NSS
Neutrosophic Sets and Systems, Vol. 73, 2024
University of New Mexico

Application of Neutrosophic Fourier Transform in solving Heat Equation and Integral Equation

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Abstract. Fourier transform is one of the oldest and well-known technique in the field of mathematics and engineering mathematical works. As the concept of uncertainty has been introduced in the mathematics, most of the works gravitate towards the use Neutrosophic set. So, it is also important to study the Fourier transform in the sense of Neutrosophic set. In this paper we have applied the Neutrosophic Fourier transform in solving heat equation, and integral equation. Where detailed examples are given to clarify each case.

Keywords: Fourier Transform, Neutrosophic Fourier Transform, Neoutrosophic Function.

Abreviation: F.T.= Fourier transform
N.F.T.=Neutrosophic Fourier Transform
F.S.T= Fourier sine transform
F.C.T.= Fourier cosine transform
N.R.N.= Neutrosophic real number
N.C.N.= Neutrosophic complex number

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1. Introduction:

Since the world is full of indeterminacy, the Neutrosophic found their place in contemporary research. Thus, Smarandache [1] put forward the definitions of Neutrosophic measure, Neutrosophic real number in standard form, and the considerations for the existence of a division of two Neutrosophic real numbers. He [2,3,4,5] also defined the Neutrosophic complex numbers in standard form and found the root index $n \geq 2$ of a Neutrosophic real and complex number, studying the concept of the Neutrosophic probability, the Neutrosophic statistics, and professor Smarandache [6,8] entered the concept of preliminary calculus of the differential and integral calculus. Madeleine [9] presented results on single-valued Neutrosophic (weak) polygroups. Edalatpanah [10] proposed a new direct algorithm to solve the neutrosophic linear programming where the variables and right-hand side are represented with triangular Neutrosophic numbers. Chakraborty [11, 12] used pentagonal Neutrosophic numbers in networking problems, and shortest-path problems.

A. Kharal[15] presents a method of multicriteria decision making using neutrosophic sets. A. A. Salama and F. Smarandache [16] used neutrosophic set to introduce new types of neutrosophic crisp sets with three types 1, 2, 3. D. Koundal, S. Gupta and S. Singh [17] demonstrates the use of neutrosophic theory in medical image denoising and segmentation, using which the performance is observed to be much better.

The concepts of Neutrosophic set have been used in different areas of Mathematics. Here we are using this concepts in Fourier integral and Fourier transform. Fourier integral represents a certain type of non periodic functions that are defined on either $(-\infty, \infty)$ or $(0, \infty)$. Fourier transform is a mathematical tool used to decompose a signal into its constituent frequency components. It breaks down signals into a combination of sines and cosines, which can be used to analyse the frequency content of a signal. Fourier analysis has been used in digital image and processing of image and for analysis of a single image into a two-dimensional wave form, and more recently has been used for magnetic resonance imaging, angiographic assessment, automated lung segmentation and image quality assessment and Mobile stethoscope [18]. Fourier transforms which is also used in frequency domain representation. Fourier analysis used as time series analysis proved its application in Quantum mechanics; Signal processing, Image Processing and filters, representation, Data Processing and Analysis and many more.

Fourier transforms are obviously very essential to conduct of Fourier spectroscopy, and that alone would justify its importance. Fourier transforms are very vital in other pursuits as well; such as electrical signal analysis, diffraction, optical testing, optical processing, imaging, holography, and also for remote sensing [13, 14]. Thus, knowledge of Fourier transforms can be a springboard to many other fields. The main idea behind Fourier transforms is that a function of direct time can be expressed as a complex valued function of reciprocal space, that is, frequency.

As we all are known that no things in this universe are certain or fully determined, there must need a detailed study of every phenomenon in the universe and try to find the uncertainty amount and then try to find out the actual value. For example, when we want to find the distance between two places, then different people will find different distances. Similarly, if we want to know the age of a person, then we cannot say the actual age of that person i.e., we cannot say accurately that the age of that person is 25 years. Because, during the time of giving the answer, that person's age has increased even by some seconds.

Fourier transform is a mathematical tool used to decompose a signal into its constituent frequency components. It breaks down signals into a combination of sines and cosines, which can be used to analyse the frequency content of a signal. During the transformation in Fourier transform, some informations are lost. It means that, when we extract the components of the signal by F.T., we can not determine the actual frequencies of each of the components of the signal. Thus, there is some indeterminacy, and thus we study F.T. by using the concept of neutrosophy. Here, the idea of neutrosophic set is used in some basics of Fourier integral. Neutrosophic Fourier transform will help to get the actual frequencies which constitutes a signal or a sound and can be filter out the unwnated frequncies in most appropriately.

The Fourier transform is beneficial in differential equations because it can reformulate them as problems which are easier to solve. In this work we shall discuss the applications of neutrosophic Fourier transform in some partial differential equations of neutrosophic sense. We have applied the Neutrosophic Fourier transform in solving heat equation. The solution of heat equation in one dimension represents the temperature of a position at a time of a thin rod or a wire. The concept of N.F.T. in solving heat equation will lead the result in another level.

We have studied neutrosophic fourier transform of the Derivatives of a neutrosophic function.

We have also used the N.F.T. in solving neutrosophic integral equatios. The neutrosophic integral equation is an equation in which an unknown neutrosophic function is to be found lies within in integral sign. It plays an pivotal role in various applications, including physics, engineering, and quantum mechanics, bridging differential equations and broader mathematical analysis. Solution of integral equations with the help of N.F.T. will give more advance results than the classical form.

2. **Preliminaries:**

In this part, definitions of N.R.N. and division of two N.R.N.s are discussed.

2.1. *Neutrosophic Real Number[4]:*

If a number that can be written in the form $p_n + q_n I$, where p_n, q_n are real numbers and I is an indeterminate number such that $I.0 = 0$ and $I^N = I$, for all natural number $\mathbf{N}$, is called N.R.N..

Here we denote the N.R.N. by w, and thus we can write $w = p_n + q_n I$ and it is knwon as the standard form of N.R.N..

2.2. *Division of two N.R.N.s[4]:*

Consider that w_1 and w_2 be two N.R.N.s where,
$w_1 = p_{n1} + q_{n1}I$ and $w_2 = p_{n2} + q_{n2}I$
Then to find $(p_{n1} + q_{n1}I) \div (p_{n2} + q_{n2}I)$, let us take

$$\frac{p_{n1} + q_{n1}I}{p_{n2} + q_{n2}I} = x + yI$$

where x and y are real numbers.

$$\Rightarrow p_{n1} + q_{n1}I = (p_{n2} + q_{n2}I)(x + yI)$$

$$\Rightarrow p_{n1} + q_{n1}I = p_{n2}x + (p_{n2}y + q_{n2}x + q_{n2}y)I$$

Equating both sides, we get

$$\begin{aligned} p_{n1} &= p_{n2}x \\ \Rightarrow x &= \frac{p_{n1}}{p_{n2}} \\ and\ q_{n1} &= p_{n2}y + q_{n2}x + q_{n2}y \\ \Rightarrow q_{n1} &= p_{n2}y + q_{n2}.\frac{p_{n1}}{p_{n2}} + q_{n2}y \\ \Rightarrow \frac{p_{n2}q_{n1} - p_{n1}q_{n2}}{p_{n2}} &= (p_{n2} + q_{n2})y \\ \Rightarrow y &= \frac{p_{n2}q_{n1} - p_{n1}q_{n2}}{p_{n2}(p_{n2} + q_{n2})} \end{aligned}$$

provided $p_{n2}(p_{n2} + q_{n2}) \neq 0$ or $p_{n2} \neq 0$ and $p_{n2} \neq -q_{n2}$

Thus

$$\frac{p_{n1} + q_{n1}I}{p_{n2} + q_{n2}I} = \frac{p_{n1}}{p_{n2}} + \frac{p_{n2}q_{n1} - p_{n1}q_{n2}}{p_{n2}(p_{n2} + q_{n2})}.I$$

3. **Application of N.F.T. to Heat Equation:**

The heat equation is a parabolic partial differential equation, describing the distribution of heat in a given space over time. The theory of heat equation was first developed by Joseph Fourier in 1822 for the purpose of modeling how a quantity such as heat diffuses through a given region. In this study we have applied the concept of neutrosophic set in solving the heat equation to minimize the vagueness or uncertainity occured in the result by the classical method.
The general form of one dimensional heat equation in neutrosophic sense is given by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Where $u = u(x, t, I)$ is the neutrosophic temperature in rod at position x at time t and the constant c^2 is called the thermal diffusivity of the rod.

The N.F.S.T. and N.F.C.T. can be applied when the range of the variable selected for exclusion is 0 to ∞. The choice of N.F.S.T. and N.F.C.T. is decided by the form of the boundary conditions at the lower limit of the variable selected for exclusion. In this connection note the following:

$$\begin{aligned}
\mathscr{F}_s\{\frac{\partial^2 u}{\partial x^2}\} &= \int_0^\infty \frac{\partial^2 u}{\partial x^2} sins(p_n + q_n I)xdx \\
&= \left[\frac{\partial u}{\partial x} sins(p_n + q_n I)x\right]_0^\infty - s(p_n + q_n I)\int_0^\infty \frac{\partial u}{\partial x} coss(p_n + q_n I)xdx \\
&= -s(p_n + q_n I)\int_0^\infty \frac{\partial u}{\partial x} coss(p_n + q_n I)xdx, \quad \text{if } \frac{\partial u}{\partial x} \to 0 \quad as \quad x \to \infty \\
&= -s(p_n + q_n I)\left\{[ucoss(p_n + q_n I)x]_0^\infty + s(p_n + q_n I)\int_0^\infty usins(p_n + q_n I)xdx\right\} \\
&= s(p_n + q_n I)(u)_{x=0} - s^2(p_n + q_n I)^2 \overline{u}_s, \quad u \to 0 \quad as \quad x \to \infty
\end{aligned}$$

$$Thus, \ \mathscr{F}_s\{\frac{\partial^2 u}{\partial x^2}\} = s(p_n + q_n I)u(0, t, I) - s^2(p_n + q_n I)^2 \overline{u}_s(s, t, I) \tag{1}$$

Where, $u(x, t, I)$ is a function of three variables x, t and I and $\overline{u}_s(s, t, I)$ is the N.F.S.T. of $u(x, t, I)$ with respect to x.

Again, by using the N.F.C.T., we get the following result:

$$\begin{aligned}
\mathscr{F}_c\{\frac{\partial^2 u}{\partial x^2}\} &= \int_0^\infty \frac{\partial^2 u}{\partial x^2} coss(p_n + q_n I)xdx \\
&= \left[\frac{\partial u}{\partial x} coss(p_n + q_n I)x\right]_0^\infty + s(p_n + q_n I)\int_0^\infty \frac{\partial u}{\partial x} sins(p_n + q_n I)xdx \\
&= -\left(\frac{\partial u}{\partial x}\right)_{x=0} + s(p_n + q_n I)\int_0^\infty \frac{\partial u}{\partial x} sins(p_n + q_n I)xdx, \quad \text{if } \frac{\partial u}{\partial x} \to 0 \quad as \quad x \to \infty \\
&= -\left(\frac{\partial u}{\partial x}\right)_{x=0} + s(p_n + q_n I)\left[[usins(p_n + q_n I)x]_0^\infty - s(p_n + q_n I)\int_0^\infty ucoss(p_n + q_n I)xdx\right] \\
&= -\left(\frac{\partial u}{\partial x}\right)_{x=0} - s^2(p_n + q_n I)^2 \overline{u}_c, \quad u \to 0 \quad as \quad x \to \infty
\end{aligned}$$

$$\text{Thus, } \mathscr{F}_c\{\frac{\partial^2 u}{\partial x^2}\} = -\left(\frac{\partial u}{\partial x}\right)_{x=0} - s^2(p_n + q_n I)^2 \overline{u}_c(s, t, I) \tag{2}$$

Where, $\overline{u}_c$(s,t,I)is the N.F.C.T. of $u(x, t, I)$ with respect to x.

Choice of N.F.S.T. & N.F.C.T. :

From (1) and (2), it can be noted that successful use of a N.F.S.T. in removing a term $\frac{\partial^2 u}{\partial x^2}$ from partial differential equation requires a knowledge of u(s,t,I) and u(0,t,I) while the use of N.F.C.T. for the same purpose requires $u_x(s, t, I)$ and $u_x(0, t, I)$.

4. **Solved Examples based on Application of N.F.S.T. & N.F.C.T. to Heat Equation:**

Using the N.F.S.T. and N.F.C.T., we can solve the boundary value problems of heat equation. The result so obtained will give more accurate result. We have discussed some problems and solutions of them.

Example 1. Let us consider the following equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ x > 0, t > 0 \text{ subject to conditions}$$
$$u(0,t,I) = 0,$$
$$u(x,0,I) = \begin{cases} 1+I & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Since $u(0,t,I) i.e., (u)_{x=0}$ is given, taking the N.F.S.T. of both sides of the given partial differential equation, we have

$$\int_0^\infty \frac{\partial u}{\partial t} sins(p_n + q_n I)xdx = \int_0^\infty \frac{\partial^2 u}{\partial t^2} sins(p_n + q_n I)xdx$$
$$\Rightarrow \frac{d}{dt}\int_0^\infty usins(p_n + q_n I)xdx = \left[\frac{\partial u}{\partial x} sins(p_n + q_n I)x\right]_0^\infty - s(p_n + q_n I)\int_0^\infty \frac{\partial u}{\partial x} coss(p_n + q_n I)xdx$$
$$\Rightarrow \frac{d}{dt}(\overline{u}_s) = -s(p_n + q_n I)\int_0^\infty \frac{\partial u}{\partial x} coss(p_n + q_n I)xdx, \quad \text{if } \frac{\partial u}{\partial x} \to 0 \quad as \quad x \to \infty$$
$$[Here \quad \overline{u}_s(s,t,I) \quad is \quad the \quad N.F.S.T. of \ u(x,t,I)]$$
$$\Rightarrow \frac{d}{dt}(\overline{u}_s) = -s(p_n + q_n I)\left[[ucoss(p_n + q_n I)x]_0^\infty + s(p_n + q_n I)\int_0^\infty usins(p_n + q_n I)xdx\right]$$
$$\Rightarrow \frac{d}{dt}(\overline{u}_s) = s(p_n + q_n I)u(0,t,I) - s^2(p_n + q_n I)^2\overline{u}_s, \quad u \to 0 \quad as \quad x \to \infty$$
$$\Rightarrow \frac{d}{dt}(\overline{u}_s) = -s^2(p_n + q_n I)^2\overline{u}_s,\ \{as\ u(0,t,I) = 0$$
$$\Rightarrow \frac{1}{\overline{u}_s} d\overline{u}_s = -s^2(p_n + q_n I)^2 dt$$

Whose solution is given by $\overline{u}_s(s,t,I) = ce^{-s^2(p_n+q_n I)^2 t}$ (3)

where, c is an arbitrary constant.

Putting $t = 0$ in (1), we get $c = \overline{u}_s(s,0,I)$

Therefore,

$$c=\overline{u}_s(s,0,I)=\int_0^\infty u(x,0,I)sins(p_n+q_nI)xdx$$
$$=\int_0^1 u(x,0,I)sins(p_n+q_nI)xdx+\int_1^\infty u(x,0,I)sins(p_n+q_nI)xdx$$
$$=\int_0^1 (1+I)sins(p_n+q_nI)xdx,\ using\ the\ given\ value\ of\ u(x,0,I)$$
$$=-\left[\frac{1+I}{s(p_n+q_nI)}.coss(p_n+q_nI)x\right]_{x=0}^2$$
$$=-\frac{1+I}{s(p_n+q_nI)}.[coss(p_n+q_nI)-1]$$

$$=\frac{(1+I)\{1-coss(p_n+q_nI)\}}{s(p_n+q_nI)}$$
$$(1)\Rightarrow\overline{u}_s(s,t,I)=\frac{(1+I)\{1-coss(p_n+q_nI)\}}{s(p_n+q_nI)}.e^{-s^2(p_n+q_nI)^2t}$$

Now, taking the inverse N.F.S.T., we get

$$u(x,t,I)=\frac{2}{\pi}\int_0^\infty\frac{(1+I)\{1-coss(p_n+q_nI)\}}{s(p_n+q_nI)}.e^{-s^2(p_n+q_nI)^2t}sins(p_n+q_nI)xds$$
$$\Rightarrow u(x,t,I)=\frac{2(1+I)}{\pi(p_n+q_nI)}\int_0^\infty\frac{\{1-coss(p_n+q_nI)\}}{s}.e^{-s^2(p_n+q_nI)^2t}sins(p_n+q_nI)xds$$

Which is the required solution.

Example 2. Let $\frac{\partial u}{\partial t}=2\frac{\partial^2 u}{\partial x^2}$, if $u(0,t,I)=0$, $u(x,0,I)=e^{-x+I}$, $x>0$, $u(x,t,I)$ is bounded where $x>0,t>0$.

here,

$$\frac{\partial u}{\partial t}=2\frac{\partial^2 u}{\partial x^2},\quad x>0,t>0 \tag{4}$$

with the boundary conditions:

$$u(0,t,I)=0,\qquad u(x,t,I)\quad is\quad bounded \tag{5}$$

and initial condition: $u(x,0,I)=e^{-x+I}\quad x>0$ (6)

Since, $u(0,t,I)$ is given, we take the N.F.S.T. of both sides of (1) and obtain

$$\int_0^\infty\frac{\partial u}{\partial t}sins(p_n+q_nI)xdx=2\int_0^\infty\frac{\partial^2 u}{\partial t^2}sins(p_n+q_nI)xdx$$
$$\Rightarrow\frac{d}{dt}\int_0^\infty usins(p_n+q_nI)xdx=2\left[\frac{\partial u}{\partial x}sins(p_n+q_nI)x\right]_0^\infty-2\int_0^\infty\frac{\partial u}{\partial x}s(p_n+q_nI)coss(p_n+q_nI)xdx$$
$$\Rightarrow\frac{d}{dt}(\overline{u}_s)=-2s(p_n+q_nI)\int_0^\infty\frac{\partial u}{\partial x}coss(p_n+q_nI)xdx,\quad \text{if }\frac{\partial u}{\partial x}\to 0\quad as\quad x\to\infty$$

$[Here\quad \overline{u}_s(s,t,I)\quad is\quad the\quad N.F.S.T.\ of\ u(x,t,I)]$

$$\Rightarrow \frac{d\overline{u}_s}{dt} = -2s(p_n+q_nI)\left[[u(x,t,I)coss(p_n+q_nI)x]_0^\infty + s(p_n+q_nI)\int_0^\infty u(x,t,I)sins(p_n+q_nI)xdx\right]$$

$$\Rightarrow \frac{d\overline{u}_s}{dt} = -2s(p_n+q_nI)\left[0 - u(0,t,I) + s(p_n+q_nI)\int_0^\infty u(x,t,I)sins(p_n+q_nI)xdx\right]$$

$$\text{Since} \quad u(x,t,I) \to 0 \quad as \quad x \to \infty$$

$$\Rightarrow \frac{d\overline{u}_s}{dt} = -2s^2(p_n+q_nI)^2\overline{u}_s, \quad as \quad u(0,t,I)=0$$

$$\Rightarrow \frac{1}{\overline{u}_s}d\overline{u}_s = -2s^2(p_n+q_nI)^2dt$$

$$Integrating, \quad log\overline{u}_s - logc = -2s^2(p_n+q_nI)^2t$$

$$\Rightarrow \overline{u}_s(s,t,I) = ce^{-2s^2(p_n+q_nI)^2t} \tag{7}$$

Taking the N.F.S.T. of both sides of (3), we get

$$\int_0^\infty u(x,0,I)sins(p_n+q_nI)xdx = \int_0^\infty e^{-x+I}sins(p_n+q_nI)xdx$$

$$\Rightarrow \overline{u}_s(s,0,I) = \left[\frac{e^{-x+I}}{1+s^2(p_n+q_nI)^2}\{-sins(p_n+q_nI)x - s(p_n+q_nI)coss(p_n+q_nI)x\}\right]_0^\infty$$

$$\Rightarrow \overline{u}_s(s,0,I) = \frac{e^I.s(p_n+q_nI)}{1+s^2(p_n+q_nI)^2} \tag{8}$$

Putting $t=0$ in (4), and using (5), we get

$$c = \overline{u}_s(s,0,I) = \frac{e^I.s(p_n+q_nI)}{1+s^2(p_n+q_nI)^2}$$

Therefore from (4), we get

$$\overline{u}_s(s,t,I) = \frac{e^I.s(p_n+q_nI)}{1+s^2(p_n+q_nI)^2}.e^{-2s^2(p_n+q_nI)^2t}$$

Now, taking the inverse N.F.S.T., we get

$$u(x,t,I) = \frac{2}{\pi}\int_0^\infty \frac{e^I.s(p_n+q_nI)}{1+s^2(p_n+q_nI)^2}.e^{-2s^2(p_n+q_nI)^2t}.sins(p_n+q_nI)xds$$

This equation gives the solution of the given heat equation.

Example 3. Let us consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x>0, t>0 \quad \text{subject} \quad \text{to} \quad \text{conditions}$$

$$u(0,t,I) = 0, u(x,t,I) \quad is \quad bounded \quad and$$

$$u(x,0,I) = \begin{cases} x+I & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Since, $u(x,t,I)$ is given, taking the N.F.C.T. of both sides of the given partial differential equation, we have

$$\int_0^\infty \frac{\partial u}{\partial t} coss(p_n+q_nI)xdx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} coss(p_n+q_nI)xdx$$
$$\Rightarrow \frac{d}{dt}\int_0^\infty ucoss(p_n+q_nI)xdx = \left[\frac{\partial u}{\partial x}coss(p_n+q_nI)x\right] + s(p_n+q_nI)\int_0^\infty \frac{\partial u}{\partial x} sins(p_n+q_nI)xdx$$
$$\Rightarrow \frac{d}{dt}\overline{u}_c = -\left(\frac{\partial u}{\partial x}\right)_{x=0} + s(p_n+q_nI)\{[usins(p_n+q_nI)x]_0^\infty - s(p_n+q_nI)\int_0^\infty ucoss(p_n+q_nI)xdx\}$$

[where, we have assumed that $\frac{\partial u}{\partial x} \to 0$ as $x\to\infty$ and $\overline{u}_c(s,t,I)$ is the N.F.C.T. of $u(x,t,I)$.]

$$\begin{aligned}
&\Rightarrow \frac{d\overline{u}_c}{dt} = -s^2(p_n+q_nI)^2\overline{u}_c,\ \{\text{ since, } u_x(0,t,I)=0 \ and \ u\to 0 \ as \ x\to\infty\}\\
&\Rightarrow \frac{1}{\overline{u}_c}d\overline{u}_c = s^2(p_n+q_nI)^2dt\\
&Integrating, \quad log\overline{u}_c - logA = -s^2(p_n+q_nI)^2t\\
&\Rightarrow \overline{u}_c(s,t,I) = Ae^{-s^2(p_n+q_nI)^2t}
\end{aligned} \tag{9}$$

Putting $t=0$ in (1) we get $A=\overline{u}_c(s,0,I)$.

Hence by the definition of N.F.C.T., we get

$$\begin{aligned}
A =& \overline{u}_c(s,0,I)\\
=& \int_0^\infty u(x,0,I)coss(p_n+q_nI)xdx\\
=& \int_0^1 u(x,0,I)coss(p_n+q_nI)xdx + \int_1^\infty u(x,0,I)coss(p_n+q_nI)xdx\\
=& \int_0^1 (x+I)coss(p_n+q_nI)xdx, \quad \text{By using the given value of u(x,0,I)}\\
=& \left[(x+I).\frac{sins(p_n+q_nI)x}{s(p_n+q_nI)}\right]_0^1 - \int_0^1 1.\frac{sins(p_n+q_nI)x}{s(p_n+q_nI)}dx\\
=& \frac{Isins(p_n+q_nI)}{s(p_n+q_nI)} + \left[\frac{coss(p_n+q_nI)x}{s^2(p_n+q_nI)^2}\right]_0^1
\end{aligned}$$

Thus,

$$A = \frac{Isins(p_n+q_nI)}{s(p_n+q_nI)} + \frac{coss(p_n+q_nI)-1}{s^2(p_n+q_nI)^2}$$

Therefore, from equation (1) we get,

$$\overline{u}_c(s,t,I) = \left[\frac{Isins(p_n+q_nI)}{s(p_n+q_nI)} + \frac{coss(p_n+q_nI)-1}{s^2(p_n+q_nI)^2}\right].e^{-s^2(p_n+q_nI)^2t}$$

. Now taking the inverse N.F.C.T., we get

$$u(x,t,I) = \frac{2}{\pi}\int_0^\infty \left[\frac{Isins(p_n+q_nI)}{s(p_n+q_nI)} + \frac{coss(p_n+q_nI)-1}{s^2(p_n+q_nI)^2}\right].e^{-s^2(p_n+q_nI)^2t}coss(p_n+q_nI)xds$$

5. **Convolution or Falting of Neutrosophic Functions:**

The convolution is a mathematical tool for combining two signals to form a third signal. Therefore, in signals and systems, the convolution is very important because it relates the input signal and the impulse response of the system to produce the output signal from the system. Here, we have been discussed the convolution in terms of neutrosophic functions and the convolution of neutrosophic functions is defined as follows:

Definition: The convolution of two neutrosophic functions $\mathscr{F}(x,I)$ and $\mathscr{G}(x,I)$, where $-\infty < x < \infty$ and I is an indterminate number, is denoted and defined as

$$\mathscr{F} * \mathscr{G} = \int_{-\infty}^{\infty} \mathscr{F}(u,I)\mathscr{G}(x-u,I)du$$

or,

$$\mathscr{F} * \mathscr{G} = \int_{-\infty}^{\infty} \mathscr{G}(u,I)\mathscr{F}(x-u,I)du$$

5.1. *The Neutrosophic Convolution Theorem or Neutrosophic Falting Theorem:*

The N.F.T. of the convolutionof two neutrosophic functions $\mathscr{F}(x,I)$ and $\mathscr{G}(x,I)$ is the product of the N.F.T.s of $\mathscr{F}(x,I)$ and $\mathscr{G}(x,I)$ i.e.,

$$\mathscr{F}\{\mathscr{F} * \mathscr{G}\} = \mathscr{F}\{\mathscr{F}(x,I)\}\mathscr{F}\{\mathscr{G}(x,I)\}$$

Proof:

$$
\begin{aligned}
L.H.S. =&\mathscr{F}\left\{\int_{-\infty}^{\infty}\mathscr{F}(u,I)\mathscr{G}(x-u,I)du\right\},\ \text{By definition of neutrosphic Convolution}\\
=&\int_{-\infty}^{\infty}e^{is(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty}\mathscr{F}(u,I)\mathscr{G}(x-u,I)du\right\}dx,\ \text{By definition of N.F.T.}\\
=&\int_{-\infty}^{\infty}\mathscr{F}(u,I)\left\{-\int_{\infty}^{\infty}e^{is(p_n+q_nI)x}\mathscr{G}(x-u,I)dx\right\}du,\ \text{Changing the order of integration}\\
=&\int_{-\infty}^{\infty}\mathscr{F}(u,I)\left\{\int_{-\infty}^{\infty}e^{is(p_n+q_nI)(u+v)}\mathscr{G}(v,I)dv\right\}du,\ \text{Putting } x-u=v\ so\ that\ dx=dv\\
=&\int_{-\infty}^{\infty}e^{is(p_n+q_nI)u}\mathscr{F}(u,I)\left\{\int_{-\infty}^{\infty}e^{is(p_n+q_nI)v}\mathscr{G}(v,I)dv\right\}du\\
=&\int_{-\infty}^{\infty}e^{is(p_n+q_nI)u}\mathscr{F}(u,I)\left\{\int_{-\infty}^{\infty}e^{is(p_n+q_nI)x}\mathscr{G}(x,I)dx\right\}du,\ \text{Changing the variable from v to x}\\
=&\int_{-\infty}^{\infty}e^{is(p_n+q_nI)u}\mathscr{F}(u,I)\mathscr{F}\{\mathscr{G}(x,I)\}du\\
=&\int_{-\infty}^{\infty}e^{is(p_n+q_nI)u}\mathscr{F}(x,I)dx\mathscr{F}\{\mathscr{G}(x,I)\},\ \text{Changing the variable from u to x}\\
=&\mathscr{F}\{\mathscr{F}(x,I)\}\mathscr{F}\{\mathscr{G}(x,I)\}
\end{aligned}
$$

The convolution theorem is certainly useful in solving differential equations, but it can also help us to solve integral equations, equations involving an integral of the unknown function, integro-differential equations, those involving both a derivative and an integral of the unknown function. As in this case, we have discussed the theorem by using the neutrosophic functions, it will give more reliable result than the classical form.

6. Neutrosophic Fourier Transforms of the Derivatives of a Neutrosophic function (The Derivative Theorem for N.F.T.):

Theorem 1: If $\mathscr{F}\{\mathscr{F}(x,I)\}=f(s,I)$, then $\mathscr{F}\{\mathscr{F}'(x,I)\}=-is(p_n+q_nI)f(s,I)$.
Or, If $\mathscr{F}(x,I)$ has the N.F.T. $f(s,I)$, then the N.F.T. of $\mathscr{F}'(x,I)$, the derivative of $\mathscr{F}(x,I)$ is $-is(p_n+q_nI)f(s,I)$

Proof:

$$
\begin{aligned}
\mathscr{F}\{\mathscr{F}'(x,I)\}=&\int_{-\infty}^{\infty}e^{is(p_n+q_nI)x}\mathscr{F}'(x,I)dx\\
=&e^{is(p_n+q_nI)x}\int_{-\infty}^{\infty}\mathscr{F}'(x,I)dx-\int_{-\infty}^{\infty}\{\frac{d}{dx}e^{is(p_n+q_nI)x}\int\mathscr{F}'(x,I)dx\}dx\\
=&\left[e^{is(p_n+q_nI)x}\mathscr{F}(x,I)\right]_{-\infty}^{\infty}-is(p_n+q_nI)\int_{-\infty}^{\infty}e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx\\
=&(0-0)-is(p_n+q_nI)f(s,I),\ \{\text{Since } \mathscr{F}(x,I)\to 0 \text{ as } x\to\pm\infty\}\\
=&-is(p_n+q_nI)f(s,I).
\end{aligned}
$$

Theorem 2 : $\mathscr{F}\{\mathscr{F}^n(x,I)\} = -is(p_n+q_nI)^n\mathscr{F}\{\mathscr{F}(x,I)\}$ i.e., the N.F.T. of the function $\frac{d^n\mathscr{F}(x,I)}{dx^n}$ is $\{-is(p_n+q_nI)\}^n$ times the N.F.T. of the function $\mathscr{F}(x,I)$, provided that first (n-1) derivatives of $\mathscr{F}(x,I)$ vanish as $x \to \pm\infty$.

Proof: From the theorem 1, we get

$$\begin{aligned}\mathscr{F}\{\mathscr{F}'(x,I)\} &= -\,is(p_n+q_nI)f(s,I)\\ &= -\,is(p_n+q_nI)\mathscr{F}\{\mathscr{F}(x,I)\}\end{aligned} \tag{10}$$

Therefore,

$$\begin{aligned}\mathscr{F}\{\mathscr{F}''(x,I)\} &= \int_{-\infty}^{\infty} e^{-is(p_n+q_nI)x}\mathscr{F}''(x,I)dx\\ &= \left[e^{is(p_n+q_nI)x}\mathscr{F}'(x,I)\right]_{-\infty}^{\infty} - is(p_n+q_nI)\int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}'(x,I)dx\\ &=(0-0) - is(p_n+q_nI)\int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}'(x,I)dx,\ \{\text{Since } \mathscr{F}'(x,I)\to 0 \text{ as } x\to\pm\infty\ \}\\ &=-is(p_n+q_nI) - is(p_n+q_nI)\mathscr{F}\{\mathscr{F}(x,I)\}\\ &=\{-is(p_n+q_nI)\}^2\mathscr{F}\{\mathscr{F}(x,I)\}\end{aligned}$$

Proceeding in this way, we get

$$\mathscr{F}\{\mathscr{F}^n(x,I)\} = \{-is(p_n+q_nI)\}^n\mathscr{F}\{\mathscr{F}(x,I)\}.$$

Theorem 3: To show that $\mathscr{F}\{\int\mathscr{F}(x,I)dx\} = \frac{f(s,I)}{-is(p_n+q_nI)}$, where $f(s,I)$ is N.F.T. of $\mathscr{F}(x,I)$.

Proof:

$$\begin{aligned}\mathscr{F}\left\{\int\mathscr{F}(x,I)dx\right\} &= \int_{-\infty}^{\infty}\left\{e^{is(p_n+q_nI)x}\int\mathscr{F}(x,I)dx\right\}dx\\ &= \left[\int\mathscr{F}(x,I)dx.\frac{e^{is(p_n+q_nI)x}}{is(p_n+q_nI)}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left\{\frac{d}{dx}\int\mathscr{F}(x,I)dx.\int e^{is(p_n+q_nI)x}dx\right\}dx\\ &= (0-0) - \int_{-\infty}^{\infty}\mathscr{F}(x,I).\frac{e^{is(p_n+q_nI)x}}{is(p_n+q_nI)}dx,\ \{\text{Since, } \mathscr{F}(x,I)\to 0\ as\ x\to\pm\infty\}\\ &= -\frac{1}{is(p_n+q_nI)}\int_{-\infty}^{\infty}\mathscr{F}(x,I)e^{is(p_n+q_nI)x}dx\\ &= \frac{f(s,I)}{-is(p_n+q_nI)}\end{aligned}$$

7. **Solution of Integral Equation using N.F.T.**

Before going to the solutions of Integral Equations with the help of N.F.T., let us discuss neutrosophic integral equations.

Neutrosophic Integral Equation:

Definition: A neutrosophic integral equation is an equation in which an unknown neutrosophic function appears under one or more integral signs.

For example, for $a \leq x \leq b$, $a \leq t \leq b$, the equations

$$\int_a^b K(x,t,I)y(t,I)dt = f(x,I) \tag{11}$$

$$y(x,I) - \lambda \int_a^b K(x,t,I)y(t,I)dt = f(x,I) \tag{12}$$

and

$$y(x,I) = \int_a^b K(x,t,I)[y(t,I)]^2 dt \tag{13}$$

Where $y(x,I)$ is the unknown neutrosophic function while $f(x,I)$ and $K(x,t,I)$ are known functions and λ, a and b are constants, are all neutrosophic integral equations.

Linear and Non-Linear Neutrosophic Integral Equations:

Definition: A neutrosophic integral equation is called linear if only linear operators are performed in it upon the unknown neutrosophic function.

A neutrosophic integral equation which is not linear is called as non-linear neutrosophic integral equation.

In this study, we shall discuss only the linear neutrosophic integral equations. In the above examples of neutrosophic integral equations, the equations (1) and (2) are linear while the equation (3) in non-linear neutrosophic integral equation.

The most general type of linear neutrosophic integral equation is defined as

$$g(x,I)y(x,I) = f(x,I) + \lambda \int_a K(x,t,I)y(t,I)dt \tag{14}$$

Where the upper limit may be either variable x or fixed. The neutrosophic functions f, g and K are known functions while y is to be determined; λ is a non-zero real or complex parameter. The function $K(x,t,I)$ is known as the neutrosophic kernel of the neutrosophic integral equation.

Remark: If $g(x,I) \neq 0$, equation (4) is known as linear neutrosophic integral equation of the third kind.

When $g(x,I) = 0$, the equation (4) reduces to

$$f(x,I) + \lambda \int_a K(x,t,I)y(t,I)dt = 0 \tag{15}$$

which is known as the linear neutrosophic integral equation of the first kind.

Again when $g(x,I) = 1$, then equation (1) reduces to

$$y(x,I) = f(x,I) + \lambda \int_a K(x,t,I)y(t,I)dt \tag{16}$$

which is konwn as the linear neutrosophic integral equation of the second kind.

In this work, we shall study the solutions of neutrosophic integral equations with the help of neutrosophic fourier transform (N.F.T.) of the first and second kind only.

The linear neutrosophic integral equation can be divided into two different types depending on the limits of the integrations.

(1) Fredholm Neutrosophic Integral Equation:

Definition 1: A linear neutrosophic integral equation of the form

$$f(x,I)+\lambda\int_a^b K(x,t,I)y(t,I)dt=0 \tag{17}$$

where a, b are both constatnts, $f(x,I)$ and $K(x,t,I)$ are known functions while $y(x,I)$ is unknown function and λ is a non-zero real or complex parameter is called Fredholm neutrosophic integral equation of the first kind.

Definition 2: A linear neutrosophic integral equation of the form

$$y(x,I)=f(x,I)+\lambda\int_a^b K(x,t,I)y(t,I)dt \tag{18}$$

is known as Fredholm neutrosophic integral equation of the second kind.

Definition 3: A linear neutrosophic integral equation of the form

$$y(x,I)=\lambda\int_a^b K(x,t,I)y(t,I)dt \tag{19}$$

is known as a homogeneous Fredholm neutrosphic integral equation of the second kind.

Volterra Neutrosophic Integral Equation:

Definition 1: A linear neutrosophic integral equation of the form

$$f(x,I)+\lambda\int_a^x K(x,t,I)y(t,I)dt=0 \tag{20}$$

is known as Volterra neutrosphic integral equation of the first kind. In this case, the upper limit of the integration is a variable.

Definition 2: A linear neutrosophic integral equation of the form

$$y(x,I)=f(x,I)+\lambda\int_a^x K(x,t,I)y(t,I)dt \tag{21}$$

is known as Volterra neutrosophic integral equation of the second kind.

Definition 3: A linear neutrosophic integral equation of the form

$$y(x,I)=\lambda\int_a^x K(x,t,I)y(t,I)dt \tag{22}$$

is known as a homogeneous Volterra neutrosphic integral equation of the second kind.

Now, we shall solve some neutrosophic integral equations with the help of N.F.T.

Exmaple 1: Let us consider the neutrosophic integral equation

$$\int_0^\infty F(x,I)coss(p_n+q_nI)xdx=\begin{cases}1-s+I, \ 0\le s\le 1\\ 0, \ s>1\end{cases}$$

Here,

$$\int_0^\infty F(x,I)coss(p_n+q_nI)xdx=f_c^N(s,I) \tag{23}$$

where,

$$f_c^N(s,I) = \begin{cases} 1 - s + I, \ 0 \leq s \leq 1 \\ 0, \ s > 1 \end{cases} \tag{24}$$

Then by the definition of $f_c^n(s,I)$ is N.F.C.T. of $F(x,I)$. In order to solve the given neutrosophic integral equation, we must obtain value of $F(x,I)$.

Now, by inversion formula for Neutrosophic Fourier cosine transform, we get

$$\begin{aligned} F(x,I) &= F_c^{-1}\{f_c(s,I)\} \\ &= \frac{2}{\pi}\int_0^\infty f_c^N(s,I)coss(p_n+q_nI)xds \\ &= \frac{2}{\pi}\left[\int_0^1 f_c^N(s,I)coss(p_n+q_nI)xds + \int_1^\infty f_c^N(s,I)coss(p_n+q_nI)xds\right] \\ &= \frac{2}{\pi}\left[\int_0^\infty (1-s+I)coss(p_n+q_nI)xds + 0\right], \ by \end{aligned} \tag{25}$$

$$\begin{aligned} &= \frac{2}{\pi}\left\{\left[(1-s+I)\frac{sins(p_n+q_nI)x}{(p_n+q_nI)x}\right]_0^1 - \int_0^1 (-1)\frac{sins(p_n+q_nI)x}{(p_n+q_nI)x}ds\right\} \\ &= \frac{2}{\pi}.I.\frac{sin(p_n+q_nI)x}{(p_n+q_nI)x} + \frac{2}{\pi(p_n+q_nI)x}\left[\frac{-coss(p_n+q_nI)x}{(p_n+q_nI)x}\right]_0^1 \\ &= \frac{2}{\pi(p_n+q_nI)x}\left[Isin(p_n+q_nI)x + \frac{1-cos(p_n+q_nI)x}{(p_n+q_nI)x}\right] \\ &= \frac{2}{\pi(p_n+q_nI)^2x^2}[I(p_n+q_nI)xsin(p_n+q_nI)x + 1 - cos(p_n+q_nI)x] \end{aligned}$$

Example 2: In the integral equation

$$\int_0^\infty f(x,I)coss(p_n+q_nI)xdx = e^{-s+I}$$

In this case we assume,

$$\int_0^\infty f(x,I)coss(p_n+q_nI)xdx = f_c(s,I) \tag{26}$$

where,

$$f_c(s,I) = e^{-s+I} \tag{27}$$

Then by definition, $f_c(s,I)$ is neutrosophic fourier cosine transform of $f(x,I)$. In order to solve the given neutrosophic integral equation, we must obtain the value of $f(x,I)$.

Now, by inversion formula for N.F.C.T., we get

$$\begin{aligned} f(x,I) &= F_c^{-1}\{f_c(s,I)\} \\ &= \frac{2}{\pi}\int_0^\infty f_c(s,I)coss(p_n+q_nI)xds \\ &= \frac{2}{\pi}\int_0^\infty e^{-s+I}coss(p_n+q_nI)xds \\ &= \frac{2}{\pi}\left[\frac{e^{-s+I}}{1+(p_n+q_nI)^2x^2}\{-coss(p_n+q_nI)x+(p_n+q_nI)xsins(p_n+q_nI)xds\}\right]_0^\infty \\ &= \frac{2}{\pi\{1+(p_n+q_nI)^2x^2\}}[0-e^I(-1+0)] \\ &= \frac{2e^I}{\pi\{1+(p_n+q_nI)^2x^2\}} \end{aligned}$$

Example 3: For the integral equation

$$\int_0^\infty F(x,I)sins(p_n+q_nI)xdx = \begin{cases} 1+I, \ 0\le s\le 1 \\ 2-I, \ 1\le s\le 2 \\ 0, \ s>2 \end{cases}$$

Here, we assume,

$$\int_0^\infty F(x,I)sins(p_n+q_nI)xdx = f_s(s,I) \tag{28}$$

where,

$$f_s(s,I) = \begin{cases} 1+I, \ 0\le s\le 1 \\ 2-I, \ 1\le s\le 2 \\ 0, \ s>2 \end{cases} \tag{29}$$

Then by definition, $f_s(s,I)$ is the N.F.S.T. of $F(x,I)$.

Now, by inversion formula for N.F.S.T., we have

$$\begin{aligned} F(x,I) &= F_s^1\{f_s(s,I)\} \\ &= \frac{2}{\pi}\int_0^\infty f_s(s,I)sins(p_n+q_nI)xds \\ &= \frac{2}{\pi}[\int_0^1 f_s(s,I)sins(p_n+q_nI)xds+\int_1^2 f_s(s,I)sins(p_n+q_nI)xds \\ &+ \int_2^\infty f_s(s,I)sins(p_n+q_nI)xds] \\ &= \frac{2}{\pi}\left[\int_0^1(1+I)sins(p_n+q_nI)xds+\int_1^2(2-I)sins(p_n+q_nI)xds+0\right] \end{aligned}$$

$$= \frac{2}{\pi}\left\{(1+I)\left[\frac{-coss(p_n+q_nI)x}{(p_n+q_nI)x}\right]_0^1 + (2-I)\left[\frac{-coss(p_n+q_nI)x}{(p_n+q_nI)x}\right]_1^2\right\}$$

$$= \frac{2}{\pi}\left[-\frac{(1+I)}{(p_n+q_nI)x}\{cos(p_n+q_nI)x-1\} - \frac{(2-I)}{(p_n+q_nI)x}\{cos2(p_n+q_nI)x - cos(p_n+q_nI)x\}\right]$$

$$= \frac{2}{\pi(p_n+q_nI)x}[cos(p_n+q_nI)x - 2Icos(p_n+q_nI)x - 2cos2(p_n+q_nI)x + I]$$

Conclusion:

In this article, we have discussed applications of Neutrosophic Fourier Transform in solving two important partial different equations. We have studied the solution Heat Equation and Integral Equations by using the Neutrosophic Fourier Transform and given detailed examples.It is observed that the application of Neutrosophic Fourier Transform in these equations give new results which are more fruitful than the results obtained in classical form.

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Received: July 10, 2024. Accepted: September 03, 2024