Neutrosophic Sets and Systems, Vol. 75, 2025
University of New Mexico

A Numerical Approach For The Algebra of Two-Fold 2×2 Fuzzy Real Matrices and Anti Fuzzy Matrices

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Abstract:

This paper is dedicated to study for the first time the concept of square 2×2 fuzzy and anti-fuzzy two-fold matrix with real entries, where we present the two-fold algebraic operations between the these matrices and obtain their special properties such as associativity, commutative properties, and the existing of additive of multiplicative inverses. Also, we provide many examples to explain the two-fold algebraic properties and the two-fold algebraic structure.

Keywords: two-fold algebra, two-fold real numbers, two-fold fuzzy matrix, two-fold inverse, anti-fuzzy matrix.

Introduction

The theory of matrices based on mathematical and algebraic logic has been studied by different authors around the world. For example, we can see many important results about the computations of plithogenic matrices in [5].

Also, neutrosophic matrices and their extended types have been defined and handled in [1-3], where many problems such as diagonalization, inverses, and algebraic representations were studied and provided [4, 6].

The two-fold algebra and two-fold algebraic structures began in [7], and then these algebras in fuzzy versions were used in many different contexts, such as mathematical analysis, algebraic elements, and in other types of mathematical structures [8-23].

In this paper we to study for the first time the concept of square 2×2 fuzzy and anti-fuzzy two-fold matrix with real entries, where we present the two-fold algebraic operations between the these matrices and obtain their special properties such as associativity, commutative properties, and the existing of additive of multiplicative inverses.

Two-Fold 2×2 -Fuzzy Real Matrices

Definition:

Let $\mathbb{R}$ be the real field, $\mu: \mathbb{R} \to [0,1]$ such that $\begin{cases} \mu(0) = 0 \\ \mu(1) = 1 \end{cases}$ be a fuzzy mapping. Let $\mathbb{R}_F = \{x_{\mu(y)} : x, y \in \mathbb{R}\}$be the corresponding two-fold fuzzy real algebra, then we define:

$$A = \begin{pmatrix} {x_1}_{\mu(y_1)} & {x_2}_{\mu(y_2)} \\ {x_3}_{\mu(y_3)} & {x_4}_{\mu(y_4)} \end{pmatrix} \ ; \ x_i, y_i \in \mathbb{R}.$$

A is called a 2×2 fuzzy two-fold matrix.

Definition:

Let $a_i = (x_i)_{\mu(y_i)} \quad 1 \le i \le 4$, $b_i = (z_i)_{\mu(t_i)}$; $1 \le i \le 4$ and $x_i, y_i, z_i, t_i \in \mathbb{R}$, $a_i, b_i \in \mathbb{R}_F$.

Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$, we define:

$$A + B = \begin{pmatrix} a_1 * b_1 & a_2 * b_2 \\ a_3 * b_3 & a_4 * b_4 \end{pmatrix} \ ; a_i * b_i = (x_i + z_i)_{\min(\mu(y_i), \mu(t_i))}$$

$$-A = \begin{pmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 \end{pmatrix}; -a_i = (-x_i)_{\mu(-x_i)}$$

$$A \times B = \begin{pmatrix} (a_1 \circ b_1) * (a_2 \circ b_3) & (a_1 \circ b_2) * (a_2 \circ b_4) \\ (a_3 \circ b_1) * (a_4 \circ b_3) & (a_3 \circ b_2) * (a_4 \circ b_4) \end{pmatrix}, \text{ where } a_i \circ b_j = (x_i z_j)_{\max(\mu(x_i), \mu(t_j))}$$

Example:S

Take: $\mu:\mathbb{R}\to[0,1]:\mu(x)=\begin{cases}0 & ;x=0\\ 1 & ;x=1\\ \frac{1}{2} & ;x>1\\ \frac{1}{3} & ;x<0\\ \frac{1}{4} & ;0<x<1\end{cases}$

And: $a_1=3_{\mu(2)}=3_{\frac{1}{2}},a_2=4_{\mu\left(\frac{1}{3}\right)}=4_{\frac{1}{4}},$

$$a_3=1_{\mu(-5)}=1_{\frac{1}{3}},a_4=0_{\mu(0)}=0_0\ ,$$

$A=\begin{pmatrix}a_1 & a_2\\ a_3 & a_4\end{pmatrix}=\begin{pmatrix}3_{\frac{1}{2}} & 4_{\frac{1}{4}}\\ 1_{\frac{1}{3}} & 0_0\end{pmatrix}.$

$$b_1=(-2)_{\mu(1)}=(-2)_1\ ,b_2=(-1)_{\mu(7)}=(-1)_{\frac{1}{2}},b_3=(1)_{\mu(0)}=1_0\ ,$$

$b_4=(2)_{\mu(\frac{1}{10})}=2_{\frac{1}{4}}\ ,B=\begin{pmatrix}b_1 & b_2\\ b_3 & b_4\end{pmatrix}=\begin{pmatrix}(-2)_1 & (-1)_{\frac{1}{2}}\\ 1_0 & 2_{\frac{1}{4}}\end{pmatrix}.$

We have:

$-A=\begin{pmatrix}(-3)_{\mu(-2)} & (-4)_{\mu\left(\frac{-1}{3}\right)}\\ (-1)_{\mu(5)} & (-0)_{\mu(-0)}\end{pmatrix}=\begin{pmatrix}(-3)_{\frac{1}{3}} & (-4)_{\frac{1}{3}}\\ (-1)_{\frac{1}{2}} & 0_0\end{pmatrix}.$

$A+B=\begin{pmatrix}a_1*b_1 & a_2*b_2\\ a_3*b_3 & a_4*b_4\end{pmatrix}=\begin{pmatrix}1_{\frac{1}{2}} & (3)_{\frac{1}{4}}\\ 2_0 & 2_0\end{pmatrix}\ ;\begin{cases}a_1*b_1=(3-2)_{\frac{1}{2}}=1_{\frac{1}{2}}\\ a_2*b_2=(4-1)_{\frac{1}{4}}=3_{\frac{1}{4}}\\ a_3*b_3=(1+1)_0=2_0\\ a_4*b_4=(0+2)_0=2_0\end{cases}$

$A\times B=\begin{pmatrix}(a_1\circ b_1)*(a_2\circ b_3) & (a_1\circ b_2)*(a_2\circ b_4)\\ (a_3\circ b_1)*(a_4\circ b_3) & (a_3\circ b_2)*(a_4\circ b_4)\end{pmatrix}$

$$=\begin{pmatrix}(-2)_{\frac{1}{4}} & 5_{\frac{1}{4}}\\ (-2)_0 & (-1)_{\frac{1}{4}}\end{pmatrix};\ \begin{cases}a_1\circ b_1=(-6)_1,a_2\circ b_3=4_{\frac{1}{4}}\\ (a_1\circ b_1)*(a_2\circ b_3)=(-2)_{\frac{1}{4}}\\ a_1\circ b_2=(-3)_{\frac{1}{2}},a_2\circ b_4=8_{\frac{1}{4}}\\ (a_1\circ b_2)*(a_2\circ b_4)=5_{\frac{1}{4}}\end{cases}$$

And: $\begin{cases}a_3\circ b_1=(-2)_1,a_4\circ b_3=(0)_0\\ (a_3\circ b_1)*(a_4\circ b_3)=(-2)_0\\ a_3\circ b_2=(-1)_{\frac{1}{2}},a_4\circ b_4=0_{\frac{1}{4}}\\ (a_3\circ b_2)*(a_4\circ b_4)=(-1)_{\frac{1}{4}}\end{cases}$

Take $C=\begin{pmatrix}c_1 & c_2\\ c_3 & c_4\end{pmatrix}=\begin{pmatrix}1_1 & 0_1\\ 1_0 & 0_0\end{pmatrix}$, then:

$B \times C = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \times \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} b_1c_1 + b_2c_3 & b_1c_2 + b_2c_4 \\ b_3c_1 + b_4c_3 & b_3c_2 + b_4c_4 \end{pmatrix}.$

$b_1c_1 = (-2)_1 \circ (1)_1 = (-2)_1, b_2c_3 = (-1)_{\frac{1}{2}} \circ (1_0) = (-1)_{\frac{1}{2}},\ b_1c_1 + b_2c_3 = (-2)_1 * (-1)_{\frac{1}{2}} = (-3)_{\frac{1}{2}},$

$b_1c_2 = (-2)_1 \circ (0)_1 = (0)_1, b_2c_4 = (-1)_{\frac{1}{2}} \circ (0)_0 = (0)_{\frac{1}{2}},\ b_1c_2 + b_2c_4 = (0)_1 * (0)_{\frac{1}{2}} = (0)_{\frac{1}{2}},$

$b_3c_1 = (1_0) \circ (1_1) = (1_1), b_4c_3 = (2_{\frac{1}{4}}) \circ (1_0) = (2_{\frac{1}{4}}),\ b_3c_1 + b_4c_3 = (1_1) * (2_{\frac{1}{4}}) = (3_{\frac{1}{4}}),$

$b_3c_2 = (1_0) \circ (0_1) = (0_1), b_4c_4 = (2_{\frac{1}{4}}) \circ (0_0) = (0_{\frac{1}{4}}),\ b_3c_2 + b_4c_4 = (0_1) * (0_{\frac{1}{4}}) = (0_{\frac{1}{4}}),$

Thus $B \times C = \begin{pmatrix} -3_{\frac{1}{2}} & 0_{\frac{1}{2}} \\ 3_{\frac{1}{4}} & 0_{\frac{1}{4}} \end{pmatrix}.$

$A \times (B \times C) = \begin{pmatrix} 3_{\frac{1}{2}} & 4_{\frac{1}{4}} \\ 1_{\frac{1}{3}} & 0_0 \end{pmatrix} \times \begin{pmatrix} -3_{\frac{1}{2}} & 0_{\frac{1}{2}} \\ 3_{\frac{1}{4}} & 0_{\frac{1}{4}} \end{pmatrix} = \begin{pmatrix} 3_{\frac{1}{4}} & 0_{\frac{1}{4}} \\ (-3)_{\frac{1}{4}} & 0_{\frac{1}{4}} \end{pmatrix},$

$\left(3_{\frac{1}{2}}\right) \circ \left(-3_{\frac{1}{2}}\right) * \left(4_{\frac{1}{4}}\right) \circ \left(3_{\frac{1}{4}}\right) = \left(-9_{\frac{1}{2}}\right) * \left(12_{\frac{1}{4}}\right) = 3_{\frac{1}{4}},$

$\left(3_{\frac{1}{2}}\right) \circ \left(0_{\frac{1}{2}}\right) * \left(4_{\frac{1}{4}}\right) \circ \left(0_{\frac{1}{4}}\right) = \left(0_{\frac{1}{2}}\right) * \left(0_{\frac{1}{4}}\right) = 0_{\frac{1}{4}},$

$\left(1_{\frac{1}{3}}\right) \circ \left(-3_{\frac{1}{2}}\right) * (0_0) \circ \left(3_{\frac{1}{4}}\right) = \left(-3_{\frac{1}{2}}\right) * \left(0_{\frac{1}{4}}\right) = (-3)_{\frac{1}{4}},$

$\left(1_{\frac{1}{3}}\right) \circ \left(0_{\frac{1}{2}}\right) * (0_0) \circ \left(0_{\frac{1}{4}}\right) = \left(0_{\frac{1}{2}}\right) * \left(0_{\frac{1}{4}}\right) = 0_{\frac{1}{4}}.$

$(A \times B) \times C = \begin{pmatrix} -2_{\frac{1}{4}} & 5_{\frac{1}{4}} \\ -3_1 & -1_{\frac{1}{4}} \end{pmatrix} \begin{pmatrix} 1_1 & 0_1 \\ 1_0 & 0_0 \end{pmatrix} = \begin{pmatrix} 3_{\frac{1}{4}} & 0_{\frac{1}{4}} \\ -3_{\frac{1}{4}} & 0_{\frac{1}{4}} \end{pmatrix},$

$\left(-2_{\frac{1}{4}}\right) \circ (1_1) * \left(5_{\frac{1}{4}}\right) \circ (1_0) = (-2_1) * \left(5_{\frac{1}{4}}\right) = 3_{\frac{1}{4}},$

$\left(-2_{\frac{1}{4}}\right) \circ (0_1) * \left(5_{\frac{1}{4}}\right) \circ (0_0) = (0_1) * \left(0_{\frac{1}{4}}\right) = 0_{\frac{1}{4}},$

$(-2_1) \circ (1_1) * \left(-1_{\frac{1}{4}}\right) \circ (1_0) = (-2_1) * \left(-1_{\frac{1}{4}}\right) = -3_{\frac{1}{4}},$

$(-3_1) \circ (0_1) * \left(-1_{\frac{1}{4}}\right) \circ (0_0) = (0_1) * \left(0_{\frac{1}{4}}\right) = 0_{\frac{1}{4}}.$

Thus $(A \times B) \times C = A \times (B \times C).$

$B \times A = \begin{pmatrix} -2_1 & -1_{\frac{1}{2}} \\ 1_0 & 2_{\frac{1}{4}} \end{pmatrix} \times \begin{pmatrix} 3_{\frac{1}{2}} & 4_{\frac{1}{4}} \\ 1_{\frac{1}{3}} & 0_0 \end{pmatrix} = \begin{pmatrix} -7_1 & m_1 \\ m_2 & m_3 \end{pmatrix},$

$(-2_1) \circ \left(3_{\frac{1}{2}}\right) * \left(-1_{\frac{1}{2}}\right) \circ \left(1_{\frac{1}{3}}\right) = (-6_1) * \left(-1_{\frac{1}{2}}\right) = -7_1$

It is clear that $B \times A \neq A \times B$.

The Algebra Properties of two-fold Matrix Operations:

Property (1):

Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$; $\begin{cases} a_i = (x_i)_{\mu(y_i)} & ; x_i, y_i \in \mathbb{R} \\ b_i = (z_i)_{\mu(t_i)} & ; z_i, t_i \in \mathbb{R} \end{cases}$

We have: $a_i * b_i = (x_i + z_i)_{\min(\mu(y_i),\mu(t_i))} = (z_i + x_i)_{\min(\mu(y_i),\mu(t_i))} = b_i * a_i$, then $A + B = B + A$.

Property (2):

Let $C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$; $c_i = (m_i)_{\mu(n_i)}$, then:

$B + C = \begin{pmatrix} b_1 * c_1 & b_2 * c_2 \\ b_3 * c_3 & b_4 * c_4 \end{pmatrix}, A + (B + C) = \begin{pmatrix} a_1 * b_1 * c_1 & a_2 * b_2 * c_2 \\ a_3 * b_3 * c_3 & a_4 * b_4 * c_4 \end{pmatrix} = (A + B) + C.$

Property (3):

$$A - A = A + (-A) = \begin{pmatrix} (x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)} \\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)} \end{pmatrix} + \begin{pmatrix} (-x_1)_{\mu(-y_1)} & (-x_2)_{\mu(-y_2)} \\ (-x_3)_{\mu(-y_3)} & (-x_4)_{\mu(-y_4)} \end{pmatrix} = \begin{pmatrix} 0_{l_1} & 0_{l_2} \\ 0_{l_3} & 0_{l_4} \end{pmatrix} ;$$

$l_i = \min(\mu(y_i), \mu(-y_i))$.

Property (4):

$A \times (B \times C) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \times \begin{pmatrix} (b_1 \circ c_1) * (b_2 \circ c_3) & (b_1 \circ c_2) * (b_2 \circ c_4) \\ (b_3 \circ c_1) * (b_4 \circ c_3) & (b_3 \circ c_2) * (b_4 \circ c_4) \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \times \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} = \begin{pmatrix} (a_1 \circ d_1) * (a_2 \circ d_3) & (a_1 \circ d_2) * (a_2 \circ d_4) \\ (a_3 \circ d_1) * (a_4 \circ d_3) & (a_3 \circ d_2) * (a_4 \circ d_4) \end{pmatrix}.$

We have:

$a_1 \circ d_1 = a_1 \circ [(b_1 \circ c_1) * (b_2 \circ c_3)] = (a_1 \circ b_1 \circ c_1) * (a_1 \circ b_2 \circ c_3)$, $a_2 \circ d_3 = a_2 \circ [(b_3 \circ c_1) * (b_4 \circ c_3)] = (a_2 \circ b_3 \circ c_1) * (a_2 \circ b_4 \circ c_3)$,

Put

$T_1 = (a_1 \circ d_1) * (a_2 \circ d_3) = (a_1 \circ b_1 \circ c_1) * (a_1 \circ b_2 \circ c_3) * (a_2 \circ b_3 \circ c_1) * (a_2 \circ b_4 \circ c_3)$.

$a_1 \circ d_2 = a_1 \circ [(b_1 \circ c_2) * (b_2 \circ c_4)] = (a_1 \circ b_1 \circ c_2) * (a_1 \circ b_2 \circ c_4)$, $a_2 \circ d_4 = a_2 \circ [(b_3 \circ c_2) * (b_4 \circ c_4)] = (a_2 \circ b_3 \circ c_2) * (a_2 \circ b_4 \circ c_4)$,

$T_2 = (a_1 \circ d_2) * (a_2 \circ d_4) = (a_1 \circ b_1 \circ c_2) * (a_1 \circ b_2 \circ c_4) * (a_2 \circ b_3 \circ c_2) * (a_2 \circ b_4 \circ c_4)$,

$a_3 \circ d_1 = a_3 \circ [(b_1 \circ c_1) * (b_2 \circ c_3)] = (a_3 \circ b_1 \circ c_1) * (a_3 \circ b_2 \circ c_3)$, $a_4 \circ d_3 = a_4 \circ [(b_3 \circ c_1) * (b_4 \circ c_3)] = (a_4 \circ b_3 \circ c_1) * (a_4 \circ b_4 \circ c_3)$,

Put

$T_3 = (a_3 \circ d_1) * (a_4 \circ d_3) = (a_3 \circ b_1 \circ c_1) * (a_3 \circ b_2 \circ c_3) * (a_4 \circ b_3 \circ c_1) * (a_4 \circ b_4 \circ c_3)$,

$a_3 \circ d_2 = a_3 \circ [(b_1 \circ c_2) * (b_2 \circ c_4)] = (a_3 \circ b_1 \circ c_2) * (a_3 \circ b_2 \circ c_4)$, $a_4 \circ d_4 = a_4 \circ [(b_3 \circ c_2) * (b_4 \circ c_4)] = (a_4 \circ b_3 \circ c_2) * (a_4 \circ b_4 \circ c_4)$,

$T_4 = (a_3 \circ d_2) * (a_4 \circ d_4) = (a_3 \circ b_1 \circ c_2) * (a_3 \circ b_2 \circ c_4) * (a_4 \circ b_3 \circ c_2) * (a_4 \circ b_4 \circ c_4)$.

Thus $A \times (B \times C) = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$.

On the other hand, we have:

$(A \times B) \times C = \begin{pmatrix} (a_1 \circ b_1) * (a_2 \circ b_3) & (a_1 \circ b_2) * (a_2 \circ b_4) \\ (a_3 \circ b_1) * (a_4 \circ b_3) & (a_3 \circ b_2) * (a_4 \circ b_4) \end{pmatrix} \times \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$, where

$k_1 = [(a_1 \circ b_1) * (a_2 \circ b_3)] \circ c_1 * [(a_1 \circ b_2) * (a_2 \circ b_4)] \circ c_3 = (a_1 \circ b_1 \circ c_1) * (a_2 \circ b_3 \circ c_1) * (a_1 \circ b_2 \circ c_3) * (a_2 \circ b_4 \circ c_3) = T_1$.

By a similar argument, we can see clearly that:

$k_2 = T_2, k_3 = T_3, k_4 = T_4$, thus $A \times (B \times C) = (A \times B) \times C$.

Property (5):

$A \times (B + C) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \times \begin{pmatrix} b_1 * c_1 & b_2 * c_2 \\ b_3 * c_3 & b_4 * c_4 \end{pmatrix} = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}$, where:

$F_1 = a_1 \circ (b_1 * c_1) * a_2 \circ (b_3 * c_3) = (a_1 \circ b_1) * (a_1 \circ c_1) * (a_2 \circ b_3) * (a_2 \circ c_3)$,

$F_2 = a_1 \circ (b_2 * c_3) * a_2 \circ (b_4 * c_4) = (a_1 \circ b_2) * (a_1 \circ c_3) * (a_2 \circ b_4) * (a_2 \circ c_4)$,

$F_3 = a_3 \circ (b_1 * c_1) * a_4 \circ (b_3 * c_3) = (a_3 \circ b_1) * (a_3 \circ c_1) * (a_4 \circ b_3) * (a_4 \circ c_3)$,

$F_4 = a_3 \circ (b_2 * c_2) * a_4 \circ (b_4 * c_4) = (a_3 \circ b_2) * (a_3 \circ c_2) * (a_4 \circ b_4) * (a_4 \circ c_4)$,

On the other hand, we have:

$(A \times B) + (A \times C) = \begin{pmatrix} (a_1 \circ b_1) * (a_2 \circ b_3) & (a_1 \circ b_2) * (a_2 \circ b_4) \\ (a_3 \circ b_1) * (a_4 \circ b_3) & (a_3 \circ b_2) * (a_4 \circ b_4) \end{pmatrix} +$

$\begin{pmatrix} (a_1 \circ c_1) * (a_2 \circ c_3) & (a_1 \circ c_2) * (a_2 \circ c_4) \\ (a_3 \circ c_1) * (a_4 \circ c_3) & (a_3 \circ c_2) * (a_4 \circ c_4) \end{pmatrix} = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}$.

Which implies that $A \times (B + C) = (A \times B) + (A \times C)$.

Result:

If TF_μ is the set of all (2×2) two-fold fuzzy matrices with two-fold real entries, then:

1] $(+)$ is a commutative and associative.

2] $(\times)$ is associative.

3] $(\times)$ is a distributive with respect to $(+)$.

Definition:

A (2×2) two-fold zero fuzzy matrix is defined as:

$O=\begin{pmatrix}0_1 & 0_1\\ 0_1 & 0_1\end{pmatrix}$.

Remark:

For any $A\in TF_{\mu}$, we have $A+O=O+A=A$.

It is clear that $A\times O=O$.

The additive inverse of $A=\begin{pmatrix}(x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)}\\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)}\end{pmatrix}$ is not existed.

Definition:

A (2×2) two-fold unitary matrix is:

$I=\begin{pmatrix}1_0 & 0_1\\ 0_1 & 1_0\end{pmatrix}$

Remark:

For $A=\begin{pmatrix}(x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)}\\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)}\end{pmatrix}$, we have:

$A\times I=\begin{pmatrix}(x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)}\\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)}\end{pmatrix}\times\begin{pmatrix}1_0 & 0_1\\ 0_1 & 1_0\end{pmatrix}=\begin{pmatrix}L_1 & L_2\\ L_3 & L_4\end{pmatrix}$;

$L_1=\left((x_1)_{\mu(y_1)}\circ 1_0\right)*\left((x_2)_{\mu(y_2)}\circ(0_1)\right)=(x_1)_{\mu(y_1)}*(0_1)=(x_1)_{\mu(y_1)}$

$L_2=\left((x_1)_{\mu(y_1)}\circ 0_1\right)*\left((x_2)_{\mu(y_2)}\circ 1_0\right)=(0_1)*(x_2)_{\mu(y_2)}=(x_2)_{\mu(y_2)}$

$L_3=(x_3)_{\mu(y_3)}$, $L_4=(x_4)_{\mu(y_4)}$,

thus $A\times I=I\times A=A$.

Finding the multiplicative inverse:

Let $A=\begin{pmatrix}(x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)}\\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)}\end{pmatrix}\in TF_{\mu}$; $x_1x_4-x_2x_3\neq 0$

Put $B=\begin{pmatrix}(x_4)_{\mu(z_4)} & (-x_2)_{\mu(z_2)}\\ (-x_3)_{\mu(z_3)} & (x_1)_{\mu(z_1)}\end{pmatrix}$, and compute:

$A\times B=\begin{pmatrix}\mu_1 & \mu_2\\ \mu_3 & \mu_4\end{pmatrix}$ such that:

$\mu_1=\left[(x_1)_{\mu(y_1)}\circ(x_4)_{\mu(z_4)}\right]*\left[(x_2)_{\mu(y_2)}\circ(-x_3)_{\mu(z_3)}\right]=(x_1x_4-x_2x_3)_{l_1}$;

$l_1 = \min[\max(\mu(y_1), \mu(z_4)), \max(\mu(y_2), \mu(z_3))] = 1,$

Thus: we can put: $\mu(z_4) = \mu(z_3) = 1$.

Also, if we put $\mu(z_2) = \mu(z_1) = 1$, then:

$$A \times B = \begin{pmatrix} (x_1x_4 - x_2x_3)_1 & 0_1 \\ 0_1 & (x_1x_4 - x_2x_3)_1 \end{pmatrix}.$$

Define:

$\det A = (x_1x_4 - x_2x_3)_1$, then:

$$\frac{1}{(x_1x_4 - x_2x_3)_1} \times \begin{pmatrix} (x_1x_4 - x_2x_3)_1 & 0_1 \\ 0_1 & (x_1x_4 - x_2x_3)_1 \end{pmatrix} = \begin{pmatrix} 1_1 & 0_1 \\ 0_1 & 1_1 \end{pmatrix} \neq I.$$

On the other hand, if we put $l_1 = 0$, then we can not choose $\mu(z_4), \mu(z_3)$to obtain this result, which means that A is not invertible.

Result:

The set of all two-fold fuzzy matrices is not a ring, that is because there are no additive inverses.

Two-Fold 2×2 anti-Fuzzy Real Matrices:

Definition:

Let $\mathbb{R}$ be the real field, $\mu: \mathbb{R} \to [0,1]$ such that $\begin{cases} \mu(0) = 0 \\ \mu(1) = 1 \end{cases}$ be a fuzzy mapping. Let $\mathbb{R}_F = \{x_{\mu(y)} : x, y \in \mathbb{R}\}$be the corresponding two-fold fuzzy real algebra, then we define:

$$A = \begin{pmatrix} {x_1}_{\mu(y_1)} & {x_2}_{\mu(y_2)} \\ {x_3}_{\mu(y_3)} & {x_4}_{\mu(y_4)} \end{pmatrix} \ ; \ x_i, y_i \in \mathbb{R}.$$

A is called a 2×2 anti-fuzzy two-fold matrix.

Definition:

Let $a_i = (x_i)_{\mu(y_i)} \quad 1 \leq i \leq 4$, $b_i = (z_i)_{\mu(t_i)}$; $1 \leq i \leq 4$ and $x_i, y_i, z_i, t_i \in \mathbb{R}$, $a_i, b_i \in \mathbb{R}_F$.

Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$, we define:

$$A + B = \begin{pmatrix} a_1 * b_1 & a_2 * b_2 \\ a_3 * b_3 & a_4 * b_4 \end{pmatrix} \ ; a_i * b_i = (x_i + z_i)_{\max(\mu(y_i), \mu(t_i))}$$

$$-A = \begin{pmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 \end{pmatrix}; -a_i = (-x_i)_{\mu(-x_i)}$$

$$A \times B = \begin{pmatrix} (a_1 \circ b_1) * (a_2 \circ b_3) & (a_1 \circ b_2) * (a_2 \circ b_4) \\ (a_3 \circ b_1) * (a_4 \circ b_3) & (a_3 \circ b_2) * (a_4 \circ b_4) \end{pmatrix}, \text{where } a_i \circ b_j = (x_i z_j)_{\min(\mu(x_i), \mu(t_j))}$$

Example:

Take: $\mu:\mathbb{R}\to[0,1]:\mu(x)=\begin{cases}0 & ;x=0\\ 1 & ;x=1\\ \frac{1}{2} & ;x>1\\ \frac{1}{3} & ;x<0\\ \frac{1}{4} & ;0<x<1\end{cases}$

And: $a_1=3_{\mu(2)}=3_{\frac{1}{2}}, a_2=4_{\mu\left(\frac{1}{3}\right)}=4_{\frac{1}{4}}$,

$$a_3=1_{\mu(-5)}=1_{\frac{1}{3}}, a_4=0_{\mu(0)}=0_0 \ ,$$

$A=\begin{pmatrix}a_1 & a_2\\ a_3 & a_4\end{pmatrix}=\begin{pmatrix}3_{\frac{1}{2}} & 4_{\frac{1}{4}}\\ 1_{\frac{1}{3}} & 0_0\end{pmatrix}$.

$$b_1=(-2)_{\mu(1)}=(-2)_1, b_2=(-1)_{\mu(7)}=(-1)_{\frac{1}{2}}, b_3=(1)_{\mu(0)}=1_0 \ ,$$

$b_4=(2)_{\mu(\frac{1}{10})}=2_{\frac{1}{4}}, B=\begin{pmatrix}b_1 & b_2\\ b_3 & b_4\end{pmatrix}=\begin{pmatrix}(-2)_1 & (-1)_{\frac{1}{2}}\\ 1_0 & 2_{\frac{1}{4}}\end{pmatrix}$.

We have:

$-A=\begin{pmatrix}(-3)_{\mu(-2)} & (-4)_{\mu\left(\frac{-1}{3}\right)}\\ (-1)_{\mu(5)} & (-0)_{\mu(-0)}\end{pmatrix}=\begin{pmatrix}(-3)_{\frac{1}{3}} & (-4)_{\frac{1}{3}}\\ (-1)_{\frac{1}{2}} & 0_0\end{pmatrix}$.

$A+B=\begin{pmatrix}a_1*b_1 & a_2*b_2\\ a_3*b_3 & a_4*b_4\end{pmatrix}=\begin{pmatrix}1_1 & (3)_{\frac{1}{2}}\\ 2_{\frac{1}{3}} & 2_{\frac{1}{4}}\end{pmatrix}$; $\begin{cases}a_1*b_1=(3-2)_1=1_1\\ a_2*b_2=(4-1)_{\frac{1}{2}}=3_{\frac{1}{2}}\\ a_3*b_3=(1+1)_{\frac{1}{3}}=2_{\frac{1}{3}}\\ a_4*b_4=(0+2)_{\frac{1}{4}}=2_{\frac{1}{4}}\end{cases}$

The Algebraic Properties of two-fold anti-fuzzy Matrix Operations:

Property (1):

Let $A=\begin{pmatrix}a_1 & a_2\\ a_3 & a_4\end{pmatrix}, B=\begin{pmatrix}b_1 & b_2\\ b_3 & b_4\end{pmatrix}$; $\begin{cases}a_i=(x_i)_{\mu(y_i)} & ;x_i,y_i\in\mathbb{R}\\ b_i=(z_i)_{\mu(t_i)} & ;z_i,t_i\in\mathbb{R}\end{cases}$

We have: $a_i*b_i=(x_i+z_i)_{\max(\mu(y_i),\mu(t_i))}=(z_i+x_i)_{\max(\mu(y_i),\mu(t_i))}=b_i*a_i$,

then $A+B=B+A$.

Property (2):

Let $C=\begin{pmatrix}c_1 & c_2\\ c_3 & c_4\end{pmatrix}$; $c_i=(m_i)_{\mu(n_i)}$, then:

$$B+C=\begin{pmatrix} b_1*c_1 & b_2*c_2 \\ b_3*c_3 & b_4*c_4 \end{pmatrix},\ A+(B+C)=\begin{pmatrix} a_1*b_1*c_1 & a_2*b_2*c_2 \\ a_3*b_3*c_3 & a_4*b_4*c_4 \end{pmatrix}=(A+B)+C.$$

Property (3):

$$A-A=A+(-A)=\begin{pmatrix} (x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)} \\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)} \end{pmatrix}+\begin{pmatrix} (-x_1)_{\mu(-y_1)} & (-x_2)_{\mu(-y_2)} \\ (-x_3)_{\mu(-y_3)} & (-x_4)_{\mu(-y_4)} \end{pmatrix}=\begin{pmatrix} 0_{l_1} & 0_{l_2} \\ 0_{l_3} & 0_{l_4} \end{pmatrix};$$

$l_i=\max(\mu(y_i),\mu(-y_i))$.

Property (4):

$$A\times(B\times C)=\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\times\begin{pmatrix} (b_1\circ c_1)*(b_2\circ c_3) & (b_1\circ c_2)*(b_2\circ c_4) \\ (b_3\circ c_1)*(b_4\circ c_3) & (b_3\circ c_2)*(b_4\circ c_4) \end{pmatrix}=\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\times$$
$$\begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}=\begin{pmatrix} (a_1\circ d_1)*(a_2\circ d_3) & (a_1\circ d_2)*(a_2\circ d_4) \\ (a_3\circ d_1)*(a_4\circ d_3) & (a_3\circ d_2)*(a_4\circ d_4) \end{pmatrix}.$$

We have:

$a_1\circ d_1=a_1\circ[(b_1\circ c_1)*(b_2\circ c_3)]=(a_1\circ b_1\circ c_1)*(a_1\circ b_2\circ c_3)$, $a_2\circ d_3=a_2\circ[(b_3\circ c_1)*(b_4\circ c_3)]=(a_2\circ b_3\circ c_1)*(a_2\circ b_4\circ c_3)$,

Put

$$T_1=(a_1\circ d_1)*(a_2\circ d_3)=(a_1\circ b_1\circ c_1)*(a_1\circ b_2\circ c_3)*(a_2\circ b_3\circ c_1)*(a_2\circ b_4\circ c_3).$$

$a_1\circ d_2=a_1\circ[(b_1\circ c_2)*(b_2\circ c_4)]=(a_1\circ b_1\circ c_2)*(a_1\circ b_2\circ c_4)$, $a_2\circ d_4=a_2\circ[(b_3\circ c_2)*(b_4\circ c_4)]=(a_2\circ b_3\circ c_2)*(a_2\circ b_4\circ c_4)$,

$$T_2=(a_1\circ d_2)*(a_2\circ d_4)=(a_1\circ b_1\circ c_2)*(a_1\circ b_2\circ c_4)*(a_2\circ b_3\circ c_2)*(a_2\circ b_4\circ c_4),$$

$a_3\circ d_1=a_3\circ[(b_1\circ c_1)*(b_2\circ c_3)]=(a_3\circ b_1\circ c_1)*(a_3\circ b_2\circ c_3)$, $a_4\circ d_3=a_4\circ[(b_3\circ c_1)*(b_4\circ c_3)]=(a_4\circ b_3\circ c_1)*(a_4\circ b_4\circ c_3)$,

Put

$$T_3=(a_3\circ d_1)*(a_4\circ d_3)=(a_3\circ b_1\circ c_1)*(a_3\circ b_2\circ c_3)*(a_4\circ b_3\circ c_1)*(a_4\circ b_4\circ c_3),$$

$a_3\circ d_2=a_3\circ[(b_1\circ c_2)*(b_2\circ c_4)]=(a_3\circ b_1\circ c_2)*(a_3\circ b_2\circ c_4)$, $a_4\circ d_4=a_4\circ[(b_3\circ c_2)*(b_4\circ c_4)]=(a_4\circ b_3\circ c_2)*(a_4\circ b_4\circ c_4)$,

$$T_4=(a_3\circ d_2)*(a_4\circ d_4)=(a_3\circ b_1\circ c_2)*(a_3\circ b_2\circ c_4)*(a_4\circ b_3\circ c_2)*(a_4\circ b_4\circ c_4).$$

Thus $A\times(B\times C)=\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$.

On the other hand, we have:

$(A \times B) \times C = \begin{pmatrix} (a_1 \circ b_1) * (a_2 \circ b_3) & (a_1 \circ b_2) * (a_2 \circ b_4) \\ (a_3 \circ b_1) * (a_4 \circ b_3) & (a_3 \circ b_2) * (a_4 \circ b_4) \end{pmatrix} \times \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$, where

$k_1 = [(a_1 \circ b_1) * (a_2 \circ b_3)] \circ c_1 * [(a_1 \circ b_2) * (a_2 \circ b_4)] \circ c_3 = (a_1 \circ b_1 \circ c_1) * (a_2 \circ b_3 \circ c_1) * (a_1 \circ b_2 \circ c_3) * (a_2 \circ b_4 \circ c_3) = T_1.$

By a similar argument, we can see clearly that:

$k_2 = T_2, k_3 = T_3, k_4 = T_4$, thus $A \times (B \times C) = (A \times B) \times C.$

Property (5):

$A \times (B + C) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \times \begin{pmatrix} b_1 * c_1 & b_2 * c_2 \\ b_3 * c_3 & b_4 * c_4 \end{pmatrix} = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}$, where:

$F_1 = a_1 \circ (b_1 * c_1) * a_2 \circ (b_3 * c_3) = (a_1 \circ b_1) * (a_1 \circ c_1) * (a_2 \circ b_3) * (a_2 \circ c_3),$

$F_2 = a_1 \circ (b_2 * c_3) * a_2 \circ (b_4 * c_4) = (a_1 \circ b_2) * (a_1 \circ c_3) * (a_2 \circ b_4) * (a_2 \circ c_4),$

$F_3 = a_3 \circ (b_1 * c_1) * a_4 \circ (b_3 * c_3) = (a_3 \circ b_1) * (a_3 \circ c_1) * (a_4 \circ b_3) * (a_4 \circ c_3),$

$F_4 = a_3 \circ (b_2 * c_2) * a_4 \circ (b_4 * c_4) = (a_3 \circ b_2) * (a_3 \circ c_2) * (a_4 \circ b_4) * (a_4 \circ c_4),$

On the other hand, we have:

$(A \times B) + (A \times C) = \begin{pmatrix} (a_1 \circ b_1) * (a_2 \circ b_3) & (a_1 \circ b_2) * (a_2 \circ b_4) \\ (a_3 \circ b_1) * (a_4 \circ b_3) & (a_3 \circ b_2) * (a_4 \circ b_4) \end{pmatrix} +$

$\begin{pmatrix} (a_1 \circ c_1) * (a_2 \circ c_3) & (a_1 \circ c_2) * (a_2 \circ c_4) \\ (a_3 \circ c_1) * (a_4 \circ c_3) & (a_3 \circ c_2) * (a_4 \circ c_4) \end{pmatrix} = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}.$

Which implies that $A \times (B + C) = (A \times B) + (A \times C).$

Result:

If TF_μ is the set of all (2×2) two-fold anti- fuzzy matrices with two-fold real entries, then:

1] $(+)$ is a commutative and associative.

2] $(\times)$ is associative.

3] $(\times)$ is a distributive with respect to $(+)$.

Definition:

A (2×2) two-fold zero anti- fuzzy matrix is defined as:

$O = \begin{pmatrix} 0_0 & 0_0 \\ 0_0 & 0_0 \end{pmatrix}.$

Remark:

For any $A \in TF_\mu$, we have $A + O = O + A = A.$

It is clear that $A \times O = O$.

The additive inverse of $A = \begin{pmatrix} (x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)} \\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)} \end{pmatrix}$ is not existed.

Definition:

A (2×2) two-fold anti-fuzzy unitary matrix is:

$I = \begin{pmatrix} 1_1 & 0_0 \\ 0_0 & 1_1 \end{pmatrix}$

Remark:

For $A = \begin{pmatrix} (x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)} \\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)} \end{pmatrix}$, we have:

$A \times I = \begin{pmatrix} (x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)} \\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)} \end{pmatrix} \times \begin{pmatrix} 1_1 & 0_0 \\ 0_0 & 1_1 \end{pmatrix} = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}$;

$L_1 = \left((x_1)_{\mu(y_1)} \circ 1_1\right) * \left((x_2)_{\mu(y_2)} \circ (0_0)\right) = (x_1)_{\mu(y_1)} * (0_0) = (x_1)_{\mu(y_1)}$

$L_2 = \left((x_1)_{\mu(y_1)} \circ 0_0\right) * \left((x_2)_{\mu(y_2)} \circ 1_1\right) = (0_0) * (x_2)_{\mu(y_2)} = (x_2)_{\mu(y_2)}$

$L_3 = (x_3)_{\mu(y_3)}$, $L_4 = (x_4)_{\mu(y_4)}$,

thus $A \times I = I \times A = A$.

Finding the multiplicative inverse:

Let $A = \begin{pmatrix} (x_1)_{\mu(y_1)} & (x_2)_{\mu(y_2)} \\ (x_3)_{\mu(y_3)} & (x_4)_{\mu(y_4)} \end{pmatrix} \in TF_\mu$; $x_1x_4 - x_2x_3 \neq 0$

Put $B = \begin{pmatrix} (x_4)_{\mu(z_4)} & (-x_2)_{\mu(z_2)} \\ (-x_3)_{\mu(z_3)} & (x_1)_{\mu(z_1)} \end{pmatrix}$, and compute:

$A \times B = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}$ such that:

$\mu_1 = \left[(x_1)_{\mu(y_1)} \circ (x_4)_{\mu(z_4)}\right] * \left[(x_2)_{\mu(y_2)} \circ (-x_3)_{\mu(z_3)}\right] = (x_1x_4 - x_2x_3)_{l_1}$;

$l_1 = \max\left[\min\left(\mu(y_1), \mu(z_4)\right), \min\left(\mu(y_2), \mu(z_3)\right)\right] = 1$,

We can not get a similar result by choosing $\mu(z_3), \mu(z_4)$

which means that A is not invertible.

Conclusion

In this paper we have studied for the first time the concept of square 2×2 fuzzy and anti-fuzzy two-fold matrix with real entries, where we presented the two-fold algebraic operations between the these matrices and obtain their special properties such as associativity, commutative properties, and the existing of additive of multiplicative inverses. Also, we provided many examples to explain the two-fold algebraic properties and the two-fold algebraic structure.

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Acknowledgments

The authors extend their appreciation to the Arab Open University for funding this work through AOU research fund no. (AOUKSA-524008).

Received: June 1, 2024. Accepted: Oct 12, 2024