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Spherical Fermatean Neutrosophic Topology

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Abstract. In this paper, we introduce Spherical Fermatean Neutrosophic Topological Spaces (SFNTS), expanding on neutrosophic sets characterized by Degrees of Membership (DoM), Degrees of Indeterminacy (DoI) and Degrees of Non-Membership (DoN). Fermatean neutrosophic sets in a universe satisfy the conditions where the sum of the cubes of the DoM and DoN is between zero and one and the cube of the DoI is between zero and one. The DoM, DoI and DoN are represented accordingly. We define a Spherical Fermatean Neutrosophic Set (SFNS) as a set where each element consists of an element in the universe, along with its DoM, DoN, DoI and a radius. The DoM, DoN, DoI and the radius are functions mapping the universe to the interval from zero to one. We extend this to a topological framework by defining an SFNTS on a set. We also studied the properties of the SFN closure and SFN interior operators, provided numerical examples and presented a geometric representation of SFNTS. Additionally, we explored the separation of two SFNSs, the intersection of two SFNSs and the overlapping of two SFNSs.

Keywords: Neutrosophic topology; SFNTS; DoN, DoM, Between zero

1. Introduction

Topology [12], as a foundational branch of mathematics, explores the properties of spaces that remain invariant under continuous transformations. It serves as a vital framework for understanding key concepts such as convergence, continuity, and compactness, which are central to various mathematical disciplines, including analysis, geometry, and mathematical physics. Neutrosophic sets [19], introduced by Florentin Smarandache in 1998, generalize traditional fuzzy sets by incorporating three distinct DoM, DoI and DoN. Neutrosophic topology [11] represents a significant advancement in the field by incorporating the concept of neutrosophic sets. Neutrosophic sets extend traditional set theory by allowing for the representation of indeterminacy and uncertainty in membership values. This capability is particularly valuable in contexts where classical binary logic fails to capture the complexity of information, such as in decision-making processes that involve ambiguous or incomplete data. Neutrosophic topological spaces thus provide a more generalized mathematical framework that accommodates these complexities, facilitating the study of continuity, convergence, and related topological properties in a broader context. The geomentric representation of collection of neutrosophic sets namely cubic spherical neutrosophic sets [7, 13] and cubic spherical neutrosophic topology [8]plays vital role in recent research of this field.

Fermatean fuzzy sets (FFS) [18], introduced by Tapan Senapati and Ronald R. Yager, extend traditional fuzzy set theory by allowing for a more nuanced representation of uncertainty through DoM and DoN. A notable extension within this field is the introduction of Fermatean fuzzy topology [10], which integrates Fermatean fuzzy sets into topological structures. This enriched framework allows for more sophisticated modeling of complex systems. The geomentric representation of collection of Fermatean fuzzy sets namely circular Fermatean fuzzy sets [1, 16, 17]. The foundation of spherical Fermatean neutrosophic sets [14] [15] lies in their ability to encapsulate varying degrees of membership that reflect the inherent vagueness and ambiguity present in many real-world scenarios. By utilizing the principles of Fermatean fuzzy sets, these sets enable a more effective representation of uncertainties, thereby facilitating improved decision-making processes. The spherical representation further enhances this framework by allowing for a more comprehensive analysis of complex systems, particularly in fields where traditional binary logic and flat geometric structures are insufficient. Gonul Bilgin et. al. introduced and studied the notion of Fermatean neutrosophic topology [9].

In this paper, we further extend the concept of Fermatean neutrosophic sets [2–5] to introduce Spherical Fermatean Neutrosophic Sets (SFNS). A SFNS is defined by a four-tuple $\langle \aleph, \zeta(\aleph), \varkappa(\aleph), F(\aleph); \kappa \rangle$, where $\zeta(\aleph)$, $\varkappa(\aleph)$, $F(\aleph)$ and $\kappa(\aleph)$ are functions mapping Υ to $[0, 1]$. The additional component κ represents the radius of a sphere with the center at $(\zeta(\aleph), \varkappa(\aleph), F(\aleph))$, encapsulating the DoM, DoI and DoN.

Acronyms Used in the Article:

The following table lists the acronyms used in the article and their meanings:

Acronym	Meaning
DoM	Degrees of Membership
DoI	Degrees of Indeterminacy
DoN	Degrees of Non-membership
NS	Neutrosophic Sets
FNS	Fermatean Neutrosophic Sets
SFNS	Spherical Fermatean Neutrosophic Sets
SFNTS	Spherical Fermatean Neutrosophic Topological Spaces

2. Preliminaries

Definition 2.1. [19] A neutrosophic set A in a universe Υ is defined as:

$$A = \{\langle \aleph, \zeta_A(\aleph), \digamma_A(\aleph), \varkappa_A(\aleph)\rangle | \aleph \in \Upsilon\}$$

where $\zeta_A : \Upsilon \to [0,1]$, $\digamma_A : \Upsilon \to [0,1]$ and $\varkappa_A : \Upsilon \to [0,1]$. Here, $\zeta_A(\aleph)$ represents the DoM, $\digamma_A(\aleph)$ the DoI and $\varkappa_A(\aleph)$ the DoN.

Definition 2.2. [18] A Fermatean Fuzzy Set (FFS) A in the universe Υ is defined as:

$$A = \{\langle \aleph, \zeta_A(\aleph), \varkappa_A(\aleph)\rangle | \aleph \in \Upsilon\}$$

with the conditions $0 \leq \varkappa_A^3(\aleph) + \zeta_A^3(\aleph) \leq 1$ for all $\aleph \in \Upsilon$. The functions $\zeta_A(\aleph)$ and $\varkappa_A(\aleph)$ represent the DoM and DoN respectively.

Definition 2.3. [20] A Fermatean Neutrosophic Set (FNS) A in the universe Υ is defined as:

$$A = \{\langle \aleph, \zeta_A(\aleph), \digamma_A(\aleph), \varkappa_A(\aleph)\rangle | \aleph \in \Upsilon\}$$

with the conditions $0 \leq \zeta_A^3(\aleph) + \varkappa_A^3(\aleph) \leq 1$ and $0 \leq \digamma_A^3(\aleph) \leq 1$. where $\zeta_A : \Upsilon \to [0,1]$, $\digamma_A : \Upsilon \to [0,1]$ and $\varkappa_A : \Upsilon \to [0,1]$. Here, $\zeta_A(\aleph)$ represents the DoM, $\digamma_A(\aleph)$ the DoI and $\varkappa_A(\aleph)$ the DoN. This definition extends the concept of Fermatean fuzzy sets by incorporating an additional DoI.

Definition 2.4. [14,15] A Spherical Fermatean Neutrosophic Set (SFNS) A_κ in the universe Υ is defined as:

$$A_\kappa = \{\langle \aleph, \zeta(\aleph), \digamma(\aleph), \varkappa(\aleph); \kappa\rangle | \aleph \in \Upsilon\}$$

where $\zeta(\aleph)$, $\varkappa(\aleph)$, $\digamma(\aleph)$ and κ are functions mapping Υ to $[0,1]$. The radius κ represents the distance from the center $(\zeta(\aleph), \digamma(\aleph), \varkappa(\aleph))$ to the boundary of the sphere within the cube.

The center of the sphere is

$$< \zeta(\aleph_i), \varkappa(\aleph_i), \digamma(\aleph_i) >=< \frac{\sum_{j=1}^{k_i} \zeta_{i,j}}{k_i}, \frac{\sum_{j=1}^{k_i} \digamma_{i,j}}{k_i}, \frac{\sum_{j=1}^{k_i} \varkappa_{i,j}}{k_i} >$$

and the radius

$$\kappa_i = \min\left\{ \max_{1 \leq j \leq k_i} \sqrt{(\zeta(\aleph_i) - \zeta_{i,j})^2 + (\digamma(\aleph_i) - \digamma_{i,j})^2 + (\varkappa(\aleph_i) - \varkappa_{i,j})^2}, 1 \right\}.$$

Definition 2.5. [15] Let $\Delta_{\kappa_1} = \{\langle \aleph, \zeta_{\Delta_1}, \varkappa_{\Delta_1}, \digamma_{\Delta_1}; \kappa_{\Delta_1}\rangle : \aleph \in \Upsilon\}$ and $\Delta_{\kappa_2} = \{\langle \aleph, \zeta_{\Delta_2}, \varkappa_{\Delta_2}, \digamma_{\Delta_2}; \kappa_{\Delta_2}\rangle : \aleph \in \Upsilon\}$ be two SFNSs over the universal set Υ. Then the following operations are defined as follows

1. The union of any two SFN sets $\Delta_{\kappa_1} \cup_{\max} \Delta_{\kappa_2} =$
$\{< \aleph, \max\{\zeta_{\Delta_1}, \zeta_{\Delta_2}\}, \min\{\digamma_{\Delta_1}, \digamma_{\Delta_2}\}, \min\{\varkappa_{\Delta_1}, \varkappa_{\Delta_2}\}; \max\{\kappa_{\Delta_1}, \kappa_{\Delta_2}\} > : \aleph \in \Upsilon\}$.

2. The intersection of any two SFN sets $\Delta_{\kappa_1} \cap_{\min} \Delta_{\kappa_2} =$
$\{< \aleph, \min\{\zeta_{\Delta_1}, \zeta_{\Delta_2}\}, \max\{F_{\Delta_1}, F_{\Delta_2}\}, \max\{\varkappa_{\Delta_1}, \varkappa_{\Delta_2}\}; \min\{\kappa_{\Delta_1}, \kappa_{\Delta_2}\} > : \aleph \in \Upsilon\}$.
3. $\Delta_{\kappa_1} = \Delta_{\kappa_2}$ IFF $\{< \aleph, \zeta_{\Delta_1} = \zeta_{\Delta_2}, F_{\Delta_1} = F_{\Delta_2}, \varkappa_{\Delta_1} = \varkappa_{\Delta_2}; \kappa_{\Delta_1} = \kappa_{\Delta_2} >: \aleph \in \Upsilon\}$.
4. $\Delta_{\kappa_1} \subseteq \Delta_{\kappa_2}$ IFF $\{< \aleph, \zeta_{\Delta_1} \subseteq \zeta_{\Delta_2}, F_{\Delta_1} \supseteq F_{\Delta_2}, \varkappa_{\Delta_1} \supseteq \varkappa_{\Delta_2}; \kappa_{\Delta_1} \subseteq \kappa_{\Delta_2} >: \aleph \in \Upsilon\}$.
5. $\Delta^c_{\kappa_1} = \{< \aleph, \varkappa_{\Delta_1}, F_{\Delta_1}, \zeta_{\Delta_1}; \kappa_{\Delta_1} >: \aleph \in \Upsilon\}$.

Example 2.6. Let $B = \{\langle \aleph, 0.5, 0.3, 0.4\rangle, \langle \aleph, 0.6, 0.2, 0.3\rangle, \langle \aleph, 0.4, 0.5, 0.2\rangle, \langle \aleph, 0.5, 0.4, 0.5\rangle\}$ be a collection of SFNSs. Then the center of the SFNS is ,

$\zeta_\aleph = \frac{0.5+0.6+0.4+0.5}{4} = 0.5$, $F_\aleph = \frac{0.3+0.2+0.5+0.4}{4} = 0.35$, $\varkappa_\aleph = \frac{0.4+0.3+0.2+0.5}{4} = 0.35$,

The radius is given by:

$$\Delta_i = \min\left\{\max_{1\leq j\leq 4} \sqrt{(\zeta_\aleph - \zeta_{i,j})^2 + (F_\aleph - F_{i,j})^2 + (\varkappa_\aleph - \varkappa_{i,j})^2}, 1\right\}$$

$$\Delta_i = \min\left\{\max\left\{\sqrt{(0.5-0.5)^2 + (0.35-0.3)^2 + (0.35-0.4)^2},\right.\right.$$
$$\sqrt{(0.5-0.6)^2 + (0.35-0.2)^2 + (0.35-0.3)^2},$$
$$\sqrt{(0.5-0.4)^2 + (0.35-0.5)^2 + (0.35-0.2)^2},$$
$$\left.\left.\sqrt{(0.5-0.5)^2 + (0.35-0.4)^2 + (0.35-0.5)^2}\right\}, 1\right\}$$
$$= \min\{\max\{0.07, 0.14, 0.15, 0.15\}, 1\} = 0.15$$

Thus, the SFNS is: $A_\kappa = \langle 0.5, 0.35, 0.35; 0.15\rangle$.

3. SPHERICAL FERMATEAN NEUTROSOPHIC TOPOLOGICAL SPACES

In this section, we study the new notion namely spherical Fermatean neutrosophic topology and its characterization. The spherical Fermatean neutrosophic $1_\odot$ and spherical Fermatean neutrosophic $0_\odot$ in Υ as follows $1_\odot = \langle 1, 0, 0; 1\rangle$, and $0_\odot = \langle 0, 1, 1; 0\rangle$.

Proposition 3.1. *Let $\Delta_{\kappa_1} = < \zeta_{\Delta_1}, \varkappa_{\Delta_1}, F_{\Delta_1}; \kappa_{\Delta_1} >$ and $\Delta_{\kappa_2} = < \zeta_{\Delta_2}, \varkappa_{\Delta_2}, F_{\Delta_2}; \kappa_{\Delta_2} >$ be two SFNSs over the universal set Υ. Then the following hold:*

(1) $\Delta_{\kappa_1} \cup \Delta_{\kappa_1} = \Delta_{\kappa_1}$ *and* $\Delta_{\kappa_1} \cap \Delta_{\kappa_1} = \Delta_{\kappa_1}$.
(2) $\Delta_{\kappa_1} \cup 0_\odot = \Delta_{\kappa_1}$ *and* $\Delta_{\kappa_1} \cap 0_\odot = 0_\odot$.
(3) $\Delta_{\kappa_1} \cup 1_\odot = 1_\odot$ *and* $\Delta_{\kappa_1} \cap 1_\odot = \Delta_{\kappa_1}$.
(4) $(\Delta^c_{\kappa_1})^c = \Delta_{\kappa_1}$.

Definition 3.2. Let $\Xi_\odot \in FN(\Upsilon)$, then $\Xi_\odot$ is called a spherical Fermatean neutrosophic topology on Υ, if the following hold

(1) $1_\odot, 0_\odot \in \Xi_\odot$.
(2) $\Delta_{\kappa_1}, \Delta_{\kappa_2} \in \Xi_\odot \Rightarrow \Delta_{\kappa_1} \cap \Delta_{\kappa_2} \in \Xi_\odot$.
(3) $\{\Delta_{\kappa_i}; i \in \triangle\} \subseteq \Xi_\odot \Rightarrow \bigcup \Delta_{\kappa_i} \in \Xi_\odot$.

The pair $(\Upsilon, \Xi_\odot)$ is called a Spherical Fermatean Neutrosophic Topological Space (FNTS) over Υ. This generalization involves considering sphere for the DoM, DoI and DoN in the context of topology. Furthermore, the members of $\Xi_{\odot}$ are said to be SFN-open sets in Υ. If $\Delta^c_{\kappa_1} \in \Xi_{\odot}$, then $\Delta_{\kappa_1} \in FN(\Upsilon)$ is said to be SFN-closed set in Υ.

The SFN-interior of set Δ_{κ_1}, denoted by $\Gamma_{\odot}(\Delta_{\kappa_1})$ is defined as the union of all SFN-open subsets of Δ_{κ_1}. Notably, $\Gamma_{\odot}(\Delta_{\kappa_1})$ represents the largest SFN-open set over Υ that containing Δ_{κ_1}. The SFN-closure of set Δ_{κ_1}, denoted by $\Delta_{\odot}(\Delta_{\kappa_1})$ is defined as the intersection of all SFN-closed supersets of Δ_{κ_1}. Notably, $\Delta_{\odot}(\Delta_{\kappa_1})$ represents the smallest SFN-closed set over Υ that contains Δ_{κ_1}.

Let $\Xi_{\odot} = \{0_{\odot}, 1_{\odot}\}$ and $\Psi_{\odot} = FN(\Upsilon)$. Then, $(\Upsilon, \Xi_{\odot})$ and $(\Upsilon, \Psi_{\odot})$ are two trivial FNTS over Υ. Additionally, they are referred to as FN-discrete topological space and FN-indiscreet topological space over Υ, respectively.

Example 3.3. Let Υ=$\{\aleph, \sqsupset\}$ and Δ_1, $\Delta_2 \in SFNS(\Upsilon)$ such that
$\Delta_1 = \{\langle \aleph, 0.7, 0.6, 0.3\rangle, \{\langle \aleph, 0.8, 0.1, 0.4\rangle, \{\langle \aleph, 0.95, 0.05, 0.3\rangle\}$ and
$\Delta_2 = \{\langle \sqsupset, 0.2, 0.1, 0.5\rangle, \langle \sqsupset, 0.4, 0.3, 0.6\rangle, \langle \sqsupset, 0.6, 0.5, 0.4\rangle\}$. Then

(1) The SFNSs are $\Delta_{\kappa_1} = \{\langle \aleph, 0.82, 0.25, 0.33; 0.37\rangle : \aleph \in \Upsilon\}$ and $\Delta_{\kappa_2} = \{\langle \sqsupset, 0.40, 0.30, 0.50; 0.30\rangle : \sqsupset \in \Upsilon\}$.
(2) The union of two SFNSs Δ_{κ_1} and Δ_{κ_2} is $\Delta_{\kappa_1} \cup_{\max} \Delta_{\kappa_2} = \{\langle \aleph, 0.82, 0.25, 0.33; 0.37\rangle : \sqsupset \in \Upsilon\}$.
(3) The intersection of two SFNSs Δ_{κ_1} and Δ_{κ_2} is $\Delta_{\kappa_1} \cap_{\min} \Delta_{\kappa_2} = \{\langle \sqsupset, 0.40, 0.30, 0.50; 0.30\rangle : \sqsupset \in \Upsilon\}$.
(4) The complement of a SFNS Δ_{κ_1} is $\Delta^c_{\kappa_1} = \{\langle \aleph, 0.33, 0.44, 0.82; 0.20\rangle : \aleph \in \Upsilon\}$.
(5) We have that $\Delta_{\kappa_2} \subset \Delta_{\kappa_1}$.
(6) The family $\Xi_{\odot} = \{0_{\odot}, 1_{\odot}, \Delta_{\kappa_1}, \Delta_{\kappa_2}\}$, of SFNSs in Υ is SFNTS.
(7) The geometric representation of Δ_1, Δ_2, Δ_{κ_1}, and Δ_{κ_2} are

Proposition 3.4. *Let* $(\Upsilon, \Xi_{1\odot})$ *and* $(\Upsilon, \Xi_{2\odot})$ *be two SFNTSs over* Υ, *then* $(\Upsilon, \Xi_{1\odot} \cap \Xi_{2\odot})$ *is a SFNTS over* Υ.

Proof. *Let* $(\Upsilon, \Xi_{1\odot})$ *and* $(\Upsilon, \Xi_{2\odot})$ *be two SFNTSs over* Υ. *It can be seen clearly that* $0_{\odot}, 1_{\odot} \in \Xi_{1\odot} \cap \Xi_{2\odot}$. *If* $\Delta_{\kappa_1}, \Delta_{\kappa_2} \in \Xi_{1\odot} \cap \Xi_{2\odot}$ *then,* $\Delta_{\kappa_1}, \Delta_{\kappa_2} \in \Xi_{1\odot}$ *and* $\Delta_{\kappa_1}, \Delta_{\kappa_2} \in \Xi_{2\odot}$. *It is given that* $\Delta_{\kappa_1} \cap \Delta_{\kappa_2} \in \Xi_{1\odot}$ *and* $\Delta_{\kappa_1} \cap \Delta_{\kappa_2} \in \Xi_{2\odot}$. *Thus,* $\Delta_{\kappa_1} \cap \Delta_{\kappa_2} \in \Xi_{1\odot} \cap \Xi_{2\odot}$. *Let* $\{\Delta_{\kappa_1 i} : i \in I\} \subseteq \Xi_{1\odot} \cap \Xi_{2\odot}$. *Then,* $\Delta_{\kappa_1 i} \in \Xi_{1\odot} \cap \Xi_{2\odot}$ *for all* $i \in I$. *Thus,* $\Delta_{\kappa_1 i} \in \Xi_{1\odot}$ *and* $\Delta_{\kappa_1} i \in \Xi_{2\odot}$ *for all* $i \in I$. *So, we have* $\bigcap_{i \in I} \Delta_{\kappa_1 i} \in \Xi_{1\odot} \cap \Xi_{2\odot}$.

Corollary 3.5. *Let* $\{(\Upsilon, \Xi_{\odot i}) : i \in I\}$ *be a family of SFNTSs over* $\mathbb{X}$. *Then,* $(\Upsilon, \bigcap_{i \in I} \Xi_{\odot i})$ *is a SFNTS over* $\mathbb{X}$.

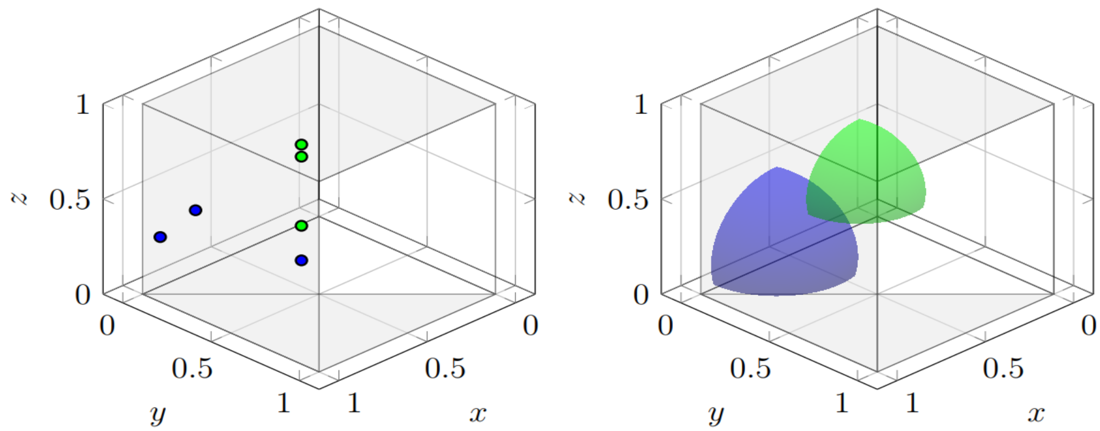


FIGURE 1

Example 3.6. Let $\Upsilon = \{\aleph\}$, and Δ_1, Δ_2, $\Delta_3 \in FNS(\Upsilon)$ such that

$\Delta_1 = \{\langle\aleph, 0.7, 0.6, 0.3\rangle, \{\langle\aleph, 0.8, 0.1, 0.4\rangle, \{\langle\aleph, 0.95, 0.05, 0.3\rangle\}$

$\Delta_2 = \{\langle\aleph 0.2, 0.1, 0.5\rangle, \langle\aleph, 0.4, 0.3, 0.6\rangle, \langle\aleph, 0.6, 0.5, 0.4\rangle\}$.

$\Delta_3 = \{\langle\aleph, 0.8, 0.1, 0.4\rangle, \langle\aleph, 0.7, 0.6, 0.3\rangle, \langle\aleph, 0.7, 0.6, 0.3\rangle\}$.

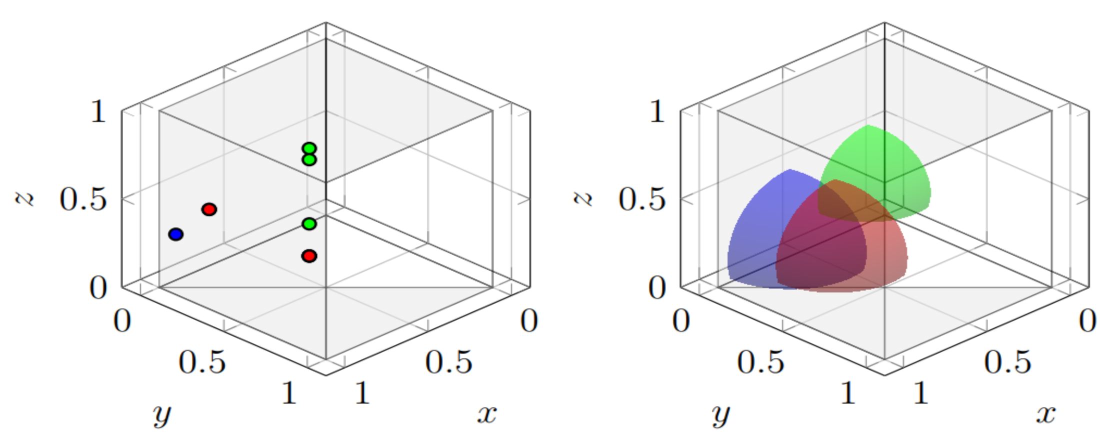
Then the SFNSs are

$\Delta_{\kappa_1} = \{\langle\aleph, 0.82, 0.25, 0.33; 0.37\rangle : \aleph \in \Upsilon\}$,

$\Delta_{\kappa_2} = \{\langle\aleph, 0.40, 0.30, 0.50; 0.30\rangle : \aleph \in \Upsilon\}$,

$\Delta_{\kappa_3} = \{\langle\aleph, 0.73, 0.43, 0.33; 0.35\rangle : \aleph \in \Upsilon\}$ and

The family $\Xi_{\odot} = \{0_{\odot}, 1_{\odot}, \Delta_{\kappa_1}, \Delta_{\kappa_2}\}$, and $\Psi_{\odot} = \{0_{\odot}, 1_{\odot}, \Delta_{\kappa_1}, \Delta_{\kappa_3}\}$, are SFNTSs but their union $\Xi_{\odot} \cup \Psi_{\odot}$ is not a SFNTS, since $\Delta_{\kappa_2} \cup \Delta_{\kappa_3} \notin \Xi_{\odot} \cup \Psi_{\odot}$.



Proposition 3.7. *Let $(\Upsilon, \Xi_{\odot})$ be a SFNTS over Υ. Then*

(1) $0_{\odot}$ *and* $1_{\odot}$ *are SFN closed sets over* Υ.

(2) *The intersection of any number of SFN-closed sets is a SFN-closed set over* Υ.

(3) *The union of any two SFN-closed sets is a SFN-closed set over* Υ.

Proposition 3.8. *Let* $(\Upsilon, \Xi_{\odot})$ *be a SFNTS over* Υ *and* $\Delta_{\kappa_1}, \Delta_{\kappa_2} \in FNS(\Upsilon)$. *Then*

(1) $\Gamma_{\odot}(0_{\odot}) = 0_{\odot}$ *and* $\Gamma_{\odot}(1_{\odot}) = 1_{\odot}$.
(2) $\Gamma_{\odot}(\Delta_{\kappa_1}) \subseteq \Delta_{\kappa_1}$.
(3) Δ_{κ_1} *is a SFN-open set if and only if* $\Delta_{\kappa_1} = \Gamma_{\odot}(\Delta_{\kappa_1})$.
(4) $\Gamma_{\odot}(\Gamma_{\odot}(\Delta_{\kappa_1})) = \Gamma_{\odot}(\Delta_{\kappa_1})$.
(5) $\Delta_{\kappa_1} \subseteq \Delta_{\kappa_2}$ *implies* $\Gamma_{\odot}(\Delta_{\kappa_1}) \subseteq \Gamma_{\odot}(\Delta_{\kappa_2})$.
(6) $\Gamma_{\odot}(\Delta_{\kappa_1}) \cup \Gamma_{\odot}(\Delta_{\kappa_2}) \subseteq \Gamma_{\odot}(\Delta_{\kappa_1} \cup \Delta_{\kappa_2})$.
(7) $\Gamma_{\odot}(\Delta_{\kappa_1} \cap \Delta_{\kappa_2}) = \Gamma_{\odot}(\Delta_{\kappa_1}) \cap \Gamma_{\odot}(\Delta_{\kappa_2})$.

Proof. *1. and 2. are obvious.*

3. If A *is a SFN-open set over* Υ, *then* A *is itself a SFN-open set over* Υ *which contains* A. *So,* A *is the largest neutrosophic open set contained in* A *and* $\Gamma_{\odot}(A) = A$. *Conversely, suppose that* $\Gamma_{\odot}(A) = A$. *Then* $A \in \Xi_{\odot}$.

4. Let $\Gamma_{\odot}(A) = B$. *Then,* $\Gamma_{\odot}(B) = B$ *from 3. and then,* $\Gamma_{\odot}(\Gamma_{\odot}(A)) = \Gamma_{\odot}(A)$.

5. Suppose that $A \subseteq B$. *As* $\Gamma_{\odot}(A) \subseteq A \subseteq B$. *By definition, we have* $\Gamma_{\odot} \subseteq \Gamma_{\odot}(B)$.

6. It is clear that $A \subseteq A \cup B$ *and* $B \subseteq A \cup B$. *Thus* $\Gamma_{\odot}(A) \subseteq \Gamma_{\odot}(A \cup B)$ *and* $\Gamma_{\odot}(B) \subseteq \Gamma_{\odot}(A \cup B)$. *So we have* $\Gamma_{\odot}(A) \cup \Gamma_{\odot}(B) \subseteq \Gamma_{\odot}(A \cup B)$.

7. It is known that $\Gamma_{\odot}(A \cap B) \subseteq \Gamma_{\odot}(A)$ *and* $\Gamma_{\odot}(A \cap B) \subseteq \Gamma_{\odot}(B)$ *by 5. So that* $\Gamma_{\odot}(A \cap B) \subseteq \Gamma_{\odot}(A) \cap \Gamma_{\odot}(B)$. *Also, from* $\Gamma_{\odot}(A) \subseteq A$ *and* $\Gamma_{\odot}(B) \subseteq B$, *we have* $\Gamma_{\odot}(A) \cap \Gamma_{\odot}(B) \subseteq A \cap B$. *These imply that* $\Gamma_{\odot}(A \cap B) = \Gamma_{\odot}(A) \cap \Gamma_{\odot}(B)$.

Proposition 3.9. *Let* $(\Upsilon, \Xi_{\odot})$ *be a SFNTS over* Υ *and* $\Delta_{\kappa_1}, \Delta_{\kappa_2} \in FNS(\Upsilon)$. *Then*

(1) $\Delta_{\odot}(0_{\odot}) = 0_{\odot}$ *and* $\Delta_{\odot}(1_{\odot}) = 1_{\odot}$.
(2) $\Delta_{\kappa_1} \subseteq \Delta_{\odot}(\Delta_{\kappa_1})$.
(3) Δ_{κ_1} *is a SFN closed set if and only if* $\Delta_{\kappa_1} = \Delta_{\odot}(\Delta_{\kappa_1})$.
(4) $\Delta_{\odot}(\Delta_{\odot}(\Delta_{\kappa_1})) = \Delta_{\odot}(\Delta_{\kappa_1})$.
(5) $\Delta_{\kappa_1} \subseteq \Delta_{\kappa_2}$ *implies* $\Delta_{\odot}(\Delta_{\kappa_1}) \subseteq \Delta_{\odot}(\Delta_{\kappa_2})$.
(6) $\Delta_{\odot}(\Delta_{\kappa_1} \cup \Delta_{\kappa_2}) = \Delta_{\odot}(\Delta_{\kappa_1}) \cup \Delta_{\odot}(\Delta_{\kappa_2})$.
(7) $\Delta_{\odot}(\Delta_{\kappa_1} \cap \Delta_{\kappa_2}) \subseteq \Delta_{\odot}(\Delta_{\kappa_1}) \cap \Delta_{\odot}(\Delta_{\kappa_2})$.

Corollary 3.10. *Let* $(\Upsilon, \Xi_{\bigodot})$ *be a SFNTS over* Υ *and* $\Delta_{\kappa_1} \in FN(\Upsilon)$. *Then* $\Gamma_{\bigodot}(\Delta^c_{\kappa_1}) = (\Delta_{\bigodot}(\Delta_{\kappa_1}))^c$. *and* $\Delta_{\bigodot}(\Delta^c_{\kappa_1}) = (\Gamma_{\bigodot}(\Delta_{\kappa_1}))^c$.

Proposition 3.11. *Let* $(\Upsilon, \Xi_{\bigodot})$ *be a FNTS over* Υ, $\Delta_{\kappa_1} * \Delta_{\kappa_2}$ *be a FN set on* Υ. *Then* $(\Delta_{\kappa_1} * \Delta_{\kappa_2}) \cup (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'}$ *is a SFN-closed set.*

Proof. To prove $(\Delta_{\kappa_1} * \Delta_{\kappa_2}) \cup (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'}$ is SFN-closed it is enough to prove that $\left((\Delta_{\kappa_1} * \Delta_{\kappa_2}) \cup (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'}\right)^{'}$ is SFN-open. If $\left((\Delta_{\kappa_1} * \Delta_{\kappa_2}) \cup (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'}\right)^{'} = \phi$ then it is SFN-open. Let $\left((\Delta_{\kappa_1} * \Delta_{\kappa_2}) \cup (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'}\right)^{'} \neq \phi$ and $\aleph \in \left((\Delta_{\kappa_1} * \Delta_{\kappa_2}) \cup (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'}\right)^{'}$ $\Rightarrow \aleph \notin (\Delta_{\kappa_1} * \Delta_{\kappa_2}) \cup (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'} \Rightarrow \aleph \notin (\Delta_{\kappa_1} * \Delta_{\kappa_2})$ and $\aleph \notin (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'} \Rightarrow \exists$ a SFN-open set $G * B \ni \aleph \in (G * B)$ and $(G * B) \cap (\Delta_{\kappa_1} * \Delta_{\kappa_2}) = \phi \Rightarrow \aleph \in (G * B) \subseteq (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'}$ Again $\aleph \notin (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'} \Rightarrow \aleph \in G * B \subseteq ((\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'})^{'}$. Therefore $\aleph \in (G * B) \subseteq \left((\Delta_{\kappa_1} * \Delta_{\kappa_2}) \cup (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'}\right)^{'}$ and hence $\left((\Delta_{\kappa_1} * \Delta_{\kappa_2}) \cup (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'}\right)^{'}$ is SFN-open set.

Proposition 3.12. $(\Upsilon, \Xi_{\bigodot})$ *be a SFNTS over* Υ *and* $\Delta_{\kappa_1} * \Delta_{\kappa_2}$ *be a SFN set over* X *then* $\Delta_{\kappa_1} * \Delta_{\kappa_2}$ *is a SFN-closed IFF* $\overline{\Delta_{\kappa_1} * \Delta_{\kappa_2}} = \Delta_{\kappa_1} * \Delta_{\kappa_2}$

Proof. If $\Delta_{\kappa_1} * \Delta_{\kappa_2}$ is an SFN-closed set then the smallest SFN-closed super set of $\Delta_{\kappa_1} * \Delta_{\kappa_2}$ is $\Delta_{\kappa_1} * \Delta_{\kappa_2}$ itself. Therefore $\overline{\Delta_{\kappa_1} * \Delta_{\kappa_2}} = \Delta_{\kappa_1} * \Delta_{\kappa_2}$. Conversely if $\overline{\Delta_{\kappa_1} * \Delta_{\kappa_2}} = \Delta_{\kappa_1} * \Delta_{\kappa_2}$ then $\overline{\Delta_{\kappa_1} * \Delta_{\kappa_2}}$ being SFN-closed so as $\Delta_{\kappa_1} * \Delta_{\kappa_2}$.

4. **Distance between Spherical Fermatean Neutrosophic Sets**

Definition 4.1. The distance between two SFNS A_κ and B_κ can be defined using a metric that incorporates the differences in their parameters:

$$D(A_\kappa, B_\kappa) = \sqrt{(\zeta_A(\aleph) - \zeta_B(y))^2 + (\varkappa_A(\aleph) - \varkappa_B(y))^2 + (F_A(\aleph) - F_B(y))^2}$$

where:

$$A_\kappa = \{\langle \aleph, \zeta_A(\aleph), \varkappa_A(\aleph), F_A(\aleph); \kappa_A \rangle | \aleph \in X\}$$

$$B_\kappa = \{\langle y, \zeta_B(y), \varkappa_B(y), F_B(y); \kappa_B \rangle | y \in Y\}$$

$D = D(A_\kappa, B_\kappa) - (\kappa_A + \kappa_B)$

Example 4.2. Consider two SFNSs:

$$A_\rho = \langle 0.7, 0.2, 0.1; 0.1 \rangle \quad \text{and} \quad B_\rho = \langle 0.4, 0.4, 0.2; 0.1 \rangle$$

The center of SFNS A_ρ is $(0.7, 0.2, 0.1)$ and for B_ρ is $(0.4, 0.4, 0.2)$, both with radius 0.1.

$$d = \sqrt{(0.7 - 0.4)^2 + (0.2 - 0.4)^2 + (0.1 - 0.2)^2}$$
$$= \sqrt{(0.3)^2 + (-0.2)^2 + (-0.1)^2}$$

$$= \sqrt{0.09 + 0.04 + 0.01} = \sqrt{0.14} \approx 0.374$$

$$D = d - (\rho_A + \rho_B) = 0.374 - (0.1 + 0.1) = 0.374 - 0.2 = 0.174$$

This indicates separation of two SFNSs.

Example 4.3. Consider two SFNSs:

$$A_\rho = \langle 0.5, 0.4, 0.1; 0.1\rangle \quad \text{and} \quad B_\rho = \langle 0.6, 0.3, 0.1; 0.1\rangle$$

The center of SFNS A_ρ is $(0.5, 0.4, 0.1)$ and for B_ρ is $(0.6, 0.3, 0.1)$, both with radius 0.1.

Distance Calculation:

$$d = \sqrt{(0.5 - 0.6)^2 + (0.4 - 0.3)^2 + (0.1 - 0.1)^2}$$

$$= \sqrt{(-0.1)^2 + (0.1)^2 + (0)^2}$$

$$= \sqrt{0.01 + 0.01 + 0} = \sqrt{0.02} \approx 0.1414$$

$$D = d - (\rho_A + \rho_B) = 0.1414 - (0.1 + 0.1) = 0.1414 - 0.2 = -0.0586$$

This indicates intersection of two SFNSs.

Example 4.4. Consider two SFNSs:

$$A_\rho = \langle 0.6, 0.3, 0.1; 0.1\rangle \quad \text{and} \quad B_\rho = \langle 0.6, 0.3, 0.1; 0.1\rangle$$

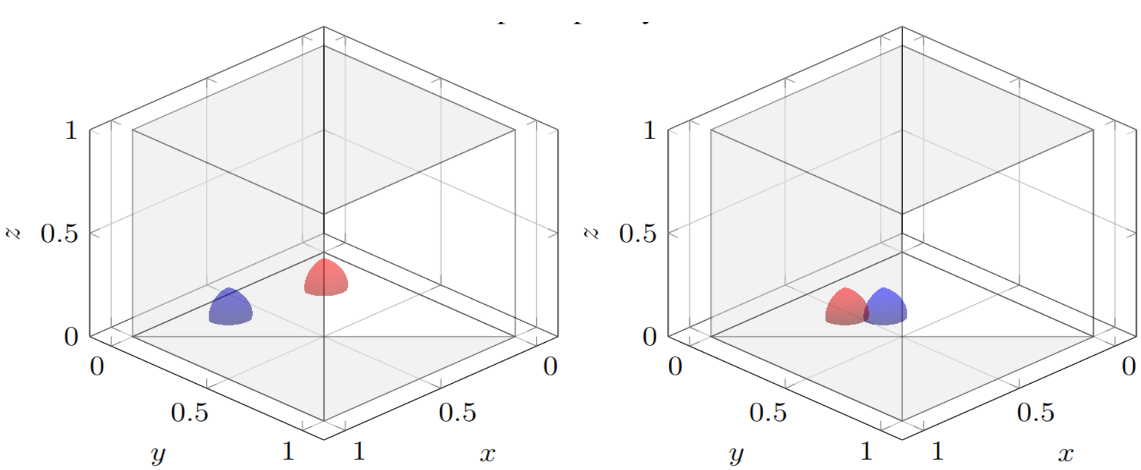
The center of both SFNSs is $(0.6, 0.3, 0.1)$ and the radius of both SFNSs is 0.1.

Distance Calculation:

$$d = \sqrt{(0.6 - 0.6)^2 + (0.3 - 0.3)^2 + (0.1 - 0.1)^2} = \sqrt{0 + 0 + 0} = 0$$

$$D = d - (\rho_A + \rho_B) = 0 - (0.1 + 0.1) = 0 - 0.2 = -0.2$$

This indicates that the two SFNSs overlap completely.



Conclusion and Future Work:

In this paper, we introduced SFNTS as an extension of neutrosophic sets and Fermatean fuzzy sets, characterized by Degrees of Membership, Degrees of Indeterminacy, and Degrees of Non-Membership, along with a radius component. We defined the properties of SFN closure and SFN interior operators, providing numerical examples and a geometric representation of SFNTS. Furthermore, we explored the SFN distance between spheres, as well as the separation, intersection, and overlapping of two SFNSs. These foundational concepts and examples demonstrate the potential of SFNTS in modeling and analyzing complex systems that require a nuanced representation of uncertainty and vagueness.

Future research can extend the study of SFNTS by exploring its applications in real-world scenarios, such as decision-making, pattern recognition, and multi-criteria analysis. Additionally, there is significant potential in examining the topological properties of SFNTS, such as continuity, connectedness, compactness, separation axioms, metrizability, homotopy, homology and other fundamental concepts. Investigating the relationships between SFNTS and other topological structures.

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