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Similarity -Based Pattern Recognition for Disease Symptom Extraction and Characterization

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Abstract. Neutrosophic Fuzzy Sets (NFS) expand upon classical fuzzy sets in the field of fuzzy set theory by including measures of truth, indeterminacy, and falsity. This paper thoroughly examines the creation and assessment of similarity measures for Single-Valued Neutrosophic Fuzzy Sets (SVNFS). The similarity measure is a crucial metric that quantifies the extent of similarity between two sets. It finds extensive application in various fields such as pattern recognition, medical diagnosis, and decision-making challenges. Nevertheless, the current similarity measures of Neutrosophic Fuzzy Sets(NFS) suffer from limited practicality and interpretation, and do not yield highly reliable outcomes. In order to tackle this issues,We provide a variety of new similarity measures, including the Hausdorff similarity measure, Membership-grade based similarity measure, and Trigonometric Hausdorff similarity measure specifically designed for Neutrosophic Fuzzy Sets(NFS). We conduct a comparison of their performance against existing measures. We verify the efficacy of these approaches by conducting thorough theoretical research and practical trials, showcasing their suitability in pattern recognition. The findings demonstrate substantial enhancements in precision and resilience, offering vital tools for academics and practitioners working with intricate and unpredictable data. The results of our research provide a foundation for future progress in the Neutrosophic Fuzzy Set theory and its practical use in several areas.

Keywords: Nuetrosophic Fuzzy Sets; Single Valued Nuetrosophic Fuzzy Sets; Hausdorff similarity measure between Nuetrosophic Fuzzy Sets; Enhanced cosine similarity measure between Nuetrosophic Fuzzy Sets

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1. **Introduction**

The fuzzy sets (FS) introduced by Zadeh [2] [6]have a wide variety of application in different fields of study. The fuzzy set handles uncertainty and ambiguity that the classical set cannot handle. To handle uncertain, incomplete, imprecise and inconsistent information,Smarandache [7]popularized the notion of a neutrosophic set (NS) from a philosophical perspective, It expands upon the principles of classical sets and fuzzy sets. In the neutrosophic set, a truth membership function $\mathcal{T}(x)$, an indeterminacy membership function $\mathcal{I}(x)$, and a falsity membership function $\mathcal{F}(x)$ are characterized independently, [3] where $\mathcal{T}(x)$, $\mathcal{I}(x)$, and $\mathcal{F}(x)$ are real standard or non-standard subsets of] -0,1 + [, such that $\mathcal{T} : X \rightarrow]-0,1+[, \mathcal{I} : X \rightarrow]-0,1+[, and \mathcal{F} : X \rightarrow]-0,1+[$. Thus, the sum of $\mathcal{T}(x), \mathcal{I}(x), and \mathcal{F}(x)$ satisfies the condition $-0 \leq sup\mathcal{T}(x) + sup\mathcal{I}(x) + sup\mathcal{F}(x) \leq 3+$ [7].The neutrosophic set possesses the advantage of representing imprecise and conflicting information, a capability that fuzzy sets (FSs) lack in addressing imprecise and inconsistent information. The neutrosophic set possesses philosophical applications; however, its use in engineering is problematic. Consequently, the specified range of $\mathcal{T}(x)$, $\mathcal{I}(x)$, and $\mathcal{F}(x)$ can be constrained to the real standard unit interval [0, 1] for practical engineering applications [8]. Consequently, a single-valued neutrosophic set (SVNS) [9] was established as a subfamily of a neutrosophic set. S. Das et al. created the concept of a neutrosophic fuzzy set (NFS) by integrating fuzzy set (FS) with neutrosophic set (NS), resulting in the emergence of novel concepts. Due to the challenges that neutrosophic fuzzy sets (NFS) encounter in addressing certain real-world issues stemming from the non-standard intervals of neutrosophic components, the concept of single-valued neutrosophic fuzzy sets (SVNFS) has been introduced [4]. Additionally, certain set-theoretic operations, numerical illustrations, and distance-based metrics for assessing the similarity of single-valued neutrosophic fuzzy sets (SVNFS) are proposed, and their properties are derived accordingly.

Similarity measurements are crucial in the application of fuzzy sets and neutrosophic sets. The research of similarity measurement is of greatest significance. A primary challenge in fuzzy set theory and neutrosophic set theory is the formulation of distance and similarity measurements. A significant amount of effort has been dedicated to the development of distance and similarity measures for fuzzy sets [10]. A degree of resemblance between sets of single-value neutrosophic is essential for addressing multi-criteria decision-making challenges [11].Numerous research studies have established numerous metrics of resemblance between collections of single-valued neutrosophic data. This work delineated Hausdorff similarity measurements of neutrosophic fuzzy sets, membership-grade-based similarity measures, and Enhanced Cosine similarity measures, and examined their application in medical diagnostics.

This paper is organized as follows: The introductory portion comprises essential concepts and findings necessary for our research. The subsequent sections contain the definitions of the Hausdorff similarity measure, the membership grade-based similarity measure, and the enhanced cosine similarity measure pertaining to neutrosophic fuzzy sets. In the subsequent section, we demonstrate that the newly established similarity metrics fulfill their essential properties. The clinical diagnosis utilizing similarity-based pattern recognition is elucidated in a subsequent section. Concluding remarks are presented in the final section.

2. Related Works

The existing similarity measures for fuzzy sets and intuitionistic fuzzy sets constituted the initial endeavors in formulating similarity measures for neutrosophic sets. Numerous research proposed modifications and expansions of these strategies to accommodate the additional dimension of indeterminacy in neutrosophic sets.Recent studies have introduced novel similarity measures tailored for Neutrosophic Sets (NS). Ye, J. [17] proposed an enhanced cosine similarity metric that incorporates the dimensions of truth, indeterminacy, and falsity in neutrosophic numbers. This metric provides a more comprehensive comparative tool for complex datasets. Combining various similarity measurements has proven to be an effective technique for leveraging the benefits of different metrics. Peng et al. [18] introduced a hybrid similarity measure that integrates the weighted Hamming distance with the Jaccard index. This integration enhances the precision and reliability of similarity assessments within the framework of Neutrosophic Set (NS).Entropy-based metrics have been examined to tackle the intrinsic uncertainty and indeterminacy within Neutrosophic Sets (NS). The similarity measure presented by Broumi et al. [19] is based on entropy and is used to assess the level of uncertainty between sets. This approach provides a fresh perspective on measuring similarity.Geometric methodologies have been modified for Neutrosophic Set(NS) in order to quantify similarity. Yang and Singh [20] proposed a novel geometric distance-based similarity measure that takes into account the spatial distribution of neutrosophic fuzzy elements. This measure has been found to be highly effective in many image processing applications. Significant progress has been made in the use of similarity measurements in medical diagnosis. In their study, Khan et al. [21] devised a similarity measure utilizing the overlap coefficient for Neutrosophic Set(NS). This enhancement resulted in improved precision in disease detection by effectively aligning patient symptoms with corresponding medical conditions.The integration of similarity measurements with machine learning algorithms has become a promising field. In a study conducted by Li et al. [22], it was shown that the combination of similarity measures for Neutrosophic Set(NS) with clustering algorithms improved the effectiveness of clustering algorithms in intricate datasets.

Fuzzy set theory is proficient in managing uncertainty, while neutrosophic set theory proves beneficial in handling indeterminate and inconsistent data. However, if the available knowledge is both unreliable and contradictory, then neither the Fuzzy Set (FS) nor the Neutrosophic Set (NS) can effectively manage it separately. A combination of Fuzzy Set (FS) and Neutrosophic Set (NS) is necessary to accomplish the objective. This particular challenge required the extension of the fuzzy set concept within the framework of NS. In fuzzy set theory, the membership grade is represented by a specific real integer [2]. A wide range of theorems has been formulated to address the problem of ambiguous membership grades, predominantly through diverse extensions of fuzzy sets. When the degree of membership is vague and inconsistent, it is crucial to delineate the membership grade utilizing neutrosophic components—namely, truth, indeterminacy, and falsity membership values—to accurately represent the uncertain and inconsistent information. Nonetheless, to the best of our knowledge, S. Das et al. [4] were the first to incorporate neutrosophic components with fuzzy membership grades to precisely represent an environment characterized by ambiguity and inconsistency. Furthermore, they introduced the notion of Euclidean distance and similarity metrics, along with their practical applications in decision-making [4]. Our work sought to provide more accurate similarity measures than those already available and to investigate their application in pattern recognition.

3. Preliminaries

This section establishes a solid framework for our study by introducing essential definitions and properties.

Definition 3.1. [2] Let X be the universal set, Then a **fuzzy set** F over X is defined by $F = \{(x, \mu_F(x))/x \in X$, where $\mu_F : X \to [0, 1]$ is called the membership function of X. The value $\mu_F(x)$ for each $x \in X$,reflects the degree to which x is a member of the fuzzy set F.

Definition 3.2. [3] Consider the universal set X and $x \in X$. A single-valued neutrosophic set (SVNS) N in X is distinguished by the function of truth membership T_N, the function of indeterminacy membership I_N and the function of falsity membership F_N. For each point x in X, $T_N(x), I_N(x), F_N(x) \in [0, 1]$. Thus, an SVNS N is denoted by $N = \{(x, T_N(x), I_N(x), F_N(x))|x \in X\}$

Definition 3.3. [4] Let X be the universal set. Then an NFS NF on X is defined by $NF = \{(x, \mu_{NF}(x), T_{NF}(x, \mu), I_{NF}(x, \mu), F_{NF}(x, \mu))/x \in X\}$ where each membership value is represented by a membership degree for truth, indeterminacy, and falsity, which are indicated as $T_{NF}(x, \mu), I_{NF}(x, \mu)$ and $F_{NF}(x, \mu)$. Furthermore, T_{NF}, I_{NF} and F_{NF} are existing standard or non-standard subsets of $]0^-, 1^+[$, That is,
$T_{NF} : E \to]0^-, 1^+[$,
$I_{NF} : E \to]0^-, 1^+[$,
$F_{NF} : E \to]0^-, 1^+[$.
where,$0^- \leq Sup(T_{NF}) + Sup(I_{NF}) + Sup(F_{NF}) \leq 3^+$.

Definition 3.4. [4] Let X be a universal set. then Single -Valued Neutrosophic Fuzzy Set SNF on X is defined by $SNF = \{(x, \mu_{SNF}(x), T_{SNF}(x, \mu), I_{SNF}(x, \mu), F_{SNF}(x, \mu))/x \in X\}$, where $T_{SNF}(x, \mu), I_{SNF}(x, \mu), F_{SNF}(x, \mu) \in [0, 1]$,and $0 \leq T_{SNF}(x, \mu)+I_{SNF}(x, \mu)+F_{SNF}(x, \mu) \leq 3$.

Definition 3.5. [1] Consider two sets $S = \{s_1, s_2, ...s_p\}$ and $T = \{t_1, t_2, ..., t_q\}$ The Hausdorff distance between S and T is denoted by $\mathcal{H}(S, T)$ and is defined as

$\mathcal{H}(S,T) = max\{h(S,T), h(T,S)\}$ where, $h(S,T) = \max_{s\in S}\min_{t\in T} d(s,t)$
Here, s and t are elements of S and T, respectively, and $d(s,t)$ is the metric between s and t.

Definition 3.6. [5] Consider two Nuetrosophic Sets K_1 and K_2 on X, then the Hausdorff distance between K_1 and K_2 is denoted by $d_{HNS}(K_1, K_2)$ and is defined as follows:
$d_{HNS}(K_1, K_2) = \frac{1}{n}\sum_{i=1}^{n}\max\{|T_{K_1}(x_i) - T_{K_2}(x_i)|, |I_{K_1}(x_i) - I_{K_2}(x_i)|, |F_{K_1}(x_i) - F_{K_2}(x_i)|\}$

Theorem 3.7. *[5] The Hausdorff Distance between K_1 and K_2 satisfies the following properties*

(1) $d_{HNS}(K_1, K_2) \geq 0$
(2) $d_{HNS}(K_1, K_2) = 0$ *if and only if* $K_1 = K_2$
(3) $d_{HNS}(K_1, K_2) = d_{HNS}(K_2, K_1)$
(4) *If* $K_1 \subseteq K_2 \subseteq K_3$ *then* $d_{HNS}(K_1, K_3) \geq d_{HNS}(K_1, K_2)$ *and* $d_{HNS}(K_1, K_3) \geq d_{HNS}(K_2, K_3)$

Definition 3.8. [16] Let $X = \{x_1, x_2, ..., x_n\}$ be the universe of the discourse. Consider$\kappa_1, \kappa_2 \in \mathcal{K}_{NFS}(X)$, where $\mathcal{K}_{NFS}(X)$ is the family of all neutrosophic fuzzy sets (NFSs) in X.
Let $\kappa_1 = \{(x_i, \mu_{\kappa_1}(x_i), \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}), \mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}), \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}))/x_i \in X\}$
$\kappa_2 = \{(x_i, \mu_{\kappa_2}(x_i), \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}), \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}), \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}))/x_i \in X\}$
The Hausdorff distance between κ_1 and κ_2 is denoted by $d_{HNFS}(\kappa_1, \kappa_2)$ and is defined as follows.
$d_{HNFS}(\kappa_1, \kappa_2) = \sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})|\}$

Definition 3.9. [16] Let $X = \{x_1, x_2, ..., x_n\}$ be the universe of the discourse. Consider$K_1, K_2 \in \mathcal{K}_{NFS}(X)$, where $\mathcal{K}_{NFS}(X)$ is the family of all neutrosophic fuzzy sets (NFS) in X.
Let $K_1 = \{(x_i, \mu_{K_1}(x_i), \mathcal{T}_{K_1}(x_i, \mu_{K_1}), \mathcal{I}_{K_1}(x_i, \mu_{K_1}), \mathcal{F}_{K_1}(x_i, \mu_{K_1}))/x_i \in X\}$
$K_2 = \{(x_i, \mu_{K_2}(x_i), \mathcal{T}_{K_2}(x_i, \mu_{K_2}), \mathcal{I}_{K_2}(x_i, \mu_{K_2}), \mathcal{F}_{K_2}(x_i, \mu_{K_2}))/x_i \in X\}$
The normalized Hausdorff distance between K_1 and K_2 is denoted by $d_{NHNFS}(K_1, K_2)$ and is defined as follows.
$d_{NHNFS}(K_1, K_2) = \frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{K_1}(x_i) - \mu_{K_2}(x_i)|, |\mathcal{T}_{K_1}(x_i, \mu_{K_1}) - \mathcal{T}_{K}(x_i, \mu_{K_2})|, |\mathcal{I}_{K_1}(x_i, \mu_{K_1}) - \mathcal{I}_{K_2}(x_i, \mu_{K_2})|, |\mathcal{F}_{K_1}(x_i, \mu_{K_1}) - \mathcal{F}_{K_2}(x_i, \mu_{K_2})|\}$

Definition 3.10. [15] A measure of similarity between two sets of neutrosophic with a single value is a mapping $\mathcal{S} : \mathcal{N}(X) \rightarrow [0, 1]$ that satisfies the following properties:

(1) $\mathcal{S}(K_1, K_2) \in [0, 1]$

(2) $\mathcal{S}(K_1, K_2) = 1 \iff K_1 = K_2$

(3) $\mathcal{S}(K_1, K_2) = \mathcal{S}(K_2, K_1)$

(4) $K_1 \subset K_2 \subset K_3 \to \mathcal{S}(K_1, K_3) \leq \mathcal{S}(K_1, K_2)$ and $\mathcal{S}(K_1, K_3) \leq \mathcal{S}(K_2, K_3)$

(1) – (4) are called Axioms of Similarity.

4. **Similarity Measure for Neutrosophic Fuzzy Sets**

A similarity measure can be used to examine differences between alternatives, making it an effective tool for multiple criteria judgment problems. Using the Hausdorff distance measure and membership grades, this section extends the similarity measure for Fuzzy Sets and Neutrosophic Sets to Neutrosophic Fuzzy Sets. In addition, a brand-new similarity metric based on the combination of Hausdorff distance and cosine similarity is suggested.

4.1. *Hausdorff Similarity Measure For Neutrosophic Fuzzy Sets*

Definition 4.1. Let $X = \{x_1, x_2, ..., x_n\}$ be the universe of discourse. Consider$\kappa_1, \kappa_2 \in \mathcal{K}_{NFS}(X)$, where $\mathcal{K}_{NFS}(X)$ is the group that includes all neutrosophic fuzzy sets (NFSs) in X.

Let $\kappa_1 = \{(x_i, \mu_{\kappa_1}(x_i), \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}), \mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}), \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}))/x_i \in X\}$

$\kappa_2 = \{(x_i, \mu_{\kappa_2}(x_i), \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}), \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}), \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}))/x_i \in X\}$

The Hausdorff Similarity Measure for κ_1 and κ_2 is denoted by $\mathcal{S}_H(\kappa_1, \kappa_2)$ and is defined as follows.

$\mathcal{S}_H(\kappa_1, \kappa_2) = 1 - \frac{1}{n}\sum_{i=1}^{n} \max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})|\}$

Theorem 4.2. *The Hausdorff Similarity Measure between two Neutrosophic Fuzzy Sets κ_1 and κ_2 has the following properties.*

(1) $\mathcal{S}_H(\kappa_1, \kappa_2) \in [0, 1]$

(2) $\mathcal{S}_H(\kappa_1, \kappa_2) = 1 \iff \kappa_1 = \kappa_2$

(3) $\mathcal{S}_H(\kappa_1, \kappa_2) = \mathcal{S}_H(\kappa_2, \kappa_1)$

(4) $\kappa_1 \subset \kappa_2 \subset \kappa_3 \to \mathcal{S}_H(\kappa_1, \kappa_3) \leq \mathcal{S}_H(\kappa_1, \kappa_2)$ *and* $\mathcal{S}_H(\kappa_1, \kappa_3) \leq \mathcal{S}_H(\kappa_2, \kappa_3)$

Proof. (1) We have $\mathcal{S}_H(\kappa_1, \kappa_2) = 1 - \frac{1}{n}\sum_{i=1}^{n} \max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})|\}$

From the definition itself it is clear that $\mathcal{S}_H(\kappa_1, \kappa_2) \in [0, 1]$

(2) $\mathcal{S}_H(\kappa_1, \kappa_2) = 1$ implies $\frac{1}{n}\sum_{i=1}^{n} \max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})|\} = 0$ that is,

$d_{NHNFS}(\kappa_1,\kappa_2)=0$. Then by the properties of Hausdorff Distance between NFSs we can conclude that $\kappa_1=\kappa_2$

(3) $\mathcal{S}_H(\kappa_1,\kappa_2) = 1 - \frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i,\mu_{\kappa_2})|, |\mathcal{I}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i,\mu_{\kappa_2})|, |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i,\mu_{\kappa_2})|\}$
$= 1 - \frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_2}(x_i) - \mu_{\kappa_1}(x_i)|, |\mathcal{T}_{\kappa_2}(x_i,\mu_{\kappa_2}) - \mathcal{T}_{\kappa_1}(x_i,\mu_{\kappa_1})|, |\mathcal{I}_{\kappa_2}(x_i,\mu_{\kappa_2}) - \mathcal{I}_{\kappa_1}(x_i,\mu_{\kappa_1})|, |\mathcal{F}_{\kappa_2}(x_i,\mu_{\kappa_2}) - \mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1})|\}$
$= \mathcal{S}_H(\kappa_2,\kappa_1)$

(4) Since $\kappa_1 \subseteq \kappa_2 \subseteq \kappa_3$ we have $\mu_{\kappa_1}(x_i) \le \mu_{\kappa_2}(x_i) \le \mu_{\kappa_3}(x_i)$,$\mathcal{T}_{\kappa_1}(x_i,\mu_{\kappa_1}) \le \mathcal{T}_{\kappa_2}(x_i,\mu_{\kappa_2}) \le \mathcal{T}_{\kappa_3}(x_i,C$,$\mathcal{I}_{\kappa_1}(x_i,\mu_{\kappa_1}) \le \mathcal{I}_{\kappa_2}(x_i,\mu_{\kappa_2}) \le \mathcal{I}_{\kappa_3}(x_i,\mu_{\kappa_3}), \mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) \ge \mathcal{F}_{\kappa_2}(x_i,\mu_{\kappa_2}) \ge \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})$.

First, we have to prove that $d_{HNFS}(\kappa_1,\kappa_3) \ge d_{HNFS}(\kappa_1,\kappa_2)$

Case-1

$|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \ge |\mathcal{T}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i,\mu_{\kappa_3})| \ge |\mathcal{I}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i,\mu_{\kappa_3})| \ge |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})|$

Then, $d_{HNFS}(\kappa_1,\kappa_3) = |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|$

(a) For all $x_i \in X$

$|\mathcal{T}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i,\mu_{\kappa_2})| \le |\mathcal{T}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|$
$|\mathcal{I}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i,\mu_{\kappa_2})| \le |\mathcal{I}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|$
$|\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i,\mu_{\kappa_2})| \le |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|$

(b) For all $x_i \in X$

$|\mathcal{T}_{\kappa_2}(x_i,\mu_{\kappa_2}) - \mathcal{T}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mathcal{T}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|$
$|\mathcal{I}_{\kappa_2}(x_i,\mu_{\kappa_2}) - \mathcal{I}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mathcal{I}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|$
$|\mathcal{F}_{\kappa_2}(x_i,\mu_{\kappa_2}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|$

(c) For all $x_i \in X$

$|\mu_{\kappa_1}(x_i)-\mu_{\kappa_2}(x_i)| \le |\mu_{\kappa_1}(x_i)-\mu_{\kappa_3}(x_i)|$ and $|\mu_{\kappa_2}(x_i)-\mu_{\kappa_3}(x_i)| \le |\mu_{\kappa_1}(x_i)-\mu_{\kappa_3}(x_i)|$

We can deduce from (1),(2) and (3) that, for all $x_i \in X$

$\frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i,\mu_{\kappa_2})|, |\mathcal{I}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i,\mu_{\kappa_2})|, |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i,\mu_{\kappa_2})|\} \le \frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i,\mu_{\kappa_3})|, |\mathcal{I}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i,\mu_{\kappa_3})|, |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})|\}$
$\rightarrow d_{HNFS}(\kappa_1,\kappa_2) \le d_{HNFS}(\kappa_1,\kappa_3)$

and

$\frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_2}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_2}(x_i,\mu_{\kappa_2}) - \mathcal{T}_{\kappa_3}(x_i,\mu_{\kappa_3})|, |\mathcal{I}_{\kappa_2}(x_i,\mu_{\kappa_2}) - \mathcal{I}_{\kappa_3}(x_i,\mu_{\kappa_3})|, |\mathcal{F}_{\kappa_2}(x_i,\mu_{\kappa_2}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})|\} \le \frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i,\mu_{\kappa_3})|, |\mathcal{I}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i,\mu_{\kappa_3})|, |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})|\}$

$\rightarrow d_{HNFS}(\kappa_2,\kappa_3) \le d_{HNFS}(\kappa_1,\kappa_3)$

Case-2

$|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \le |\mathcal{T}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mathcal{I}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})|$

Then, $d_{HNFS}(\kappa_1,\kappa_3) = |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})|$

(a) For all $x_i \in X$

$|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)| \le |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \le |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})|$

$|\mathcal{T}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i,\mu_{\kappa_2})| \le |\mathcal{T}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})|$

$|\mathcal{I}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i,\mu_{\kappa_2})| \le |\mathcal{I}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})|$

(b) For all $x_i \in X$

$|\mu_{\kappa_2}(x_i) - \mu_{\kappa_3}(x_i)| \le |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \le |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})|$

$|\mathcal{T}_{\kappa_2}(x_i,\mu_{\kappa_2}) - \mathcal{T}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mathcal{T}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})|$

$|\mathcal{I}_{\kappa_2}(x_i,\mu_{\kappa_2}) - \mathcal{I}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mathcal{I}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})|$

(c) For all $x_i \in X$

$|\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i,\mu_{\kappa_2})| \le |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})|$

and $|\mathcal{F}_{\kappa_2}(x_i,\mu_{\kappa_2}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})|$

We can deduce from (1),(2) and (3) that, for all $x_i \in X$

$\frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i,\mu_{\kappa_2})|, |\mathcal{I}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i,\mu_{\kappa_2})|, |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i,\mu_{\kappa_2})|\} \le \frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i,\mu_{\kappa_3})|, |\mathcal{I}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i,\mu_{\kappa_3})|, |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})|\}$

$\rightarrow d_{HNFS}(\kappa_1,\kappa_2) \le d_{HNFS}(\kappa_1,\kappa_3)$ and

$\frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_2}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_2}(x_i,\mu_{\kappa_2}) - \mathcal{T}_{\kappa_3}(x_i,\mu_{\kappa_3})|, |\mathcal{I}_{\kappa_2}(x_i,\mu_{\kappa_2}) - \mathcal{I}_{\kappa_3}(x_i,\mu_{\kappa_3})|, |\mathcal{F}_{\kappa_2}(x_i,\mu_{\kappa_2}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})|\} \le \frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i,\mu_{\kappa_3})|, |\mathcal{I}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i,\mu_{\kappa_3})|, |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})|\}$

$\rightarrow d_{HNFS}(\kappa_2,\kappa_3) \le d_{HNFS}(\kappa_1,\kappa_3)$

Case-3

$|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \le |\mathcal{T}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mathcal{F}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i,\mu_{\kappa_3})| \le |\mathcal{I}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i,\mu_{\kappa_3})|$

Then, $d_{HNFS}(\kappa_1,\kappa_3) = |\mathcal{I}_{\kappa_1}(x_i,\mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i,\mu_{\kappa_3})|$

(a) For all $x_i \in X |\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)| \leq |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|$
$|\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|$
$|\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

(b) For all $x_i \in X$
$|\mu_{\kappa_2}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|$
$|\mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|$
$|\mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

(c) For all $x_i \in X$
$|\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|$
and $|\mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

We can deduce from (1),(2) and (3) that, for all $x_i \in X$

$\frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})|\} \leq \frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|\}$
$\to d_{HNFS}(\kappa_1, \kappa_2) \leq d_{HNFS}(\kappa_1, \kappa_3)$

and

$\frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_2}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|\} \leq \frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|\}$
$\to d_{HNFS}(\kappa_2, \kappa_3) \leq d_{HNFS}(\kappa_1, \kappa_3)$

Case-4

$|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

Then, $d_{HNFS}(\kappa_1, \kappa_3) = |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

(a) For all $x_i \in X |\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)| \leq |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$
$|\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$
$|\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

(b) For all $x_i \in X$ $|\mu_{\kappa_2}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

$|\mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

$|\mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

(c) For all $x_i \in X$

$|\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

and $|\mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})| \leq |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|$

We can deduce from (1),(2) and (3) that, for all $x_i \in X$

$\frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_2}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})|\} \leq \frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|\}$

$\rightarrow d_{HNFS}(\kappa_1, \kappa_2) \leq d_{HNFS}(\kappa_1, \kappa_3)$

and

$\frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_2}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|\} \leq \frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_{\kappa_1}(x_i) - \mu_{\kappa_3}(x_i)|, |\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})|, |\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) - \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})|\}$

$\rightarrow d_{HNFS}(\kappa_2, \kappa_3) \leq d_{HNFS}(\kappa_1, \kappa_3)$

Thus, in all possible cases, we have proved $d_{HNFS}(\kappa_1, \kappa_2) \leq d_{HNFS}(\kappa_1, \kappa_3)$ and $d_{HNFS}(\kappa_2, \kappa_3) \leq d_{HNFS}(\kappa_1, \kappa_3)$. Therefore, $1 - d_{HNFS}(\kappa_1, \kappa_2) \geq 1 - d_{HNFS}(\kappa_1, \kappa_3)$ and $1 - d_{HNFS}(\kappa_2, \kappa_3) \geq 1 - d_{HNFS}(\kappa_1, \kappa_3)$. $\rightarrow \mathcal{S}(\kappa_1, \kappa_3) \leq \mathcal{S}(\kappa_1, \kappa_2)$ and $\mathcal{S}(\kappa_1, \kappa_3) \leq \mathcal{S}(\kappa_2, \kappa_3)$ Therefore, the Hausdorff similarity measure, as defined above, is a measure of similarity between two neutrosophic fuzzy sets. $\square$

4.2. *Membership Grade-based Similarity Measure for Neutrosophic Fuzzy Sets*

We are trying to define the similarity measure between two neptrosophic fuzzy sets based on the membership grades in this section.

Definition 4.3. Let $X = \{x_1, x_2, ..., x_n\}$ be the universe of discourse. Consider$\kappa_1, \kappa_2 \in \mathcal{K}_{NFS}(X)$, where $\mathcal{K}_{NFS}(X)$ is the group that includes all neutrosophic fuzzy sets (NFSs) in

X.

Let $\kappa_1 = \{(x_i, \mu_{\kappa_1}(x_i), \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}), \mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}), \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}))/x_i \in X\}$
$\kappa_2 = \{(x_i, \mu_{\kappa_2}(x_i), \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}), \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}), \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}))/x_i \in X\}$

The Similarity Measure based on the membership grades for κ_1 and κ_2 is denoted by $\mathcal{S}_M(\kappa_1, \kappa_2)$ and is defined as follows.

$$\mathcal{S}_M(\kappa_1, \kappa_2) = \frac{\sum_{i=1}^n min(\mu_{\kappa_1}(x_i),\mu_{\kappa_2}(x_i))+min(\mathcal{T}_{\kappa_1}(x_i),\mathcal{T}_{\kappa_2}(x_i))+min(\mathcal{I}_{\kappa_1}(x_i),\mathcal{I}_{\kappa_2}(x_i))+min(\mathcal{F}_{\kappa_1}(x_i),\mathcal{F}_{\kappa_2}(x_i))}{\sum_{i=1}^n max(\mu_{\kappa_1}(x_i),\mu_{\kappa_2}(x_i))+max(\mathcal{T}_{\kappa_1}(x_i),\mathcal{T}_{\kappa_2}(x_i))+max(\mathcal{I}_{\kappa_1}(x_i),\mathcal{I}_{\kappa_2}(x_i))+max(\mathcal{F}_{\kappa_1}(x_i),\mathcal{F}_{\kappa_2}(x_i))}$$

Theorem 4.4. *The similarity measure based on membership grades between two neutrosophic fuzzy sets κ_1 and κ_2 has the following properties.*

(1) $\mathcal{S}_M(\kappa_1, \kappa_2) \in [0, 1]$
(2) $\mathcal{S}_M(\kappa_1, \kappa_2) = 1 \iff \kappa_1 = \kappa_2$
(3) $\mathcal{S}_M(\kappa_1, \kappa_2) = \mathcal{S}_M(\kappa_2, \kappa_1)$
(4) $\kappa_1 \subset \kappa_2 \subset \kappa_3 \to \mathcal{S}_M(\kappa_1, \kappa_3) \leq \mathcal{S}_M(\kappa_1, \kappa_2)$ *and* $\mathcal{S}_M(\kappa_1, \kappa_3) \leq \mathcal{S}_M(\kappa_2, \kappa_3)$

Proof. Properties (1) and (3) hold directly from the definition itself. We have to prove properties (2) and (4)

2. If $\kappa_1 = \kappa_2$ then by definition itself it is clear that $\mathcal{S}_M(\kappa_1, \kappa_2) = 1$

Conversely, let $\mathcal{S}_M(\kappa_1, \kappa_2) = 1$

$$\to \frac{\sum_{i=1}^n min(\mu_{\kappa_1}(x_i),\mu_{\kappa_2}(x_i))+min(\mathcal{T}_{\kappa_1}(x_i),\mathcal{T}_{\kappa_2}(x_i))+min(\mathcal{I}_{\kappa_1}(x_i),\mathcal{I}_{\kappa_2}(x_i))+min(\mathcal{F}_{\kappa_1}(x_i),\mathcal{F}_{\kappa_2}(x_i))}{\sum_{i=1}^n max(\mu_{\kappa_1}(x_i),\mu_{\kappa_2}(x_i))+max(\mathcal{T}_{\kappa_1}(x_i),\mathcal{T}_{\kappa_2}(x_i))+max(\mathcal{I}_{\kappa_1}(x_i),\mathcal{I}_{\kappa_2}(x_i))+max(\mathcal{F}_{\kappa_1}(x_i),\mathcal{F}_{\kappa_2}(x_i))} = 1$$

$\to \sum_{i=1}^n min(\mu_{\kappa_1}(x_i), \mu_{\kappa_2}(x_i)) + min(\mathcal{T}_{\kappa_1}(x_i), \mathcal{T}_{\kappa_2}(x_i)) + min(\mathcal{I}_{\kappa_1}(x_i), \mathcal{I}_{\kappa_2}(x_i)) + min(\mathcal{F}_{\kappa_1}(x_i), \mathcal{F}_{\kappa_2}(x_i)) =$
$\sum_{i=1}^n max(\mu_{\kappa_1}(x_i), \mu_{\kappa_2}(x_i)) + max(\mathcal{T}_{\kappa_1}(x_i), \mathcal{T}_{\kappa_2}(x_i)) + max(\mathcal{I}_{\kappa_1}(x_i), \mathcal{I}_{\kappa_2}(x_i)) + max(\mathcal{F}_{\kappa_1}(x_i), \mathcal{F}_{\kappa_2}(x_i))$

$\to \sum_{i=1}^n [min(\mu_{\kappa_1}(x_i), \mu_{\kappa_2}(x_i)) - max(\mu_{\kappa_1}(x_i), \mu_{\kappa_2}(x_i))] + [min(\mathcal{T}_{\kappa_1}(x_i), \mathcal{T}_{\kappa_2}(x_i)) - max(\mathcal{T}_{\kappa_1}(x_i), \mathcal{T}_{\kappa_2}(x_i))] + [min(\mathcal{I}_{\kappa_1}(x_i), \mathcal{I}_{\kappa_2}(x_i)) - max(\mathcal{I}_{\kappa_1}(x_i), \mathcal{I}_{\kappa_2}(x_i))] + [min(\mathcal{F}_{\kappa_1}(x_i), \mathcal{F}_{\kappa_2}(x_i)) - max(\mathcal{F}_{\kappa_1}(x_i), \mathcal{F}_{\kappa_2}(x_i))] = 0$

$\to [min(\mu_{\kappa_1}(x_i), \mu_{\kappa_2}(x_i)) - max(\mu_{\kappa_1}(x_i), \mu_{\kappa_2}(x_i))] = 0$
$[min(\mathcal{T}_{\kappa_1}(x_i), \mathcal{T}_{\kappa_2}(x_i)) - max(\mathcal{T}_{\kappa_1}(x_i), \mathcal{T}_{\kappa_2}(x_i))] = 0$
$[min(\mathcal{I}_{\kappa_1}(x_i), \mathcal{I}_{\kappa_2}(x_i)) - max(\mathcal{I}_{\kappa_1}(x_i), \mathcal{I}_{\kappa_2}(x_i))] = 0$
$[min(\mathcal{F}_{\kappa_1}(x_i), \mathcal{F}_{\kappa_2}(x_i)) - max(\mathcal{F}_{\kappa_1}(x_i), \mathcal{F}_{\kappa_2}(x_i))] = 0$, for each x.Therefore, $\mu_{\kappa_1}(x) = \mu_{\kappa_2}(x)$,$\mathcal{T}_{\kappa_1}(x) = \mathcal{T}_{\kappa_2}(x)$,$\mathcal{I}_{\kappa_1}(x) = \mathcal{I}_{\kappa_2}(x)$, $\mathcal{F}_{\kappa_1}(x) = \mathcal{F}_{\kappa_2}(x)$ for each x.

$\rightarrow \kappa_1 = \kappa_2$

4. Let $\kappa_1 \subset \kappa_2 \subset \kappa_3$, Then

$\mu_{\kappa_1}(x_i) \leq \mu_{\kappa_2}(x_i) \leq \mu_{\kappa_3}(x_i)$,$\mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) \leq \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) \leq \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3})$,$\mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) \leq \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) \leq \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3})$,

$\mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) \geq \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}) \geq \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})$.

Therefore, we have

$\mu_{\kappa_1}(x_i) + \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{I}_{\kappa_1}(x_i), \mu_{\kappa_1}) + \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}) \geq \mu_{\kappa_1}(x_i) + \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})$

$\mu_{\kappa_2}(x_i) + \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) + \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) + \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) \leq \mu_{\kappa_3}(x_i) + \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1})$

$\mathcal{S}_M(\kappa_1, \kappa_2) = \frac{\sum_{i=1}^{n} \mu_{\kappa_1(x_i)} + \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})}{\sum_{i=1}^{n} \mu_{\kappa_2(x_i)} + \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) + \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) + \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1})}$
$\geq \frac{\sum_{i=1}^{n} \mu_{\kappa_1(x_i)} + \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})}{\sum_{i=1}^{n} \mu_{\kappa_3(x_i)} + \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1})} = \mathcal{S}_M(\kappa_1, \kappa_3)$

Similarly, we have

$\mu_{\kappa_2}(x_i) + \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) + \mathcal{I}_{\kappa_2}(x_i), \mu_{\kappa_2}) + \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3}) \geq \mu_{\kappa_1}(x_i) + \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})$

$\mu_{\kappa_3}(x_i) + \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{I}_{\kappa_3}(x_i), \mu_{\kappa_3}) + \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}) \geq \mu_{\kappa_3}(x_i) + \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})$

$\mathcal{S}_M(\kappa_2, \kappa_3) = \frac{\sum_{i=1}^{n} \mu_{\kappa_2(x_i)} + \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}) + \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}) + \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})}{\sum_{i=1}^{n} \mu_{\kappa_3(x_i)} + \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2})}$
$\geq \frac{\sum_{i=1}^{n} \mu_{\kappa_1(x_i)} + \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}) + \mathcal{F}_{\kappa_3}(x_i, \mu_{\kappa_3})}{\sum_{i=1}^{n} \mu_{\kappa_3(x_i)} + \mathcal{T}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{I}_{\kappa_3}(x_i, \mu_{\kappa_3}) + \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1})} = \mathcal{S}_M(\kappa_1, \kappa_3)$

Therefore, the membership grade-based similarity measure, as defined above, is a measure of similarity between two neutrosophic fuzzy sets. □

5. **Enhanced Cosine Similarity Measure for Single Valued Neutrosophic Fuzzy Sets**

5.1. *Cosine Similarity*

Definition 5.1. Cosine similarity is a basic angle-based measure of similarity. Here, we are comparing two n-dimensional vectors for the similarity between them, expressed as a cosine. Assesses how similar things are. The comparison between two vectors is based only on their direction and omits the effect of their distance from each other. Consider the vectors $P = (p_1, p_2, ..., p_n)$ and $Q = (q_1, q_2, ..., q_n)$, the cosine similarity [12] is defined as

$$\cos\theta = \frac{\sum_{i=1}^{n} x_i y_i}{\sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}}$$

The measure of cosine similarity between two fuzzy sets κ_1 and κ_2 [13] is defined as follows.

$$C_F(\kappa_1.\kappa_2) = \frac{\sum_{i=1}^{n} \mu_{\kappa_1}(x_i)\mu_{\kappa_2}(x_i)}{\sqrt{\sum_{i=1}^{n}(\mu_{\kappa_1}(x_i))^2}\sqrt{\sum_{i=1}^{n}(\mu_{\kappa_2}(x_i))^2}}$$

The cosine similarity measure among two single valued neutrosophic sets κ_1 and κ_2 [14] is defined as follows

$$C_{SVNS}(\kappa_1, \kappa_2) = \frac{\sum_{i=1}^{n} \mathcal{T}_{\kappa_1}(x_i)\mathcal{T}_{\kappa_2}(x_i)+\mathcal{I}_{\kappa_1}(x_i)\mathcal{I}_{\kappa_2}(x_i)+\mathcal{F}_{\kappa_1}(x_i)\mathcal{F}_{\kappa_2}(x_i)}{\sqrt{\sum_{i=1}^{n} \mathcal{T}_{\kappa_1}(x_i)^2+\mathcal{I}_{\kappa_1}(x_i)^2+\mathcal{F}_{\kappa_1}(x_i)^2}\sqrt{\sum_{i=1}^{n} \mathcal{T}_{\kappa_2}(x_i)^2+\mathcal{I}_{\kappa_2}(x_i)^2+\mathcal{F}_{\kappa_2}(x_i)^2}}$$

Based on the above definitions, we can define the measure of cosine similarity between two single-valued neutrosophic fuzzy sets κ_1 and κ_2 as follows.

$$C_{SVNFS}(\kappa_1, \kappa_2) = \frac{1}{n}\sum_{i=1}^{n} \frac{\mu_{\kappa_1}(x_i)\mu_{\kappa_2}(x_i)+\mathcal{T}_{\kappa_1}(x_i,\mu)\mathcal{T}_{\kappa_2}(x_i,\mu)+\mathcal{I}_{\kappa_1}(x_i,\mu)\mathcal{I}_{\kappa_2}(x_i,\mu)+\mathcal{F}_{\kappa_1}(x_i,\mu)\mathcal{F}_{\kappa_2}(x_i,\mu)}{\sqrt{\mu_{\kappa_1}(x_i)^2+\mathcal{T}_{\kappa_1}(x_i,\mu)^2+\mathcal{I}_{\kappa_1}(x_i,\mu)^2+\mathcal{F}_{\kappa_1}(x_i,\mu)^2}\sqrt{\mu_{\kappa_2}(x_i)^2+\mathcal{T}_{\kappa_2}(x_i,\mu)^2+\mathcal{I}_{\kappa_2}(x_i,\mu)^2+\mathcal{F}_{\kappa_2}(x_i,\mu)^2}}$$

But the above-defined similarity measure sometimes fails to satisfy the basic properties of a similarity measure, that is,

If 1. $\mathcal{T}_{\kappa_1}(x_i, \mu) = 2\mathcal{T}_{\kappa_2}(x_i, \mu), \mathcal{I}_{\kappa_1}(x_i, \mu) = 2\mathcal{I}_{\kappa_2}(x_i, \mu), \mathcal{F}_{\kappa_1}(x_i, \mu) = 2\mathcal{F}_{\kappa_2}(x_i, \mu)$
or
2.$\mathcal{T}_{\kappa_1}(x_i, \mu) = \mathcal{T}_{\kappa_2}(x_i, \mu), 2\mathcal{I}_{\kappa_1}(x_i, \mu) = \mathcal{I}_{\kappa_2}(x_i, \mu), 2\mathcal{F}_{\kappa_1}(x_i, \mu) = \mathcal{F}_{\kappa_2}(x_i, \mu)$

That is, when $\kappa_1 \neq \kappa_2$, $C_{SVNFS}(\kappa_1, \kappa_2) = 1$, which means that the cosine similarity measure defined above does not satisfy the necessary condition for a similarity measure. Therefore, based on the improved cosine similarity measure proposed by [15] we are going to define the Enhanced Cosine Similarity Measure for Neutrosophic Fuzzy Sets(EC_{SVNFS}) as follows.

5.2. *Enhanced Cosine Similarity Measure between Neutrosophic Fuzzy Sets*

Definition 5.2. Let $X = \{x_1, x_2, ..., x_n\}$ be the universe of discourse. Consider$\kappa_1, \kappa_2 \in \mathcal{K}_{NFS}(X)$, where $\mathcal{K}_{NFS}(X)$ is the group that includes all neutrosophic fuzzy sets (NFSs) in X.
Let $\kappa_1 = \{(x_i, \mu_{\kappa_1}(x_i), \mathcal{T}_{\kappa_1}(x_i, \mu_{\kappa_1}), \mathcal{I}_{\kappa_1}(x_i, \mu_{\kappa_1}), \mathcal{F}_{\kappa_1}(x_i, \mu_{\kappa_1}))/x_i \in X\}$
$\kappa_2 = \{(x_i, \mu_{\kappa_2}(x_i), \mathcal{T}_{\kappa_2}(x_i, \mu_{\kappa_2}), \mathcal{I}_{\kappa_2}(x_i, \mu_{\kappa_2}), \mathcal{F}_{\kappa_2}(x_i, \mu_{\kappa_2}))/x_i \in X\}$
Then, $EC_{SVNFS}(\kappa_1, \kappa_2) = \frac{1}{2}[C_{SVNFS}(\kappa_1, \kappa_2) + 1 - d_{NHNFS}(\kappa_1, \kappa_2)]$

Theorem 5.3. *The Enhanced Cosine Similarity Measure between two Neutrosophic Fuzzy Sets κ_1 and κ_2 has the following properties.*

(1) $EC_{SVNFS}(\kappa_1, \kappa_2) \in [0, 1]$

(2) $EC_{SVNFS}(\kappa_1, \kappa_2) = 1 \iff \kappa_1 = \kappa_2$

(3) $EC_{SVNFS}(\kappa_1, \kappa_2) = EC_{SVNFS}(\kappa_2, \kappa_1)$

Proof. (1) Since $d_{NHNFS}(A, B)$ is the normalized Hausdorff distance between (A, B), we have $0 \leq d_{NHNFS}(A, B) \leq 1$ Also by [15] we have $0 \leq C_{SNFS}(A, B) \leq 1$, Therefore $0 \leq \frac{1}{2}[C_{SNFS}(A, B) + 1 - d_{NHNFS}(A, B)] \leq 1$, That is $0 \leq EC_{SNFS} \leq 1$.

(2) Let $EC_{SNFS}(A, B) = 1$
$\to C_{SNFS}(A, B) + 1 - d_{NHNFS}(A, B) = 2$
$\to C_{SNFS}(A, B) = 2 - 1 + d_{NHNFS}(A, B) = 1 + d_{NHNFS}(A, B)$
According to [15] $0 \leq C_{SNFS}(A, B) \leq 1$. Since $d_{NHNFS}(A, B)$ is the normalized Hausdorff distance between A, B we have $0 \leq d_{NHNFS}(A, B) \leq 1$ Therefore, $C_{SNFS}(A, B) = 0 + 1 = 1$ and $d_{NHNFS}(A, B) = 0$
When $C_{SNFS}(A, B) = 1$
$\to \mu_A(x_i) = K\mu_B(x_i)$
$\to T_A(x_i, \mu) = KT_B(x_i, \mu)$
$\to I_A(x_i, \mu) = KI_B(x_i, \mu)$
$\to F_A(x_i, \mu) = KF_B(x_i, \mu)$, where K is any constant.
Also $d_{NHNFS}(A, B) = 0 \to A = B$
In contrast, let $A = B$ then $d_{NHNFS}(A, B) = 0$ and $C_{SNFS}(A, B) = 1$
$\to EC_{SNFS}(A, B) = 1$

(3) $EC_{SNFS}(A, B) = \frac{1}{2}[C_{SNFS}(A, B) + 1 - d_{NHNFS}(A, B)] = \frac{1}{2}[C_{SNFS}(B, A) + 1 - d_{NHNFS}(B, A)] = EC_{SNFS}(B, A)$ □

6. Clinical Diagnosis using Similarity-Based Pattern Recognition

An individual afflicted by an illness will exhibit a range of symptoms, including fever, cough, weariness, sore throat, headache, and sneezing. Furthermore, each viral infection will exhibit various symptoms. Dengue, typhoid, and chikungunya might induce illness, chronic weariness, pharyngitis, coughing, and more symptoms. The symptoms of coronavirus infection encompass fever, cough, nasal congestion, and chronic weariness, among others. The primary symptoms of each disease can be identified by similarity-based pattern recognition involving neutrosophic fuzzy sets, and from the maximum similarity measure or core symptoms, we can ascertain the patient's disease kind. A medical diagnosis is generally established primarily on enduring symptoms, while fleeting symptoms do not yield conclusive insights. The ambiguity

	Temperature	Headache	Sore Throat	Diarrhoea	Shortness of Breath
Dengue Fever	(0.4,0.6,0.2,0.1)	(0.8,0.9,0.1,0.1)	(0.2,0.6.0.3,0.1)	(0.7,0.8,0.0,0.1)	(0.2,0.8,0.3,0.2)
Covid-19	(0.7,0.6,0.2,0.2)	(0.5,0.6,0.2,0.2)	(0.8,0.7.0.1,0.1)	(0.5,0.3,0.4,0.2)	(0.8,0.6,0.3,0.2)
Viral Fever	(0.8,0.7,0.2,0.2)	(0.6,0.6,0.1,0.1)	(0.7,0.5.0.3,0.3)	(0.4,0.2,0.3,0.4)	(0.2,0.3,0.4,0.3)
Rat Fever	(0.8,0.6,0.2,0.2)	(0.7,0.8,0.2,0.1)	(0.2,0.7.0.1,0.2)	(0.3,0.2,0.4,0.5)	(0.2,0.3,0.4,0.4)
Chikungunya	(0.9,0.6,0.5,0.1)	(0.6,0.9,0.1,0.0)	(0.2,0.6.0.1,0.3)	(0.3,0.3,0.5,0.4)	(0.3,0.6,0.2,0.2)

Table 1. Ideal values for symptoms of each disease

and uncertainties in the available medical data must be considered when addressing persistent symptoms; however, employing Neutrosophic Fuzzy Sets (NFS) to characterize this data might significantly mitigate these distractions. The similarity metric among neutrosophic fuzzy sets enables the identification of the principal symptoms of each ailment and the deduction of the patient's condition.

For our case study, let us select five patients $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5\}$, each patient having multiple symptoms such as
$\mathbb{S}$={$\mathbb{S}_1$(ProlongedTemperature),$\mathbb{S}_2$(ProlongedHeadache),$\mathbb{S}_3$(Prolonged SoreThroat),$\mathbb{S}_4$(ProlongedDiarrhoea),$\mathbb{S}_5$(Prolonged Shortness of Breath)}.
Now employing the medical data represented by neutrsophic fuzzy sets, we have to deduce the type of disease that affects the person such as
$\mathcal{D}$={$\mathcal{D}_1$(Dengue), $\mathcal{D}_2$(Covid-19), $\mathcal{D}_3$(Viral Fever),$\mathcal{D}_4$(Rat Fever), $\mathcal{D}_5$(Chikungunya)}

Consider the following NFS to be the patient's representation of all symptoms: $\mathcal{P}_1$=$\{(\mathbb{S}_1, 0.8, 0.7, 0.2, 0.1)$,
$(\mathbb{S}_2, 0.7, 0.6, 0.2, 0.2)$,$(\mathbb{S}_3, 0.5, 0.6.0.1, 0.1)$,$(\mathbb{S}_4, 0.2, 0.2, 0.0, 0.1)$,$(\mathbb{S}_5, 0.7, 0.3, 0.6, 0.4)\}$

We can establish an ideal value for each symptom associated with a given disease when we analyze the medical records of numerous patients who have that disease.**Table 1** provides an ideal value for each symptom associated with each disease.
It is our intention to place pattern$\mathcal{P}_1$ in one of the classes $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, or \mathcal{D}_5$.Accurate diagnosis $\mathcal{D}_k$ for patient $\mathcal{P}_1$ is derived according to $k = argMax_{1\leq i\leq 5} S.M(\mathcal{P}_1, \mathcal{D}_i)$ where S.M stands for similarity measure. The similarity measures between patient $\mathcal{P}_1$ and disease $\mathcal{D}_i$ are given in **Table 2**.
The maximum similarity measure obtained by employing three different similarity measures according to the given neutrophic fuzzy data is shown in bold in **Table2**. From **Table 2** and **FIGURE-1** we can conclude that patient $\mathcal{P}_1$ is affected by viral fever.

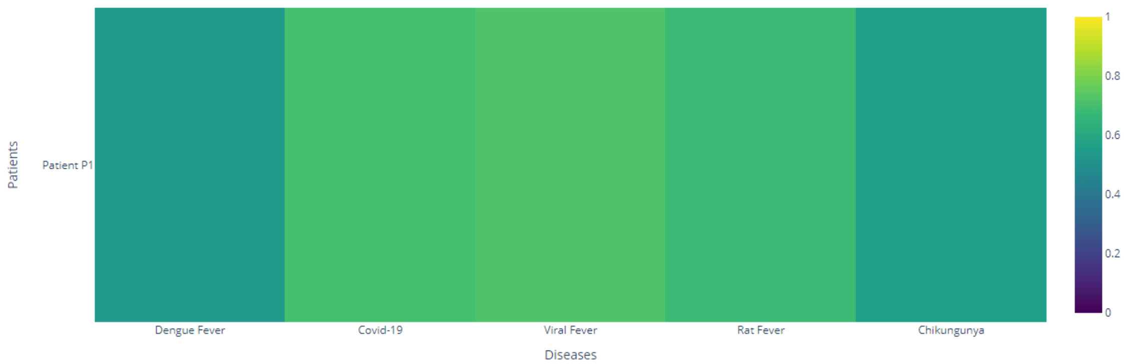

FIGURE 1. Heat map showing patient-disease mapping

	$(\mathcal{P}_1, \mathcal{D}_1)$	$(\mathcal{P}_1, \mathcal{D}_2)$	$(\mathcal{P}_1, \mathcal{D}_3)$	$(\mathcal{P}_1, \mathcal{D}_4)$	$(\mathcal{P}_1, \mathcal{D}_5)$
$S_H(\mathcal{P}_1, \mathcal{D}_i)$	0.58	0.74	**0.76**	0.70	0.64
$S_M(\mathcal{P}_1, \mathcal{D}_i)$	0.56	0.69	**0.70**	0.68	0.58
$EC_{SVNFS}(\mathcal{P}_1, \mathcal{D}_i)$	0.54	0.70	**0.72**	0.68	0.56
$S_E(\mathcal{P}_1, \mathcal{D}_i)$	0.8648	0.9168	**0.9250**	0.9121	0.8833

TABLE 2. Similarity measures between patient $\mathcal{P}_1$ and disease $\mathcal{D}_i$

7. Comparative Evaluations with Existing Similarity Measures

The initial similarity measure established for the Neutrosophic Fuzzy Set was the Euclidean similarity measure (S_E), devised by S.Das et al. [4].In this section, we compare and evaluate recent definitions of similarity measures with the traditional Euclidean similarity measure. In **TABLE-2**, we have compiled a summary of various similarity measures between the patient P_1 and each disease D_i. The heat map displayed in **FIGURE-3** will enable us to assess the efficacy of each similarity measure in the given modeling problem.

The recently introduced Hausdorff similarity measure for Neutrosophic Fuzzy Sets (NFS) presents notable benefits in managing the intricacy and multidimensional characteristics of Neutrosophic Fuzzy Sets (NFS), offering a more resilient and all-encompassing evaluation of similarity. However, the Euclidean similarity measure is favored because of its simplicity, computing efficiency, and ability to directly assess similarity between individual points. The selection among the four commonalities is based on the particular demands and limitations of the application. The comparative diagram presented in **FIGURE-2** will provide a comprehensive understanding of the benefits and constraints associated with each similarity.

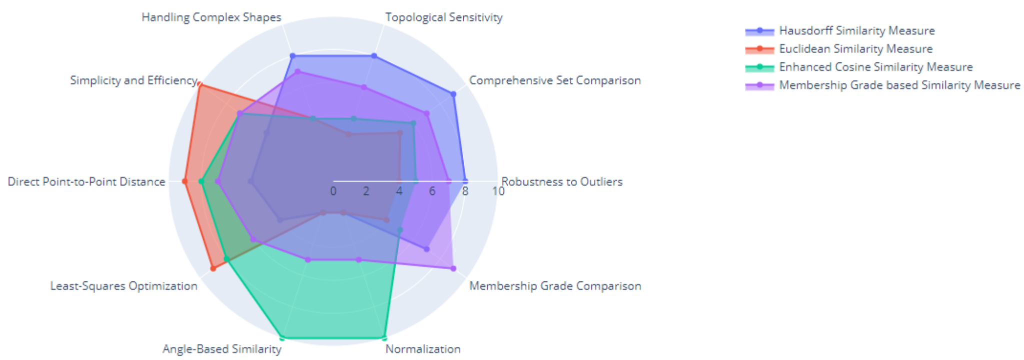

FIGURE 2. Comparison diagram for various similarity measures

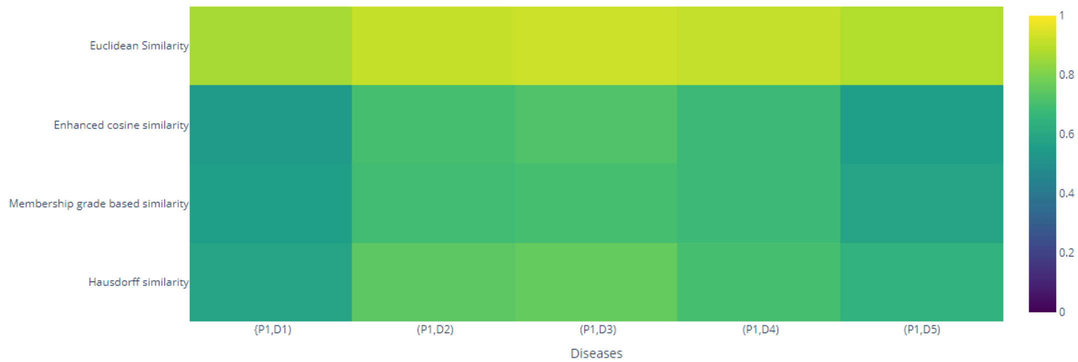

FIGURE 3. Comparison Map

8. Conclusion

In Neutrosophic Fuzzy Set theory, the measurement of distance and similarity is a key research area since these concepts are useful tools for handling incomplete and ambiguous information. In the present study, we derive Hausdorff Similarity Measures, Membership-Grade-Based Similarity Measure, and Trigonometric Similarity Measure in a single-valued neutrosophic environment. the use of these similarity measures in pattern recognition represents a significant advancement in the field of medical data analysis.Future work will focus on extending the application of Neutrosophic Fuzzy Sets to more complex medical datasets and exploring their integration with advanced machine learning techniques, such as deep learning, to further enhance the precision and efficiency of symptom extraction and disease characterization. Additionally, incorporating these models into real-time clinical decision support systems can provide substantial benefits to healthcare practitioners by improving diagnostic accuracy and patient outcomes.

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