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A Study on the Number of Neutrosophic Crisp Topological Spaces in a Finite Set

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Abstract. The computation of formulae for the number of topological spaces is one of the challenging areas of study. The present work aims to find formulae to compute the number of neutrosophic crisp topological spaces having 2-NCrOSs, 3-NCrOSs, and 4-NCrOSs.

Keywords: Topological spaces; neutrosophic topological spaces, neutrosophic crisp topological spaces.

1. Introduction

Zadeh [1] introduced the fuzzy set and Atanassov [2] introduced the intuitionistic fuzzy set. By generalizing the crisp and fuzzy counterparts, neutrosophy has established the groundwork for an entire family of new mathematical theories. The concept of a "neutrosophic set" is introduced by Smarandache [3–5]. Later, some possible definitions for basic concepts of the neutrosophic crisp set and its operations have been investigated by Hanafy et al. [6] and Salama [7].

A topology tells how elements of the set are related to each other. From the literature, it is found that the explicit formula for finding the number of topologies in a set is still not obtained. This is one of the fascinating research areas of topology. Let τ_n denotes the number of topologies on a finite set $\mathcal{X}$ with $|S| = n$. Krishnamurty [8] computed a sharper bound namely $2^{n(n-1)}$ for τ_n. Sharp [9] shows that only discrete topology has cardinal greater than $\frac{3}{4}2^n$ and derived bounds for the cardinality of topologies which are connected, nonconnected, non-T_0, and some more. Several authors [**?**, 10–19, 21] also worked in this interesting and difficult research area. Recently Basuma tary et al. [22, 23] started research work on number of neutrosophic topological spaces. They computed some results for finding the number of neutosophic topologies for $k \leq 4$ open sets, the number of neutrosophic clopen topological spaces having small ($k = 2, 3, 4, 5$) open sets with neutrosophic values in M, and the number of neutrosophic bitopological spaces and tritopological spaces. Salama et al. [24] introduced the basic concept of neutosophic crisp topological spaces. In this paper, the formulae for computation of the number of neutrosophic crisp toplogical spaces for small ($k = 2, 3, 4$) open sets are initiated.

2. Materials and Methods

Definition 2.1. [24] Let $\mathcal{X}$ be a non-empty fixed set. A neutrosophic crisp set (NCrS) A is an object having the form $A = \langle A_1, A_2, A_3\rangle$, where A_1, A_2, and A_3 are subsets of $\mathcal{X}$ satisfying $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$, and $A_2 \cap A_3 = \phi$.

Remark 2.2. [24] A NCrS $A = \langle A_1, A_2, A_3\rangle$ can be identified as an ordered triple $\langle A_1, A_2, A_3\rangle$, where A_1, A_2, and A_3 are subsets of $\mathcal{X}$.

Definition 2.3. [24] $\phi_{\mathcal{N}}$ may be defined in many ways as an NCrS as follows:

(1) $\phi_{\mathcal{N}} = \langle \phi, \phi, \mathcal{X}\rangle$.
(2) $\phi_{\mathcal{N}} = \langle \phi, \mathcal{X}, \mathcal{X}\rangle$.
(3) $\phi_{\mathcal{N}} = \langle \phi, \mathcal{X}, \phi\rangle$.
(4) $\phi_{\mathcal{N}} = \langle \phi, \phi, \phi\rangle$.

$\mathcal{X}_{\mathcal{N}}$ may also be defined in many ways as an NCrS as follows:

(1) $\mathcal{X}_{\mathcal{N}} = \langle \mathcal{X}, \phi, \phi\rangle$.
(2) $\mathcal{X}_{\mathcal{N}} = \langle \mathcal{X}, \mathcal{X}, \phi\rangle$.
(3) $\mathcal{X}_{\mathcal{N}} = \langle \mathcal{X}, \mathcal{X}, \mathcal{X}\rangle$.

Definition 2.4. [24] Let $\mathcal{X}$ be a non-empty set, and the NCrSs A and B be in the form $A = \langle A_1, A_2, A_3\rangle$, $B = \langle B_1, B_2, B_3\rangle$ respectively. Then the following two possible definitions may be considered for subsets ($A \subseteq B$):

(1) $A \subseteq B \iff A_1 \subseteq B_1$, $A_1 \subseteq B_2$, and $A_3 \supseteq B_3$.
(2) $A \subseteq B \iff A_1 \subseteq B_1$, $A_2 \supseteq B_2$, and $A_3 \supseteq B_3$.

Definition 2.5. [24] Let $\mathcal{X}$ is a non-empty set, and the NCrSs A and B be in the form $A = \langle A_1, A_2, A_3\rangle$ and $B = \langle B_1, B_2, B_3\rangle$ respectively. Then,

(1) $A \cap B$ may be defined in two ways:
  (a) $A \cap B = \langle A_1 \cap B_1, A_2 \cap B_2, A_3 \cup B_3\rangle$.
  (b) $A \cap B = \langle A_1 \cap B_1, A_2 \cup B_2, A_3 \cup B_3\rangle$.

(2) $A \cup B$ may also be defined in two ways:

(a) $A \cup B = \langle A_1 \cup B_1, A_2 \cap B_2, A_3 \cap B_3 \rangle$.

(b) $A \cup B = \langle A_1 \cup B_1, A_2 \cup B_2, A_3 \cap B_3 \rangle$.

Definition 2.6. [17] The number of partitions of a finite set with n elements into k blocks, is the Stirling number of the second kind. It is denoted by $S(n, k)$ or $S_{n,k}$. The explicit formula for Stirling numbers of the second kind is

$$S(n, k) = S_{n,k} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k - j)^n.$$

Definition 2.7. [25] The set of all neutrosophic crisp subsets of a non-empty finite set $\mathcal{X}$ is called the neutrosophic crisp power set of $\mathcal{X}$.

The notation for the neutrosophic crisp power set of $\mathcal{X}$ is $\mathcal{P}_{\mathcal{NC}r}(\mathcal{X})$ and its cardinality is denoted by $|\mathcal{P}_{\mathcal{NC}r}(\mathcal{X})|$.

Proposition 2.8. *[25] A set $\mathcal{X}$ with $|\mathcal{X}| = n$ has*

$$(3.2^n - 4) + 3! \left\{ \sum_{i=2}^{n} \mathcal{S}(i, 2) \binom{n}{i} + \sum_{j=3}^{n} \mathcal{S}(i, 3) \binom{n}{j} \right\}$$

neutrosophic crisp subsets.

Corollary 2.9. *[25] If $|\mathcal{X}| = n$, then the cardinality of the power set of NCrS of $\mathcal{X}$ is*

$$|\mathcal{P}_{\mathcal{NC}r}(\mathcal{X})| = (3.2^n - 4) + 3! \left\{ \sum_{i=2}^{n} \mathcal{S}(i, 2) \binom{n}{i} + \sum_{j=3}^{n} \mathcal{S}(j, 3) \binom{n}{j} \right\}.$$

Definition 2.10. [24] A neutrosophic crisp topology (NCrT) on a non-empty set $\mathcal{X}$ is a family τ^{NC} of neutrosophic crisp subsets in $\mathcal{X}$ satisfying the following axioms

(1) $\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}} \in \tau^{NC}$.

(2) $A_1 \cap A_2 \in \tau^{NC}$; for any $A_1, A_2 \in \tau^{NC}$.

(3) $\cup A_j \in \tau^{NC}$; $\forall \{A_j : j \in J\} \subseteq \tau^{NC}$.

In this case, the pair $(\mathcal{X}, \tau^{NC})$ is called a neutrosophic crisp topological space (NCrTS) in $\mathcal{X}$. The elements in τ^{NC} are called neutrosophic crisp open sets (NCrOSs) in $\mathcal{X}$. A NCrS F is closed if and only if its complement F^c is an open NCrS.

3. **Neutrosophic Crisp Topological Spaces**

Definition 3.1. An NCrT having k-NCrOSs on a non-empty set $\mathcal{X}$ is said to be an NCrT of cardinality k. The number of NCrTs of cardinality k on $\mathcal{X}$ with $|\mathcal{X}| = n$ will be denoted by $\mathcal{T}_{\mathcal{C}r}(n, k)$.

Example 3.2. Let $\mathcal{X} = \{u, v, w\}$ and $\mathcal{A}_1 = \langle \emptyset, \emptyset, \{u\} \rangle$, then $\tau^{NCr} = \{\phi_\mathcal{N}, \mathcal{X}_\mathcal{N}, \mathcal{A}_1\}$ form an NCrT on $\mathcal{X}$. So, τ^{NCr} is an NCrT of cardinality 3 as it has 3-NCrOSs.

Proposition 3.3. *For a non-empty finite set $\mathcal{X}$ with $|\mathcal{X}| = n$,*

(1) $\mathcal{T}_{\mathcal{C}r}(n, 2) = 1$,

(2) $\mathcal{T}_{\mathcal{C}r}(n, k) = 1$, *where* $k = |\mathcal{P}_{\mathcal{N}\mathcal{C}r}(\mathcal{X})|$, *the cardinality of the neutrosophic crisp power set on $\mathcal{X}$.*

Proof:

(1) The NCrT having 2-NCrOSs is the indiscrete NCrT which is $\mathcal{T}_\mathcal{N} = \{\phi_\mathcal{N}, \mathcal{X}_\mathcal{N}\}$. Therefore, $(\mathcal{X}, \mathcal{T}_\mathcal{N})$ is the only NCrTS having 2-NCrOSs as $\mathcal{T}_\mathcal{N}$ contains only two members $\phi_\mathcal{N}$ and $\mathcal{X}_\mathcal{N}$. Hence, the number of neutrosophic crisp topological spaces (NCrTSs) having 2-NCrOSs is 1 i.e., $\mathcal{T}_{\mathcal{C}r}(n, 2) = 1$.

(2) The NCrT of cardinality $k = |\mathcal{P}_{\mathcal{N}\mathcal{C}r}(\mathcal{X})|$ is the discrete NCrT only. Hence, $\mathcal{T}_{\mathcal{C}r}(n, k) = 1$, for $k = |\mathcal{P}_{\mathcal{N}\mathcal{C}r}(\mathcal{X})|$.

Example 3.4. Let $\mathcal{X} = \{u, v\}$, then, $|\mathcal{X}| = n = 2$. Here, the neutrosophic crisp subsets on $\mathcal{X}$ are

$\phi_\mathcal{N}$, $\mathcal{X}_\mathcal{N}$, $\mathcal{A}_1 = \langle \emptyset, \emptyset, \{u\} \rangle$, $\mathcal{A}_2 = \langle \emptyset, \{u\}, \emptyset \rangle$, $\mathcal{A}_3 = \langle \{u\}, \emptyset, \emptyset \rangle$,
$\mathcal{A}_4 = \langle \emptyset, \emptyset, \{v\} \rangle$, $\mathcal{A}_5 = \langle \emptyset, \{v\}, \emptyset \rangle$, $\mathcal{A}_6 = \langle \{v\}, \emptyset, \emptyset \rangle$,
$\mathcal{A}_7 = \langle \emptyset, \{u\}, \{v\} \rangle$, $\mathcal{A}_8 = \langle \{u\}, \emptyset, \{v\} \rangle$, $\mathcal{A}_9 = \langle \{u\}, \{v\}, \emptyset \rangle$,
$\mathcal{A}_{10} = \langle \emptyset, \{v\}, \{u\} \rangle$, $\mathcal{A}_{11} = \langle \{v\}, \emptyset, \{u\} \rangle$, $\mathcal{A}_{12} = \langle \{v\}, \{u\}, \emptyset \rangle$.

In this case, the only NCrT having 2-NCrOSs is $\{\phi_\mathcal{N}, \mathcal{X}_\mathcal{N}\}$ and hence $\mathcal{T}_{\mathcal{C}r}(n, 2) = 1$. Also, the NCrT having $k = |\mathcal{P}_{\mathcal{N}\mathcal{C}r}(\mathcal{X})| = 14$-NCrOSs is

$$\{\phi_\mathcal{N}, \mathcal{X}_\mathcal{N}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \mathcal{A}_8, \mathcal{A}_9, \mathcal{A}_{10}, \mathcal{A}_{11}, \mathcal{A}_{12}\}$$

and hence, $\mathcal{T}_{\mathcal{C}r}(n, k) = 1$, for $k = |\mathcal{P}_{\mathcal{N}\mathcal{C}r}(\mathcal{X})| = 14$.

4. Neutrosophic Crisp Topological Spaces with 3-NCrOSs

Proposition 4.1. *The number of NCrTs of cardinality* 3 *on a non-empty finite set $\mathcal{X}$ with $|\mathcal{X}| = n$ is given by the formula*

$$\begin{aligned}\mathcal{T}_{\mathcal{C}r}(n, 3) &= |\mathcal{P}_{\mathcal{N}\mathcal{C}r}(\mathcal{X})| - 2 \\ &= 3(2^n - 2) + 3! \left[\sum_{i=2}^{n} \mathcal{S}(i, 2)\binom{n}{i} + \sum_{j=3}^{n} \mathcal{S}(j, 3)\binom{n}{j}\right].\end{aligned}$$

Proof:

The NCrTs having 3-NCrOSs necessarily consists of a chain containing $\phi_\mathcal{N}, \mathcal{X}_\mathcal{N}$ and any other neutrosophic crisp subset $\mathcal{A}_\mathcal{N}$ of $\mathcal{X}$ other than $\phi_\mathcal{N}$ and $\mathcal{X}_\mathcal{N}$. Clearly, $\phi_\mathcal{N} \subset \mathcal{A}_\mathcal{N} \subset \mathcal{X}_\mathcal{N}$. It is observed that the number of such $\mathcal{A}_\mathcal{N}$ is equal to $|\mathcal{P}_{\mathcal{N}\mathcal{C}r}(\mathcal{X})| - 2$. Since the set $\{\phi_\mathcal{N}, \mathcal{A}_\mathcal{N}, \mathcal{X}_\mathcal{N}\}$ form an NCrT and the total number of such NCrTs is $|\mathcal{P}_{\mathcal{N}\mathcal{C}r}(\mathcal{X})| - 2$.

Now, $|\mathcal{P}_{\mathcal{NC}r}(\mathcal{X})| = (3.2^n - 4) + 3!\left\{\sum_{i=2}^{n}\mathcal{S}(i,2)\binom{n}{i} + \sum_{j=3}^{n}\mathcal{S}(j,3)\binom{n}{j}\right\}$.

Therefore,

$$\begin{aligned}|\mathcal{P}_{\mathcal{NC}r}(\mathcal{X})| - 2 &= \left[(3.2^n - 4) + 3!\left\{\sum_{i=2}^{n}\mathcal{S}(i,2)\binom{n}{i} + \sum_{j=3}^{n}\mathcal{S}(j,3)\binom{n}{j}\right\}\right] - 2\\ &= (3.2^n - 6) + 3!\left\{\sum_{i=2}^{n}\mathcal{S}(i,2)\binom{n}{i} + \sum_{j=3}^{n}\mathcal{S}(j,3)\binom{n}{j}\right\}\\ &= 3(2^n - 2) + 3!\left\{\sum_{i=2}^{n}\mathcal{S}(i,2)\binom{n}{i} + \sum_{j=3}^{n}\mathcal{S}(j,3)\binom{n}{j}\right\}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{T}_{\mathcal{C}r}(n,3) &= |\mathcal{P}_{\mathcal{NC}r}(\mathcal{X})| - 2\\ &= 3(2^n - 2) + 3!\left[\sum_{i=2}^{n}\mathcal{S}(i,2)\binom{n}{i} + \sum_{j=3}^{n}\mathcal{S}(j,3)\binom{n}{j}\right].\end{aligned}$$

Example 4.2. Let $\mathcal{X} = \{u, v\}$, then

$$\mathcal{T}_{\mathcal{C}r}(2,3) = 3(2^2 - 2) + 3!\left\{\sum_{i=2}^{2}\mathcal{S}(i,2)\binom{2}{i} + \sum_{j=3}^{2}\mathcal{S}(j,3)\binom{2}{j}\right\}.$$

Clearly, $\sum_{j=3}^{2}\mathcal{S}(j,3)\binom{2}{j} = 0$.

So, $\mathcal{T}_{\mathcal{C}r}(2,3) = 6 + 6\left\{\mathcal{S}(2,2)\binom{2}{2} + 0\right\} = 12$.

Consequently, $\mathcal{T}_{\mathcal{C}r}(2,3) = 12$ and these NCrTs having 3-NCrOSs are listed below

$\{\phi_{\mathcal{N}}, \mathcal{A}_1, \mathcal{X}_{\mathcal{N}}\}$, $\{\phi_{\mathcal{N}}, \mathcal{A}_2, \mathcal{X}_{\mathcal{N}}\}$,$\{\phi_{\mathcal{N}}, \mathcal{A}_3, \mathcal{X}_{\mathcal{N}}\}$, $\{\phi_{\mathcal{N}}, \mathcal{A}_4, \mathcal{X}_{\mathcal{N}}\}$,
$\{\phi_{\mathcal{N}}, \mathcal{A}_5, \mathcal{X}_{\mathcal{N}}\}$,$\{\phi_{\mathcal{N}}, \mathcal{A}_6, \mathcal{X}_{\mathcal{N}}\}$, $\{\phi_{\mathcal{N}}, \mathcal{A}_7, \mathcal{X}_{\mathcal{N}}\}$, $\{\phi_{\mathcal{N}}, \mathcal{A}_8, \mathcal{X}_{\mathcal{N}}\}$,
$\{\phi_{\mathcal{N}}, \mathcal{A}_9, \mathcal{X}_{\mathcal{N}}\}$, $\{\phi_{\mathcal{N}}, \mathcal{A}_{10}, \mathcal{X}_{\mathcal{N}}\}$, $\{\phi_{\mathcal{N}}, \mathcal{A}_{11}, \mathcal{X}_{\mathcal{N}}\}$,$\{\phi_{\mathcal{N}}, \mathcal{A}_{12}, \mathcal{X}_{\mathcal{N}}\}$.

5. Neutrosophic Crisp Topological Spaces with 4-NCrOSs

The NCrT having 4-NCrOSs must have the form $\mathcal{T} = \{\phi_{\mathcal{N}}, \mathcal{A}, \mathcal{B}, \mathcal{X}_{\mathcal{N}}\}$, where $\mathcal{A} \neq \mathcal{B}$ such that $\mathcal{A}\cap\mathcal{B}, \mathcal{A}\cup\mathcal{B} \in \mathcal{T}$. To compute the number of NCrTs with exactly 4-NCrOSs, we need to compute formulae for following cases:

Case 1: $\mathcal{A}\cap\mathcal{B} = \phi_{\mathcal{N}}, \mathcal{A}\cup\mathcal{B} = \mathcal{X}_{\mathcal{N}}$
Case 2: $\mathcal{A}\cap\mathcal{B} = \phi_{\mathcal{N}},\ \mathcal{A}\cup\mathcal{B} = \phi_{\mathcal{N}}$
Case 3: ($\mathcal{A}\cap\mathcal{B} = \mathcal{A}$ or $\mathcal{B}$, $\mathcal{A}\cup\mathcal{B} = \phi_{\mathcal{N}}$) or
($\mathcal{A}\cap\mathcal{B} = \phi_{\mathcal{N}}, \mathcal{A}\cup\mathcal{B} = \mathcal{A}$ or $\mathcal{B}$)
Case 4: ($\mathcal{A}\cap\mathcal{B} = \mathcal{A}, \mathcal{A}\cup\mathcal{B} = \mathcal{A}$) or ($\mathcal{A}\cap\mathcal{B} = \mathcal{B}, \mathcal{A}\cup\mathcal{B} = \mathcal{B}$)
Case 5: ($\mathcal{A}\cap\mathcal{B} = \mathcal{A}, \mathcal{A}\cup\mathcal{B} = \mathcal{B}$) or ($\mathcal{A}\cap\mathcal{B} = \mathcal{B}, \mathcal{A}\cup\mathcal{B} = \mathcal{A}$).

Proposition 5.1.

For a non-empty finite set $\mathcal{X}$ with $|\mathcal{X}| = n$, the number of NCrTs having 4-NCrOSs satisfying the condition in case 1 is obtained by the formula

$$\mathcal{S}(n,2)(2^n + 1).$$

Proof:

In general, the number of partitions of a non-empty set $\mathcal{X}$ with $|\mathcal{X}| = n$ into two blocks is given by $\mathcal{S}(n,2)$. To obtain $\mathcal{A}\cap\mathcal{B}=\phi_{\mathcal{N}}$ and $\mathcal{A}\cup\mathcal{B}=\mathcal{X}_{\mathcal{N}}$, clearly $\mathcal{A}$ and $\mathcal{B}$ must have the following two forms:

(1) $\mathcal{A}=\langle\mathcal{A}_1,\mathcal{A}_2,\emptyset\rangle$ & $\mathcal{B}=\langle\mathcal{B}_1,\mathcal{B}_2,\emptyset\rangle$,

(2) $\mathcal{A}=\langle\mathcal{A}_1,\emptyset,\mathcal{A}_3\rangle$ & $\mathcal{B}=\langle\mathcal{B}_1,\emptyset,\mathcal{B}_3\rangle$.

Let us count the ways that they can be chosen.

(1) We have, $\mathcal{A}\cap\mathcal{B}=\langle\mathcal{A}_1\cap\mathcal{B}_1,\mathcal{A}_2\cap\mathcal{B}_2,\emptyset\rangle$, and $\mathcal{A}\cup\mathcal{B}=\langle\mathcal{A}_1\cup\mathcal{B}_1,\mathcal{A}_2\cap\mathcal{B}_2,\emptyset\rangle$. Now, to get $\mathcal{A}\cap\mathcal{B}=\phi_{\mathcal{N}}$ and $\mathcal{A}\cup\mathcal{B}=\mathcal{X}_{\mathcal{N}}$, we must have, $\mathcal{A}_1\cap\mathcal{B}_1=\emptyset$, $\mathcal{A}_1\cup\mathcal{B}_1=\mathcal{X}$ and $\mathcal{A}_2\cap\mathcal{B}_2=\emptyset$. This implies that $\mathcal{A}_1$, $\mathcal{B}_1$ is a partition of $\mathcal{X}$ and so, $\mathcal{B}_1=\mathcal{X}-\mathcal{A}_1$. Therefore, $\mathcal{A}_1$, $\mathcal{B}_1$ can be chosen in $\mathcal{S}(n,2)$ ways. Now, if $|\mathcal{A}_1|=i$ then $|\mathcal{B}_1|=n-i$. Since $\mathcal{A}_2\cap\mathcal{B}_2=\emptyset$, then the neutrosophic crisp subset $\mathcal{A}_2$ can be chosen out of $n-i$ elements in $\binom{n-i}{k}, k=0,1,2,\ldots,n-i$ ways with $k=0$ representing the empty set. Therefore, $\mathcal{A}_2$ can be chosen in $\sum_{k=0}^{n-i}\binom{n-i}{k}=2^{n-i}$ ways. Similarly, $\mathcal{B}_2$ can be chosen out of $n-(n-i)=i$ elements in $\sum_{k=0}^{i}\binom{i}{k}=2^i$ ways. Hence, the total number of ways is $\mathcal{S}(n,2).2^{n-i}.2^i=\mathcal{S}(n,2).2^n$.

(2) We have, $\mathcal{A}\cap\mathcal{B}=\langle\mathcal{A}_1\cap\mathcal{B}_1,\emptyset,\mathcal{A}_3\cup\mathcal{B}_3\rangle$, and $\mathcal{A}\cup\mathcal{B}=\langle\mathcal{A}_1\cup\mathcal{B}_1,\emptyset,\mathcal{A}_3\cap\mathcal{B}_3\rangle$. Now, to get $\mathcal{A}\cap\mathcal{B}=\phi_{\mathcal{N}}$ and $\mathcal{A}\cup\mathcal{B}=\mathcal{X}_{\mathcal{N}}$, we must have, $\mathcal{A}_1\cap\mathcal{B}_1=\emptyset$, $\mathcal{A}_3\cup\mathcal{B}_3=\mathcal{X}$ and $\mathcal{A}_1\cup\mathcal{B}_1=\mathcal{X}$, $\mathcal{A}_3\cap\mathcal{B}_3=\emptyset$ simultaneously. This shows that $\mathcal{A}_1$ and $\mathcal{B}_1$ is a partition of $\mathcal{X}$ and $\mathcal{A}_3=\mathcal{A}_1^C=\mathcal{B}_1$, $\mathcal{B}_3=\mathcal{B}_1^C=\mathcal{A}_1$. Therefore, we can take $\mathcal{A}_1$ and $\mathcal{B}_1$ or $\mathcal{A}_3$ and $\mathcal{B}_3$ in $\mathcal{S}(n,2)$ ways.

From (i) and (ii), the total number of ways is $\mathcal{S}(n,2)(2^n+1)$.

Hence, the number of NCrTs having 4-NCrOSs satisfying the condition in case 1 is obtained by the formula

$$\mathcal{S}(n,2)(2^n+1).$$

Proposition 5.2.

The number of NCrTs having 4-NCrOSs on a non-empty set $\mathcal{X}$ satisfying the condition in case 2 is obtained by the formula

$$\frac{n(n-1)}{2}+\{\mathcal{S}(n,2)\times 2^n\}+\sum_{i=3}^{n}\left\{\binom{n}{i}\mathcal{S}(i,2)\right\}$$

where $|\mathcal{X}|=n$.

Proof:

To obtain $\mathcal{A}\cap\mathcal{B}=\phi_{\mathcal{N}}$ and $\mathcal{A}\cup\mathcal{B}=\phi_{\mathcal{N}}$, clearly, $\mathcal{A}$ and $\mathcal{B}$ must have the following two forms

(1) $\mathcal{A}=\langle\emptyset,\mathcal{A}_2,\mathcal{A}_3\rangle$ & $\mathcal{B}=\langle\emptyset,\mathcal{B}_2,\mathcal{B}_3\rangle$ such that $\mathcal{A}_3\cup\mathcal{B}_3=\mathcal{X}$ and $\mathcal{A}_3\cap\mathcal{B}_3=\emptyset$ and $\mathcal{A}_2\cap\mathcal{B}_2=\emptyset$.

(2) $\mathcal{A}=\langle\emptyset,\mathcal{A}_2,\emptyset\rangle$ & $\mathcal{B}=\langle\emptyset,\mathcal{B}_2,\emptyset\rangle$ such that $\mathcal{A}_2\cap\mathcal{B}_2=\emptyset$.

From (i), $\mathcal{A}\cap\mathcal{B} = \langle\emptyset, \mathcal{A}_2\cap\mathcal{B}_2, \mathcal{A}_3\cup\mathcal{B}_3\rangle$, and $\mathcal{A}\cup\mathcal{B} = \langle\emptyset, \mathcal{A}_2\cap\mathcal{B}_2, \mathcal{A}_3\cap\mathcal{B}_3\rangle$. Since, $\mathcal{A}_3\cup\mathcal{B}_3 = \mathcal{X}$ and $\mathcal{A}_3\cap\mathcal{B}_3 = \emptyset$, which implies that $\mathcal{A}_3$ and $\mathcal{B}_3$ is a partition of $\mathcal{X}$ and say $\mathcal{B}_3 = \mathcal{X} - \mathcal{A}_3$. Therefore, $\mathcal{A}_3$ and $\mathcal{B}_3$ can be chosen in $\mathcal{S}(n, 2)$ ways. Now, if $|\mathcal{A}_3| = i$, $|\mathcal{B}_3| = n-i$, $1 \leq i \leq n-1$, and $\mathcal{A}_2\cap\mathcal{B}_2 = \emptyset$, then $\mathcal{A}_2$ can be chosen in $\sum_{k=0}^{n-i}\binom{n-i}{k} = 2^{n-i}$ ways, and similarly, $\mathcal{B}_2$ can be chosen out of $n-(n-i) = i$ elements in $\sum_{k=0}^{i}\binom{i}{k} = 2^i$ ways.

Therefore, the total number of ways is $\mathcal{S}(n, 2)\times 2^{n-i}\times 2^i$ i.e., $\mathcal{S}(n, 2)\times 2^n$.

From (ii), $\mathcal{A}\cap\mathcal{B} = \langle\emptyset, \mathcal{A}_2\cap\mathcal{B}_2, \emptyset\rangle$, $\mathcal{A}\cup\mathcal{B} = \langle\emptyset, \mathcal{A}_2\cap\mathcal{B}_2, \emptyset\rangle$, and $\mathcal{A}_2\cap\mathcal{B}_2 = \emptyset$. If $|\mathcal{A}_2\cup\mathcal{B}_2| = i$, $2 \leq i \leq n$, then $\mathcal{A}_2\cup\mathcal{B}_2$ is chosen in $\binom{n}{i}$ different ways and then it is partitioned into two disjoint blocks: this is done in $\mathcal{S}(i, 2)$ different ways. Therefore, the number of ways for form (ii) is $\sum_{i=2}^{n}\binom{n}{i}\mathcal{S}(i, 2)$.

Hence, the number of NCrTs having 4-NCrOSs satisfying condition in case 2 is obtained by the formula

$$\{\mathcal{S}(n, 2)\times 2^n\} + \sum_{i=2}^{n}\left\{\binom{n}{i}\mathcal{S}(i, 2)\right\}$$

i.e.,

$$\frac{n(n-1)}{2} + \{\mathcal{S}(n, 2)\times 2^n\} + \sum_{i=3}^{n}\left\{\binom{n}{i}\mathcal{S}(i, 2)\right\}.$$

Proposition 5.3.

For a non-empty finite set $\mathcal{X}$ with $|\mathcal{X}| = n$, the number of NCrTs having 4-NCrOSs satsifying conditions in case 3 is obtained by the formula $2(2^n-2)^2$.

Proof:

There are two forms

(i) $\mathcal{A} = \langle\emptyset, \emptyset, \mathcal{A}_3\rangle$ & $\mathcal{B} = \langle\emptyset, \mathcal{B}_2, \emptyset\rangle$,

(ii) $\mathcal{A} = \langle\mathcal{A}_1, \emptyset, \emptyset\rangle$ & $\mathcal{B} = \langle\emptyset, \mathcal{B}_2, \emptyset\rangle$.

Let us count the ways that they can be chosen.

Clearly, these two forms agree with the conditions in case 3 i.e., for the first kind, we have, $\mathcal{A}\cap\mathcal{B} = \langle\emptyset, \emptyset, \mathcal{A}_3\rangle = \mathcal{A}$ and $\mathcal{A}\cup\mathcal{B} = \langle\emptyset, \emptyset, \emptyset\rangle = \phi_{\mathcal{N}}$, and for the second kind $\mathcal{A}\cap\mathcal{B} = \langle\emptyset, \emptyset, \emptyset\rangle = \phi_{\mathcal{N}}$ and $\mathcal{A}\cup\mathcal{B} = \langle\mathcal{A}_1, \emptyset, \emptyset\rangle = \mathcal{A}$. Now, since $\emptyset\subset\mathcal{A}_3\subset\mathcal{X}$, $\emptyset\subset\mathcal{B}_2\subset\mathcal{X}$ such that $|\mathcal{A}_3| = |\mathcal{B}_2| = i, 1 \leq i \leq n-1$ so, $\mathcal{A}_3$ and $\mathcal{B}_2$ are chosen in $\binom{n}{i}$ different ways. This implies that $\mathcal{A}$ and $\mathcal{B}$ are chosen in $\binom{n}{i}$ different ways. Therefore, the number of ways in this kind is $\left\{\sum_{i=1}^{n-1}\binom{n}{i}\right\}\times\left\{\sum_{i=1}^{n-1}\binom{n}{i}\right\} = \left(\sum_{i=1}^{n-1}\binom{n}{i}\right)^2 = (2^n-2)^2$.

Similarly, the second kind is computed and is equal to $(2^n-2)^2$.

Finally, the desired number of ways is $2\left(2^n-2\right)^2$.

Proposition 5.4.

For a non-empty set $\mathcal{X}$ with $|\mathcal{X}| = n$, the number of NCrTs having 4-NCrOSs satisfying *condition in case 4 is obtained by the formula*

$$\sum_{i=1}^{n-2}\left\{\binom{n}{i}\left(2^{n-i}-2\right)\right\}+2\mathcal{T}_1+6\left(\mathcal{T}_2+\mathcal{T}_3\right),$$

where $\mathcal{T}_k=\sum_{i=k}^{n-1}\left\{\binom{n}{i}\mathcal{S}(i,k)\left(2^{n-i}-1\right)\right\}$, $k=1,2,3.$

Proof:

Let $\mathcal{A}=\langle\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3\rangle$ and $\mathcal{B}=\langle\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3\rangle$. Then to satisfy the condition ($\mathcal{A}\cap\mathcal{B}=\mathcal{A},\mathcal{A}\cup\mathcal{B}=\mathcal{A}$) or ($\mathcal{A}\cap\mathcal{B}=\mathcal{B},\mathcal{A}\cup\mathcal{B}=\mathcal{B}$), we must have, $\mathcal{A}_1=\mathcal{B}_1,\mathcal{A}_2\subset\mathcal{B}_2,\mathcal{A}_3=\mathcal{B}_3$ or $\mathcal{A}_1=\mathcal{B}_1,\mathcal{B}_2\subset\mathcal{A}_2,\mathcal{A}_3=\mathcal{B}_3$ respectively . Then, we obtain four forms

(1) $\mathcal{A}=\langle\emptyset,\mathcal{A}_2,\emptyset\rangle$ & $\mathcal{B}=\langle\emptyset,\mathcal{B}_2,\emptyset,\rangle$ such that $\mathcal{A}_2\subset\mathcal{B}_2$ or $\mathcal{B}_2\subset\mathcal{A}_2$,
(2) $\mathcal{A}=\langle\emptyset,\emptyset,\mathcal{A}_3\rangle$ & $\mathcal{B}=\langle\emptyset,\mathcal{B}_2,\mathcal{A}_3\rangle$ and $\mathcal{A}=\langle\mathcal{A}_1,\emptyset,\emptyset\rangle$ & $\mathcal{B}=\langle\mathcal{A}_1,\mathcal{B}_2,\emptyset\rangle$,
(3) $\mathcal{A}=\langle\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3\rangle$ & $\mathcal{B}=\langle\mathcal{A}_1,\mathcal{B}_2,\mathcal{A}_3\rangle$; exactly one of $\mathcal{A}_i,i=1,2,3$ is $\emptyset$ and $\mathcal{A}_2\subset\mathcal{B}_2$.
(4) $\mathcal{A}=\langle\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3\rangle$ & $\mathcal{B}=\langle\mathcal{A}_1,\mathcal{B}_2,\mathcal{A}_3\rangle$ such that all $\mathcal{A}_i,i=1,2,3$ are non-empty and $\mathcal{A}_2\subset\mathcal{B}_2$.

Let us count the ways that they can be chosen.

(1) Let $\mathcal{A}_2\subset\mathcal{B}_2$ and if $|\mathcal{A}_2|=i,1\leq i\leq n-2$ then $i<|\mathcal{B}_2|=k\leq n-1$. Therefore, $\mathcal{A}_2$ is chosen in $\binom{n}{i}$ ways and $\mathcal{B}_2$ is chosen in $\sum_{j=1}^{(n-i)-1}\binom{n-i}{j}=2^{n-i}-2$ different ways. Since, i varies from 1 to $n-2$, $\mathcal{A}_2$ and $\mathcal{B}_2$ are chosen in $\sum_{i=1}^{n-2}\left\{\binom{n}{i}\left(2^{n-i}-2\right)\right\}$ different ways. Hence, the neutrosophic crisp subsets $\mathcal{A}$ and $\mathcal{B}$ are chosen in $\sum_{i=1}^{n-2}\left\{\binom{n}{i}\left(2^{n-i}-2\right)\right\}$ different ways.
(2) Let $|\mathcal{A}_3|=i,1\leq i\leq n-1$ then $\mathcal{A}_3$ is chosen in $\binom{n}{i}$ different ways then it is partitioned into one block: this is done in $\mathcal{S}(i,1)$ different ways and hence $\mathcal{A}$. Next, in $\mathcal{B}$, $\mathcal{A}_3\cap\mathcal{B}_2=\emptyset$ and so, $\mathcal{B}_2$ is chosen from $n-i$ elements in $\sum_{j=1}^{n-i}\binom{n-i}{j}=2^{n-i}-1$ different ways and hence $\mathcal{B}$. Since i varies from 1 to $n-1$, we obtain $\sum_{i=1}^{n-1}\binom{n}{i}\mathcal{S}(i,1)(2^{n-i}-1)$ different ways for $\mathcal{A}$ and $\mathcal{B}$.

   Similarly, for $\mathcal{A}=\langle\mathcal{A}_1,\emptyset,\emptyset\rangle$ & $\mathcal{B}=\langle\mathcal{A}_1,\mathcal{B}_2,\emptyset\rangle$, we have $\sum_{i=1}^{n-1}\binom{n}{i}\mathcal{S}(i,1)(2^{n-i}-1)$ different ways.

(3) We have, $\mathcal{A}_1\cap\mathcal{A}_3=\emptyset$. If $|\mathcal{A}_1\cup\mathcal{A}_3|=i,2\leq i\leq n-1$ then $\mathcal{A}_1,\mathcal{A}_3$ is chosen in $\binom{n}{i}\mathcal{S}(i,2)$ different ways. Since $\mathcal{A}_1\cap\mathcal{B}_2=\mathcal{A}_3\cap\mathcal{B}_2=\emptyset$, so, $\mathcal{B}_2$ is chosen in $\binom{n-i}{j},1\leq j\leq n-i$ different ways. Therefore, $\mathcal{B}_2$ is chosen in $\sum_{j=1}^{n-i}\binom{n-i}{j}=2^{n-i}-1$ different ways. Together $\mathcal{A}$ and $\mathcal{B}$ is chosen in $\sum_{i=2}^{n-1}\binom{n}{i}\mathcal{S}(i,2)(2^{n-i}-1)$ different ways. It is known that we can arrange three element into three places in six different ways, so, $\mathcal{A}$ has six forms, as three components of $\mathcal{A}$ are the neutrosophic crisp subsets $\mathcal{A}_1,\mathcal{A}_3$ and $\emptyset$.

Hence, the total number of ways to choose $\mathcal{A}$ and $\mathcal{B}$ is

$$6\sum_{i=2}^{n-1}\binom{n}{i}\mathcal{S}(i,2)(2^{n-i}-1).$$

(4) We have, $\mathcal{A}_1\cap\mathcal{A}_2=\mathcal{A}_1\cap\mathcal{A}_3=\mathcal{A}_2\cap\mathcal{A}_3=\emptyset$. If $|\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{A}_3|=i, 3\le i\le n-1$ then $\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3$ are chosen in $\binom{n}{i}\mathcal{S}(i,3)$ different ways. Since $\mathcal{A}_1\cap\mathcal{B}_2=\mathcal{A}_3\cap\mathcal{B}_2=\emptyset$, so, $\mathcal{B}_2$ is chosen in $\binom{n-i}{j}, 1\le j\le n-i$ different ways. Therefore, $\mathcal{B}_2$ is chosen in $\sum_{j=1}^{n-i}\binom{n-i}{j}=2^{n-i}-1$ different ways. Together $\mathcal{A}$ and $\mathcal{B}$ are chosen in $\sum_{i=3}^{n-1}\binom{n}{i}\mathcal{S}(i,3)2^{n-i}-1$ different ways. It is known that we can arrange three elements into the three places in six different ways, so, $\mathcal{A}$ has 6 forms, as three components of $\mathcal{A}$ are different neutrosophic crisp subsets $\mathcal{A}_1,\mathcal{A}_2$ and $\mathcal{A}_3$.

Hence, the total number of ways to choose $\mathcal{A}$ and $\mathcal{B}$ is

$$6\sum_{i=3}^{n-1}\binom{n}{i}\mathcal{S}(i,3)(2^{n-i}-1).$$

Hence, we have the total

$$\sum_{i=1}^{n-2}\left\{\binom{n}{i}\left(2^{n-i}-2\right)\right\}+\sum_{i=1}^{n-1}\binom{n}{i}\mathcal{S}(i,1)(2^{n-i}-1)+$$

$$6\sum_{i=2}^{n-1}\binom{n}{i}\mathcal{S}(i,2)(2^{n-i}-1)+6\sum_{i=3}^{n-1}\binom{n}{i}\mathcal{S}(i,3)(2^{n-i}-1).$$

i.e.,

$$\sum_{i=1}^{n-2}\left\{\binom{n}{i}\left(2^{n-i}-2\right)\right\}+2\mathcal{T}_1+6\left(\mathcal{T}_2+\mathcal{T}_3\right),$$

where $\mathcal{T}_k=\sum_{i=k}^{n-1}\left\{\binom{n}{i}\mathcal{S}(i,k)\left(2^{n-i}-1\right)\right\}$, $k=1,2,3$. This formula gives the number of NCrTs having 4-NCrOSs satisfying condition in case 4.

Proposition 5.5.

For a non-empty set $\mathcal{X}$ with $|\mathcal{X}|=n$, the number of NCrTs having 4-NCrOSs satisfying condition in case 5 is obtained by the formula

$$\sum_{i=1}^{n-1}\binom{n}{i}\left[(2^n-2)+2\left\{\left(\sum_{j=1}^{i-1}\binom{i}{j}2^{n-j}\right)+(2^{n-i}-1)\right\}\right]+$$

$$2\sum_{i=1}^{n-1}\binom{n}{i}(2^{n-i}-1)+\sum_{i=1}^{n-1}\binom{n}{i}(2^{n-i}-1)^2+2\mathcal{T}_n+\sum_{i=0}^{n-2}\binom{n}{i}\mathcal{T}_{n-i},$$

where $\mathcal{T}_n=\sum_{i=1}^{n-2}\binom{n}{i}\left\{\sum_{k=1}^{n-(i+1)}\binom{n-i}{k}(2^{n-(i+k)}-1)\right\}+$
$\sum_{i=1}^{n-2}\binom{n}{i}\left[\sum_{j=1}^{n-i}\binom{n-i}{j}\left\{\sum_{k=1}^{j}\binom{j}{k}(2^{n-(i+j)}-1)\right\}\right]$
or

$\mathcal{T}_n = \sum_{i=1}^{n-2} \binom{n}{i} \left\{ \sum_{k=1}^{n-(i+1)} \binom{n-i}{k} (2^{n-(i+k)} - 1) \right\} +$
$\sum_{i=1}^{n-2} \binom{n}{i} \left[\sum_{j=1}^{n-i} \left\{ \sum_{k=j}^{n-i} \binom{n-i}{j} \binom{j}{k} \right\} (2^{n-(i+j)} - 1) \right].$

Proof:

Here, the second component must always match to satisfy the conditions in case 5.

For $\mathcal{A} = \langle \emptyset, \emptyset, \mathcal{A}_3 \rangle$ we can choose $\mathcal{B}$ in two forms which are $\mathcal{B} = \langle \mathcal{B}_1, \emptyset, \emptyset \rangle$ and $\mathcal{B} = \langle \mathcal{B}_1, \emptyset, \mathcal{B}_3 \rangle$ such that $\mathcal{B}_3 \subseteq \mathcal{A}_3$. For this kind of $\mathcal{A}$ we have $\binom{n}{i}$ different ways. For each $\mathcal{A}$, we can choose $\mathcal{B} = \langle \mathcal{B}_1, \emptyset, \emptyset \rangle$ in $2^n - 2$ different ways. Next if $\mathcal{B}_3 \subset \mathcal{A}_3$, say $|\mathcal{B}_3| = j < i = |\mathcal{A}_3|$, we can choose $\mathcal{B}$ in $\sum_{j=1}^{i-1} \binom{i}{j} 2^{n-j}$ different ways and if $\mathcal{B}_3 = \mathcal{A}_3$, say $|\mathcal{B}_3| = |\mathcal{A}_3| = i$, then $\mathcal{B}$ can be chosen in $2^{n-i} - 1$ different ways. Similarly, for $\mathcal{A} = \langle \mathcal{A}_1, \emptyset, \emptyset \rangle$, we have same number of choices for $\mathcal{B}$ satisfying conditions in case 5.

Therefore, in this part we have

$$\sum_{i=1}^{n-1} \binom{n}{i} \left[(2^n - 2) + 2 \left\{ \left(\sum_{j=1}^{i-1} \binom{i}{j} 2^{n-j} \right) + (2^{n-i} - 1) \right\} \right]$$

different ways.

For $\mathcal{A} = \langle \emptyset, \mathcal{A}_2, \emptyset \rangle$, we can choose $\mathcal{B} = \langle \emptyset, \mathcal{A}_2, \mathcal{B}_3 \rangle$ and $\mathcal{B} = \langle \mathcal{B}_1, \mathcal{A}_2, \emptyset \rangle$. Since $\mathcal{A}_2$ can be chosen in $\binom{n}{i}, i = 1, 2, \ldots, n-1$ different ways then $\mathcal{B}_3$ can be chosen in $2^{n-i} - 1$ different ways for each i and therefore, $\mathcal{B}$. As we have two forms of $\mathcal{B}$ and are symmetric, and i varies from 1 to $n-1$, we have the total $2 \sum_{i=1}^{n-1} \binom{n}{i} (2^{n-i} - 1)$.

For $\mathcal{A} = \langle \emptyset, \mathcal{A}_2, \mathcal{A}_3 \rangle$, we can choose $\mathcal{B} = \langle \mathcal{B}_1, \mathcal{A}_2, \mathcal{B}_3 \rangle, \mathcal{A}_3 \subseteq \mathcal{B}_3$ and $\mathcal{B}_1$ is any subset of $\mathcal{X}$ different from $\mathcal{A}_2$ and $\mathcal{B}_3$. Then $\mathcal{A}$ and $\mathcal{B}$ can be chosen in $\sum_{i=1}^{n-2} \binom{n}{i} \left\{ \sum_{k=1}^{n-(i+1)} \binom{n-i}{k} (2^{n-(i+k)} - 1) \right\} +$ $\sum_{i=1}^{n-2} \binom{n}{i} \left[\sum_{j=1}^{n-i} \binom{n-i}{j} \left\{ \sum_{k=1}^{j} \binom{j}{k} (2^{n-(i+j)} - 1) \right\} \right]$ different ways. Let us take $\sum_{i=1}^{n-2} \binom{n}{i} \left\{ \sum_{k=1}^{n-(i+1)} \binom{n-i}{k} (2^{n-(i+k)} - 1) \right\} +$ $\sum_{i=1}^{n-2} \binom{n}{i} \left[\sum_{j=1}^{n-i} \binom{n-i}{j} \left\{ \sum_{k=1}^{j} \binom{j}{k} (2^{n-(i+j)} - 1) \right\} \right] = \mathcal{T}_n$ for further use. Also, for $\mathcal{A} = \langle \mathcal{A}_1, \mathcal{A}_2, \emptyset \rangle$, we have equal number of choices as it is symmetric to $\mathcal{A} = \langle \emptyset, \mathcal{A}_2, \mathcal{A}_3 \rangle$. Hence, a total of $2\mathcal{T}_n$ different ways.

For $\mathcal{A} = \langle \emptyset, \mathcal{A}_2, \mathcal{A}_3 \rangle$, we can also choose $\mathcal{B} = \langle \mathcal{B}_1, \mathcal{A}_2, \emptyset \rangle$ which can be done in $\sum_{i=1}^{n-1} \binom{n}{i} (2^{n-i} - 1)^2$ different ways.

For $\mathcal{A} = \langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \rangle$, we can choose $\mathcal{B} = \langle \mathcal{B}_1, \mathcal{A}_2, \mathcal{B}_3 \rangle$ such that $\mathcal{A}_1 \subseteq \mathcal{B}_1$, $\mathcal{B}_3 \subseteq \mathcal{A}_3$ and $|\mathcal{A}_2| = i, 0 \leq i \leq n-2$. If $|\mathcal{A}_2| = 0$ i.e., $\mathcal{A}_2 = \emptyset$ then $\mathcal{B}$ can be chosen in $\binom{n}{0} \mathcal{T}_n$ different ways. Further, if $|\mathcal{A}_2| = 1$ then $\mathcal{B}$ can be chosen in $\binom{n}{1} \mathcal{T}_{n-1}$ different ways. Continuing in the similar way for $|\mathcal{A}_2| = n-2$, we have $\binom{n}{n-2} \mathcal{T}_{n-(n-2)}$ i.e., $\binom{n}{n-2} \mathcal{T}_2$ different ways. Thus, for $\mathcal{A} = \langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \rangle$, we can choose $\mathcal{B}$ in $\sum_{i=0}^{n-2} \binom{n}{i} \mathcal{T}_{n-i}$ different ways.

Hence, the number of NCrTs having 4-NCrOSs satisfying conditions in case 5 is obtained by the formula

$$\sum_{i=1}^{n-1}\binom{n}{i}\left[(2^n-2)+2\left\{\left(\sum_{j=1}^{i-1}\binom{i}{j}2^{n-j}\right)+(2^{n-i}-1)\right\}\right]+$$
$$2\sum_{i=1}^{n-1}\binom{n}{i}(2^{n-i}-1)+\sum_{i=1}^{n-1}\binom{n}{i}(2^{n-i}-1)^2+2\mathcal{T}_n+\sum_{i=0}^{n-2}\binom{n}{i}\mathcal{T}_{n-i}.$$

Example 5.6.

The follwing table gives the number of NCrTs having 4-NCrOSs for $\mathcal{X} \leq 5$. Suppose, $\mathcal{X} =$

TABLE 1. Number of NCrTSs having 4-NCrOSs on $\mathcal{X}$

$\mathcal{A}\cap\mathcal{B}, \mathcal{A}\cup\mathcal{B}$	Number of NCrTSs having 4-NCrOSs on $\mathcal{X}$				
	$\lvert\mathcal{X}\rvert = 1$	$\lvert\mathcal{X}\rvert = 2$	$\lvert\mathcal{X}\rvert = 3$	$\lvert\mathcal{X}\rvert = 4$	$\lvert\mathcal{X}\rvert = 5$
Case 1: $\mathcal{A}\cap\mathcal{B} = \phi_{\mathcal{N}}$, $\mathcal{A}\cup\mathcal{B} = \mathcal{X}_{\mathcal{N}}$	0	5	27	119	495
Case 2: $\mathcal{A}\cap\mathcal{B} = \phi_{\mathcal{N}}$, $\mathcal{A}\cup\mathcal{B} = \phi_{\mathcal{N}}$	0	5	30	137	570
Case 3: $\mathcal{A}\cap\mathcal{B} = \mathcal{A}$, $\mathcal{A}\cup\mathcal{B} = \phi_{\mathcal{N}}$	0	8	72	392	1800
Case 4: $\mathcal{A}\cap\mathcal{B} = \mathcal{A}$, $\mathcal{A}\cup\mathcal{B} = \mathcal{A}$	0	4	48	340	2040
Case 5: $\mathcal{A}\cap\mathcal{B} = \mathcal{A}$, $\mathcal{A}\cup\mathcal{B} = \mathcal{B}$	0	14	216	1958	15240
The total number of NCrTSs having 4-NCrOSs on $\mathcal{X}$	0	36	393	2946	20145

$\{a, b\}$ i.e., $|\mathcal{X}| = 2$, then from Table 1, we have, $\mathcal{T}_{\mathcal{C}r}(2, 4) = 36$. These are

For Case 1:

$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_3 = \langle\{a\}, \emptyset, \emptyset\rangle, A_6 = \langle\{b\}, \emptyset, \emptyset\},$
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_3 = \langle\{a\}, \emptyset, \emptyset, A_{12} = \langle\{b\}, \{a\}, \emptyset\rangle\},$
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_6 = \langle\{b\}, \emptyset, \emptyset\rangle, A_9 = \langle\{a\}, \{b\}, \emptyset\rangle\},$
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_8 = \langle\{a\}, \emptyset, \{b\}\rangle, A_{11} = \langle\{b\}, \emptyset, \{a\}\rangle\},$
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_9 = \langle\{a\}, \{b\}, \emptyset\rangle, A_{12} = \langle\{b\}, \{a\}, \emptyset\rangle\}.$

For Case 2:

$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_2 = \langle\emptyset, \{a\}, \emptyset\rangle, A_5 = \langle\emptyset, \{b\}, \emptyset\rangle\},$
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_1 = \langle\emptyset, \emptyset, \{a\}\rangle, A_4 = \langle\emptyset, \emptyset, \{b\}\rangle\},$
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_1 = \langle\emptyset, \emptyset, \{a\}\rangle, A_7 = \langle\emptyset, \{a\}, \{b\}\rangle\},$
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_4 = \langle\emptyset, \emptyset, \{b\}\rangle, A_{10} = \langle\emptyset, \{b\}, \{a\}\rangle\},$
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_7 = \langle\emptyset, \{a\}, \{b\}\rangle, A_{10} = \langle\emptyset, \{b\}, \{a\}\rangle\}.$

For Case 3:

$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_1 = \langle\emptyset, \emptyset, \{a\}\rangle, A_2 = \langle\emptyset, \{a\}, \emptyset\rangle\},$
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_1 = \langle\emptyset, \emptyset, \{a\}\rangle, A_5 = \langle\emptyset, \{b\}, \emptyset\rangle\},$

$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_4 = \langle \emptyset, \emptyset, \{b\}\rangle, A_2 = \langle \emptyset, \{a\}, \emptyset\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_4 = \langle \emptyset, \emptyset, \{b\}\rangle, A_5 = \langle \emptyset, \{b\}, \emptyset\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_3 = \langle \{a\}, \emptyset, \emptyset\rangle, A_2 = \langle \emptyset, \{a\}, \emptyset\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_3 = \langle \{a\}, \emptyset, \emptyset\rangle, A_5 = \langle \emptyset, \{b\}, \emptyset\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_6 = \langle \{b\}, \emptyset, \emptyset\rangle, A_2 = \langle \emptyset, \{a\}, \emptyset\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_6 = \langle \{b\}, \emptyset, \emptyset\rangle, A_5 = \langle \emptyset, \{b\}, \emptyset\rangle\}.$$

For Case 4:

$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_1 = \langle \emptyset, \emptyset, \{a\}\rangle, A_{10} = \langle \emptyset, \{b\}, \{a\}\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_3 = \langle \{a\}, \emptyset, \emptyset\rangle, A_9 = \langle \{a\}, \{b\}, \emptyset\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_4 = \langle \emptyset, \emptyset, \{b\}\rangle, A_7 = \langle \emptyset, \{a\}, \{b\}\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_6 = \langle \{b\}, \emptyset, \emptyset\rangle, A_{12} = \langle \{b\}, \{a\}, \emptyset\rangle\}.$$

For Case 5:

$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_1 = \langle \emptyset, \emptyset, \{a\}\rangle, A_3 = \langle \{a\}, \emptyset, \emptyset\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_1 = \langle \emptyset, \emptyset, \{a\}\rangle, A_6 = \langle \{b\}, \emptyset, \emptyset\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_4 = \langle \emptyset, \emptyset, \{b\}\rangle, A_3 = \{a\}, \emptyset, \emptyset\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_4 = \langle \emptyset, \emptyset, \{b\}\rangle, A_6 = \langle \{b\}, \emptyset, \emptyset\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_7 = \langle \emptyset, \{a\}, \{b\}\rangle, A_2 = \langle \emptyset, \{a\}, \emptyset\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_7 = \langle \emptyset, \{a\}, \{b\}\rangle, A_{12} = \langle \{b\}, \{a\}, \emptyset\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_{12} = \langle \{b\}, \{a\}, \emptyset\rangle, A_2 = \langle \emptyset, \{a\}, \emptyset\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_9 = \langle \{a\}, \{b\}, \emptyset\rangle, A_5 = \langle \emptyset, \{b\}, \emptyset\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_9 = \langle \{a\}, \{b\}, \emptyset\rangle, A_{10} = \langle \emptyset, \{b\}, \{a\}\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_{10} = \langle \emptyset, \{b\}, \{a\}\rangle, A_5 = \langle \emptyset, \{b\}, \emptyset\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_8 = \langle \{a\}, \emptyset, \{b\}\rangle, A_3 = \langle \{a\}, \emptyset, \emptyset\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_8 = \langle \{a\}, \emptyset, \{b\}\rangle, A_4 = \langle \emptyset, \emptyset, \{b\}\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_{11} = \langle \{b\}, \emptyset, \{a\}\rangle, A_1 = \langle \emptyset, \emptyset, \{a\}\rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_{11} = \langle \{b\}, \emptyset, \{a\}\rangle, A_6 = \langle \{b\}, \emptyset, \emptyset\rangle\}.$$

As a result, we have the total $\mathcal{T}_{\mathcal{C}r}(2, 4) = 36$.

Conclusion

This paper computes the formulae for the number of neutrosophic crisp topological spaces having 2, 3, and 4 open sets. This work is the foundation for computation of the formulae to find the number of neutrosophic crisp topological spaces.

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