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On The Continuous and Differentiable Two-Fold Neutrosophic and Fuzzy Real Functions

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Abstract

This paper is dedicated to defining the concepts of two-fold neutrosophic continuous functions, two-fold fuzzy continuous functions, and two-fold neutrosophic/fuzzy differentiable functions for the first time.

The elementary properties of these novel concepts will be studied and handled through many theorems, as well as many clear examples that clarify the validity of these analytical concepts.

Keywords: two-fold algebra, two-fold neutrosophic function, two-fold fuzzy function, two-fold differentiability.

Introduction

Two-fold neutrosophic algebras are new algebraic structures presented by Smarandache [1] by combining neutrosophic values of truth, falsity, and indeterminacy with classical algebraic sets. These ideas were used by many authors to generalize other famous algebraic structures such as two-fold fuzzy number theoretical systems [2-3], two-fold modules and spaces [4], and two-fold fuzzy rings [5]. Also, they were used in the study of some special two-fold complex functions such as Gamma function [7], and in extending n-refined neutrosophic rings [6].

Neutrosophic real functions and their applications were studied by many authors [8-25] in many different ways.

This has motivated us to introduce the concept of two-fold neutrosophic continuous functions, two-fold fuzzy continuous functions, and two-fold neutrosophic/fuzzy differentiable functions for the first time**.** The elementary properties of these novel concepts will be studied and handled through many theorems, as well as many clear examples that clarify the validity of these analytical concepts.

This study will be very important in the near future to generalize classical real analysis into an extended version based on two-fold algebra of real numbers.

Main Discussion

Definition 1.

Let $i, f, t: \mathbb{R} \to [0,1]$ be three real functions denote to the neutrosophic ordinary real values of truth, indeterminacy and falsity. Let $g: \mathbb{R} \to \mathbb{R}$ be a real function in one variable $g = g(x); x \in \mathbb{R}$. We define the corresponding two-fold neutrosophic real function as follows:

$g_N: R \to R_{(t,i,f)}$: $g_N(x) = (g(x))_{(t(x),i(x),f(x))}$

Example 1.

Let

$$i, t, f: \mathbb{R} \to [0,1],$$

such that

$$\begin{cases} i(x) = \min\left(|x|, \dfrac{1}{|x|}\right) ; x \neq 0 \\ t(x) = \min\left(|x^3|, \dfrac{1}{|x^3|}\right) ; x \neq 0 \\ f(x) = \min\left(|x^2|, \dfrac{1}{|x^2|}\right) ; x \neq 0 \end{cases} \text{ and } \begin{cases} i(0) = 1 \\ t(0) = 0 \\ f(0) = \dfrac{1}{2} \end{cases}.$$

Consider

$$g: \mathbb{R} \to \mathbb{R} ; g(x) = x^2 + 1,$$

then

$$g_N: \mathbb{R} \to \mathbb{R}_{(t,i,f)},$$

such that:

$$g_N(x) = \begin{cases} (x^2 + 1)_{(t(x),i(x),f(x))} & ; x \neq 0 \\ (1)_{(0,1,\frac{1}{2})} & ; x = 0 \end{cases}.$$

For example if $x = 2$, then

$$t(x) = \frac{1}{8}, i(x) = \frac{1}{2}, f(x) = \frac{1}{4}, g(x) = 5,$$

$$g_N(x) = (5)_{(\frac{1}{8},\frac{1}{2},\frac{1}{4})}.$$

Definition 2.

Let

$$g_N: \mathbb{R} \to \mathbb{R}_{(t,i,f)},$$

be a two-fold neutrosophic real function, we say that:

1) g_N is fully differential at $x_0 \in \mathbb{R}$ if:

$$\begin{cases} g'(x_0) \text{ is existed} \\ i'(x_0), f'(x_0), t'(x_0) \text{ are existed} \\ i'(x_0), f'(x_0), t'(x_0) \in [0,1] \end{cases},$$

2) g_N is T-differential (differential with respect to truth component) at $x_0 \in \mathbb{R}$ if:

$$\begin{cases} g'(x_0) \text{ is existed} \\ t'(x_0) \in [0,1] \\ t'(x_0) \text{ is existed} \end{cases}$$

3) g_N is I-differential (differential with respect to the indeterminacy component) at $x_0 \in \mathbb{R}$ if:

$$\begin{cases} g'(x_0) \text{ is existed} \\ i'(x_0) \text{ is existed} \\ i'(x_0) \in [0,1] \end{cases},$$

4) g_N is F-differential (differential with respect to the falisty component) at $x_0 \in \mathbb{R}$ if:

$$\begin{cases} g'(x_0) \text{ is existed} \\ f'(x_0) \text{ is existed} \\ f'(x_0) \in [0,1] \end{cases}.$$

Example 2.

Take: $f(x) = \min\left(\frac{1}{x^2}, x^2\right)$, $i(x) = \min\left(\frac{1}{x^4}, x^4\right)$, $t(x) = \min\left(|x|, \frac{1}{|x|}\right)$, $t(0) = i(0) = f(0) = \frac{1}{2}$.

$g(x) = x^2 + x + 1$.

Hence,

$$g_N(x): \mathbb{R} \to \mathbb{R}_{(t,i,f)},$$

$$g_N(x) = \begin{cases} (x^2 + x + 1)_{(t(x), i(x), f(x))} & ; x \neq 0 \\ (1)_{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)} & ; x = 0 \end{cases}.$$

It is easy to see that :

$g'(x_0)$ is existed for all $x_0 \neq 0$,

$t'(x_0), f'(x_0), i'(x_0)$ are existed for all $x_0 \neq 0$,

$t'(x_0), f'(x_0), i'(x_0)$ are not existed for all $x_0 = 0$.

Hence, g_N is not differential for $x_0 = 0$.

For $x_0 = 2$, we have:

$$g'(2) = 5, f(2) = \frac{1}{4}, f'(2) < 0, i(2) = \frac{1}{16}, i'(2) < 0, t(2) = \frac{1}{2}.$$

$$\lim_{x \to 2} \frac{t(x) - t(2)}{x - 2} = \lim_{x \to 2} \frac{\frac{1}{|x|} - \frac{1}{2}}{x - 2} = \lim_{x \to 2} \frac{\frac{2 - |x|}{2|x|}}{x - 2} = \lim_{x \to 2} \frac{-1}{2|x|} = -\frac{1}{4} < 0.$$

Hence g_N is not fully differentiable for $x_0 = 2$.

Example 3.

Let $g(x) = x^3 - x - 1$, $i(x) = f(x) = t(x) = \frac{1}{4}$, then $i'(x) = f'(x) = t'(x) = 0$ for all $x \in \mathbb{R}$.

Hence g_N is fully differentiable for all $x \in \mathbb{R}$.

Remark1.

The set of all points $x \in \mathbb{R}$ for which $g_N(x)$ is fully differentiable is denoted by $FD(g_N)$.

Example 4.

Try to find $FD(g_N)$ for:

$$g(x) = 3x^2 + x - 5 \,, i(x) = \begin{cases} \frac{1}{2} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases} \,, t(x)$$

$$= \begin{cases} \frac{1}{x} & x \geq 1 \\ \frac{-1}{x} & x \leq -1 \\ |x| & -1 < x < 1 \end{cases} \,, f(x) = \begin{cases} \frac{1}{x^2} & x \geq 1, x \leq -1 \\ x^2 & ; -1 < x < 1 \end{cases} ,$$

$$g_N(x) = \begin{cases} (3x^2 + x - 5\,)_{(\frac{1}{x},\frac{1}{2},\frac{1}{x^2})} & x \geq 1 \\ (3x^2 + x - 5\,)_{(|x|,\frac{1}{2},x^2)} & -1 < x < 1\,, x \neq 0 \\ (3x^2 + x - 5\,)_{(\frac{-1}{x},\frac{1}{2},\frac{1}{x^2})} & x \leq -1 \end{cases} .$$

We remark the following:

The function (g) is differentiable on $\mathbb{R}$, and $g'^{(x)} = 6x + 1$.

The function $i(x)$ is differentiable an $\mathbb{R}^*$ with $i'(x) = 0$ for all $x \neq 0$.

The function $t(x)$ is differentiable on R^* with:

$$t'(x) = \begin{cases} \frac{-1}{x^2} & x \geq 1 \\ \frac{1}{x^2} & x \leq -1 \\ 1 & 0 < x < 1 \\ -1 & -1 < x < 0 \end{cases} .$$

The function $f(x)$ is differentiable an $\mathbb{R}$ with:

$$f'(x) = \begin{cases} \frac{-2}{x^3} & x \geq 1, x \leq -1 \\ 2x & -1 < x < 1 \end{cases} .$$

We can see that:

$$t'(x) \in [0,1] \Leftrightarrow x \in \,]0,1[\, \cup \,]-\infty, -1] \,,$$

$$f'(x) \in [0,1] \Leftrightarrow x \in \left]0, \tfrac{1}{2}\right[\cup \left]-\infty, -\sqrt[3]{2}\right[.$$

$$i'(x) \in [0,1] \Leftrightarrow x \in \mathbb{R}^*.$$

Thus

$$FD(g_N) = \left]0, \tfrac{1}{2}\right[\cup \left]-\infty, -\sqrt[3]{2}\right[.$$

Definition 3.

Let g_N be a differential two-fold neutrosophic function on $I \subseteq R$, we define

$$g_N'(x) = \left(g'(x)\right)_{\left(t'(x), i'(x), f'(x)\right)}\ ; x \in I.$$

In the previous example, we can see:

$$g_N'(x) = \begin{cases} (6x+1)_{(1,0,2x)} & x \in \left]0, \frac{1}{2}\right[\\ (6x+1)_{(\frac{1}{x^2}, 0, \frac{-2}{x^3})} & x \in \left]-\infty, -\sqrt[3]{2}\right[\end{cases}.$$

Definition 4.

Let $g_N, h_N : \mathbb{R} \to \mathbb{R}_{(t,i,f)}$ be two-fold neutrosophic real functions, we define:

1) $(g_N + h_N)(x) = \left(g(x) + h(x)\right)_{\left(t(x), i(x), f(x)\right)'}$

where

$$\begin{cases} t(x) = \min(t_1(x), t_2(x)) \\ f(x) = \max(f_1(x), f_2(x)) \\ i(x) = \max(i_1(x), i_2(x)) \end{cases},$$

2) $(-g_N)(x) = \left(-g(x)\right)_{\left(t_1(x), i_1(x), f_1(x)\right)}$,

3) $(g_N . h_N)(x) = \left(g(x) . h(x)\right)_{\left(t(x), i(x), f(x)\right)'}$

where,

$$\begin{cases} t(x) = \max(t_1(x), t_2(x)) \\ f(x) = \min(f_1(x), f_2(x)) \\ i(x) = \min(i_1(x), i_2(x)) \end{cases},$$

4) For $h_N(x) \neq 0, \left(\frac{1}{h_N}\right)(x) = \left(\frac{1}{h(x)}\right)_{\left(t(x), i(x), f(x)\right)'}$

where,

$$\begin{cases} t(x) = 1 - t_2(x) \\ f(x) = 1 - f_2(x) \\ i(x) = 1 - i_2(x) \end{cases}.$$

Result 1.

From the definition, we get directly:

1) $(g_N - h_N)(x) = \left(g(x) - h(x)\right)_{\left(t(x), i(x), f(x)\right)'}$

where,

$$\begin{cases} t(x) = \min(t_1(x), t_2(x)) \\ f(x) = \max(f_1(x), f_2(x)) \\ i(x) = \max(i_1(x), i_2(x)) \end{cases},$$

2) $\left(\frac{g_N}{h_N}\right)(x) = \left(\frac{g(x)}{h(x)}\right)_{(t(x),i(x),f(x))}{}'$

where,

$$\begin{cases} t(x) = \max(t_1(x), 1 - t_2(x)) \\ f(x) = \min(f_1(x), 1 - f_2(x)) \\ i(x) = \min(i_1(x), 1 - i_2(x)) \end{cases},$$

3) $[g_N(x)]^n = ([g(x)]^n)_{(t_1(x),i_1(x),f_1(x))}$.

Theorem 1.

Let $g_N, h_N: \mathbb{R} \to \mathbb{R}_{(t,i,f)}$ be two-fold neutrosophic real functions, then

1) $(g_N + h_N)' = g'_N + h'_N$,

2) $(g_N - h_N)' = g'_N - h'_N$,

3) $(g_N^n)' = n.\, g_N^{n-1}{}_{(0,1,1)} g'_N$.

Proof.

1) $[(g+h)(x)]' = g'(x) + h'(x), t'(x) = \min\big(t'_1(x), t'_2(x)\big), f'(x) = \max\big(f'_1(x), f'_2(x)\big),\ i'(x) = \max\big(i'_1(x), i'_2(x)\big)$,

so that:

$$(g_N + h_N)'(x) = \big(g'(x) + h'(x)\big)_{\left(t'(x),i'(x),f'(x)\right)} = g'_N(x) + h'_N(x).$$

2) it can be proved by a similar argument.

3) $[g^n(x)]' = n.\, g'(x).\, g^{n-1}(x)$,and

$$\left(g_N^n(x)\right)' = \left(n.\, g'(x) g_N^{n-1}(x)\right)_{\left(t'_1(x),i'_1(x),f'_1(x)\right)} = n.\left(g'(x)\right)_{\left(t'_1(x),i'_1(x),f'_1(x)\right)} \cdot \left(g_N^{n-1}(x)\right)_{(0,1,1)}$$

$$= n.\, g'_N(x).\left(g_N^{n-1}(x)\right)_{(0,1,1)}$$

Remark 2.

If $g_N, h_N$are fully differentiable on $I \subseteq R$, then

$$\left(g_N, h_N(x)\right)' = (g' h(x), h' g(x))_{(t'(x),i'(x),f'(x))},$$

where,

$$\begin{cases} t'(x) = \max(t'_1(x), t'_2(x)) \\ i'(x) = \min(i'_1(x), i'_2(x)) \\ f'(x) = \min(f'_1(x), f'_2(x)) \end{cases}.$$

Definition 5.

We define

$$\left(\frac{1}{h_N}\right)'(x) = \left(\frac{-h'(x)}{h^2(x)}\right)_{(1-t'_2(x),1-i'_2(x),1-f'_2(x))}.$$

Result 2.

The derivative $\left(\frac{g_N}{h_N}\right)'(x) = \left(\frac{(g'.h-h'g)(x)}{h^2(x)}\right)_{(t(x),i(x),f(x))}$,

where,

$$\begin{cases} t(x) = \max(t_1'(x), 1 - t_2'(x)) \\ i(x) = \min(i_1'(x), 1 - i_2'(x)) \\ f(x) = \min(f_1'(x), 1 - f_2'(x)) \end{cases}.$$

Example 5.

Take: $g(x) = 3x^2 + 1, h(x) = x^3$as two real functions.

$$t_1(x) = \begin{cases} 0\,;\ x > 0, x < 0 \\ 1\ ; x = 0 \end{cases} \quad t_2(x) = \begin{cases} 1\ \ x \geq 0 \\ \frac{1}{2}\ \ ; x < 0 \end{cases},$$

$$i_1(x) = \begin{cases} \frac{-1}{x} & ; x \leq -1 \\ 0 & ; x > -1 \end{cases} \quad i_2(x) = \begin{cases} \frac{1}{3} & ; x \geq 0 \\ \frac{1}{4} & ; x < 0 \end{cases},$$

$$f_1(x) = \{0 \quad ; x \in \mathbb{R} \quad f_2(x) = \begin{cases} \frac{1}{4} & ; x > 1 \\ \frac{1}{3} & ; -1 \leq x \leq 1 \\ \frac{1}{2} & ; x < -1 \end{cases},$$

$g_N, h_N: \mathbb{R} \to \mathbb{R}_{(t,i,f)}$,

such that:

$g_N(x) = (g(x))_{(t_1,i_1,f_1)} = (3x^2 + 1)_{(t_1(x),i_1(x),f_1(x))}$,

$h_N(x) = (h(x))_{(t_2,i_2,f_2)} = (x^3)_{(t_2(x),i_2(x),f_2(x))}$,

$g(x)$ is differentiable on $\mathbb{R}$, and $g'(x) = 6x$,

$t_1(x)$ is differentiable on $\mathbb{R}^*$, and $t_1'(x) = 0\,; x \in \mathbb{R}^*$,

$i_1(x)$ is differentiable on $\mathbb{R}|\{-1\}$, and $i_1'(x) = \begin{cases} \frac{1}{x^2} & x < -1 \\ 0 & x > -1 \end{cases}$,

$f_1(x)$ is differentiable on $\mathbb{R}$, and $f_1'(x) = 0$.

So that $g_N(x)$ is differentiable on $\mathbb{R}|\{0, -1\}$, and

$$g_N'(x) = \begin{cases} (6x)_{\left(0,\frac{1}{x^2},0\right)} & x \in \left]-\infty, -1\right[\\ (6x)_{(0,0,0)} & x \in \left]-1,0\right[\cup \left]0, \infty\right[\end{cases}.$$

On the other hand, we have:

$h(x)$ is differentiable on $\mathbb{R}$, and $h'(x) = 3x^2$,

$t_2(x)$ is differentiable on $\mathbb{R}^*$with $t_2'(x) = 0\,; x \neq 0$,

$i_2(x)$ is differentiable on $\mathbb{R}^*$ with $i_2'(x) = 0\ ; x \neq 0$,

$f_2(x)$ is differentiable on $\mathbb{R}|\{-1,1\}$, and $f_2'(x) = 0\ ; x \in \mathbb{R}|\{-1,1\}$.

Thus $h_N(x)$ is fully differentiable on $\mathbb{R}|\{0,1,-1\}$, and

$h_N' = (3x^2)_{(0,0,0)}\ ;\ x \in \mathbb{R}|\{0,1,-1\}$,

$(g_N + h_N)'(x) = (6x + 3x^2)_{(0,\frac{1}{x^2},0)}\ ; x \in \mathbb{R}|\{0,1,-1\}$,

$(g_N - h_N)'(x) = (6x - 3x^2)_{(0,\frac{1}{x^2},0)}\ ; x \in \mathbb{R}|\{0,1,-1\}$,

$(g_N . h_N)'(x) = (6x . x^3 + (3x^2 + 1). 3x^2)_{(0,0,0)} = (15x^4 + 3x^2)_{(0,0,0)}$,

$\left(\frac{g_N}{h_N}\right)'(x) = \left(\frac{6x(x^3)-(3x^2)(3x^2+1)}{x^6}\right)_{(1,0,0)} = \left(\frac{-3x^4-3x^2}{x^6}\right)_{(1,0,0)}\ ; x \in \mathbb{R}|\{0,1,-1\}$.

Definition 6.

Let $\mu: \mathbb{R} \to [0,1]$ be a real fuzzy function, and $g: \mathbb{R} \to \mathbb{R}$ be a real function in one variable $g = g(x)$.

We define the corresponding two-fold fuzzy real function $g_\mu(x) = (g(x))_{\mu(x)}\ x \in \mathbb{R}$.

Example 6.

Consider $\mu: \mathbb{R} \to [0,1]$, $g: \mathbb{R} \to \mathbb{R}$ such that:

$$\mu(x) = \begin{cases} \frac{-1}{x} & x \leq -1 \\ x^2 & -1 < x < 0 \\ x & 0 \leq x \leq 1 \\ \frac{1}{x} & x > 1 \end{cases},$$

and $g(x) = 7x^2 + \cos x$, then: $g_\mu: \mathbb{R} \to \mathbb{R}_F$ such that:

$$g_\mu(x) = \begin{cases} (7x^2 + \cos x)_{\frac{-1}{x}} & x \leq -1 \\ (7x^2 + \cos x)_{x^2} & -1 < x < 0 \\ (7x^2 + \cos x)_x & 0 \leq x \leq 1 \\ (7x^2 + \cos x)_{\frac{1}{x}} & x > 1 \end{cases}.$$

For $x = \pi$, we have

$$g_\mu(\pi) = (7\pi^2 - 1)_{\frac{1}{\pi}},$$

For $x = 0$, we have

$$g_\mu(0) = (1)_0.$$

Definition 7.

Let $g_N: \mathbb{R} \to \mathbb{R}_F$ be a two-fold fuzzy real function in one variable, we say that g_N is differentiable at $x_0 \in \mathbb{R}$ if and only if:

$g'(x_0), \mu'(x_0)$ are existed, and $\mu'(x_0) \in [0,1]$.

Example 7.

Take $\mu(x): \mathbb{R} \to [0,1], g(x): \mathbb{R} \to \mathbb{R}$ such that:

$$\mu(x) = \begin{cases} \frac{1}{2} & x \le 0 \\ \frac{1}{x^2} & x > 0 \end{cases}, g(x) = 1 + x^2,$$

$$g_\mu(x): \mathbb{R} \to \mathbb{R}_F: g_\mu(x) = \begin{cases} (1+x^2)_{\frac{1}{2}} & ; x \le 0 \\ (1+x^2)_{\frac{1}{x^2}} & ; x > 0 \end{cases},$$

g_μ is not differentiable on $[0, \infty[$, that is because $g'(0)$ is not existed, $\mu'(x) \notin [0,1]$ $; x > 0$, g_μ is differentiable on $]-\infty, 0[$, that is because $g'(x), \mu'(x)$ are existed for all $x \in]-\infty, 0[$ and $\mu'(x) = 0$.

Thus $g'(x) = \{(2x)_0 \quad ; x < 0$.

Definition 8.

Let $(\mu, \alpha): \mathbb{R} \to [0,1], (g, h): \mathbb{R} \to \mathbb{R}, (g_\mu, h_\alpha): \mathbb{R} \to \mathbb{R}_F$ such that: $g_\mu(x) = (g(x))_{\mu(x)}, h_\alpha(x) = (h(x))_{\alpha(x)}$, then:

1) $(g_\mu + h_\alpha)(x) = (g(x) + h(x))_{\min(\mu(x),\alpha(x))}$,
2) $(g_\mu - h_\alpha)(x) = (g(x) - h(x))_{\min(\mu(x),\alpha(x))}$,
3) $(g_\mu)^n(x) = (g^n(x))_{\mu(x)}$,
4) $(g_\mu . h_\alpha)(x) = (g(x). h(x))_{\max(\mu(x),\alpha(x))}$,
5) $\left(\frac{1}{g_\mu}\right)(x) = \left(\frac{1}{g_\mu}\right)_{(1-\mu(x))}$,
6) $\left(\frac{h_\alpha}{g_\mu}\right)(x) = \left(\frac{h(x)}{g_\mu}\right)_{\max(\alpha(x),1-\mu(x))}$.

On the other hand, we define:

1) $(g'(x) + h'(x))_{\min(\mu'(x),\alpha'(x))} = (g_\mu + h_\alpha)'$,
2) $(g'(x) - h'(x))_{\min(\mu'(x),\alpha'(x))} = (g_\mu - h_\alpha)'$,
3) $.(g'(x). g^{n-1}(x))_{\mu'(x)} = (g_\mu^n)'$,
4) $(g'(x). h(x) + h'(x). g(x))_{\max(\mu'(x),\alpha'(x))}$,
5) $\left(\frac{-g'(x)}{g^2(x)}\right)_{(1-\mu'(x))} = \left(\frac{1}{g_\mu}\right)'$,
6) $\left(\frac{h_\alpha}{g_\mu}\right)' = \left(\frac{h'(x)g(x) - g'(x)h(x)}{g^2(x)}\right)_{\max(\alpha'(x),1-\mu'(x))}$.

Example 8.

Take $(\mu, \alpha): \mathbb{R} \to [0,1], (g, h): \mathbb{R} \to \mathbb{R}, (g_\mu, h_\alpha): \mathbb{R} \to \mathbb{R}_F$,

such that:

$$g(x) = x + 1, h(x) = x^2, \mu(x) = \begin{cases} \frac{-1}{x} & x \leq -1 \\ 1 & x > -1 \end{cases}, \alpha(x) = \begin{cases} \frac{1}{3} & x \leq -2 \\ 0 & -2 < x \leq 0 \\ \frac{1}{2} & x > 0 \end{cases},$$

$$g_\mu(x) = \begin{cases} (x+1)_{\frac{-1}{x}} & ; x \leq -1 \\ (x+1)_1 & ; x > -1 \end{cases}, \ h_\alpha(x) = \begin{cases} (x^2)_{\frac{1}{3}} & ; x \leq -2 \\ (x^2)_0 & ; -2 < x \leq 0 \\ (x^2)_{\frac{1}{2}} & ; x > 0 \end{cases},$$

$h(x), g(x)$ are differentiable on R, with $g'(x) = 1, h'(x) = 2x$.

$\alpha(x)$ is differentiable on $\mathbb{R}|\{-2,0\}$with $\alpha'(x) = 0$ for all $x \in \mathbb{R}|\{-2,0\}$.

For $\mu(x)$ at $x = -1$, we have:

$$\frac{\mu(x) - \mu(-1)}{x+1} = \frac{\mu(x) - 1}{x+1},$$

$$\lim_{x \to -1^+} \frac{\mu(x) - \mu(-1)}{x+1} = \lim_{x \to -1^+} \frac{1 - 1}{x+1} = 0,$$

$$\lim_{x \to -1^-} \frac{\mu(x) - \mu(-1)}{x+1} = \lim_{x \to -1^-} \frac{\frac{-1}{x} - 1}{x+1} = \lim_{x \to -1^-} \frac{\frac{-1-x}{x}}{x+1} = \lim_{x \to -1^-} \frac{-1}{x} = 1,$$

hence $\mu(x)$ is differentiable at $x = -1$,

$\mu(x)$ is differentiable on $\mathbb{R}|\{-1\}$, and

$$\mu'^{(x)} = \begin{cases} \frac{1}{x^2} & x \leq -1 \\ 0 & ; x > -1 \end{cases},$$

$$\left(g_\mu + h_\alpha\right)'(x) = (x + 1 + x^2)'_{\min(\mu'(x),0)} = (2x+1)_0 \quad ; x \in \mathbb{R}|\{-2,0,-1\},$$

$$\left(g_\mu - h_\alpha\right)(x) = (1 - 2x)_{\min(\mu'(x),0)} = (1 - 2x)_0 \quad ; x \in \mathbb{R}|\{-2,0,-1\},$$

$$\left(g_\mu^n\right)'(x) = n.(x+1)^{n-1}_{\mu'(x)} = \begin{cases} n.(x+1)^{n-1}_{\frac{1}{x^2}} & ; x \leq -1 \\ n.(x+1)^{n-1}_0 & ; x > -1 \end{cases},$$

$$\left(g_\mu . h_\alpha\right)'(x) = (3x^2 + 2x)_{\max(\mu'(x),\alpha'(x))} = \begin{cases} (3x^2 + 2x)_{\frac{1}{x^2}} & ; x \leq -1, x \neq -2 \\ (3x^2 + 2x)_0 & ; x > -1, x \neq 0 \end{cases},$$

and so on.

Definition 9.

Let $g_N: \mathbb{R} \to \mathbb{R}_{(t,i,f)}$ be a two-fold neutrosophic real function, we say that:

1) g_N is fully continuous at $x_0 \in \mathbb{R}$ if:

$$\begin{cases} g \text{ is continuous at } x_0 \\ i, f \text{ are continuous at } x_0 \\ t \text{ is continuous at } x_0 \end{cases},$$

2) g_N is T- continuous at x_0 (continuous at x_0 with respect to truth component) at $x_0 \in \mathbb{R}$ if:

$$\begin{cases} g \text{ is continuous at } x_0 \\ t \text{ is continuous at } x_0 \end{cases},$$

3) g_N is I- $continuous$ ($continuous$ with respect to the indeterminacy component) at $x_0 \in \mathbb{R}$ if:

$$\begin{cases} g \text{ is continuous at } x_0 \\ i \text{ is continuous at } x_0 \end{cases},$$

4) g_N is F- continuous (continuous with respect to the falisty component) at $x_0 \in \mathbb{R}$ if:

$$\begin{cases} g \text{ is continuous at } x_0 \\ f \text{ is continuous at } x_0 \end{cases}.$$

Definition 10.

Let $g_N: \mathbb{R} \to \mathbb{R}_F$ be a two-fold fuzzy real function in one variable, we say that g_N is $continuous$ at $x_0 \in \mathbb{R}$ if and only if:

g, μ are continuous at, $(x_0) \in R$.

Theorem 2.

Let $g_N: \mathbb{R} \to \mathbb{R}_{(t,i,f)}$ be a two-fold neutrosophic real function.If g_N is fully differentiable, then it is fully continuous.

Proof.

Assume that g_N is fully differentiable, then we have:

$g'(x_0)$ is existed
$i'(x_0), f'(x_0), t'(x_0)$ are existed
$i'(x_0), f'(x_0), t'(x_0) \in [0,1]$,

Thus

$$\begin{cases} g \text{ is continuous at } x_0 \\ i, f \text{ are continuous at } x_0 \ , \\ t \text{ is continuous at } x_0 \end{cases},$$

so that it is fully continuous.

Theorem 3.

Let $g_N: \mathbb{R} \to \mathbb{R}_{(t,i,f)}$ be a two-fold neutrosophic real function.If g_N is T- differentiable, then it is T- continuous.

Proof:

Assume that g_N is T- differentiable, then we have:

$g'(x_0)$ is existed,
$t'(x_0)$ is existed,
$t'(x_0) \in [0,1]$.

Thus, g is continuous at x_0, and the function *t* is continuous at x_0 , so that it is T- continuous

Theorem 4.

Let $g_N: \mathbb{R} \to \mathbb{R}_{(t,i,f)}$ be a two-fold neutrosophic real function.If g_N is F-differentiable, then it is F-continuous.

Proof.

Assume that g_N is F- differentiable, then we have:

$g'(x_0)$ is existed,
$f'(x_0)$ is existed,
$f'(x_0) \in [0,1]$.

Thus, g is continuous at x_0, and the function f is continuous at x_0 , so that it is *F*- continuous.

Theorem 5.

Let $g_N: \mathbb{R} \rightarrow \mathbb{R}_{(t,i,f)}$ be a two-fold neutrosophic real function.If g_N is *I*- differentiable, then it is *I*- continuous.

Proof.

Assume that g_N is I- differentiable, then we have:

$g'(x_0)$ is existed,
$i'(x_0)$ is existed,
$i'(x_0) \in [0,1]$.

Thus, g is continuous at x_0, and the function i is continuous at x_0 , so that it is *I*-continuous

Conclusion

In this paper, we defined the concepts of two-fold neutrosophic continuous functions, two-fold fuzzy continuous functions, and two-fold neutrosophic/fuzzy differentiable functions for the first time.

The elementary properties of these novel concepts are studied and handled through many theorems, as well as many clear examples that clarify the validity of these analytical concepts.

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