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On Neutrosophic Crisp $g^{\#}\alpha$ Closed Set Operators

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Abstract. The concept of $g^{\#}\alpha$ closed set in general topological spaces was first introduced by Muthukumaraswamy K et. al.,. Recently, kokilavani V, Tharani K et. al., introduced neutrosophic crisp $g^{\#}\alpha$ closed set in neutrosophic crisp topological space. Now, in this present paper, we introduced and study the neutrosophic crisp topological properties of neutrosophic crisp $g^{\#}\alpha$ interior, neutrosophic crisp $g^{\#}\alpha$ closure, neutrosophic crisp $g^{\#}\alpha$ frontier, neutrosophic crisp $g^{\#}\alpha$ border, neutrosophic crisp $g^{\#}\alpha$ exterior via the concept of neutrosophic crisp $g^{\#}\alpha$ open set.

Keywords: $NCg^{\#}\alpha int(\mathcal{A})$; $NCg^{\#}\alpha cl(\mathcal{A})$; $NCg^{\#}\alpha Fr(\mathcal{A})$; $NCg^{\#}\alpha Br(\mathcal{A})$; $NCg^{\#}\alpha Ext(\mathcal{A})$

1. Introduction

Zadeh [13] proposed the concept of a fuzzy set that provides a degree of membership function in 1965. Chang [3] first proposed the idea of fuzzy topological space, in 1968. Atanassov [2] created the next stage of fuzzy sets in 1983. These sets, known as intuitionistic fuzzy sets provide a degree of membership and a degree of non-membership functions. Coker [4] introduced the idea of intuitionistic fuzzy topological space in intuitionistic fuzzy sets in 1997. Salama and Alblowi [5] defined neutrosophic topological space and many of its applications. The concept of neutrosophic crisp set and neutrosophic set was investigated by Smaradache [7] [10] [11] in 2005. Since the invention of the neutrosophic set, numerous mathematical topics and applications have been developed. The neutrosophic closed sets and neutrosophic continuous functions were introduced by Salama et.al. [6] in 2014. Salama, et, al. [9] proposed an innovative mathematical model called " Neutrosophic crisp sets and Neutrosophic crisp topological spaces ".

Salama, et, al., [8] expand the notion of neutrosophic crisp topological spaces to neutrosophic crisp α-topological spaces in 2016. V. Kokilavani , K.Tharani et. al., [12] presented neutrosophic crisp $g^{\#}\alpha$ closed set in neutrosophic crisp topological space. Riad K. Al-Hamido [1] introduced new operators like neutrosophic crisp frontier, neutrosophic crisp border and neutrosophic crisp exterior using neutrosophic crisp open set in 2023. In this paper, we use the neutrosophic crisp sets to introduce neutrosophic crisp $g^{\#}\alpha$ interior, neutrosophic crisp $g^{\#}\alpha$ closure, neutrosophic crisp $g^{\#}\alpha$ frontier, neutrosophic crisp $g^{\#}\alpha$ border, neutrosophic crisp $g^{\#}\alpha$ exterior and discuss their properties in neutrosophic crisp topological space.

2. Preliminaries

Definition 2.1. [9] Let (X,Γ) be a $NCTS$ on X and $\mathcal{A}$ be a NCS on X. Then the neutrosophic crisp closure of $\mathcal{A}$ (shortly $NC\text{cl}(\mathcal{A})$) and neutrosophic crisp interior (shortly $NC\text{int}(\mathcal{A})$) of $\mathcal{A}$ are defined by

$NC\text{cl}(\mathcal{A})=\cap$ {$\mathcal{C}$:$\mathcal{A} \subseteq \mathcal{C}$ & $\mathcal{C}$ is a $NCCS$ in X}

$NC\text{int}(\mathcal{A})=\cup$ {$\mathcal{F}$:$\mathcal{F} \subseteq \mathcal{A}$ & $\mathcal{F}$ is a $NCOS$ in X}

Definition 2.2. Let $\mathcal{A}$ be a neutrosophic crisp subset, and let $\mathcal{F}$ be a $NCgOS$ in a $NCTS(X,\Gamma)$ where $\mathcal{A} \subseteq \mathcal{F}$ then $\mathcal{A}$ is called neutrosophic crisp g$^{\#}\alpha$-closed set (briefly,$NCg^{\#}\alpha CS$) if $NC\alpha cl(\mathcal{A}) \subseteq \mathcal{F}$ and the complement of a $NCg^{\#}\alpha CS$ is a $NCg^{\#}\alpha OS$ in (X,Γ).

3. Neutrosophic Crisp $g^{\#}\alpha$ Interior

In this section, we introduce neutrosophic crisp $g^{\#}\alpha$ interior and discuss their properties in neutrosophic crisp topological spaces.

Definition 3.1. Let (X,Γ) be a $NCTS$ and let $x \in$ X. A subset $\mathcal{A}$ of X is said to be $NCg^{\#}\alpha$-neighbourhood of x if there exists a $NCg^{\#}\alpha$ open set $\mathcal{F}$ such that $x \in \mathcal{F} \subset \mathcal{A}$.

Definition 3.2. Let (X,Γ) be a $NCTS$ and let $\mathcal{A} \subset$ X. A point $x \in \mathcal{A}$ is said to be $NCg^{\#}\alpha$ interior point of $\mathcal{A}$ if and only if $\mathcal{A}$ is a $NCg^{\#}\alpha$-neighbourhood of x.

Remark 3.1. Let $\mathcal{A}$ be a neutrosophic crisp subset of the $NCTS(X,\Gamma)$. Then the set of all $NCg^{\#}\alpha$ interior points of $\mathcal{A}$ is called the $NCg^{\#}\alpha$ interior of $\mathcal{A}$ and is denoted by $NCg^{\#}\alpha int(\mathcal{A})$.

Theorem 3.1. *If $\mathcal{A}$ be a neutrosophic crisp subset of $NCTS(X,\Gamma)$. Then $NCg^{\#}\alpha int(\mathcal{A}) = \cup\{\mathcal{F}$: $\mathcal{F}$ is a $NCg^{\#}\alpha$ open, $\mathcal{F} \subset \mathcal{A}\}$.*

Proof. Let $\mathcal{A}$ be a neutrosophic crisp subset of $NCTS(X,\Gamma)$.

$$\begin{aligned} \text{Then } x \in NCg^{\#}\alpha int(\mathcal{A}) &\Leftrightarrow x \text{ is a } NCg^{\#}\alpha \text{ interior point of } \mathcal{A}. \\ &\Leftrightarrow \mathcal{A} \text{ is a } NCg^{\#}\alpha \text{ nbhd of point } x. \\ &\Leftrightarrow \text{there exists } NCg^{\#}\alpha \text{ open set } \mathcal{F} \text{ such that } x \in \mathcal{F} \subset \mathcal{A}. \\ &\Leftrightarrow x \in \cup\{\mathcal{F} : \mathcal{F} \text{ is a } NCg^{\#}\alpha \text{ open}, \mathcal{F} \subset \mathcal{A}\} \end{aligned}$$

Hence $NCg^{\#}\alpha int(\mathcal{A}) = \cup\{\mathcal{F} : \mathcal{F}$ is a $NCg^{\#}\alpha$ *open*, $\mathcal{F} \subset \mathcal{A}\}$. □

Theorem 3.2. *If $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then*

(i) $NCg^{\#}\alpha int(X_N) = X_N$ *and* $NCg^{\#}\alpha int(\phi_N) = \phi_N$

(ii) $NCg^{\#}\alpha int(\mathcal{A}) \subset \mathcal{A}$.

(iii) *If $\mathcal{D}$ is any $NCg^{\#}\alpha OS$ contained in $\mathcal{A}$, then $\mathcal{D} \subset NCg^{\#}\alpha int(\mathcal{A})$*

(iv) *If $\mathcal{A} \subset \mathcal{D}$, then $NCg^{\#}\alpha int(\mathcal{A}) \subset NCg^{\#}\alpha int(\mathcal{D})$*

(v) $NCg^{\#}\alpha int(NCg^{\#}\alpha int(\mathcal{A}))= NCg^{\#}\alpha int(\mathcal{A})$

Proof.

(i) Since X_N and ϕ_N are $NCg^{\#}\alpha$ open sets,

$$\begin{aligned} NCg^{\#}\alpha int(X_N) &= \cup\{\mathcal{F} : \mathcal{F} \text{ is a } NCg^{\#}\alpha open, \mathcal{F} \subset X\} \\ &= X \cup \mathcal{F} \text{ is } NCg^{\#}\alpha OS \\ &= X_N \end{aligned}$$

(ie) $NCg^{\#}\alpha int(X_N)$=X_N. Since ϕ_N is the only $NCg^{\#}\alpha OS$ contained in ϕ_N, $NCg^{\#}\alpha int(\phi_N)$=$\phi_N$

(ii) Let $X \in NCg^{\#}\alpha int(\mathcal{A}) \Rightarrow x$ is a interior point of $\mathcal{A}$.

$$\begin{aligned} &\Rightarrow \mathcal{A} \text{ is a nbhd of } x \\ &\Rightarrow x \in \mathcal{A} \end{aligned}$$

Thus, $x \in NCg^{\#}\alpha int(\mathcal{A}) \Rightarrow x \in \mathcal{A}$. Hence, $NCg^{\#}\alpha int(\mathcal{A}) \subset \mathcal{A}$.

(iii) Let $\mathcal{D}$ be any $NCg^{\#}OS$ such that $\mathcal{D} \subset \mathcal{A}$. Let $x \in \mathcal{D}$. Since $\mathcal{D}$ is a $NCg^{\#}OS$ contained in $\mathcal{A}$. x is a $NCg^{\#}\alpha$ interior point of $\mathcal{A}$. (ie) $x \in NCg^{\#}\alpha int(\mathcal{A})$. Hence $\mathcal{D} \subset NCg^{\#}\alpha int(\mathcal{A})$.

(iv) Let $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$ such that $\mathcal{A} \subset \mathcal{D}$. Let $x \in NCg^{\#}\alpha int(\mathcal{A})$. Then x is a $NCg^{\#}\alpha$ interior point of $\mathcal{A}$ and so $\mathcal{A}$ is a $NCg^{\#}\alpha$-nbhd of x. Since $\mathcal{D} \supset \mathcal{A}, \mathcal{D}$ is also $NCg^{\#}\alpha$-nbhd of x. $\Rightarrow x \in NCg^{\#}\alpha int(\mathcal{D})$. Thus we have shown that $x \in NCg^{\#}\alpha int(\mathcal{A}) \Rightarrow x \in NCg^{\#}\alpha int(\mathcal{D})$.

(v) Let $\mathcal{A}$ be a neutrosophic crisp subset of $NCTS(X,\Gamma)$. $NCg^{\#}\alpha int(\mathcal{A})$ is $NCg^{\#}\alpha OS$ and hence $NCg^{\#}\alpha int(NCg^{\#}\alpha int(\mathcal{A})) = NCg^{\#}\alpha int(\mathcal{A})$

Theorem 3.3. *If a neutrosophic crisp subset $\mathcal{A}$ of $NCTS(X,\Gamma)$ is $NCg^{\#}\alpha$open, then $NCg^{\#}\alpha int(\mathcal{A})$=$\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be $NCg^{\#}\alpha$open subset of $NCTS(X,\Gamma)$. We know that $NCg^{\#}\alpha int(\mathcal{A}) \subset \mathcal{A}$. Also, $\mathcal{A}$ is $NCg^{\#}\alpha OS$ contained in $\mathcal{A}$. From Theorem 3.2 (iii) $\mathcal{A} \subset NCg^{\#}\alpha int(\mathcal{A})$. Hence $NCg^{\#}\alpha int(\mathcal{A})$=$\mathcal{A}$. □

Theorem 3.4. *If $\mathcal{A}$ and $\mathcal{D}$ are neutrosophic crisp subset of $NCTS(X,\Gamma)$, then $NCg^{\#}\alpha int(\mathcal{A}) \cup NCg^{\#}\alpha int(\mathcal{D}) \subset NCg^{\#}\alpha int(\mathcal{A} \cup \mathcal{D})$.*

Proof. We know that $\mathcal{A} \subset \mathcal{A} \cup \mathcal{D}$ and $\mathcal{D} \subset \mathcal{A} \cup \mathcal{D}$. The result from Theorem 3.2 (iv) that $NCg^{\#}\alpha int(\mathcal{A}) \subset NCg^{\#}\alpha int(\mathcal{A} \cup \mathcal{D})$ and also we have $NCg^{\#}\alpha int(\mathcal{D}) \subset NCg^{\#}\alpha int(\mathcal{A} \cup \mathcal{D})$ This implies that $NCg^{\#}\alpha int(\mathcal{A}) \cup NCg^{\#}\alpha int(\mathcal{D}) \subset NCg^{\#}\alpha int(\mathcal{A} \cup \mathcal{D})$. □

Theorem 3.5. *If $\mathcal{A}$ and $\mathcal{D}$ are neutrosophic crisp subset of $NCTS(X,\Gamma)$, then $NCg^{\#}\alpha int(\mathcal{A} \cap \mathcal{D}) = NCg^{\#}\alpha int(\mathcal{A}) \cap NCg^{\#}\alpha int(\mathcal{D})$.*

Proof. We know that $\mathcal{A} \cap \mathcal{D} \subset \mathcal{A}$ and $\mathcal{A} \cap \mathcal{D} \subset \mathcal{D}$. The result from Theorem 3.2 (iv) that $NCg^{\#}\alpha int(\mathcal{A} \cap \mathcal{D}) \subset NCg^{\#}\alpha int(\mathcal{A})$ and $NCg^{\#}\alpha int(\mathcal{A} \cap \mathcal{D}) \subset NCg^{\#}\alpha int(\mathcal{D})$. This implies that

$$NCg^{\#}\alpha int(\mathcal{A} \cap \mathcal{D}) \subset NCg^{\#}\alpha int(\mathcal{A}) \cap NCg^{\#}\alpha int(\mathcal{D}). \qquad (1)$$

Let $x \in NCg^{\#}\alpha int(\mathcal{A}) \cap NCg^{\#}\alpha int(\mathcal{D})$. Then $x \in NCg^{\#}\alpha int(\mathcal{A})$ and $x \in NCg^{\#}\alpha int(\mathcal{D})$. Hence x is a $NCg^{\#}\alpha$-int point of each of sets $\mathcal{A}$ and $\mathcal{D}$. It follows that $\mathcal{A}$ and $\mathcal{D}$ is $NCg^{\#}\alpha$-nbhds of x, so that their intersection $\mathcal{A} \cap \mathcal{D}$ is also a $NCg^{\#}\alpha$-nbhds of x. Hence $x \in NCg^{\#}\alpha int(\mathcal{A} \cap \mathcal{D})$. Thus $x \in NCg^{\#}\alpha int(\mathcal{A}) \cap NCg^{\#}\alpha int(\mathcal{D})$ implies that $x \in NCg^{\#}\alpha int(\mathcal{A} \cap \mathcal{D})$. Therefore

$$NCg^{\#}\alpha int(\mathcal{A}) \cap NCg^{\#}\alpha int(\mathcal{D}) \subset NCg^{\#}\alpha int(\mathcal{A} \cap \mathcal{D}) \qquad (2)$$

From (1) and (2), We get $NCg^{\#}\alpha int(\mathcal{A} \cap \mathcal{D}) = NCg^{\#}\alpha int(\mathcal{A}) \cap NCg^{\#}\alpha int(\mathcal{D})$. □

Theorem 3.6. *If $\mathcal{A}$ neutrosophic crisp subset of a $NCTS(X,\Gamma)$, then $NCint(\mathcal{A}) \subset NCg^{\#}\alpha int(\mathcal{A})$.*

Proof. Let $\mathcal{A}$ neutrosophic crisp subset of a $NCTS(X, \Gamma)$.
Let $x \in NCint(\mathcal{A}) \Rightarrow x \in \cup \{\mathcal{F}\text{: } \mathcal{F} \text{ is } NCOS,\ \mathcal{F} \subset \mathcal{A}\}$

$\Rightarrow$ there exists an NCOS $\mathcal{F}$ such that $x \in \mathcal{F} \subset \mathcal{A}$.

$\Rightarrow$ there exist a $NCg^{\#}\alpha OS$ $\mathcal{F}$ such that $x \in \mathcal{F} \subset \mathcal{A}$,

as every NCOS is a $NCg^{\#}\alpha OS$ in x.

$\Rightarrow x \in \cup\{\mathcal{F} : \mathcal{F} \text{ is } NCg^{\#}\alpha OS,\ \mathcal{F} \subset \mathcal{A}\}$

$\Rightarrow x \in NCg^{\#}\alpha int(\mathcal{A})$

Thus $x \in NCint(\mathcal{A}) \Rightarrow x \in NCg^{\#}\alpha int(\mathcal{A})$. Hence $NCint(\mathcal{A}) \subset NCg^{\#}\alpha int(\mathcal{A})$. □

Remark 3.2. If $\mathcal{A}$ is a neutrosophic crisp subset of $NCTS(X, \Gamma)$, then

(i) $NC\alpha int(\mathcal{A}) \subset NCg^{\#}\alpha int(\mathcal{A})$
(ii) $NCg\alpha gint(\mathcal{A}) \subset NCg^{\#}\alpha int(\mathcal{A})$
(iii) $NCg^{\#}\alpha int(\mathcal{A}) \subset NC\alpha gint(\mathcal{A})$
(iv) $NCg^{\#}\alpha int(\mathcal{A}) \subset NCgsint(\mathcal{A})$

4. Neutrosophic Crisp $g^{\#}\alpha$ Closure

Definition 4.1. Let $\mathcal{A}$ be a neutrosophic crisp subset of $NCTS(X, \Gamma)$. We define the $NCg^{\#}\alpha$ closure of $\mathcal{A}$ to be the intersection of all $NCg^{\#}\alpha CS's$ containing $\mathcal{A}$. It denotes, $NCg^{\#}\alpha cl(\mathcal{A})$ $= \cap\ \{\mathcal{C} : \mathcal{C}$ is a $NCg^{\#}\alpha CS$ and $\mathcal{A} \subset \mathcal{C}\}$.

Theorem 4.1. *If $\mathcal{A}$ and $\mathcal{D}$ are neutrosophic crisp subset of $NCTS(X, \Gamma)$. Then,*

(i) $NCg^{\#}\alpha cl(X_N) = X_N$ *and* $NCg^{\#}\alpha cl(\phi_N) = \phi_N$
(ii) $\mathcal{A} \subset NCg^{\#}\alpha cl(\mathcal{A})$
(iii) *If $\mathcal{D}$ is any $NCg^{\#}\alpha$ closed set containing $\mathcal{A}$, then $NCg^{\#}\alpha cl(\mathcal{A}) \subset \mathcal{D}$*
(iv) *If $\mathcal{A} \subset \mathcal{D}$ then $NCg^{\#}\alpha cl(\mathcal{A}) \subset NCg^{\#}\alpha cl(\mathcal{D})$*
(v) $NCg^{\#}\alpha cl(\mathcal{A} \cap \mathcal{D}) \subset NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{D})$
(vi) $NCg^{\#}\alpha cl(\mathcal{A} \cup \mathcal{D}) = NCg^{\#}\alpha cl(\mathcal{A}) \cup NCg^{\#}\alpha cl(\mathcal{D})$
(vii) $NCg^{\#}\alpha cl(NCg^{\#}\alpha cl(\mathcal{A})) = NCg^{\#}\alpha cl(\mathcal{A})$

Proof.

(i) By the definition of $NCg^{\#}\alpha cl(\mathcal{A})$, X is the only $NCg^{\#}\alpha$ closed set containing X.

$\therefore\ NCg^{\#}\alpha cl(X_N) =$ Intersection of all the $NCg^{\#}\alpha$ closed sets containing X.

$= \cap\{X\} = X_N$

That is $NCg^{\#}\alpha cl(X_N)$=X_N.

Consequently,

$$NCg^{\#}\alpha cl(\phi_N) = \text{Intersection of all the } NCg^{\#}\alpha \text{ closed sets containing } \phi$$

$$= \cap\{\phi\} = \phi_N.$$

That is $NCg^{\#}\alpha cl(\phi_N) = \phi_N$.

(ii) By the definition of $NCg^{\#}$ closure of $\mathcal{A}$, it is obvious that $\mathcal{A} \subset NCg^{\#}\alpha cl(A)$.

(iii) Let $\mathcal{D}$ be any $NCg^{\#}\alpha$CS containing A. Since $NCg^{\#}\alpha cl(A)$ is the intersection of all $NCg^{\#}\alpha CS's$ containing $\mathcal{A}$, $NCg^{\#}\alpha cl(A)$ is contained in every $NCg^{\#}\alpha CS$ containing $\mathcal{A}$. Hence in particular, $NCg^{\#}\alpha cl(A) \subset \mathcal{D}$.

(iv) Let $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of (X, Γ) such that $\mathcal{A} \subset \mathcal{D}$. By the definition $NCg^{\#}\alpha cl(D)$ $=\cap$ $\{\mathcal{C}$: $\mathcal{D} \subset \mathcal{C} \in NCg^{\#}\alpha C(X)$ $\}$. If $\mathcal{D} \subset \mathcal{C} \in NCg^{\#}\alpha C(X)$, then $NCg^{\#}\alpha cl(\mathcal{D}) \subset \mathcal{C}$. Since $\mathcal{A} \subset \mathcal{D}$, and by the definition, if $\mathcal{D} \subset \mathcal{C}$, then $\mathcal{A} \subset \mathcal{C}$ for any $\mathcal{C} \in NCg^{\#}\alpha C(X)$, we have $NCg^{\#}\alpha cl(\mathcal{A}) \subset \mathcal{C}$. Therefore $NCg^{\#}\alpha cl(\mathcal{A}) \subset \cap$ $\{\mathcal{C}$: $\mathcal{D} \subset \mathcal{C} \in NCg^{\#}\alpha C(X)\}$= $NCg^{\#}\alpha cl(\mathcal{D})$.

(i.e) $NCg^{\#}\alpha cl(\mathcal{A}) \subset NCg^{\#}\alpha cl(\mathcal{D})$.

(v) Let $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of (X, Γ). Clearly $\mathcal{A} \cap \mathcal{D} \subset \mathcal{A}$ and $\mathcal{A}$ $\cap$ $\mathcal{D} \subset \mathcal{D}$. By theorem $NCg^{\#}\alpha cl(\mathcal{A} \cap \mathcal{D}) \subset NCg^{\#}\alpha cl(\mathcal{A})$ and $NCg^{\#}\alpha cl(\mathcal{A} \cap \mathcal{D}) \subset NCg^{\#}\alpha cl(\mathcal{D})$. Hence $NCg^{\#}\alpha cl(\mathcal{A} \cap \mathcal{D}) \subset NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{D})$.

(vi) Let $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of (X, Γ). Clearly $\mathcal{A} \subset \mathcal{A} \cup \mathcal{D}$ and $\mathcal{D} \subset \mathcal{A} \cup \mathcal{D}$. We have $NCg^{\#}\alpha cl(\mathcal{A}) \subset NCg^{\#}\alpha cl(\mathcal{A} \cup \mathcal{D})$ and $NCg^{\#}\alpha cl(\mathcal{D}) \subset NCg^{\#}\alpha cl(\mathcal{A} \cup \mathcal{D})$. Hence,

$$NCg^{\#}\alpha cl(\mathcal{A}) \cup NCg^{\#}\alpha cl(\mathcal{D}) \subset NCg^{\#}\alpha cl(\mathcal{A} \cup \mathcal{D}) \tag{1}$$

Since $NCg^{\#}\alpha cl(\mathcal{A})$, $NCg^{\#}\alpha cl(\mathcal{D})$ are NCCS's. $NCg^{\#}\alpha cl(\mathcal{A}) \cup NCg^{\#}\alpha cl(\mathcal{D})$ are NCCS. Also, $\mathcal{A} \subset NCg^{\#}\alpha cl(\mathcal{A})$ and $\mathcal{D} \subset NCg^{\#}\alpha cl(\mathcal{D})$, which implies $\mathcal{A} \cup \mathcal{D} \subset NCg^{\#}\alpha cl(\mathcal{A}) \cup NCg^{\#}\alpha cl(\mathcal{D})$. Thus, $NCg^{\#}\alpha cl(\mathcal{A}) \cup NCg^{\#}\alpha cl(\mathcal{D})$ is a NCCS containing $\mathcal{A} \cup \mathcal{D}$. Since, $NCg^{\#}\alpha cl(\mathcal{A} \cup \mathcal{D})$ is the smallest NCCS containing $\mathcal{A} \cup \mathcal{D}$, we have

$$NCg^{\#}\alpha cl(\mathcal{A} \cup \mathcal{D}) \subset NCg^{\#}\alpha cl(\mathcal{A}) \cup NCg^{\#}\alpha cl(\mathcal{D}) \tag{2}$$

from (1) and (2) we have, $NCg^{\#}\alpha cl(\mathcal{A} \cup \mathcal{D}) = NCg^{\#}\alpha cl(\mathcal{A}) \cup NCg^{\#}\alpha cl(\mathcal{D})$.

Theorem 4.2. *If $\mathcal{A} \subset X$ is $NCg^{\#}\alpha$ closed, then $NCg^{\#}\alpha cl(\mathcal{A})$ =$\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be $NCg^{\#}\alpha$ closed neutrosophic crisp subset of (X, Γ). We know that $\mathcal{A} \subset NCg^{\#}\alpha cl(\mathcal{A})$. Also $\mathcal{A}$ is $NCg^{\#}\alpha$ closed set containing $\mathcal{A}$. By theorem (iii) $NCg^{\#}\alpha cl(\mathcal{A}) \subset \mathcal{A}$. Hence, $NCg^{\#}\alpha cl(\mathcal{A}) = \mathcal{A}$.

Theorem 4.3. *If $\mathcal{A}$ is a neutrosophic crisp subset of a space (X, Γ), then $NCg^{\#}\alpha cl(\mathcal{A}) \subset NCcl(\mathcal{A})$.*

Proof. Let $\mathcal{A}$ is a neutrosophic crisp subset of a space (X, Γ). By the definition of Neutrosophic crisp closure,$NCcl(\mathcal{A})$=$\cap$ {$\mathcal{C}$: $\mathcal{C}$ is NC closed, $\mathcal{A} \subset \mathcal{C}$}. If $\mathcal{A} \subset \mathcal{C}$ and $\mathcal{C}$ is a neutrosophic crisp closed subset of X, Then $\mathcal{A} \subset \mathcal{C} \in NCg^{\#}\alpha cl(X)$, because every neutrosophic crisp closed set is $NCg^{\#}\alpha$ closed set. That is $NCg^{\#}\alpha cl(\mathcal{A}) \subset \mathcal{C}$. Therefore $NCg^{\#}\alpha cl(\mathcal{A}) \subset \cap \{\mathcal{C} : \mathcal{A} \subset \mathcal{C}$ and $\mathcal{C}$ is a neutrosophic crisp closed in $X\}= NCcl(\mathcal{A})$. Hence $NCg^{\#}\alpha cl(\mathcal{A}) \subset NCcl(\mathcal{A})$. □

Remark 4.1. Let $\mathcal{A}$ be any neutrosophic crisp subset of X. Then

(i) $(NCg^{\#}\alpha int(\mathcal{A}))^c = NCg^{\#}\alpha cl(\mathcal{A}^c)$
(ii) $NCg^{\#}\alpha int(\mathcal{A}) = (NCg^{\#}\alpha cl(\mathcal{A}^c))^c$
(iii) $NCg^{\#}\alpha cl(\mathcal{A}) = (NCg^{\#}\alpha int(\mathcal{A}^c))^c$

5. Neutrosophic Crisp $g^{\#}\alpha$ Frontier

Definition 5.1. Let $\mathcal{A}$ be a neutrosophic crisp subset of $NCTS(X, \Gamma)$. Then $NCg^{\#}\alpha$ frontier of $\mathcal{A}$ is defined as $NCg^{\#}\alpha Fr(\mathcal{A}) = NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c)$.

Theorem 5.1. *If $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X, \Gamma)$. Then*

(i) $NCg^{\#}\alpha Fr(X_N) = \phi_N$ *and* $NCg^{\#}\alpha Fr(\phi_N) = \phi_N$
(ii) $NCg^{\#}\alpha Fr(\mathcal{A}) = NCg^{\#}\alpha Fr(\mathcal{A}^c)$
(iii) $NCg^{\#}\alpha Fr(\mathcal{A}) = NCg^{\#}\alpha cl(\mathcal{A})$ - $NCg^{\#}\alpha int(\mathcal{A})$
(iv) *If $\mathcal{A}$ is $NCg^{\#}\alpha CS$ in X if and only if $NCg^{\#}\alpha Fr(\mathcal{A}) \subseteq \mathcal{A}$*
(v) *If $\mathcal{A}$ is $NCg^{\#}\alpha OS$ in X, then $NCg^{\#}\alpha Fr(\mathcal{A}) \subseteq \mathcal{A}^c$*
(vi) $(NCg^{\#}\alpha Fr(\mathcal{A}))^c = NCg^{\#}\alpha int(\mathcal{A}) \cup NCg^{\#}\alpha int(\mathcal{A}^c)$
(vii) $\mathcal{A} \cup NCg^{\#}\alpha Fr(\mathcal{A}) \subseteq NCg^{\#}\alpha cl(\mathcal{A})$
(viii) $NCg^{\#}\alpha Fr(NCg^{\#}\alpha int(\mathcal{A})) \subseteq NCg^{\#}\alpha Fr(\mathcal{A})$
(ix) $NCg^{\#}\alpha Fr(NCg^{\#}\alpha cl(\mathcal{A})) \subseteq NCg^{\#}\alpha Fr(\mathcal{A})$
(x) $NCg^{\#}\alpha int(\mathcal{A}) \subseteq \mathcal{A}$ - $NCg^{\#}\alpha Fr(\mathcal{A})$
(xi) $NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(\mathcal{A})) \subseteq NCg^{\#}\alpha Fr(\mathcal{A})$
(xii) $NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(\mathcal{A}))) \subseteq NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(\mathcal{A}))$

Proof.

(i) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X, \Gamma)$.

$$
\begin{aligned}
NCg^{\#}\alpha Fr(X_N) =& NCg^{\#}\alpha cl(X_N) \cap NCg^{\#}\alpha cl({X_N}^c) \\
&= NCg^{\#}\alpha cl(X_N) \cap NCg^{\#}\alpha cl(\phi_N) \\
&= X_N \cap \phi_N \\
&= \phi_N \\
NCg^{\#}\alpha Fr(\phi_N) &= NCg^{\#}\alpha cl(\phi_N) \cap NCg^{\#}\alpha cl({\phi_N}^c) \\
&= NCg^{\#}\alpha cl(\phi_N) \cap NCg^{\#}\alpha cl(X_N) \\
&= \phi_N \cap X_N \\
&= \phi_N
\end{aligned}
$$

(ii) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X, \Gamma)$. Then by definition of $NCg^{\#}\alpha$ frontier,

$$
\begin{aligned}
NCg^{\#}\alpha Fr(\mathcal{A}) &= NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c) \\
&= NCg^{\#}\alpha cl(\mathcal{A}^c) \cap NCg^{\#}\alpha cl(\mathcal{A}) \\
&= NCg^{\#}\alpha cl(\mathcal{A}^c) \cap (NCg^{\#}\alpha cl(\mathcal{A}^c))^c \\
&= NCg^{\#}\alpha Fr(\mathcal{A}^c) \text{ [But, by Definition 5.1]}
\end{aligned}
$$

(iii) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X, \Gamma)$. Since, $(NCg^{\#}\alpha cl(\mathcal{A}))^c = NCg^{\#}\alpha int(\mathcal{A}^c)$, then $(NCg^{\#}\alpha cl(\mathcal{A}^c))^c = NCg^{\#}\alpha int(\mathcal{A})$

By Definition 5.1, $NCg^{\#}\alpha Fr(\mathcal{A}) = NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c)$

$= NCg^{\#}\alpha cl(\mathcal{A}) \cap (NCg^{\#}\alpha int(\mathcal{A}))^c$

By using, $\mathcal{A} - \mathcal{D} = \mathcal{A} \cap \mathcal{D}^c$

$= NCg^{\#}\alpha cl(\mathcal{A})$ - $NCg^{\#}\alpha int(\mathcal{A})$

Hence, $NCg^{\#}\alpha Fr(\mathcal{A}) = NCg^{\#}\alpha cl(\mathcal{A})$ - $NCg^{\#}\alpha int(\mathcal{A})$

(iv) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X, \Gamma)$.

By Definition 5.1, $NCg^{\#}\alpha Fr(\mathcal{A}) = NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c)$

$\subseteq NCg^{\#}\alpha cl(\mathcal{A})$

$=\mathcal{A}$

Therefore, $NCg^{\#}\alpha Fr(\mathcal{A}) \subseteq \mathcal{A}$

Conversely,

Assume that, $NCg^{\#}\alpha Fr(\mathcal{A}) \subseteq \mathcal{A}$. Then $NCg^{\#}\alpha cl(\mathcal{A})$ - $NCg^{\#}\alpha int(\mathcal{A}) \subseteq \mathcal{A}$. Since, $NCg^{\#}\alpha int(\mathcal{A}) \subseteq \mathcal{A}$. We conclude that, $NCg^{\#}\alpha cl(\mathcal{A}) = \mathcal{A}$ and hence $\mathcal{A}$ is $NCg^{\#}\alpha CS$.

(v) Let $\mathcal{A}$ be a $NCg^{\#}\alpha OS$ in $NCTS(X, \Gamma)$. Then $\mathcal{A}^c$ is $NCg^{\#}\alpha CS$ in $NCTS(X, \Gamma)$.
By the Theorem 5.1 (iv), $NCg^{\#}\alpha Fr(\mathcal{A}^c) \subseteq \mathcal{A}^c$.
and by Theorem 5.1 (ii), $NCg^{\#}\alpha Fr(\mathcal{A}) \subseteq \mathcal{A}^c$.

(vi) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X, \Gamma)$.

By Defintion 5.1, $(NCg^{\#}\alpha Fr(\mathcal{A}))^c = (NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c))^c$

$= (NCg^{\#}\alpha cl(\mathcal{A}))^c \cup (NCg^{\#}\alpha cl(\mathcal{A}^c))^c$

$= (NCg^{\#}\alpha int(\mathcal{A}^c)) \cup (NCg^{\#}\alpha int(\mathcal{A}))$

Hence, $(NCg^{\#}\alpha Fr(\mathcal{A}))^c = (NCg^{\#}\alpha int(\mathcal{A}^c)) \cup (NCg^{\#}\alpha int(\mathcal{A}))$

(vii) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X, \Gamma)$.

By Defintion 5.1, $\mathcal{A} \cup NCg^{\#}\alpha Fr(\mathcal{A}) = \mathcal{A} \cup (NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c))$

$= (\mathcal{A} \cup NCg^{\#}\alpha cl(\mathcal{A})) \cap (\mathcal{A} \cup (NCg^{\#}\alpha cl(\mathcal{A}^c))$

$\subseteq NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c)$

$\subseteq NCg^{\#}\alpha cl(\mathcal{A}^c)$

Hence, $\mathcal{A} \cup NCg^{\#}\alpha Fr(\mathcal{A}) \subseteq NCg^{\#}\alpha cl(\mathcal{A}^c)$

(viii) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X, \Gamma)$. Then by Definition 5.1,

$$NCg^{\#}\alpha Fr(NCg^{\#}\alpha int(\mathcal{A})) = NCg^{\#}\alpha cl(NCg^{\#}\alpha int(\mathcal{A})) \cap NCg^{\#}\alpha cl(NCg^{\#}\alpha int(\mathcal{A}))^c$$

$$= NCg^{\#}\alpha cl(NCg^{\#}\alpha int(\mathcal{A})) \cap NCg^{\#}\alpha cl(NCg^{\#}\alpha cl(\mathcal{A}^c))$$

$$[(NCg^{\#}\alpha int(\mathcal{A}))^c = NCg^{\#}\alpha cl(\mathcal{A}^c)]$$

$$= NCg^{\#}\alpha cl(NCg^{\#}\alpha int(\mathcal{A})) \cap NCg^{\#}\alpha cl(\mathcal{A}^c)$$

$$[NCg^{\#}\alpha cl(\mathcal{A}^c) \text{ is } NCg^{\#}\alpha CS]$$

$$\subseteq NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c) \quad \text{[by Theorem 3.2 (ii)]}$$

$$= NCg^{\#}\alpha Fr(\mathcal{A}) \quad \text{[again by Definition 5.1]}$$

$NCg^{\#}\alpha Fr(NCg^{\#}\alpha int(\mathcal{A})) \subseteq NCg^{\#}\alpha Fr(\mathcal{A})$

(ix) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X, \Gamma)$. Then by Definition 5.1,

$$NCg^{\#}\alpha Fr(NCg^{\#}\alpha cl(\mathcal{A})) = NCg^{\#}\alpha cl(NCg^{\#}\alpha cl(\mathcal{A})) \cap NCg^{\#}\alpha cl(NCg^{\#}\alpha cl(\mathcal{A}))^c$$

$$= NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(NCg^{\#}\alpha int(\mathcal{A}^c))$$

$$[(NCg^{\#}\alpha int(\mathcal{A}))^c = NCg^{\#}\alpha cl(\mathcal{A}^c)]$$

$$\subseteq NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c)$$

$$= NCg^{\#}\alpha Fr(\mathcal{A}) \quad \text{[again by Definition 5.1]}$$

$NCg^{\#}\alpha Fr(NCg^{\#}\alpha cl(\mathcal{A})) \subseteq NCg^{\#}\alpha Fr(\mathcal{A})$

(x) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X,\Gamma)$.

$$\begin{aligned}\mathcal{A}-NCg^{\#}\alpha Fr(\mathcal{A}) &= \mathcal{A}\cap(NCg^{\#}\alpha Fr(\mathcal{A}))^c\\ &= \mathcal{A}\cap(NCg^{\#}\alpha cl(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{A}^c))^c \quad \text{[by Definition 5.1]}\\ &= \mathcal{A}\cap(NCg^{\#}\alpha int(\mathcal{A}^c)\cup NCg^{\#}\alpha int(\mathcal{A}))\\ &= (\mathcal{A}\cap NCg^{\#}\alpha int(\mathcal{A}^c))\cup(\mathcal{A}\cap NCg^{\#}\alpha int(\mathcal{A}))\\ &= (\mathcal{A}\cap NCg^{\#}\alpha int(\mathcal{A}^c))\cup NCg^{\#}\alpha int(\mathcal{A})\supseteq NCg^{\#}\alpha int(\mathcal{A})\end{aligned}$$

Hence, $NCg^{\#}\alpha int(\mathcal{A})\subseteq \mathcal{A}-NCg^{\#}\alpha Fr(\mathcal{A})$

(xi) Let $\mathcal{A}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then by Definition 5.1,

$$\begin{aligned}NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(\mathcal{A})) &= NCg^{\#}\alpha cl(NCg^{\#}\alpha Fr(\mathcal{A}))\cap NCg^{\#}\alpha cl(NCg^{\#}\alpha Fr(\mathcal{A}))^c\\ &= NCg^{\#}\alpha cl(NCg^{\#}\alpha cl(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{A}^c))\\ &\quad \cap NCg^{\#}\alpha cl(NCg^{\#}\alpha cl(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{A}^c))^c\\ &\subseteq (NCg^{\#}\alpha cl(NCg^{\#}\alpha cl(\mathcal{A}))\cap NCg^{\#}\alpha cl(NCg^{\#}\alpha cl(\mathcal{A}^c)))\\ &\quad \cap (NCg^{\#}\alpha cl(NCg^{\#}\alpha int(\mathcal{A}^c)\cup NCg^{\#}\alpha int(\mathcal{A})))\\ &\subseteq (NCg^{\#}\alpha cl(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{A}^c))\\ &\quad \cap (NCg^{\#}\alpha cl(NCg^{\#}\alpha int(\mathcal{A}^c)\cup (NCg^{\#}\alpha cl(NCg^{\#}\alpha int(\mathcal{A}))))\\ &\subseteq (NCg^{\#}\alpha cl(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{A}^c))\cap (NCg^{\#}\alpha cl(\mathcal{A}^c)\\ &\quad \cup NCg^{\#}\alpha cl(\mathcal{A}))\\ &\subseteq NCg^{\#}\alpha cl(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{A}^c)\\ &= NCg^{\#}\alpha Fr(\mathcal{A})\end{aligned}$$

Therefore, $NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(\mathcal{A}))\subseteq NCg^{\#}\alpha Fr(\mathcal{A})$

(xii) Let $\mathcal{A}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then by Definition 5.1,

$$\begin{aligned}NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(\mathcal{A}))) &= NCg^{\#}\alpha cl(NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(\mathcal{A})))\\ &\quad \cap NCg^{\#}\alpha cl(NCg^{\#}\alpha Fr((NCg^{\#}\alpha Fr(\mathcal{A})))^c)\\ &\subseteq NCg^{\#}\alpha cl(NCg^{\#}\alpha Fr(\mathcal{A}))\\ &\quad \cap NCg^{\#}\alpha cl(NCg^{\#}\alpha Fr(\mathcal{A})^c)\\ &\subseteq NCg^{\#}\alpha cl(NCg^{\#}\alpha Fr(\mathcal{A}))\end{aligned}$$

Hence, $NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(\mathcal{A})))\subseteq NCg^{\#}\alpha cl(NCg^{\#}\alpha Fr(\mathcal{A}))$

□

Theorem 5.2. *If $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then $NCg^{\#}\alpha Fr(\mathcal{A}\cup\mathcal{D})\subseteq NCg^{\#}\alpha Fr(\mathcal{A})\cup NCg^{\#}\alpha Fr(\mathcal{D})$*

Proof. Let $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then by Definition 5.1

$$\begin{aligned}
NCg^{\#}\alpha Fr(\mathcal{A}\cup\mathcal{D}) &= NCg^{\#}\alpha cl(\mathcal{A}\cup\mathcal{D})\cap NCg^{\#}\alpha cl(\mathcal{A}\cup\mathcal{D})^{c}\\
&= NCg^{\#}\alpha cl(\mathcal{A}\cup\mathcal{D})\cap NCg^{\#}\alpha cl(\mathcal{A}^{c}\cap\mathcal{D}^{c})\\
&\subseteq (NCg^{\#}\alpha cl(\mathcal{A})\cup NCg^{\#}\alpha cl(\mathcal{D}))\cap(NCg^{\#}\alpha cl(\mathcal{A}^{c})\cap NCg^{\#}\alpha cl(\mathcal{D}^{c}))\\
&\qquad\qquad\qquad\text{by Theorem 4.1 (v) and (vi)}\\
&= ((NCg^{\#}\alpha cl(\mathcal{A})\cup NCg^{\#}\alpha cl(\mathcal{D}))\cap(NCg^{\#}\alpha cl(\mathcal{A}^{c})))\\
&\qquad\cap((NCg^{\#}\alpha cl(\mathcal{A})\cup NCg^{\#}\alpha cl(\mathcal{D}))\cap(NCg^{\#}\alpha cl(\mathcal{D}^{c})))\\
&= ((NCg^{\#}\alpha cl(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{A}^{c}))\cup(NCg^{\#}\alpha cl(\mathcal{D})\cap NCg^{\#}\alpha cl(\mathcal{A}^{c})))\\
&\quad\cap((NCg^{\#}\alpha cl(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{D}^{c}))\cup(NCg^{\#}\alpha cl(\mathcal{A}))\cap NCg^{\#}\alpha cl(\mathcal{D}^{c})))\\
&= (NCg^{\#}\alpha Fr(\mathcal{A})\cup(NCg^{\#}\alpha cl(\mathcal{D})\cap NCg^{\#}\alpha cl(\mathcal{A}^{c})))\\
&\qquad\cap(NCg^{\#}\alpha Fr(\mathcal{D})\cup(NCg^{\#}\alpha cl(\mathcal{A}))\cap NCg^{\#}\alpha cl(\mathcal{D}^{c})))\\
&= (NCg^{\#}\alpha Fr(\mathcal{A})\cup NCg^{\#}\alpha Fr(\mathcal{D}))\cap((NCg^{\#}\alpha cl(\mathcal{D})\cap NCg^{\#}\alpha cl(\mathcal{A}^{c}))\\
&\qquad\cup(NCg^{\#}\alpha cl(\mathcal{A}))\cap NCg^{\#}\alpha cl(\mathcal{D}^{c})))\\
&\subseteq NCg^{\#}\alpha cl(\mathcal{A})\cup NCg^{\#}\alpha cl(\mathcal{D})
\end{aligned}$$

□

Theorem 5.3. *If $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then $NCg^{\#}\alpha Fr(\mathcal{A}\cap\mathcal{D}) \subseteq (NCg^{\#}\alpha Fr(\mathcal{A}) \cap NCg^{\#}\alpha Fr(\mathcal{D})) \cup (NCg^{\#}\alpha Fr(\mathcal{D}) \cap NCg^{\#}\alpha Fr(\mathcal{A}))$*

Proof. Let $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then by Definition 5.1

$$\begin{aligned}
NCg^{\#}\alpha Fr(\mathcal{A}\cap\mathcal{D}) &= NCg^{\#}\alpha cl(\mathcal{A}\cap\mathcal{D})\cap NCg^{\#}\alpha cl(\mathcal{A}\cap\mathcal{D})^{c}\\
&= NCg^{\#}\alpha cl(\mathcal{A}\cap\mathcal{D})\cap NCg^{\#}\alpha cl(\mathcal{A}^{c}\cup\mathcal{D}^{c})\\
&\subseteq (NCg^{\#}\alpha cl(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{D}))\cap(NCg^{\#}\alpha cl(\mathcal{A}^{c})\cup NCg^{\#}\alpha cl(\mathcal{D}^{c}))\\
&\qquad\qquad\qquad\text{by Theorem 4.1 (v) and (vi)}\\
&= ((NCg^{\#}\alpha cl(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{D}))\cap(NCg^{\#}\alpha cl(\mathcal{A}^{c})))\\
&\qquad\cup((NCg^{\#}\alpha cl(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{D}))\cap(NCg^{\#}\alpha cl(\mathcal{D}^{c})))\\
&= (NCg^{\#}\alpha Fr(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{D}))\cup(NCg^{\#}\alpha Fr(\mathcal{D})\cap NCg^{\#}\alpha cl(\mathcal{A}))
\end{aligned}$$

□

Corollary 5.4. *Let $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$, $NCg^{\#}\alpha Fr(\mathcal{A}\cap\mathcal{D}) \subseteq NCg^{\#}\alpha cl(\mathcal{A})\cup NCg^{\#}\alpha cl(\mathcal{D})$*

Proof.

$$
\begin{aligned}
NCg^{\#}\alpha Fr(\mathcal{A}\cap\mathcal{D}) &= NCg^{\#}\alpha cl(\mathcal{A}\cap\mathcal{D})\cap NCg^{\#}\alpha cl(\mathcal{A}\cap\mathcal{D})^c \\
&= NCg^{\#}\alpha cl(\mathcal{A}\cap\mathcal{D})\cap NCg^{\#}\alpha cl(\mathcal{A}^c\cup\mathcal{D}^c) \\
&\subseteq (NCg^{\#}\alpha cl(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{D}))\cap(NCg^{\#}\alpha cl(\mathcal{A}^c)\cup NCg^{\#}\alpha cl(\mathcal{D}^c)) \\
&\qquad\qquad\text{by Theorem 4.1 (v) and (vi)} \\
&= ((NCg^{\#}\alpha cl(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{D}))\cap(NCg^{\#}\alpha cl(\mathcal{A}^c))) \\
&\qquad\cup((NCg^{\#}\alpha cl(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{D}))\cap(NCg^{\#}\alpha cl(\mathcal{D}^c))) \\
&= (NCg^{\#}\alpha Fr(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{D}))\cup(NCg^{\#}\alpha Fr(\mathcal{D})\cap NCg^{\#}\alpha cl(\mathcal{A})) \\
&\subseteq NCg^{\#}\alpha cl(\mathcal{A})\cup NCg^{\#}\alpha cl(\mathcal{D})
\end{aligned}
$$

□

6. Neutrosophic Crisp $g^{\#}\alpha$ Border

Definition 6.1. Let $\mathcal{A}$ be a neutrosophic crisp subset of $NCTS(X,\Gamma)$. Then $NCg^{\#}\alpha$ border of $\mathcal{A}$ is defined as $NCg^{\#}\alpha Br(\mathcal{A}) = \mathcal{A}$ - $NCg^{\#}\alpha int(\mathcal{A})$.

Theorem 6.1. *If $\mathcal{A}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then*

(i) $\mathcal{A}$ *is a* $NCg^{\#}\alpha OS$ *iff* $NCg^{\#}\alpha Br(\mathcal{A}) = \phi_N$

(ii) $NCg^{\#}\alpha Br(X_N) = NCg^{\#}\alpha Br(\phi_N) = \phi_N$

(iii) $\mathcal{A} = NCg^{\#}\alpha int(\mathcal{A}) \cup NCg^{\#}\alpha Br(\mathcal{A})$

(iv) $NCg^{\#}\alpha int(\mathcal{A}) \cap NCg^{\#}\alpha Br(\mathcal{A}) = \phi_N$

(v) $NCg^{\#}\alpha int(NCg^{\#}\alpha Br(\mathcal{A})) = \phi_N$

(vi) $NCg^{\#}\alpha Br(NCg^{\#}\alpha int(\mathcal{A})) = \phi_N$

(vii) $NCg^{\#}\alpha Br(NCg^{\#}\alpha Br(\mathcal{A})) = NCg^{\#}\alpha Br(\mathcal{A})$

(viii) $NCg^{\#}\alpha Br(\mathcal{A}) = \mathcal{A} \cap NCg^{\#}\alpha cl(\mathcal{A}^c)$

(ix) $NCg^{\#}\alpha Br(\mathcal{A}) \subseteq NCg^{\#}\alpha Fr(\mathcal{A})$

Proof. Let $\mathcal{A}$ be a neutrosophic crisp subset of $NCTS(X,\Gamma)$.

(i) Necessity: Suppose $\mathcal{A}$ is $NCg^{\#}\alpha OS$. Then $NCg^{\#}\alpha int(\mathcal{A}) = \mathcal{A}$.
Now, $NCg^{\#}\alpha Br(\mathcal{A}) = \mathcal{A}$ - $NCg^{\#}\alpha int(\mathcal{A}) = \mathcal{A}$ - $\mathcal{A} = \phi_N$.
Sufficiency: Suppose $NCg^{\#}\alpha Br(\mathcal{A})=\phi_N$. This implies, $\mathcal{A}$ - $NCg^{\#}\alpha int(\mathcal{A})=\phi_N$. Therefore $\mathcal{A} = NCg^{\#}\alpha int(\mathcal{A})$ and hence $\mathcal{A}$ is $NCg^{\#}\alpha OS$.

(ii) Since ϕ_N and X_N are $NCg^{\#}\alpha OS$, by Theorem 6.1 (i), $NCg^{\#}\alpha Br(\phi_N) =\phi_N$ and $NCg^{\#}\alpha Br(X_N)=\phi_N$.

(iii) Let $x \in \mathcal{A}$. If $x \in NCg^{\#}\alpha int(A)$, then the result is obvious.
If $x \notin NCg^{\#}\alpha int(A)$, then by the definition of $NCg^{\#}\alpha Br(A)$, x $\in NCg^{\#}\alpha Br(A)$. Hence $x \in NCg^{\#}\alpha int(A) \cup NCg^{\#}\alpha Br(A)$ and so $\mathcal{A} \subseteq NCg^{\#}\alpha int(A) \cup$

$NCg^{\#}\alpha Br(A)$.

On the other hand, Since $NCg^{\#}\alpha int(A) \subseteq \mathcal{A}$ and $NCg^{\#}\alpha Br(A) \subseteq \mathcal{A}$, we have $NCg^{\#}\alpha int(A) \cup NCg^{\#}\alpha Br(A) \subseteq \mathcal{A}$

(iv) Suppose $NCg^{\#}\alpha int(\mathcal{A}) \cap NCg^{\#}\alpha Br(\mathcal{A}) \neq \phi_N$.

Let $x \in NCg^{\#}\alpha int(A) \cap NCg^{\#}\alpha Br(A)$. Then $x \in NCg^{\#}\alpha int(A)$ and $x \in NCg^{\#}\alpha Br(A)$. Since $NCg^{\#}\alpha Br(A) = \mathcal{A} \cap NCg^{\#}\alpha int(A)$, then $x \in \mathcal{A}$. But $x \in NCg^{\#}\alpha int(A)$ and $x \in \mathcal{A}$, there is a contradiction.

Hence, $NCg^{\#}\alpha int(\mathcal{A}) \cap NCg^{\#}\alpha Br(\mathcal{A}) = \phi_N$.

(v) Let $x \in X$ and assume that $x \in NCg^{\#}\alpha int(NCg^{\#}\alpha Br(A))$.

Then $x \in NCg^{\#}\alpha Br(A)$, Since $NCg^{\#}\alpha Br(A) \subseteq \mathcal{A}$, $x \in NCg^{\#}\alpha int(NCg^{\#}\alpha Br(A)) \subseteq NCg^{\#}\alpha int(\mathcal{A})$. Therefore $x \in NCg^{\#}\alpha int(A) \cap NCg^{\#}\alpha Br(A)$, this leads to a contradiction to Theorem 6.1 (iv).

Hence $NCg^{\#}\alpha int(NCg^{\#}\alpha Br(A)) = \phi_N$.

(vi) By the Definition 6.1,

$NCg^{\#}\alpha Br(NCg^{\#}\alpha int(A)) = NCg^{\#}\alpha int(A)$ - $NCg^{\#}\alpha int(NCg^{\#}\alpha int(A))$.

But Theorem 3.2 (v) we have, $NCg^{\#}\alpha int(NCg^{\#}\alpha int(\mathcal{A}))= NCg^{\#}\alpha int(\mathcal{A})$.

Hence, $NCg^{\#}\alpha Br(NCg^{\#}\alpha int(\mathcal{A})) = \phi_N$.

(vii) By the Definition 6.1,

$NCg^{\#}\alpha Br(NCg^{\#}\alpha Br(A)) = NCg^{\#}\alpha Br(A)$ - $NCg^{\#}\alpha int(NCg^{\#}\alpha Br(A))$.

By Theorem 6.1 (v), $NCg^{\#}\alpha int(NCg^{\#}\alpha Br(\mathcal{A})) = \phi_N$.

And hence $NCg^{\#}\alpha Br(NCg^{\#}\alpha Br(\mathcal{A})) = NCg^{\#}\alpha Br(\mathcal{A})$

(viii) Since,
$$\begin{aligned} NCg^{\#}\alpha Br(\mathcal{A}) &= \mathcal{A} - NCg^{\#}\alpha int(\mathcal{A}) \\ &= \mathcal{A} \cap (NCg^{\#}\alpha int(\mathcal{A}))^c \\ &= \mathcal{A} \cap NCg^{\#}\alpha cl(\mathcal{A})^c \end{aligned}$$

(ix) Since, $\mathcal{A} \subseteq NCg^{\#}\alpha cl(\mathcal{A})$, $\mathcal{A}$ - $NCg^{\#}\alpha int(\mathcal{A}) \subseteq NCg^{\#}\alpha cl(\mathcal{A})$ - $NCg^{\#}\alpha int(\mathcal{A})$, that implies, $NCg^{\#}\alpha Br(\mathcal{A}) \subseteq NCg^{\#}\alpha Fr(\mathcal{A})$

□

7. Neutrosophic Crisp $g^{\#}\alpha$ Exterior

Definition 7.1. Let $\mathcal{A}$ be a neutrosophic crisp subset of $NCTS(X, \Gamma)$. Then $NCg^{\#}\alpha$ exterior of $\mathcal{A}$ is defined as $NCg^{\#}\alpha Ext(\mathcal{A}) = NCg^{\#}\alpha int(\mathcal{A}^c)$.

Theorem 7.1. *If $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X, \Gamma)$. Then*

(i) $NCg^{\#}\alpha Ext(X_N) = \phi_N$ *and* $NCg^{\#}\alpha Ext(\phi_N) = X_N$

(ii) $NCg^{\#}\alpha Ext(\mathcal{A}) = NCg^{\#}\alpha cl(\mathcal{A})^c$

(iii) $NCg^{\#}\alpha Ext(NCg^{\#}\alpha Ext(\mathcal{A})) = NCg^{\#}\alpha int(NCg^{\#}\alpha cl(\mathcal{A})) \supseteq NCg^{\#}\alpha int(\mathcal{A})$

(iv) *If* $\mathcal{A} \subseteq \mathcal{D}$*, then* $NCg^{\#}\alpha Ext(\mathcal{D}) \subseteq NCg^{\#}\alpha Ext(\mathcal{A})$

(v) *If* $\mathcal{A}$ *is a* $NCg^{\#}\alpha CS$ *iff* $NCg^{\#}\alpha Ext(\mathcal{A}) = \phi_N$

(vi) $NCg^{\#}\alpha Ext(\mathcal{A}) = NCg^{\#}\alpha Ext(NCg^{\#}\alpha Ext(\mathcal{A}))^c$

(vii) $NCg^{\#}\alpha Ext(\mathcal{A} \cup \mathcal{D}) \subseteq NCg^{\#}\alpha Ext(\mathcal{A}) \cap NCg^{\#}\alpha Ext(\mathcal{D})$

(viii) $NCg^{\#}\alpha Ext(\mathcal{A} \cap \mathcal{D}) \supseteq NCg^{\#}\alpha Ext(\mathcal{A}) \cup NCg^{\#}\alpha Ext(\mathcal{D})$

Proof.

(i) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X, \Gamma)$. By Definition 7.1,
$NCg^{\#}\alpha Ext(X_N) = NCg^{\#}\alpha int({X_N}^c) = NCg^{\#}\alpha int(\phi_N)$
$NCg^{\#}\alpha Ext(\phi_N) = NCg^{\#}\alpha int({\phi_N}^c) = NCg^{\#}\alpha cl(X_N)$
Since, ϕ_N and X_N are $NCg^{\#}\alpha OS$, then $NCg^{\#}\alpha int(\phi_N) = \phi_N$, $NCg^{\#}\alpha int(X_N) = X_N$. Hence $NCg^{\#}\alpha Ext(X_N) = \phi_N$ and $NCg^{\#}\alpha Ext(\phi_N) = X_N$

(ii) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X, \Gamma)$. Then by definition of $NCg^{\#}\alpha$ Extontier, $(NCg^{\#}\alpha cl(\mathcal{A}))^c = NCg^{\#}\alpha int(\mathcal{A}^c)$, then $NCg^{\#}\alpha Ext(\mathcal{A}) = NCg^{\#}\alpha int(\mathcal{A}^c)$ $= (NCg^{\#}\alpha cl(\mathcal{A}))^c$

(iii) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X, \Gamma)$. Then by definition of $NCg^{\#}\alpha$ Extontier,

$$\begin{aligned} NCg^{\#}\alpha Ext(NCg^{\#}\alpha Ext(\mathcal{A})) &= NCg^{\#}\alpha Ext(NCg^{\#}\alpha int(\mathcal{A}^c)) \\ &= NCg^{\#}\alpha int(NCg^{\#}\alpha int(\mathcal{A}^c))^c \\ &= NCg^{\#}\alpha int(NCg^{\#}\alpha cl(\mathcal{A})) \\ &\supseteq NCg^{\#}\alpha int(\mathcal{A}) \end{aligned}$$

(iv) Let $\mathcal{A} \subseteq \mathcal{D}$. Then by Definition 7.1, $NCg^{\#}\alpha Ext(\mathcal{D}) = NCg^{\#}\alpha int(\mathcal{D}^c) \subseteq NCg^{\#}\alpha int(\mathcal{A}^c) = NCg^{\#}\alpha Ext(\mathcal{A})$

(v) Necessity: Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X, \Gamma)$. Then its complement $\mathcal{A}^c$ is $NCg^{\#}\alpha OS$. By Definition 7.1, $NCg^{\#}\alpha Ext(\mathcal{A}) = NCg^{\#}\alpha int(\mathcal{A}^c)$

Since $\mathcal{A}^c$ is $NCg^{\#}\alpha OS$, $NCg^{\#}\alpha int(\mathcal{A}^c) = \mathcal{A}^c$. Thus, $NCg^{\#}\alpha Ext(\mathcal{A}) = \mathcal{A}^c$.

If $NCg^{\#}\alpha Ext(\mathcal{A}) = \phi_N$, then $\mathcal{A}^c = \phi_N$, which implies $\mathcal{A} = X$.

Hence, $\mathcal{A}$ is $NCg^{\#}\alpha CS$.

Sufficient: If $NCg^{\#}\alpha Ext(\mathcal{A}) = \phi_N$, then $NCg^{\#}\alpha int(\mathcal{A})$. So, $\mathcal{A}^c = \phi_N$. This implies $\mathcal{A} = X$, and X is $NCg^{\#}\alpha CS$. Hence, $\mathcal{A}$ is $NCg^{\#}\alpha CS$.

(vi)

$$\begin{aligned} NCg^{\#}\alpha Ext(NCg^{\#}\alpha Ext(\mathcal{A}))^c &= NCg^{\#}\alpha Ext(NCg^{\#}\alpha int(\mathcal{A}^c))^c \\ &= NCg^{\#}\alpha int((NCg^{\#}\alpha int(\mathcal{A}^c))^c)^c \\ &= NCg^{\#}\alpha int(NCg^{\#}\alpha int(\mathcal{A}^c)) \\ &= NCg^{\#}\alpha int(\mathcal{A}^c) \\ &= NCg^{\#}\alpha Ext(\mathcal{A}) \end{aligned}$$

(vii) Let $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then by Definition 7.1,

$$\begin{aligned} NCg^{\#}\alpha Ext(\mathcal{A}\cup\mathcal{D}) &= NCg^{\#}\alpha int((\mathcal{A}\cup\mathcal{D})^c) \\ &= NCg^{\#}\alpha int((\mathcal{A})^c\cap(\mathcal{D})^c) \\ &\subseteq NCg^{\#}\alpha int(\mathcal{A})^c\cap NCg^{\#}\alpha int(\mathcal{D})^c \\ &= NCg^{\#}\alpha Ext(\mathcal{A})\cap NCg^{\#}\alpha Ext(\mathcal{D}) \end{aligned}$$

(viii) Let $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then by Definition 7.1,

$$\begin{aligned} NCg^{\#}\alpha Ext(\mathcal{A}\cap\mathcal{D}) &= NCg^{\#}\alpha int((\mathcal{A}\cap\mathcal{D})^c) \\ &= NCg^{\#}\alpha int((\mathcal{A})^c\cup(\mathcal{D})^c) \\ &\supseteq NCg^{\#}\alpha int(\mathcal{A})^c\cup NCg^{\#}\alpha int(\mathcal{D})^c \\ &= NCg^{\#}\alpha Ext(\mathcal{A})\cup NCg^{\#}\alpha Ext(\mathcal{D}) \end{aligned}$$

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