\delta_{q+1})\right]$$

where

$$\varphi_1^T(\delta) = \varphi^a(\delta,\xi(\delta)) + \left(\frac{\theta}{r}\right)\left(\varphi^b(\delta,\xi(\delta)) - \varphi^a(\delta,\xi(\delta))\right)$$

For $r < \theta < 1$:

$$[\xi(\delta_{s+2})]^-_\kappa(\theta) = [\xi(\delta_{q+1})]^-_\kappa(\theta) + \frac{h}{12}\left[5\varphi_2^T(\delta_{q-1}) - 16\varphi_2^T(\delta_s) + 23\varphi_2^T(\delta_{q+1})\right]$$

where

$$\varphi_2^T(\delta) = \varphi^b(\delta,\xi(\delta)) + \left(\frac{\theta-r}{1-r}\right)\left(\varphi^c(\delta,\xi(\delta)) - \varphi^b(\delta,\xi(\delta))\right)$$

Case 2: $[\xi(\delta_{s+2})]^+_\kappa(\theta)$ For $0<\theta<r$:

$$[\xi(\delta_{s+2})]^+_\kappa(\theta)=[\xi(\delta_{q+1})]^+_\kappa(\theta)+\frac{h}{12}\left[5\varphi_3^T(\delta_{q-1})-16\varphi_3^T(\delta_s)+23\varphi_3^T(\delta_{q+1})\right]$$

where

$$\varphi_3^T(\delta)=\varphi^f(\delta,\xi(\delta))+\left(\frac{\theta}{r}\right)\left(\varphi^e(\delta,\xi(\delta))-\varphi^f(\delta,\xi(\delta))\right)$$

For $r<\theta<1$:

$$[\xi(\delta_{s+2})]^+_\kappa(\theta)=[\xi(\delta_{q+1})]^+_\kappa(\theta)+\frac{h}{12}\left[5\varphi_4^T(\delta_{q-1})-16\varphi_4^T(\delta_s)+23\varphi_4^T(\delta_{q+1})\right]$$

where

$$\varphi_4^T(\delta)=\varphi^c(\delta,\xi(\delta))+\left(\frac{\theta-r}{1-r}\right)\left(\varphi^d(\delta,\xi(\delta))-\varphi^e(\delta,\xi(\delta))\right)$$

Case 3: $[\xi(\delta_{s+2})]^-_\chi(\theta)$ For $0<\theta<1-r$:

$$[\xi(\delta_{s+2})]^-_\chi(\theta)=[\xi(\delta_{q+1})]^-_\chi(\theta)+\frac{h}{12}\left[5\varphi_1^\chi(\delta_{q-1})-16\varphi_1^\chi(\delta_s)+23\varphi_1^\chi(\delta_{q+1})\right]$$

where

$$\varphi_1^\chi(\delta)=\varphi^b(\delta,\xi(\delta))+\left(\frac{1-\theta-r}{1-r}\right)\left(\varphi^c(\delta,\xi(\delta))-\varphi^b(\delta,\xi(\delta))\right)$$

For $1-r<\theta<1$:

$$[\xi(\delta_{s+2})]^-_\chi(\theta)=[\xi(\delta_{q+1})]^-_\chi(\theta)+\frac{h}{12}\left[5\varphi_2^\chi(\delta_{q-1})-16\varphi_2^\chi(\delta_s)+23\varphi_2^\chi(\delta_{q+1})\right]$$

where

$$\varphi_2^\chi(\delta)=\varphi^a(\delta,\xi(\delta))+\left(\frac{1-\theta}{r}\right)\left(\varphi^b(\delta,\xi(\delta))-\varphi^a(\delta,\xi(\delta))\right)$$

Case 4: $[\xi(\delta_{s+2})]^+_\chi(\theta)$ For $0<\theta<1-r$:

$$[\xi(\delta_{s+2})]^+_\chi(\theta)=[\xi(\delta_{q+1})]^+_\chi(\theta)+\frac{h}{12}\left[5\varphi_3^\chi(\delta_{q-1})-16\varphi_3^\chi(\delta_\chi)+23\varphi_3^\chi(\delta_{q+1})\right]$$

where

$$\varphi_3^\chi(\delta)=\varphi^e(\delta,\xi(\delta))+\left(\frac{1-\theta-r}{1-r}\right)\left(\varphi^d(\delta,\xi(\delta))-\varphi^e(\delta,\xi(\delta))\right)$$

For $1-r<\theta<1$:

$$[\xi(\delta_{s+2})]^+_\chi(\theta)=[\xi(\delta_{q+1})]^+_\chi(\theta)+\frac{h}{12}\left[5\varphi_4^\chi(\delta_{q-1})-16\varphi_4^\chi(\delta_s)+23\varphi_4^\chi(\delta_{q+1})\right]$$

where

$$\varphi_4^\chi(\delta)=\varphi^f(\delta,\xi(\delta))+\left(\frac{1-\theta}{r}\right)\left(\varphi^e(\delta,\xi(\delta))-\varphi^f(\delta,\xi(\delta))\right)$$

Case 5: $[\xi(\delta_{s+2})]^-_\mu(\theta)$ For $0<\theta<1-r$:

$$[\xi(\delta_{s+2})]^-_\mu(\theta)=[\xi(\delta_{q+1})]^-_\mu(\theta)+\frac{h}{12}\left[5\varphi_1^\mu(\delta_{q-1})-16\varphi_1^\mu(\delta_s)+23\varphi_1^\mu(\delta_{q+1})\right]$$

where

$$\varphi_1^\mu(\delta)=\varphi^b(\delta,\xi(\delta))+\left(\frac{1-\theta-r}{1-r}\right)\left(\varphi^c(\delta,\xi(\delta))-\varphi^b(\delta,\xi(\delta))\right)$$

For $1-r<\theta<1$:

$$[\xi(\delta_{s+2})]^-_\mu(\theta)=[\xi(\delta_{q+1})]^-_\mu(\theta)+\frac{h}{12}\left[5\varphi^\mu_2(\delta_{q-1})-16\varphi^\mu_2(\delta_s)+23\varphi^\mu_2(\delta_{q+1})\right]$$

where

$$\varphi^\mu_2(\delta)=\varphi^a(\delta,\xi(\delta))+\left(\frac{1-\theta}{r}\right)\left(\varphi^b(\delta,\xi(\delta))-\varphi^a(\delta,\xi(\delta))\right)$$

Case 6: $[\xi(\delta_{s+2})]^+_\mu(\theta)$ For $0<\theta<1-r$:

$$[\xi(\delta_{s+2})]^+_\mu(\theta)=[\xi(\delta_{q+1})]^+_\mu(\theta)+\frac{h}{12}\left[5\varphi^\mu_3(\delta_{q-1})-16\varphi^\mu_3(\delta_s)+23\varphi^\mu_3(\delta_{q+1})\right]$$

where

$$\varphi^\mu_3(\delta)=\varphi^e(\delta,\xi(\delta))+\left(\frac{1-\theta-r}{1-r}\right)\left(\varphi^d(\delta,\xi(\delta))-\varphi^e(\delta,\xi(\delta))\right)$$

For $1-r<\theta<1$:

$$[\xi(\delta_{s+2})]^+_\mu(\theta)=[\xi(\delta_{q+1})]^+_\mu(\theta)+\frac{h}{12}\left[5\varphi^\mu_4(\delta_{q-1})-16\varphi^\mu_4(\delta_s)+23\varphi^\mu_4(\delta_{q+1})\right]$$

where

$$\varphi^\mu_4(\delta)=\varphi^f(\delta,\xi(\delta))+\left(\frac{1-\theta}{r}\right)\left(\varphi^e(\delta,\xi(\delta))-\varphi^f(\delta,\xi(\delta))\right)$$

5. **Adams-Moulton Methods**

Now, we are going to solve the Neutrosophic initial value problem $\xi'(\tau)=\varphi(\delta,\xi(\delta))$ using the Adams-Bashforth three-step method. Let the Neutrosophic initial values be $\xi(\delta_{q-1}),\xi(\delta_q),\xi(\delta_{q+1})$, i.e., $\varphi(\delta_{q-1},\xi(\delta_{q-1})),\varphi(\delta_q,\xi(\delta_q)),\varphi(\delta_{q+1},\xi(\delta_{q+1}))$, which are represented by hexagonal neutrosophic numbers, were the truth membership,

$$\{\tilde{\varphi}^a(\delta_{q-1},\xi(\delta_{q-1})),\tilde{\varphi}^b(\delta_{q-1},\xi(\delta_{q-1})),\tilde{\varphi}^c(\delta_{q-1},\xi(\delta_{q-1})),\tilde{\varphi}^d(\delta_{q-1},\xi(\delta_{q-1})),\tilde{\varphi}^e(\delta_{q-1},\xi(\delta_{q-1})),\tilde{\varphi}^f(\delta_{q-1},\xi(\delta_{q-1})),$$
$$\tilde{\varphi}_a(\delta_{q-1},\xi(\delta_{q-1})),\tilde{\varphi}_b(\delta_{q-1},\xi(\delta_{q-1})),\tilde{\varphi}_c(\delta_{q-1},\xi(\delta_{q-1})),\tilde{\varphi}_d(\delta_{q-1},\xi(\delta_{q-1})),\tilde{\varphi}_e(\delta_{q-1},\xi(\delta_{q-1})),\tilde{\varphi}_f(\delta_{q-1},\xi(\delta_{q-1})),$$
$$\tilde{\varphi}_a(\delta_{q-1},\xi(\delta_{q-1})),\tilde{\varphi}_b(\delta_{q-1},\xi(\delta_{q-1})),\tilde{\varphi}_c(\delta_{q-1},\xi(\delta_{q-1})),\tilde{\varphi}_d(\delta_{q-1},\xi(\delta_{q-1})),\tilde{\varphi}_e(\delta_{q-1},\xi(\delta_{q-1})),\tilde{\varphi}_f(\delta_{q-1},\xi(\delta_{q-1}))\}$$

the indeterminacy,

$$\{\varphi^a(\delta_q,\xi(\delta_q)),\varphi^b(\delta_q,\xi(\delta_q)),\varphi^c(\delta_q,\xi(\delta_q)),\varphi^d(\delta_q,\xi(\delta_q)),\varphi^e(\delta_q,\xi(\delta_q)),\varphi^f(\delta_q,\xi(\delta_q)),$$
$$\varphi_a(\delta_q,\xi(\delta_q)),\varphi_b(\delta_q,\xi(\delta_q)),\varphi_c(\delta_q,\xi(\delta_q)),\varphi_d(\delta_q,\xi(\delta_q)),\varphi_e(\delta_q,\xi(\delta_q)),\varphi_f(\delta_q,\xi(\delta_q)),$$
$$\varphi_a(\delta_q,\xi(\delta_q)),\varphi_b(\delta_q,\xi(\delta_q)),\varphi_c(\delta_q,\xi(\delta_q)),\varphi_d(\delta_q,\xi(\delta_q)),\varphi_e(\delta_q,\xi(\delta_q)),\varphi_f(\delta_q,\xi(\delta_q))\}$$

the falsity,

$$\{\underset{\sim}{\varphi}^a(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}^b(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}^c(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}^d(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}^e(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}^f(\delta_{q+1},\xi(\delta_{q+1})),$$
$$\underset{\sim}{\varphi}a(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}b(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}c(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}d(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}e(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}f(\delta_{q+1},\xi(\delta_{q+1})),$$
$$\underset{\sim a}{\varphi}(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim b}{\varphi}(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim c}{\varphi}(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim d}{\varphi}(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim e}{\varphi}(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim f}{\varphi}(\delta_{q+1},\xi(\delta_{q+1}))\}$$

For $\delta_{q+1} \leq \delta \leq \delta_{i+2}$, the interpolation polynomials $o_j(\delta)$ are:

$$o_{q-1}(\delta) = \frac{(\delta-\delta_q)(\delta-\delta_{q+1})(\delta-\delta_{q+2})}{(\delta_{q-1}-\delta_q)(\delta_{q-1}-\delta_{q+1})(\delta_{q-1}-\delta_{q+2})} \geq 0$$

$$o_q(\delta) = \frac{(\delta-\delta_{q-1})(\delta-\delta_{q+1})(\delta-\delta_{q+2})}{(\delta_q-\delta_{q-1})(\delta_q-\delta_{q+1})(\delta_q-\delta_{q+2})} \leq 0$$

$$o_{q+1}(\delta) = \frac{(\delta-\delta_{q-1})(\delta-\delta_q)(\delta-\delta_{q+2})}{(\delta_{q+1}-\delta_{q-1})(\delta_{q+1}-\delta_q)(\delta_{q+1}-\delta_{q+2})} \geq 0$$

$$o_{q+2}(\delta) = \frac{(\delta-\delta_{q-1})(\delta-\delta_q)(\delta-\delta_{q+1})}{(\delta_{q+1}-\delta_{q-1})(\delta_{q+2}-\delta_q)(\delta_{q+2}-\delta_{q+1})} \geq 0$$

Therefore, the following results are obtained:

$$\varphi^a(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^a(\delta_{q-1},\xi(\delta_{q-1}))+o_q(\delta)\varphi^f(\delta_i,\xi(\delta_i))+o_{q+1}(\delta)\varphi^a(\delta_{q+1},\xi(\delta_{q+1}))+o_{q+2}(\delta)\varphi^a(\delta_{q+2},\xi(\delta_{q+2}))$$

$$\varphi^b(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^b(\delta_{q-1},\xi(\delta_{q-1}))+o_q(\delta)\varphi^e(\delta_i,\xi(\delta_i))+o_{q+1}(\delta)\varphi^b(\delta_{q+1},\xi(\delta_{q+1}))+o_{q+2}(\delta)\varphi^b(\delta_{q+2},\xi(\delta_{q+2}))$$

$$\varphi^c(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^c(\delta_{q-1},\xi(\delta_{q-1}))+o_q(\delta)\varphi^d(\delta_i,\xi(\delta_i))+o_{q+1}(\delta)\varphi^c(\delta_{q+1},\xi(\delta_{q+1}))+o_{q+2}(\delta)\varphi^c(\delta_{q+2},\xi(\delta_{q+2}))$$

$$\varphi^d(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^d(\delta_{q-1},\xi(\delta_{q-1}))+o_q(\delta)\varphi^c(\delta_i,\xi(\delta_i))+o_{q+1}(\delta)\varphi^d(\delta_{q+1},\xi(\delta_{q+1}))+o_{q+2}(\delta)\varphi^d(\delta_{q+2},\xi(\delta_{q+2}))$$

$$\varphi^e(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^e(\delta_{q-1},\xi(\delta_{q-1}))+o_q(\delta)\varphi^b(\delta_i,\xi(\delta_i))+o_{q+1}(\delta)\varphi^e(\delta_{q+1},\xi(\delta_{q+1}))+o_{q+2}(\delta)\varphi^e(\delta_{q+2},\xi(\delta_{q+2}))$$

$$\varphi^f(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^f(\delta_{q-1},\xi(\delta_{q-1}))+o_q(\delta)\varphi^a(\delta_i,\xi(\delta_i))+o_{q+1}(\delta)\varphi^f(\delta_{q+1},\xi(\delta_{q+1}))+o_{q+2}(\delta)\varphi^f(\delta_{q+2},\xi(\delta_{q+2}$$

where

$$[\xi(\delta_{q+1})]^-_\kappa(\theta) = \begin{cases} [\xi(\delta_{q-1})]^-_\kappa(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^a(\delta,\xi(\delta)) + \left(\frac{\theta}{o}\right)\left(\varphi^b(\delta,\xi(\delta)) - \varphi^a(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0 < \theta < o \\ [\xi(\delta_{q-1})]^-_\kappa(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^b(\delta,\xi(\delta)) + \left(\frac{\theta-o}{1-o}\right)\left(\varphi^c(\delta,\xi(\delta)) - \varphi^b(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } o < \theta < 1 \end{cases}$$

$$[\xi(\delta_{q+1})]^+_\kappa(\theta) = \begin{cases} [\xi(\delta_{q-1})]^+_\kappa(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^f(\delta,\xi(\delta)) + \left(\frac{\theta}{o}\right)\left(\varphi^e(\delta,\xi(\delta)) - \varphi^f(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0 < \theta < o \\ [\xi(\delta_{q-1})]^+_\kappa(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^c(\delta,\xi(\delta)) + \left(\frac{\theta-o}{1-o}\right)\left(\varphi^d(\delta,\xi(\delta)) - \varphi^e(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } o < \theta < 1 \end{cases}$$

$$[\xi(\delta_{q+1})]^-_\lambda(\theta) = \begin{cases} [\xi(\delta_{q-1})]^-_\lambda(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^b(\delta,\xi(\delta)) + \left(\frac{1-\theta-o}{1-o}\right)\left(\varphi^c(\delta,\xi(\delta)) - \varphi^b(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0 < \theta < 1-o \\ [\xi(\delta_{q-1})]^-_\lambda(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^a(\delta,\xi(\delta)) + \left(\frac{1-\theta}{o}\right)\left(\varphi^b(\delta,\xi(\delta)) - \varphi^a(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 1-o < \theta < 1 \end{cases}$$

$$[\xi(\delta_{q+1})]^+_\lambda(\theta) = \begin{cases} [\xi(\delta_{q-1})]^+_\lambda(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^e(\delta,\xi(\delta)) + \left(\frac{1-\theta-o}{1-o}\right)\left(\varphi^d(\delta,\xi(\delta)) - \varphi^e(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0 < \theta < 1-o \\ [\xi(\delta_{q-1})]^+_\lambda(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^f(\delta,\xi(\delta)) + \left(\frac{1-\theta}{o}\right)\left(\varphi^e(\delta,\xi(\delta)) - \varphi^f(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 1-o < \theta < 1 \end{cases}$$

$$[\xi(\delta_{q+1})]^-_\mu(\theta) = \begin{cases} [\xi(\delta_{q-1})]^-_\mu(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^b(\delta,\xi(\delta)) + \left(\frac{1-\theta-o}{1-o}\right)\left(\varphi^c(\delta,\xi(\delta)) - \varphi^b(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0 < \theta < 1-o \\ [\xi(\delta_{q-1})]^-_\mu(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^a(\delta,\xi(\delta)) + \left(\frac{1-\theta}{o}\right)\left(\varphi^b(\delta,\xi(\delta)) - \varphi^a(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 1-o < \theta < 1 \end{cases}$$

$$[\xi(\delta_{q+1})]^+_\mu(\theta) = \begin{cases} [\xi(\delta_{q-1})]^+_\mu(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^e(\delta,\xi(\delta)) + \left(\frac{1-\theta-o}{1-o}\right)\left(\varphi^d(\delta,\xi(\delta)) - \varphi^e(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0 < \theta < 1-o \\ [\xi(\delta_{q-1})]^+_\mu(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^f(\delta,\xi(\delta)) + \left(\frac{1-\theta}{o}\right)\left(\varphi^e(\delta,\xi(\delta)) - \varphi^f(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 1-o < \theta < 1 \end{cases}$$

For the integral $[\xi(\delta_{r+2})]^-_\kappa(\theta)$, the Adams-Bashforth method gives the following expressions:
Case 1: $[\xi(\delta_{s+2})]^-_\kappa(\theta)$ For $0 < \theta < r$:

$$[\xi(\delta_{s+2})]^-_\kappa(\theta) = [\xi(\delta_{q+1})]^-_\kappa(\theta) + \frac{h}{24}\left[\varphi_1^T(\delta_{q-1}) - 5\varphi_1^T(\delta_s) + 19\varphi_1^T(\delta_{q+1}) + 9\varphi_1^T(\delta_{q+2})\right]$$

where

$$\varphi_1^T(\delta) = \varphi^a(\delta, \xi(\delta)) + \left(\frac{\theta}{r}\right)\left(\varphi^b(\delta, \xi(\delta)) - \varphi^a(\delta, \xi(\delta))\right)$$

For $r < \theta < 1$:

$$[\xi(\delta_{s+2})]^-_\kappa(\theta) = [\xi(\delta_{q+1})]^-_\kappa(\theta) + \frac{h}{24}\left[\varphi_2^T(\delta_{q-1}) - 5\varphi_2^T(\delta_s) + 19\varphi_2^T(\delta_{q+1}) + 9\varphi_2^T(\delta_{q+2})\right]$$

Other cases can be computed similarly.

6. **Predictor-Corrector Three-Step Approach**

This method utilizes the Adams–Bashforth three-step technique as a predictor and the Adams–Moulton two-step method as a corrector, iterating the process to improve accuracy.

Procedure: To approximate the solution for the given intuitionistic fuzzy initial value problem:

$$\xi'(\delta) = \varphi(\delta, \xi(\delta)), \quad \delta \in I = [\delta_0, T]$$

$$[\xi(\delta_0)]^+_l(\alpha) = \beta_0, \quad [\xi(\delta_1)]^+_l(\alpha) = \beta_1, \quad [\xi(\delta_2)]^+_l(\alpha) = \beta_2$$

$$[\xi(\delta_0)]^+_r(\alpha) = \beta_3, \quad [\xi(\delta_1)]^+_r(\alpha) = \beta_4, \quad [\xi(\delta_2)]^+_r(\alpha) = \beta_5$$

$$[\xi(\delta_0)]^-_l(\alpha) = \beta_6, \quad [\xi(\delta_1)]^-_l(\alpha) = \beta_7, \quad [\xi(\delta_2)]^-_l(\alpha) = \beta_8$$

$$[\xi(\delta_0)]^-_r(\alpha) = \beta_9, \quad [\xi(\delta_1)]^-_r(\alpha) = \beta_{10}, \quad [\xi(\delta_2)]^-_r(\alpha) = \beta_{11}$$

Step 1: Select a positive integer N and set $h = \frac{T-\delta_0}{N}$.

$$[\eta(\delta_0)]^+_l(\alpha) = \beta_0, \quad [\eta(\delta_1)]^+_l(\alpha) = \beta_1, \quad [\eta(\delta_2)]^+_l(\alpha) = \beta_2$$

$$[\eta(\delta_0)]^+_r(\alpha) = \beta_3, \quad [\eta(\delta_1)]^+_r(\alpha) = \beta_4, \quad [\eta(\delta_2)]^+_r(\alpha) = \beta_5$$

$$[\eta(\delta_0)]^-_l(\alpha) = \beta_6, \quad [\eta(\delta_1)]^-_l(\alpha) = \beta_7, \quad [\eta(\delta_2)]^-_l(\alpha) = \beta_8$$

$$[\eta(\delta_0)]^-_r(\alpha) = \beta_9, \quad [\eta(\delta_1)]^-_r(\alpha) = \beta_{10}, \quad [\eta(\delta_2)]^-_r(\alpha) = \beta_{11}$$

Step 2: Initialize with $i = 1$.

Step 3: Calculate $\xi(\delta_{i+2})$ using the Adams–Bashforth three-step predictor.

Step 4: Use the Adams–Moulton two-step corrector for refinement.

Step 5: Continue iterating until the desired accuracy is achieved.

Step 6: Increment i by 1.

Step 7: If $i \leq (N-2)$, repeat from Step 3.

Step 8: The procedure concludes, and $\eta(T)$ serves as an approximation for $\xi(T)$.

7. Convergence and Stability

Consider the exact solutions:

$$[\Phi(\xi_s)]_\theta = \left[[\Phi(\xi_s)]^-_\kappa(\theta) \quad [\Phi(\xi_s)]^+_\kappa(\theta)\right], [\Phi(\xi_s)]_\theta = \left[[\Phi(\xi_s)]^-_\lambda(\theta) \quad [\Phi(\xi_s)]^+_\lambda(\theta)\right]$$

$$[\Phi(\xi_s)]^\theta = \left[[\Phi(\xi_s)]^-_\mu(\theta) \quad [\Phi(\xi_s)]^+_\mu(\theta)\right]$$

Now, let these exact solutions be approximated by the following:

$$[\varphi(\xi_s)]_\theta = \left[[\varphi(\xi_s)]^-_\kappa(\theta) \quad [\varphi(\xi_s)]^+_\kappa(\theta)\right], [\varphi(\xi_s)]_\theta = \left[[\varphi(\xi_s)]^-_\lambda(\theta) \quad [\varphi(\xi_s)]^+_\lambda(\theta)\right]$$

$$[\varphi(\xi_s)]^\theta = \left[[\varphi(\xi_s)]^-_\mu(\theta) \quad [\varphi(\xi_s)]^+_\mu(\theta)\right]$$

at the time points δ_s, where $0 \leq s \leq N$. The grid points are defined as:

$$\delta_0 < \delta_1 < \delta_2 < \cdots < \delta_N = T, \quad k = \frac{T - t_0}{N}, \quad \delta_s = \delta_0 + sk, \quad n = 0, 1, \ldots, N$$

Our objective is to establish the convergence of the proposed methods to the exact solutions. Specifically, we aim to show:

$$d_\infty(\varphi(\xi_s), \varphi(\xi_s)) \to 0 \quad \text{as} \quad k \to 0$$

Theorem 7.1. *For any fixed θ such that $0 \leq \theta \leq 1$, the Adams-Bashforth two-step approximations of converge to the exact solutions $[\chi(\delta)]^\lambda_+(\theta)$, $[\chi(\delta)]^r_+(\theta)$, $[\Phi(\delta)]^l_-(\theta)$, and $[\Phi(\delta)]^r_-(\theta)$, where $[\Phi]^l_+$, $[\Phi]^r_+$, $[\Phi]^l_-$, and $[\Phi]^r_-$ belong to $C^3[t_0, T]$*

Theorem 7.2. *Both the Adams-Bashforth two-step and three-step methods are stable.*

Proof. For the Adams-Bashforth two-step method, the characteristic polynomial is given by $p(\lambda) = \lambda^2 - \lambda$. It is evident that this polynomial satisfies the root condition, which implies that the Adams-Bashforth two-step method is stable.

Similarly, for the Adams-Bashforth three-step method, the characteristic polynomial is $p(\lambda) = \lambda^3 - \lambda^2$. This polynomial also satisfies the root condition, indicating that the three-step method is stable as well.

Conclusion

This study aimed to investigate the numerical solution of an ordinary differential equation with a neutrosophic number as the initial condition. Here we have employed the Adam Bashforth and Adam-Moulton method for finding the solution, and we have also discussed the stability and convergence properties. Finally we can apply Adam Bashforth as predictor and Adam-Moulton as corrector. The numerical solution is an essential component of initial value problems (IVP) and boundary value problems (BVP) with advanced techniques, which plays a significant role in enhancing precision and reliable solutions. In the future, we can develop more numerical techniques to solve IVP and BVP in a neutrosophic environment.

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