Neutrosophic Sets and Systems, Vol. 75, 2025
University of New Mexico

Deciphering the Geometric Bonferroni Mean Operator in Pythagorean Neutrosophic Sets Framework

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Abstract: The Geometric Bonferroni Mean (GBM), is an extension of The Bonferroni mean (BM), that combines both BM and the geometric mean, allowing for the representation of correlations among the combined factors while acknowledging the inherent uncertainty within the decision-making process. Within the framework of Pythagorean neutrosophic set (PNS) that encompasses truth, indeterminacy, and falsity-membership degrees, each criterion can be integrated into a unified PNS value, portraying the overall evaluation of that criterion by employing the Geometric Bonferroni mean. This study aims to enhance decision-making in Pythagorean neutrosophic framework by introducing an aggregation operator to PNS using the Geometric Bonferroni Mean. Additionally, it proposes a normalized approach to resolve decision-making quandaries within the realm of PNS, striving for improved solutions. The novel Pythagorean Neutrosophic Normalized Weighted Geometric Bonferroni Mean (PNNWGBM) aggregating operator has been tested in a case of multi-criteria decision-making (MCDM) problem involving the selection of Halal products suppliers with several criteria. The result shows that this aggregating operator is offering dependable and pragmatic method for intricate decision-making challenges and able to effectively tackle uncertainty and ambiguity in MCDM problem.

Keywords: aggregating operator; Bonferroni Mean (BM); Geometric Bonferroni Mean (GBM); Pythagorean neutrosophic set (PNS); multi-criteria decision-making (MCDM).

1. Introduction

Zadeh [1] proposed the concept of fuzzy sets as a category of entities characterized by a spectrum of membership grades. He broadened the concepts of inclusion, union, intersection, complement, relation, convexity, and others to apply to these sets, and elucidated several properties associated with these concepts within the framework of fuzzy sets. Intuitionistic Fuzzy Sets (IFS), first introduced by Atanassov [2], assigns both membership and non-membership functions to elements of a universe, ensuring their combined sum is less than or equal to one. Consequently, IFS offers a more precise and definitive description compared to fuzzy sets. However, it's limited to handling incomplete and uncertain information, unable to address the indeterminate and inconsistent information frequently encountered in real-world scenarios. Hence, as an expansion of fuzzy sets and IFS, Smarandache introduced neutrosophic sets (NS) in 1995 and published his results in 1998 [3]. Smarandache's definition outlines that a NS, denoted as A within a universal set X, is distinguished by three distinct functions: a truth-membership function, denoted as $T_A(x)$, an indeterminacy-membership function, denoted as $I_A(x)$, and a falsity-membership function, denoted as $F_A(x)$. The main strength of the neutrosophic set lies in its enhancement of fuzzy set theory by integrating membership, non-membership, and indeterminacy parameters, which are crucial for effectively managing uncertainty in the decision-making process. Smarandache [4] and H. Wang et al. [5] additionally introduced the concept of a single-valued neutrosophic set (SVNS) through adjustments to the established conditions such that $T_A(x)$, $I_A(x)$ and $F_A(x)$ $\in [0,1]$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$, which are better suited for addressing scientific and engineering problems.

Throughout the years, numerous extensions of neutrosophic sets have been developed by other researchers because of their broad range of descriptive scenarios frequently encountered in various real-life situations. Example of neutrosophic set extensions are interval neutrosophic set [6], simplified neutrosophic set [7], neutrosophic soft set [8], multi-valued neutrosophic set [9] and rough neutrosophic set [10]. Over the past decade, a wealth of intriguing studies on neutrosophic sets has emerged across diverse domains within multi-criteria decision-making (MCDM), demonstrating their relevance and impact [11]-[17].

Another novel expansion of neutrosophic set is the Pythagorean neutrosophic sets (PNS). Jansi et al. [18] extends the theory of correlation coefficient from neutrosophic sets (NS) to Pythagorean neutrosophic sets (PNS), where 'T' and 'F' represent dependent neutrosophic components. This extension relaxes the constraint condition requiring the square sum of membership, non-membership, and indeterminacy to be less than two. The Pythagorean constraint helps in better modeling and representing complex situations where the interplay between truth, indeterminacy, and falsity is more intricate. This can be particularly useful in scenarios with high degrees of uncertainty or where traditional neutrosophic sets might be too rigid. Hence, PNS can provide more accurate and refined decision-making capabilities. They allow for more sophisticated aggregation and comparison techniques, leading to potentially better outcomes in decision-making processes involving uncertain or vague information. This makes PNS a valuable tool in fields where robust decision-making under uncertainty is essential [19].

In multi-criteria decision-making (MCDM), an aggregating operator is a mathematical function or method used to combine multiple criteria or attributes into a single composite score or decision value. The purpose of these operators is to synthesize the diverse information provided by the different criteria to facilitate decision-making. Originally introduced as an enhancement of the arithmetic mean, the Bonferroni Mean (BM) is an aggregating operator celebrated for its unique ability to factor in the significance and interplay between element pairs during aggregation [20]. The geometric mean serves as an aggregation operator in various fields, particularly in situations where multiplication or compounding of values is relevant. As an aggregation operator, it combines multiple values into a single representative value that retains essential information from the original dataset. Xia et al. [21] introduces Geometric Bonferroni Mean (GBM), an integration of geometric mean with BM. The Geometric Bonferroni Mean (GBM) extends the concept of BM by applying geometric aggregation functions, offering a more flexible approach to handling multiplicative interactions among variables. The Geometric Bonferroni Mean (GBM) specifically aggregates multiple criteria in decision-making by capturing their interrelationships, enhancing the accuracy and robustness of evaluations, particularly in complex scenarios with interdependent attributes. Moreover, within the realm of Pythagorean fuzzy environments, extensions like the Pythagorean fuzzy GBM operator have emerged to delineate the connections between parameters and explore their unique characteristics [22]. These advancements underscore the flexibility and versatility of the GBM concept across different decision-making contexts. To increase its adaptability and usefulness, the GBM has been expanded through various modifications. For example, the introduction of weighted GBM (WGBM) operators assign weights to indicate the relative importance of certain criteria or attributes, enhancing the aggregation precision. Additionally, the GBM aggregating operator harbors vast potential for application across a multitude of domains within the realm of fuzzy and neutrosophic sets [23]-[27]. In addition, many research works have addressed the significance of Pythagorean fuzzy environments with some mathematical technicals, for example, Edalatpanah [32-34] discussed the impact of some applications of Pythagorean fuzzy in feature selection. Dirik and others [35-39] used Pythagorean fuzzy to develop decision models. Mohammed et al. [40,41] combined the above model with developed topological concepts in topological spaces. Al-sharqi et al [42] introduced the MCDM method for decision-making based on multi-mathematical structures like complex fuzzy structure [43,44], fuzzy graph structure [45,46], and some algebraic structures [47,48]. Al-Quran et al and other [49-52] proposed approaches for the selection of MCDM technology by using the fuzzy set and its extension method,

The motivation for this study arises from the capabilities of the Pythagorean neutrosophic set, which has garnered significant attention from researchers in MCDM techniques due to its impressive performance. A new aggregating operator that combines the strengths of the Geometric Bonferroni Mean with the versatile capabilities of Pythagorean neutrosophic sets would be a valuable addition to the field of multi-criteria decision-making. To the best of the authors' knowledge, such an aggregating operator has not yet been explored, thereby addressing a current gap in the research.

The main objectives of this study are (i) To develop Pythagorean Neutrosophic Geometric Bonferroni Mean (PNGBM), a novel aggregating operator that integrates PNS methodology with the classic GBM aggregating operator. (ii) To develop PNNWGBM, a normalized weighted Geometric Bonferroni Mean aggregating operator within the PNS framework. (iii) To test the applicability of the developed aggregating operator in a multi-criteria decision-making (MCDM) problem involving the selection of Halal product suppliers.

The paper is structured as follows: Section 2 introduces fundamental PNS theories. Section 3 thoroughly explains the proposed methodology. Section 4 demonstrates the application of the suggested methodology through a case study focusing on the selection of Halal products suppliers with several criteria for the decision makers. Lastly, Section 5 functions as the concluding remarks.

2. Experimental Section

2.1 Preliminaries

This section covers the fundamental theories essential for the development of PNNWGBM.

Definition 1. [20] Let $p, q \geq 0$, and a_i be a collection of positive real numbers $(i = 1,2, \ldots, n)$, then BM is defined as

$$BM^{p,q}(a_1, a_2, \ldots, a_n) = \left[\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i^p a_j^q\right]^{\frac{1}{p+q}} \tag{1}$$

Definition 2. [21] Let $p, q > 0$, and a_i be a set of non-negative numbers $(i = 1,2, \ldots, n)$. Then:

$$GB^{p,q}(a_1, a_2, \ldots, a_n) = \frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(pa_i + qa_j\right)^{\frac{1}{n(n-1)}} \tag{2}$$

Equation (2) is called Geometric Bonferroni Mean (GBM). The GBM possesses the following characteristics:

1. $GB^{p,q}(0,0, \ldots, 0) = 0$.
2. $GB^{p,q}(a, a, \ldots, a) = a \; if \; a_i = a$, for all i.
3. $GB^{p,q}(a_1, a_2, \ldots, a_n) \geq GB^{p,q}(d_1, d_2, \ldots, d_n), i.e., GB^{p,q}$ is monotonic, if $a_i \geq d_i$, for all i.
4. $min_i\{a_i\} \leq GB^{p,q}(a_1, a_2, \ldots, a_n) \leq max_i\{a_i\}$.

Definition 3. [28] Let $p, q > 0$, and a_i be a set of non-negative numbers $(i = 1,2, \ldots, n)$. w_i indicates the importance degree of a_i, satisfying $w_i > 0 \; (i = 1,2, \ldots, n)$, $\sum_{j=1}^{n} \omega_i = 1$, for $j \neq i$, v_{ij} indicates the importance degree of a_i for a_j, satisfying $v_{ij} > 0 \; (j \neq i, j = 1,2, \ldots, n)$, $\sum_{j=1}^{n} v_{ij} = 1$. Then we call

$$WGBM^{p,q}(a_1, a_2, \ldots, a_n) = \frac{1}{p+q} \prod_{i=1}^{n} \left(pa_i + \prod_{j \neq i}^{n} \left(qa_j\right)^{v_{ij}}\right)^{w_i} \tag{3}$$

the weighted Geometric Bonferroni Mean (WGBM).

Definition 4. [29] Let X be a non-empty set or a universe. We define a Pythagorean fuzzy set A as

$$A = \{(x, \tau_A(x), \eta_A(x)) | x \in X\} \tag{4}$$

Where $\tau_A(x), \eta_A(x) \in [0,1]$ indicate the truth membership and false membership respectively for each element $x \in X$ to the set A, and $0 \leq {\tau_A}^2(x) + {\eta_A}^2(x) \leq 1$ for each $x \in X$. The indeterminacy membership is given by $\xi_A(x) = \sqrt{1 - {\tau_A}^2(x) - {\eta_A}^2(x)}$.

Definition 5. [5] Let X be a universe or non-empty set. A single valued neutrosophic set β in X is given by:

$$\beta = \{(x, \tau_\beta(x), \xi_\beta(x), \eta_\beta(x))|x \in X\} \tag{5}$$

Where $\tau_\beta(x), \xi_\beta(x), \eta_\beta(x) \in [0,1]$ and with no limitations on the sum of the components where $0 \le \tau_\beta(x) + \xi_\beta(x) + \eta_\beta(x) \le 3$.

Definition 6. [30] Let X be a universe or non-empty set. A Pythagorean neutrosophic set with $\tau_A(x)$ and $\eta_A(x)$ are dependent neutrosophic components that is given by:

$$A = \{(x, \tau_A(x), \xi_A(x), \eta_A(x))|x \in X\} \tag{6}$$

Where τ_A represent the degree of membership, ξ_A represent the degree of indeterminacy and η_A represent the degree of non-membership respectively such that $\tau_A(x), \xi_A(x), \eta_A(x) \in [0,1]$ and satisfying

$$0 \le (\tau_A(x))^2 + (\eta_A(x))^2 \le 1 \tag{7}$$

$$0 \le (\tau_A(x))^2 + (\xi_A(x))^2 + (\eta_A(x))^2 \le 2 \tag{8}$$

Definition 7. [31] Let $x_1 = (\tau_{x_1}, \xi_{x_1}, \eta_{x_1})$, $x_2 = (\tau_{x_2}, \xi_{x_2}, \eta_{x_2})$ and $x = (\tau_x, \xi_x, \eta_x)$ are any two PNSs, then the following definitions apply to the operating rules for PNSs, which include addition, multiplication, scalar multiplication, and power operations:

i. $$x_1 \oplus x_2 = \left(\sqrt{\tau_{x_1}{}^2 + \tau_{x_2}{}^2 - \tau_{x_1}{}^2\tau_{x_2}{}^2}, \xi_{x_1}\xi_{x_2}, \eta_{x_1}\eta_{x_2}\right) \tag{9}$$

ii. $$x_1 \otimes x_2 = \left(\tau_{x_1}\tau_{x_2}, \xi_{x_1} + \xi_{x_2} - \xi_{x_1}\xi_{x_2}\sqrt{\eta_{x_1}{}^2 + \eta_{x_2}{}^2 - \eta_{x_1}{}^2\eta_{x_2}{}^2}\right) \tag{10}$$

iii. $$\mu x = \left(\sqrt{1-(1-\tau_x{}^2)^\mu}, \xi_x{}^\mu, \eta_x{}^\mu\right) \text{ where } \mu \in \Re \text{ and } \mu \ge 0 \tag{11}$$

iv. $$x^\mu = \left(\tau_x{}^\mu, 1-(1-\xi_x)^\mu, \sqrt{1-(1-\eta_x{}^2)^\mu}\right) \text{ where } \mu \in \Re \text{ and } \mu \ge 0. \tag{12}$$

2.2 Proposed Method

In this section, our goal is to enhance the capabilities of the GBM operator to accommodate situations where PNS are employed as input parameters. Therefore, our study involves implementing the PNGBM operator within the Pythagorean neutrosophic framework.

Definition 8. Let $p, q \ge 0$ and $x_i = (\tau_i(x), \xi_i(x), \eta_i(x))$ be a set of non-negative numbers $(i = 1,2, \dots, n)$. Then:

$$PNGBM^{p,q}(x_1,x_2,\dots,x_n)=\frac{1}{p+q}\left(\underset{\substack{i,j=1\\ i\neq j}}{\overset{n}{\otimes}}\left(px_i\oplus qx_j\ \right)^{\frac{1}{n(n-1)}}\right) \tag{13}$$

Equation (13) was formed based on the operating rules of PNS as mentioned in Definition 7.

Proposition 1. Let $p,q\geq 0$ with $x_i=(\tau_i(x),\xi_i(x),\eta_i(x)\,)$ where a PNS set consist of $(i=1,2,3,\dots,n)$. For any i,j *and* $i\neq j$:

$$px_i\oplus qx_j=\left(\sqrt{1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q},\xi_i{}^p\ \xi_j{}^q\ ,\eta_i{}^p\eta_j{}^q\right) \tag{14}$$

Proof Let two PNS sets $x_i=(\tau_i(x),\xi_i(x),\eta_i(x)\,)$ and $x_j=\left(\tau_j(x),\xi_j(x),\eta_j(x)\,\right)$.
Using the operating rules (iii) in Definition 7, we have

$$px_i=\left(\sqrt{1-(1-\tau_i{}^2)^p},\xi_i{}^p\ ,\eta_i{}^p\right)$$

$$qx_j=\left(\sqrt{1-\left(1-\tau_j{}^2\right)^q},\xi_j{}^q\ ,\eta_j{}^q\right)$$

By commencing the operating rules (i) based on Definition 7, we get:

$$px_i\oplus qx_j=\left(\sqrt{1-(1-\tau_i{}^2)^p},\xi_i{}^p\ ,\eta_i{}^p\right)\oplus\left(\sqrt{1-\left(1-\tau_j{}^2\right)^q},\xi_j{}^q\ ,\eta_j{}^q\right)$$

$$=\left(\sqrt{1-(1-\tau_i{}^2)^p+1-\left(1-\tau_j{}^2\right)^q-(1-(1-\tau_i{}^2)^p)\left(1-\left(1-\tau_j{}^2\right)^q\right)},\xi_i{}^p\ \xi_j{}^q\ ,\eta_i{}^p\eta_j{}^q\right)$$

$$=\left(\sqrt{1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q},\xi_i{}^p\ \xi_j{}^q\ ,\eta_i{}^p\eta_j{}^q\right)$$

Thus, Proposition 1 holds.

Proposition 2. Let $p,q\geq 0$ and $x_i=(\tau_i(x),\xi_i(x),\eta_i(x)\,)$ where a PNS set consist of $(i=1,2,3,\dots,n)$. For any i,j *and* $i\neq j$:

$$\left(px_i\oplus qx_j\right)\otimes\left(px_j\oplus qx_i\right)=\begin{pmatrix}\sqrt{\left(1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q\right)\left(1-\left(1-\tau_j{}^2\right)^p(1-\tau_i{}^2)^q\right)},\\ 1-\left(1-\xi_i{}^p\ \xi_j{}^q\right)\left(1-\xi_j{}^p\ \xi_i{}^q\right),\\ \sqrt{1-\left(1-\left(\eta_i{}^{2p}\eta_j{}^{2q}\right)\right)\left(1-\left(\eta_j{}^{2p}\eta_i{}^{2q}\right)\right)}\end{pmatrix} \tag{15}$$

Proof By referring to the result from Proposition 1, we get:

$$px_i\oplus qx_j=\left(\sqrt{1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q},\xi_i{}^p\ \xi_j{}^q\ ,\eta_i{}^p\eta_j{}^q\right)$$ and

$$px_j\oplus qx_i=\left(\sqrt{1-\left(1-\tau_j{}^2\right)^p(1-\tau_i{}^2)^q},\xi_j{}^p\ \xi_i{}^q\ ,\eta_j{}^p\eta_i{}^q\right)$$

By commencing the operating rules (ii) based on Definition 7, we get:

$$\left(px_i \oplus qx_j\right) \otimes \left(px_j \oplus qx_i\right) = \begin{pmatrix} \sqrt{1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q}\sqrt{1-\left(1-\tau_j{}^2\right)^p(1-\tau_i{}^2)^q}, \\ \xi_i{}^p\ \xi_j{}^q + \xi_j{}^p\ \xi_i{}^q - \left(\xi_i{}^p\ \xi_j{}^q\right)\left(\xi_j{}^p\ \xi_i{}^q\right), \\ \sqrt{\left(\eta_i{}^p\eta_j{}^q\right)^2 + \left(\eta_j{}^p\eta_i{}^q\right)^2 - \left(\eta_i{}^p\eta_j{}^q\right)^2\left(\eta_j{}^p\eta_i{}^q\right)^2} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\left(1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q\right)\left(1-\left(1-\tau_j{}^2\right)^p(1-\tau_i{}^2)^q\right)}, \\ 1-\left(1-\xi_i{}^p\ \xi_j{}^q\right)\left(1-\xi_j{}^p\ \xi_i{}^q\right), \\ \sqrt{1-\left(1-\left(\eta_i{}^p\eta_j{}^q\right)^2\right)\left(1-\left(\eta_j{}^p\eta_i{}^q\right)^2\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\left(1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q\right)\left(1-\left(1-\tau_j{}^2\right)^p(1-\tau_i{}^2)^q\right)}, \\ 1-\left(1-\xi_i{}^p\ \xi_j{}^q\right)\left(1-\xi_j{}^p\ \xi_i{}^q\right), \\ \sqrt{1-\left(1-\left(\eta_i{}^{2p}\eta_j{}^{2q}\right)\right)\left(1-\left(\eta_j{}^{2p}\eta_i{}^{2q}\right)\right)} \end{pmatrix}$$

Therefore, Proposition 2 is valid.

Proposition 3. Let $p, q \geq 0$ and $x_i = (\tau_i(x), \xi_i(x), \eta_i(x))$ where a PNS set consist of $(i = 1,2,3, \dots, n)$. Provided the value of f at which $1 \leq f < n$, we have:

$$\underset{i=1}{\overset{f}{\otimes}} \left(px_i \oplus qx_{f+1}\right) = \begin{pmatrix} \sqrt{\prod_{i=1}^{f}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_{f+1}{}^2\right)^q\right)}, 1-\prod_{i=1}^{f}\left(1-\xi_i{}^p\ \xi_{f+1}{}^q\right), \\ \sqrt{1-\prod_{i=1}^{f}\left(1-\left(\eta_i{}^{2p}\eta_{f+1}{}^{2q}\right)\right)} \end{pmatrix} \quad (16)$$

Proof Referring to Equation (14) in Proposition 1, when $f = 2$, we obtain:

$$px_i \oplus qx_{2+1} = \left(\sqrt{1-(1-\tau_i{}^2)^p(1-\tau_{2+1}{}^2)^q}, \xi_i{}^p\ \xi_{2+1}{}^q, \eta_i{}^p\eta_{2+1}{}^q\right)$$
$$= \left(\sqrt{1-(1-\tau_i{}^2)^p(1-\tau_3{}^2)^q}, \xi_i{}^p\ \xi_3{}^q, \eta_i{}^p\eta_3{}^q\right)$$

Then, $\underset{i=1}{\overset{2}{\otimes}} \left(px_i \oplus qx_3\right) = \left(px_1 \oplus qx_3\right) \otimes \left(px_2 \oplus qx_3\right)$

$$= \begin{pmatrix} \sqrt{(1-(1-\tau_1{}^2)^p(1-\tau_3{}^2)^q)(1-(1-\tau_2{}^2)^p(1-\tau_3{}^2)^q)}, \\ 1-(1-\xi_1{}^p\ \xi_3{}^q)(1-\xi_2{}^p\ \xi_3{}^q), \\ \sqrt{1-\left(1-(\eta_1{}^{2p}\eta_3{}^{2q})\right)\left(1-(\eta_2{}^{2p}\eta_3{}^{2q})\right)} \end{pmatrix}$$

By referring Equation (15), let $f = f_0$ to obtain the following general form:

$$\underset{i=1}{\overset{f_0}{\otimes}} \left(px_i \oplus qx_{f_0+1}\right) = \left(\begin{array}{c} \sqrt{\prod_{i=1}^{f_0}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_{f_0+1}{}^2\right)^q\right)}, 1-\prod_{i=1}^{f_0}\left(1-\xi_i{}^p\,\xi_{f_0+1}{}^q\right), \\ \sqrt{1-\prod_{i=1}^{f_0}\left(1-\left(\eta_i{}^{2p}\eta_{f_0+1}{}^{2q}\right)\right)} \end{array} \right)$$

Thus, Equation (16) as stated in Proposition 3 is applicable when $f = f_0$.

Next, let $f = f_0 + 1$

$$\underset{i=1}{\overset{f_0+1}{\otimes}} \left(px_i \oplus qx_{f_0+1+1}\right) = \underset{i=1}{\overset{f_0+1}{\otimes}} \left(px_i \oplus qx_{f_0+2}\right)$$

$$= \underset{i=1}{\overset{f_0}{\otimes}} \left(px_i \oplus qx_{f_0+2}\right) \otimes \left(px_{f_0+1} \oplus qx_{f_0+2}\right)$$

$$= \left(\begin{array}{c} \sqrt{\prod_{i=1}^{f_0}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_{f_0+1}{}^2\right)^q\right)}, 1-\prod_{i=1}^{f_0}\left(1-\xi_i{}^p\,\xi_{f_0+1}{}^q\right), \\ \sqrt{1-\prod_{i=1}^{f_0}\left(1-\left(\eta_i{}^{2p}\eta_{f_0+1}{}^{2q}\right)\right)} \end{array} \right) \otimes$$

$$\left(\sqrt{1-\left(1-\tau_{f_0+1}{}^2\right)^p\left(1-\tau_{f_0+2}{}^2\right)^q}, \xi_{f_0+1}{}^p\,\xi_{f_0+2}{}^q, \eta_{f_0+1}{}^p\eta_{f_0+2}{}^q\right)$$

$$= \left(\begin{array}{c} \sqrt{\prod_{i=1}^{f_0}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_{f_0+2}{}^2\right)^q\right)}, 1-\prod_{i=1}^{f_0}\left(1-\xi_i{}^p\,\xi_{f_0+2}{}^q\right), \\ \sqrt{1-\prod_{i=1}^{f_0}\left(1-\left(\eta_i{}^{2p}\eta_{f_0+2}{}^{2q}\right)\right)} \end{array} \right)$$

This proof also applies to the case where $f = f_0 + 1$. Thus, Proposition 3 remains valid. Consequently, we can infer the following Proposition 4 directly from Proposition 3 as stated earlier.

Proposition 4. Let $p, q \geq 0$ and $x_i = (\tau_i(x), \xi_i(x), \eta_i(x))$ where a PNS set consist of $(i = 1,2,3, \dots, n)$. Provided the value of f at which $1 \leq f < n$, we have:

$$\underset{j=1}{\overset{f}{\otimes}} \left(px_{f+1} \oplus qx_j\right) = \left(\begin{array}{c} \sqrt{\prod_{i=1}^{f}\left(1-\left(1-\tau_{f+1}{}^2\right)^p\left(1-\tau_j{}^2\right)^q\right)}, 1-\prod_{i=1}^{f}\left(1-\xi_{f+1}{}^p\,\xi_j{}^q\right), \\ \sqrt{1-\prod_{i=1}^{f}\left(1-\left(\eta_{f+1}{}^{2p}\eta_j{}^{2q}\right)\right)} \end{array} \right) \quad (17)$$

Proposition 5. Let $p, q \geq 0$ and $x_i = (\tau_i(x), \xi_i(x), \eta_i(x))$ where a PNS set consist of $(i = 1,2,3, \dots, n)$. For any i, j and $i \neq j$:

$$\underset{\substack{i,j=1 \\ i\neq j}}{\overset{n}{\otimes}} (px_i \oplus qx_j) = \begin{pmatrix} \sqrt{\prod_{i=1}^{n}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q\right)}, 1-\prod_{i=1}^{n}\left(1-\xi_i{}^p\,\xi_j{}^q\right), \\ \sqrt{1-\prod_{i=1}^{n}\left(1-\left(\eta_i{}^{2p}\eta_j{}^{2q}\right)\right)} \end{pmatrix} \tag{18}$$

Proof Proposition 5 can be proven as follows by combining the results from Propositions 2 to 4.
By using Equation (15) and letting $n = 2$, we get:

$$\underset{\substack{i,j=1 \\ i\neq j}}{\overset{2}{\otimes}} (px_i \oplus qx_j) = (px_1 \oplus qx_2) \otimes (px_2 \oplus qx_1)$$

$$= \begin{pmatrix} \sqrt{(1-(1-\tau_1{}^2)^p(1-\tau_2{}^2)^q)(1-(1-\tau_2{}^2)^p(1-\tau_1{}^2)^q)}, \\ 1-(1-\xi_1{}^p\,\xi_2{}^q)(1-\xi_2{}^p\,\xi_1{}^q), \\ \sqrt{1-\left(1-(\eta_1{}^{2p}\eta_2{}^{2q})\right)\left(1-(\eta_2{}^{2p}\eta_1{}^{2q})\right)} \end{pmatrix}$$

and if $n = f$, the equation becomes:

$$\underset{\substack{i,j=1 \\ i\neq j}}{\overset{f}{\otimes}} (px_i \oplus qx_j) = \begin{pmatrix} \sqrt{\prod_{\substack{i,j=1 \\ i\neq j}}^{f}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q\right)}, 1-\prod_{\substack{i,j=1 \\ i\neq j}}^{f}\left(1-\xi_i{}^p\,\xi_j{}^q\right), \\ \sqrt{1-\prod_{\substack{i,j=1 \\ i\neq j}}^{f}\left(1-\eta_i{}^{2p}\eta_j{}^{2q}\right)} \end{pmatrix}$$

Next, let $n = f + 1$

$$\underset{\substack{i,j=1 \\ i\neq j}}{\overset{f+1}{\otimes}} (px_i \oplus qx_j) = \underset{\substack{i,j=1 \\ i\neq j}}{\overset{f}{\otimes}} (px_i \oplus qx_j) \underset{i=1}{\overset{f}{\otimes}} (px_i \oplus qx_{f+1}) \underset{j=1}{\overset{f}{\otimes}} (px_{f+1} \oplus qx_j)$$

where $\underset{i=1}{\overset{f}{\otimes}} (px_i \oplus qx_{f+1})$ can be referred from Proposition 3, which is

$$\left(\begin{array}{c}\sqrt{\prod_{i=1}^{f}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_{f+1}{}^2\right)^q\right)}, 1-\prod_{i=1}^{f}\left(1-\xi_i{}^p\ \xi_{f+1}{}^q\right), \\ \sqrt{1-\prod_{i=1}^{f}\left(1-\left(\eta_i{}^{2p}\eta_{f+1}{}^{2q}\right)\right)}\end{array}\right)$$

and $\underset{j=1}{\overset{f}{\otimes}} \left(px_{f+1} \oplus qx_j\right)$ can be referred from Proposition 4, which is

$$\left(\begin{array}{c}\sqrt{\prod_{i=1}^{f}\left(1-\left(1-\tau_{f+1}{}^2\right)^p\left(1-\tau_j{}^2\right)^q\right)}, 1-\prod_{i=1}^{f}\left(1-\xi_{f+1}{}^p\ \xi_j{}^q\right), \\ \sqrt{1-\prod_{i=1}^{f}\left(1-\left(\eta_{f+1}{}^{2p}\eta_j{}^{2q}\right)\right)}\end{array}\right)$$

Therefore, operations are performed for $\underset{\substack{i,j=1\\ i\neq j}}{\overset{f+1}{\otimes}} \left(px_i \oplus qx_j\ \right)$:

$$\underset{\substack{i,j=1\\ i\neq j}}{\overset{f+1}{\otimes}} \left(px_i \oplus qx_j\ \right) = \left(\begin{array}{c}\sqrt{\prod_{i=1}^{f+1}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q\right)}, 1-\prod_{i=1}^{f+1}\left(1-\xi_i{}^p\ \xi_j{}^q\right), \\ \sqrt{1-\prod_{i=1}^{f+1}\left(1-\left(\eta_i{}^{2p}\eta_j{}^{2q}\right)\right)}\end{array}\right)$$

Given that Equation (18) remains applicable for $n = f + 1$. Thus, Proposition 5 holds.

Proposition 6. Let $p, q \geq 0$ and $x_i = (\tau_i(x), \xi_i(x), \eta_i(x)\ )$ where a PNS set consist of $(i = 1,2,3, \ldots, n)$. For any $i, j \text{ and } i \neq j$:

$$\frac{1}{p+q}\left(\underset{\substack{i,j=1\\ i\neq j}}{\overset{n}{\otimes}} \left(px_i \oplus qx_j\ \right)\right) = \left(\begin{array}{c}\sqrt{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q\right)^{\frac{1}{p+q}}}, \\ 1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\xi_i{}^p\ \xi_j{}^q\right)^{\frac{1}{p+q}}, \\ \sqrt{1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\eta_i{}^{2p}\eta_j{}^{2q}\right)^{\frac{1}{p+q}}}\end{array}\right) \tag{19}$$

Proof Referring to Proposition 5, we possess the following:

$$\underset{\substack{i,j=1\\ i\neq j}}{\overset{n}{\otimes}} \left(px_i \oplus qx_j\right) = \left(\begin{array}{c}\sqrt{\prod_{i=1}^{n}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q\right)}, 1-\prod_{i=1}^{n}\left(1-\xi_i{}^p\,\xi_j{}^q\right), \\ \sqrt{1-\prod_{i=1}^{n}\left(1-\left(\eta_i{}^{2p}\eta_j{}^{2q}\right)\right)}\end{array}\right)$$

By commencing the operating rules (iii) based on Definition 6, we get:

$$\frac{1}{p+q}\left(\underset{\substack{i,j=1\\ i\neq j}}{\overset{n}{\otimes}} \left(px_i \oplus qx_j\right)\right) = \left(\begin{array}{c}\sqrt{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q\right)^{\frac{1}{p+q}}}, \\ 1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\xi_i{}^p\,\xi_j{}^q\right)^{\frac{1}{p+q}}, \\ \sqrt{1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\eta_i{}^{2p}\eta_j{}^{2q}\right)^{\frac{1}{p+q}}}\end{array}\right)$$

Hence, Proposition 6 holds. Subsequently, through the application of PNGBM as outlined in Definition 8, we embark on deducing the PNNWGBM operator in the following steps:

Definition 9. Let $p,q \geq 0$ and $x_i = (\tau_i(x), \xi_i(x), \eta_i(x))$ where a PNS set consist of $(i = 1,2,3,\dots,n)$, then PNNWGBM (Pythagorean Neutrosophic Normalized Weighted Geometric Bonferroni Mean) is defined for all sets of PNS as follows:

$$PNNWGBM^{p,q}(x_1,x_2,\dots,x_n) = \left(\underset{\substack{i,j=1\\ i\neq j}}{\overset{n}{\otimes}} \frac{\omega_i\omega_j}{1-\omega_i}\left(px_i \oplus qx_j\right)^{\frac{1}{n(n-1)}}\right) \tag{20}$$

where the weight vector, $W = (\omega_1,\omega_2,\omega_3,\dots,\omega_n)^T$ signifies the level of importance of x_i for $(i = 1,2,3,\dots,n)$ with the condition $\sum_{j=1}^{n}\omega_j = 1$ and $\omega_j \in [0,1]$.

Theorem 1. Let $p,q \geq 0$ and $x_i = (\tau_i(x), \xi_i(x), \eta_i(x))$ where a PNS set consist of $(i = 1,2,3,\dots,n)$, the value computed by the PNNWGBM operator in Equation (20) represents a **Pythagorean neutrosophic number** with the following components:

$$\tau_i(x) = \left(\sqrt{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}\right)^{\frac{1}{n(n-1)}}, \xi_i(x) = 1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\xi_i{}^p\,\xi_j{}^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{n(n-1)}},$$

$$and\ \eta_i(x) = \sqrt{1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\eta_i{}^{2p}\eta_j{}^{2q}\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{n(n-1)}}}.$$

Proof Theorem 1 can be established by using the derived equations from Propositions 1 to 6:

We possess $px_i \oplus qx_j = \left(\sqrt{1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q}, \xi_i{}^p\ \xi_j{}^q, \eta_i{}^p\eta_j{}^q\right)$, then

$$\frac{\omega_i\omega_j}{1-\omega_i}\left(px_i \oplus qx_j\right) = \left(\sqrt{\left(1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}, \left(\xi_i{}^p\ \xi_j{}^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}, \left(\eta_i{}^p\eta_j{}^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)$$

Next, we get:

$$\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}\left(px_i \oplus qx_j\ \right) = \left(\begin{array}{c} \sqrt{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}, \\ 1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\xi_i{}^p\ \xi_j{}^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}, \\ \sqrt{1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\eta_i{}^{2p}\eta_j{}^{2q}\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}} \end{array}\right)$$

Therefore,

$$PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) = \left(\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}\left(px_i \oplus qx_j\ \right)^{\frac{1}{n(n-1)}}\right)$$

$$= \left(\begin{array}{c} \left(\sqrt{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}\right)^{\frac{1}{n(n-1)}}, \\ 1-\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\xi_i{}^p\ \xi_j{}^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)\right)^{\frac{1}{n(n-1)}}, \\ \sqrt{1-\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\eta_i{}^{2p}\eta_j{}^{2q}\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)\right)^{\frac{1}{n(n-1)}}} \end{array}\right)$$

Furthermore, we can simplify the equation to get:

$$PNNWGBM^{p,q}(x_1,x_2,\dots,x_n)=\left(\begin{array}{c}\left(\sqrt{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-{\tau_i}^2)^p\left(1-{\tau_j}^2\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}\right)^{\frac{1}{n(n-1)}},\\ 1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-{\xi_i}^p\ {\xi_j}^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{n(n-1)}},\\ \sqrt{1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left({\eta_i}^{2p}{\eta_j}^{2q}\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{n(n-1)}}}\end{array}\right) \tag{21}$$

$$where\ 0\le\left(\sqrt{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-{\tau_i}^2)^p\left(1-{\tau_j}^2\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}\right)^{\frac{1}{n(n-1)}}\le 1,$$

$$0\le 1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-{\xi_i}^p\ {\xi_j}^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{n(n-1)}}\le 1\ and\ 0\le\sqrt{1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left({\eta_i}^{2p}{\eta_j}^{2q}\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{n(n-1)}}}\le 1$$

satisfying Equation (7) and (8). Hence, this validates Theorem 1. Building on the established Theorem 1, we can additionally deduce the important properties of the $PNGBM^{p,q}$, namely reducibility, commutativity, idempotency, monotonicity, and boundedness.

Theorem 2. Reducibility: Let $\omega=\left(\frac{1}{n},\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n}\right)$ then

$$PNNWGBM^{p,q}(x_1,x_2,\dots,x_n)=PNGBM^{p,q}(x_1,x_2,\dots,x_n) \tag{22}$$

Proof Considering $\omega=\left(\frac{1}{n},\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n}\right)$, then in line with Definition 9, it follows that

$$\begin{aligned}PNNWGBM^{p,q}(x_1,x_2,\dots,x_n)&=\left(\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}\left(px_i\oplus qx_j\ \right)^{\frac{1}{n(n-1)}}\right)\\&=\left(\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n}\frac{1}{p+q}\left(px_i\oplus qx_j\ \right)^{\frac{1}{n(n-1)}}\right)\\&=\frac{1}{p+q}\left(\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n}\left(px_i\oplus qx_j\ \right)^{\frac{1}{n(n-1)}}\right)\\&=PNGBM^{p,q}(x_1,x_2,\dots,x_n)\end{aligned}$$

thus, completing the proof of Theorem 2.

Theorem 3. Idempotency: Let $x_i = x$ where $(i = 1,2,3, \dots, n)$ then

$$PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) = x \tag{23}$$

Proof Given that $x_i = x$ for every i, it follows that

$$PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) = \left(\underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\otimes}} \frac{\omega_i \omega_j}{1-\omega_i} \left(px_i \oplus qx_j \right)^{\frac{1}{n(n-1)}} \right)$$
$$= \left(\underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\otimes}} \frac{\omega_i \omega_j}{1-\omega_i} \left(px \oplus qx \right)^{\frac{1}{n(n-1)}} \right)$$
$$= \left(\underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\otimes}} \frac{\omega_i \omega_j}{1-\omega_i} \left(x(p \oplus q) \right)^{\frac{1}{n(n-1)}} \right)$$
$$= x \left(\underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\otimes}} \frac{\omega_i \omega_j}{1-\omega_i} \right)^{\frac{1}{n(n-1)}} = x$$

thus, concluding the proof of Theorem 3.

Theorem 4. Commutativity: $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ is any permutation of Pythagorean neutrosophic numbers $(x_1, x_2, \dots, x_n)$. For any $p, q > 0$,

$$PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) = PNNWGBM^{p,q}(\widetilde{x_1}, \widetilde{x_2}, \dots, \widetilde{x_n}) \tag{24}$$

Proof Let $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ be any permutation of Pythagorean neutrosophic numbers $(x_1, x_2, \dots, x_n)$. Then

$$PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) = \left(\underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\otimes}} \frac{\omega_i \omega_j}{1-\omega_i} \left(px_i \oplus qx_j \right)^{\frac{1}{n(n-1)}} \right)$$
$$= \left(\underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\otimes}} \frac{\omega_i \omega_j}{1-\omega_i} \left(p\tilde{x}_i \oplus q\tilde{x}_j \right)^{\frac{1}{n(n-1)}} \right)$$

thus, completing the proof of Theorem 4.

Theorem 5. Monotonicity: Let $x_i = (\tau_i(x), \xi_i(x), \eta_i(x))$ where $(i = 1,2,3, \dots, n)$ and $y_i = (\tau_i(y), \xi_i(y), \eta_i(y))$ where $(i = 1,2,3, \dots, n)$ be two collections of PNSs. For any i, $\tau_i(x) \geq \tau_i(y)$, $\xi_i(x) \leq \xi_i(y)$ and $\eta_i(x) \leq \eta_i(y)$, then

$$PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) \geq PNNWGBM^{p,q}(y_1, y_2, \dots, y_n) \tag{25}$$

Proof For the degree of truth, we have $\tau_i(x) \geq \tau_i(y)$ for all i and $p, q > 0$. Thus, $p\tau_i(x) \geq p\tau_i(y)$ and $q\tau_i(x) \geq q\tau_i(y)$. Following that

$$1 - p\tau_i(x)\, q\tau_j(x) \leq 1 - p\tau_i(y)\, q\tau_j(y)$$

$$\prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - p\tau_i(x)\, q\tau_j(x)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}} \leq \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - p\tau_i(y)\, q\tau_j(y)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}$$

$$\sqrt{\left(1 - \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - p\tau_i(x)\, q\tau_j(x)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}}} \geq \sqrt{\left(1 - \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - p\tau_i(y)\, q\tau_j(y)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}}}$$

For the degree of indeterminacy, since $\xi_i(x) \leq \xi_i(y)$ for all i and $p, q > 0$, then we get $\left(1 - p\xi_i(x)\right) \geq \left(1 - p\xi_i(y)\right)$ and $\left(1 - q\xi_j(x)\right) \geq \left(1 - q\xi_j(y)\right)$. Therefore,

$$\left(1 - p\xi_i(x)\right)\left(1 - q\xi_j(x)\right) \geq \left(1 - p\xi_i(y)\right)\left(1 - q\xi_j(y)\right)$$

$$\prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - \left(1 - p\xi_i(x)\right)\left(1 - q\xi_j(x)\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}} \leq \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - \left(1 - p\xi_i(y)\right)\left(1 - q\xi_j(y)\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}$$

$$\left(1 - \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - \left(1 - p\xi_i(x)\right)\left(1 - q\xi_j(x)\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}} \geq \left(1 - \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - \left(1 - p\xi_i(y)\right)\left(1 - q\xi_j(y)\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}}$$

$$1 - \left(1 - \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - \left(1 - p\xi_i(x)\right)\left(1 - q\xi_j(x)\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}} \leq 1 - \left(1 - \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - \left(1 - p\xi_i(y)\right)\left(1 - q\xi_j(y)\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}}$$

Likewise, for the degree of falsity, we can observe that:

$$\sqrt{1 - \left(1 - \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - \left(1 - p\xi_i(x)\right)\left(1 - q\xi_j(x)\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}}} \leq \sqrt{1 - \left(1 - \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - \left(1 - p\xi_i(y)\right)\left(1 - q\xi_j(y)\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}}}$$

Hence, considering that $\tau_i(x) \geq \tau_i(y)$, $\xi_i(x) \leq \xi_i(y)$ and $\eta_i(x) \leq \eta_i(y)$, we managed to conclude that $x \geq y$, thereby concluding the proof of Theorem 5.

Theorem 6. Boundedness: Let $x_i = (\tau_i(x), \xi_i(x), \eta_i(x))$ where a PNS set consist of $(i = 1,2,3,\dots,n)$ and $x^- = (min_i\{\tau_i\}, max_i\{\xi_i\}, max_i\{\eta_i\})$, $x^+ = (max_i\{\tau_i\}, min_i\{\xi_i\}, min_i\{\eta_i\})$, then

$$x^- \leq PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) \leq x^+ \tag{26}$$

Proof Since $x_i \geq x^-$, using Theorem 3 and 5 as a basis, we obtain:

$$PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) \geq PNNWGBM^{p,q}(x^-, x^-, \dots, x^-) = x^-$$

In a similar manner, we can derive:

$$PNNWGBM^{p,q}(x_1, x_2, \dots, x_n) \leq PNNWGBM^{p,q}(x^+, x^+, \dots, x^+) = x^+$$

Thus, boundedness is obtained, and Theorem 6 proof has been concluded.

3. Results and Discussion

This section may be divided by subheadings. It should provide a concise and precise description of the experimental results, their interpretation as well as the experimental conclusions that can be drawn.

3.1. The Multi-Criteria Decision-Making Method Based on PNNWGBM Operator

In this section, we introduce a Multiple Criteria Decision-Making (MCDM) problem within a Pythagorean neutrosophic setting. Let $S = \{S_1, S_2, S_3, \dots, S_n\}$ as the list of suppliers and $C = \{C_1, C_2, C_3, \dots, C_n\}$ as the list of criteria. Then ω_j is the weight assigned to criterion C_j for $j = 1,2,\dots,m$ such that $0 \leq \omega_j \leq 1$ and $\sum_{j=1}^{n} \omega_j = 1$. We propose the PNNWGBM operator to consolidate the overall criteria for each supplier into a singular, aggregated preference. The computational steps for this method are detailed below.

Step 1. Construct the decision matrix in the form of a Pythagorean neutrosophic set (PNS), denoted as $(x_{ij})_{m\times n} = (\tau_{ij}(x), \xi_{ij}(x), \eta_{ij}(x))_{m\times n}$. A direct-relation matrix, incorporating the criteria score for each supplier, is formed. Afterward, each criterion is converted to Pythagorean neutrosophic numbers. The rating scale for PNS numbers is determined by using seven linguistic scores ranging from negligible to exceptionally significant effect, employing Pythagorean neutrosophic linguistic variables, as detailed in Table 1.

Table 1. The new Pythagorean neutrosophic linguistic variable [31].

Score	**Linguistic Variable**	**Rating Scale in Pythagorean Neutrosophic Set**

1	No Effect	$\langle 0.10, 0.80, 0.90\rangle$
2	Low Effect	$\langle 0.20, 0.70, 0.80\rangle$
3	Medium Low Effect	$\langle 0.35, 0.60, 0.60\rangle$
4	Medium Effect	$\langle 0.50, 0.40, 0.45\rangle$
5	Medium High Effect	$\langle 0.65, 0.30, 0.25\rangle$
6	High Effect	$\langle 0.80, 0.20, 0.15\rangle$
7	Very High Effect	$\langle 0.90, 0.10, 0.10\rangle$

Step 2. Each criterion for each supplier is aggregated into a unified value by using PNNWGBM aggregating operator and the weight vector $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_m)^n$ that represents the importance of each criterion x_i with i varying from 1 to n. The weight vector satisfies the condition $\sum_{j=1}^{n} \omega_j = 1$ and $\omega_j \in [0,1]$.

$$PNNWGBM^{p,q}\left(a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^m\right) = \left(\begin{array}{c} \left(\sqrt{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\tau_i{}^2)^p\left(1-\tau_j{}^2\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}\right)^{\frac{1}{n(n-1)}}, \\ 1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\xi_i{}^p\,\xi_j{}^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{n(n-1)}}, \\ \sqrt{1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\eta_i{}^{2p}\eta_j{}^{2q}\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{n(n-1)}}} \end{array} \right)$$

Step 3. The aggregated values should adhere to the PNS number conditions outlined in Definition 6 such that

$$0 \le \left(\tau_A(x)\right)^2 + (\eta_A(x))^2 \le 1$$
$$0 \le \left(\tau_A(x)\right)^2 + \left(\xi_A(x)\right)^2 + (\eta_A(x))^2 \le 2$$

Step 4. Deneutrosophicate the PNS numbers, $(\tau_A(x), \xi_A(x), \eta_A(x))$ into a crisp value by using the following formula:

$$K = \frac{\tau_A(x) + \xi_A(x) + \eta_A(x)}{3}$$

Step 5. According to the crisp value, each supplier is ranked from highest to lowest value which represents the most ideal supplier for the decision makers to the least ideal.

3.2. Illustrative Example

To illustrate the feasibility of the proposed method, we present an example of a multi-criteria decision-making problem. A company wants to choose a Halal products supplier for their business.

Then there are four different suppliers to choose from. Each of these suppliers is going to be evaluated based on four criteria which are, (C_1) Quality of Products, (C_2) Product Variety, (C_3) Cost and Pricing, (C_4) Customer Service and Support and (C_5) Location and Delivery Options.

Step 1. By using the five criteria $C_i(i = 1,2,3,4,5)$ and four potential Halal products suppliers $S_i(i = 1,2,3,4)$, the 4×5 initial direct-relation matrix, X is obtained Table 2. These values then are transformed into Pythagorean neutrosophic numbers, which encompass the degrees of truth, indeterminacy, and falsity.

Table 2. Initial direct-relation matrix, X.

	C1	**C2**	**C3**	**C4**	**C5**
S1	6	3	1	7	7
S2	5	3	6	6	1
S3	5	4	5	6	1
S4	2	7	3	4	2

Step 2. The aggregated value was computed using the PNNWGBM operator from Equation (21) to depict the criterion selection for each supplier. The decision-makers employ a weighting vector $w = (0.25, 0.18, 0.32,0.10,0.15)$ and the result obtained is shown in Table 3.

Table 3. The aggregated value using PNNWGBM operator.

	PNNWGBM
S1	(0.9867,0.0091,0.055)
S2	(0.9872,0.0082,0.0417)
S3	(0.9856,0.008,0.0377)
S4	(0.9712,0.0165,0.0901)

Step 3. The aggregated PNS set has been verified and it satisfies the conditions outlined in Definition 6 where $0 \leq \left(\tau_A(x)\right)^2 + (\eta_A(x))^2 \leq 1$ and $0 \leq \left(\tau_A(x)\right)^2 + \left(\xi_A(x)\right)^2 + (\eta_A(x))^2 \leq 2$ as shown in Table 4.

Table 4. PNS number verification.

	$\boldsymbol{\tau}$	$\boldsymbol{\xi}$	$\boldsymbol{\eta}$	$\boldsymbol{\tau^2 + \eta^2}$	$\boldsymbol{\tau^2 + \xi^2 + \eta^2}$
S1	0.98673	0.00909	0.05501	0.97666	0.97674
S2	0.98718	0.00823	0.04171	0.97626	0.97633
S3	0.98557	0.00795	0.03768	0.97277	0.97284
S4	0.97118	0.01652	0.09007	0.9513	0.95157

Step 4. The aggregated PNS set has been deneutrosophicated into a crisp value to represent the overall criterion for each supplier.

Table 5. Crisp value.

S1	0.3503
S2	0.3457
S3	0.3437
S4	0.3593

Step 5. Table 6 presents the ranking of the suppliers according to the crisp value that has been aggregated by PNNWGBM operator. Therefore, Supplier 4 is the most recommended alternative for the company to choose.

Table 6. The rankings of suppliers.

	PNNWGBM	Rank
S1	0.3503	2
S2	0.3457	3
S3	0.3437	4
S4	0.3593	1

4. Conclusions

This paper has discussed the application of Geometric Bonferroni Mean (GBM) operator to Pythagorean neutrosophic set framework. The primary aim was to introduce and verify a novel normalized weighted Geometric Bonferroni Mean, termed PNNWGBM tailored for Pythagorean neutrosophic sets. The integration of GBM into PNS setting provides a new reliable means for decision makers in the MCDM problems, offering a nuanced approach to capturing interactions among variables in decision-making processes. The proposed PNNWGBM method exhibited promising results, validated through the illustrative example of case study pertaining to Halal products supplier selection. Moreover, this study contributes to existing literature by broadening the applications of the Geometric Bonferroni Mean operator and assessing its efficacy in Pythagorean neutrosophic settings. The findings underscore the potential of PNNWGBM to be applied to existing MCDM methodologies such as TOPSIS, AHP, PROMETHEE and DEMATEL.

The advantages of this newly developed PNNWGBM aggregating operator compared to existing operators lie in Pythagorean neutrosophic sets, which provide a more flexible framework for representing uncertainty than traditional fuzzy sets. By incorporating Pythagorean neutrosophic sets, the new operator can better capture and process degrees of truth, indeterminacy, and falsity, leading to more accurate decision-making. Moreover, the combination of PNS methodology with the GBM aggregating operator enables effective aggregation of information while considering interdependencies and correlations between different criteria, resulting in a more comprehensive assessment. Lastly, this new operator can be applied to a wide range of multi-criteria decision-making (MCDM) problems in fields such as engineering, economics, and social sciences. Its ability to handle complex and uncertain information makes it suitable for real-world applications where traditional methods may fall short.

Future research for the GBM operator in Pythagorean neutrosophic set theory encompasses algorithm development, decision-making applications, uncertainty modelling, integration with other operators, real-world applications, extensions to fuzzy and neutrosophic sets, theoretical analysis, robustness and sensitivity analysis, machine learning integration, and comparative studies. These investigations aim to enhance computational efficiency, explore practical applications, understand theoretical properties, and assess performance in uncertain environments, ultimately advancing the utility and understanding of this operator. Overall, this study lays the groundwork for utilizing the Geometric Bonferroni Mean aggregation operator in Pythagorean neutrosophic decision-making.

The integration of the Geometric Bonferroni Mean (GBM) with Pythagorean neutrosophic set, despite its advantages, has limitations including increased computational complexity, sensitivity to parameter selection, and high data quality requirements. It can also suffer from reduced interpretability and scalability issues as the number of criteria and interactions increase. The subjectivity in defining neutrosophic membership functions introduces potential biases, and there are limited practical applications and case studies validating this approach. Additionally, validating the outcomes can be challenging due to the abstract and complex nature of the method.

Funding: We would like to acknowledge the Ministry of Higher Education Malaysia for their sponsorship of the Fundamental Research Grant Scheme (Project Code: FRGS/1/2023/STG06/UITM/02/5). This financial support has been crucial in advancing our research efforts, and we are grateful for their assistance.

Conflicts of Interest: The authors declare no conflict of interest.

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Received: June 27, 2024. Accepted: Oct 12, 2024